Generalized Global Symmetries

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Contents

CHAPTER 1

Introduction

Understanding the symmetries of a theory is a powerful tool in the study of its physical properties. For example, Noether's theorem tells us how symmetries imply the existence of conserved quantities in classical systems. In quantum mechanical systems, symmetries of the Hamiltonian help us organize the spectrum of the theory through quantum numbers and constrain transition amplitudes via selection rules. In the context of quantum field theory (QFT), identifying the global symmetries of a system is fundamental in order to understand its dynamics and other aspects which we will now review.

First of all, let us extend the known results mentioned above for classical and quantum systems to the framework of QFT. Here global symmetries will impose selection rules on correlation functions by requiring that

$$\begin{aligned}\langle \mathcal{O}_1(\phi(x_1)) \cdots \mathcal{O}_n(\phi(x_n)) \rangle &\xrightarrow{g} \langle \mathcal{O}_1(\phi'(x_1)) \cdots \mathcal{O}_n(\phi'(x_n)) \rangle \\ &= \langle \mathcal{O}_1(\phi(x_1)) \cdots \mathcal{O}_n(\phi(x_n)) \rangle \,. \end{aligned} \tag{1.1}$$

This implies that correlation functions of operators with non-zero total charge under the global symmetry must vanish. On the other hand, it can also be proven that current conservation holds within correlation functions, except at points x_i where other operators might be inserted. This relation is known as the *Ward-Takahashi identity* [1].

It is important to note that relations such as Eq. 1.1 are *non-perturbative* statements — in this sense, global symmetries are one of the few tools that allow us to study the dynamics of strongly-coupled systems. For instance, another important result which follows from symmetry arguments is *Goldstone's theorem* — it states that for every spontaneously broken continuous symmetry, there exists a massless particle in the theory (known as a Nambu-Goldstone boson). Again, Goldstone's theorem is a non-perturbative statement which provides us with valuable information about the spectrum of the theory at low energies.

The reader should now be convinced that symmetries take a fundamental role in the study of quantum field theory. For this reason, one might find useful to consider generalizations of the concept of symmetry — if they were to exist, they would certainly help us to obtain further information on the dynamics and general physical properties of QFTs. In fact, such "generalized symmetries" have been and are being studied extensively in the physics community — some examples are non-invertible symmetries [2, 3] and higher-group symmetries [4, 5]. In this dissertation, we will focus on the concept of *higher-form symmetries*, which was studied systematically by Gaiotto, Kapustin, Seiberg and Willett in their seminal 2015 paper *"Generalized Global Symmetries"* [6]. It turns out that, just as in the case of ordinary global symmetries, higher-form symmetries can also be gauged, they can be spontaneously broken, and they can present 't Hooft anomalies which obstruct their gauging. Furthermore, they also provide us with selection rules and new guiding principles for the study of a theory's dynamics.

The present document is organized as follows. In Chapter 2 we review the usual concept of symmetry in quantum field theory, rephrased in a language which will make it easy to generalize to the higher-form case later on. Additionally, we will describe the less familiar concept of gauging discrete symmetries. In Chapter 3 we start developing the ideas behind higher-form symmetries. We will also understand how these symmetries can be gauged, in analogy to the case of ordinary global symmetries. We then proceed to give two important examples of higher-form symmetries in the context of 4d gauge theory. Finally, we briefly discuss some of the applications of these new symmetries on the general study of quantum field theories. We will closely follow [6] along the whole of this document, while trying to give thorough explanations on the ideas developed in the paper. Furthermore, we also aim to provide the reader with the necessary background knowledge to understand these new developments. In particular, Appendix A discusses magnetic monopoles and their corresponding line operators in gauge theory — these concepts will be important to understand our examples of higher-form symmetries in said context.

CHAPTER 2

Ordinary symmetries in QFT

We consider our field theories to be defined in general d-dimensional spacetime manifolds X. Most of the time we will equip them with extra structure such as a fixed (pseudo-)Riemannian metric g and its corresponding Levi-Civita connection ∇, etc.

2.1 Global symmetries

Let's consider a field theory described by an action $S[\phi_i, \partial_\mu \phi_i, \dots]$, which depends on various fields $\phi_i(x)$ and a finite number of their derivatives. We can implement the action of a group G on the fields by placing them in non-trivial representations $R(G)$ of the group. For example, under the action of $g \in G$ the field ϕ would transform as

$$\phi(x)^a \xrightarrow{g} R(g)^a{}_b \phi(x)^b. \tag{2.1}$$

We say that the theory has a **global** G **symmetry** if the action S is invariant under the above field transformations, that is,

$$S[\phi_i, \dots] \xrightarrow{g} S[R_i(g)\phi_i, \dots] = S[\phi_i, \dots], \forall g \in G. \tag{2.2}$$

The symmetry is called *global* because we apply the same transformation g on the fields at every point in spacetime.

In the case that G is a continuous group, Noether's theorem provides us with a set of conserved currents $j^{\mu,a}$, $a = 1, \dots \dim(G)$, which satisfy $\nabla_\mu j^{\mu,a} = 0$. For our purposes, it will be more useful to talk in the language of differential forms, so that j is a 1-form equal to $j_\mu dx^\mu$. Then the conservation equation is expressed as[1]

$$d(*j) = 0. \tag{2.3}$$

[1]For a quick review on differential forms and the basic operations between them, see [7].

For a given Noether current j we can define the corresponding conserved charge as

$$Q(\mathcal{M}^{d-1}) = \int_{\mathcal{M}^{d-1}} dx^{d-1}\sqrt{|g|}j^0 = \int_{\mathcal{M}^{d-1}} *j, \tag{2.4}$$

where $\mathcal{M}^{d-1}$ is a spatial slice (assuming spacetime can be divided into space and time as in $X = \mathcal{M}^{d-1} \times \mathbb{R}$) and g is the determinant of the metric evaluated along $\mathcal{M}^{d-1}$. Equation 2.3 ensures that Q is a constant in time.

We could be more general and define the charge $Q(M^{d-1})$ by integrating $*j$ over any codimension-1 submanifold M^{d-1}, not just spatial slices. The condition that Q is conserved is now generalized to the property that Q is *topological*, i.e. it is invariant under small deformations of M^{d-1}. For example, consider deforming M^{d-1} slightly to $\widetilde{M}^{d-1}$ as shown in Figure 2.1. Then

$$\begin{aligned}\int_{\widetilde{M}^{d-1}} *j - \int_{M^{d-1}} *j &= \int_{\partial N^d} *j \\ &= \int_{N^d} d(*j) = 0,\end{aligned} \tag{2.5}$$

where we used Stokes' theorem to go from the first to the second line. This shows that $Q(M^{d-1})$ only depends on M^{d-1} up to continuous deformations. In particular, if spacetime is of the form $X = \mathcal{M}^{d-1} \times \mathbb{R}$, a spatial slice $\mathcal{M}^{d-1}$ at time t can be continuously deformed into another spatial slice at a different time t'. In that way, one recovers the condition that Q is time-independent.

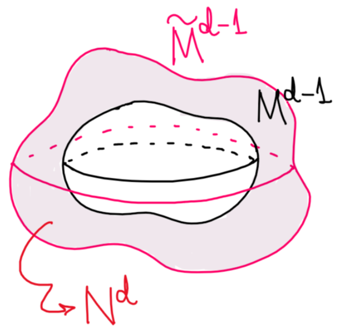

Figure 2.1: After deforming M^{d-1} continuously to $\widetilde{M}^{d-1}$, the two manifolds enclose a volume N^d such that $\partial N^d = \widetilde{M}^{d-1} - M^{d-1}$.

So far we have described global symmetries at the classical level. At the quantum level, symmetries of the action are not necessarily symmetries of the full theory. Instead, we should search for symmetries of the whole partition function:

$$\mathcal{Z} = \int [\mathcal{D}\phi]\, e^{iS[\phi_i, \partial_\mu \phi_i, \ldots]} \tag{2.6}$$

On one side, the requirement that S is invariant under symmetry transformations is relaxed to the requirement that S is invariant modulo shifts of 2π. Moreover, one should also check that the measure $[\mathcal{D}\phi]$ is invariant under 2.1. If these two conditions are satisfied, G is guaranteed to be a true global symmetry of the theory. If not, we say that the symmetry is *anomalous* as it does not survive the quantization process.

From the canonical quantization point of view, the action of a compact group G on the field operators is implemented by symmetry operators U_g defined at spatial slices $\mathcal{M}^{d-1}$. These operators constitute a unitary representation of the group[2]:

$$U_g(\mathcal{M}^{d-1})U_{g'}(\mathcal{M}^{d-1}) = U_{gg'}(\mathcal{M}^{d-1}), \tag{2.7}$$

with $g, g' \in G$. The symmetry operators act on (charged) fields by applying the transformations given in Eq. 2.1, that is,

$$U_g(\mathcal{M}^{d-1})\phi(x)^a U_g^{-1}(\mathcal{M}^{d-1}) = R(g)^a{}_b \phi(x)^b. \tag{2.8}$$

Alternatively, we can think of the U_g operators as implementing the symmetry transformation on the Hilbert space of the theory $\mathcal{H}(\mathcal{M}^{d-1})$ at a fixed time.

Just as we did for the charges, we can consider our symmetry operators to be defined on general codimension-1 submanifolds M^{d-1}. In that case, the U_g operators act by applying a symmetry transformation on the Hilbert space associated to M^{d-1} [6]. These operators will also have the property of being topological — in the case of continuous G, it is clear from their definition as exponentials of the charges:

$$U_g(M^{d-1}) = \exp\left(i\alpha^a Q^a(M^{d-1})\right). \tag{2.9}$$

Here $a = 1, \ldots, \dim(G)$, $\{\alpha^a\}$ labels the group element via the exponential map from the Lie algebra $\mathfrak{g} \to G$, and Q^a is the charge associated to the a-th generator of $\mathfrak{g}$. The action of the symmetry operators on the fields is also generalized from Eq. 2.8 to

$$U_g(S^{d-1})\phi(x)^a = R(g)^a{}_b \phi(x)^b, \tag{2.10}$$

where S^{d-1} is a $(d-1)$-dimensional sphere, surrounding the point x where the charged field operator is inserted[3]. Following the physics literature, we will refer to the U_g operators as *symmetry operators* or *defects* interchangeably.

Relation 2.10 admits a pictorial interpretation as follows (see Fig. 2.2). The operator $U_g(S^{d-1})$ is unaffected if we deform the sphere S^{d-1} slightly, as mentioned before. However, if we deform it enough so that it crosses the charged operator $\phi(x)$, the latter will pick up a factor of $R(g)$ while the sphere becomes "empty". We can then contract the sphere to a point, in which case the U_g operator becomes trivial.

We can further relate this picture back to the canonical quantization relation in Eq. 2.8. We can imagine stretching the sphere along the spatial directions all the way to infinity (or to ∂M)

[2] In general, one may allow for *projective representations* of the symmetry group. Such is the case if the symmetry has a 't Hooft anomaly, for example.

[3] We should note that relations such as Eq. 2.10 are interpreted as being true inside correlation functions.

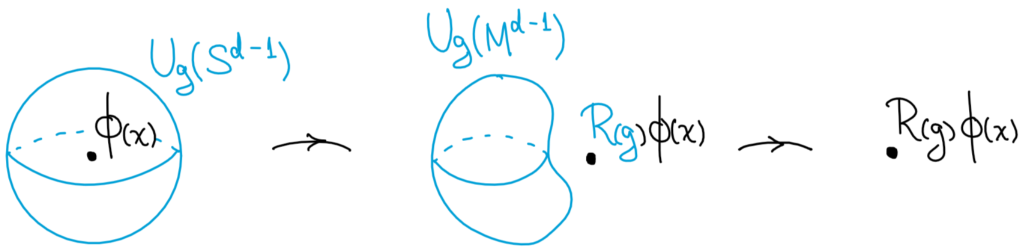

Figure 2.2: Graphical representation of the action of symmetry operators on charged fields.

as in Fig. 2.3, such that it becomes a pair of two spatial slices enclosing the charged operator $\phi(x)$. Inside correlation functions, the order of the three operators is determined by the usual time-ordering, thus

$$U_g(S^{d-1})\phi(x) \longrightarrow U_g(\mathcal{M}^{d-1}|_{x^0+\epsilon})\phi(x)U_g^{-1}(\mathcal{M}^{d-1}|_{x^0-\epsilon}), \tag{2.11}$$

where $\mathcal{M}^{d-1}|_t$ denotes an spatial slice at a fixed time t. Note that the operator defined at $t = x^0 - \epsilon$ picks up an inverse because the orientation of the sphere induces a negative orientation on it. Equating this relation with Eq. 2.10 and taking the limit $\epsilon \to 0$ gives us back the action of the symmetry operators in the canonical quantization formalism.

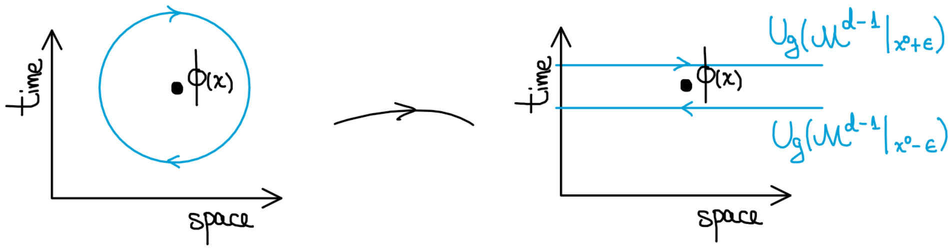

Figure 2.3: By using the topological properties of the U_g defects, we can recover their action on charged fields in the canonical quantization formalism.

If the symmetry group is discrete, we do not have any conserved currents or charges. However, we can still implement the symmetry transformation by topological operators $U_g(M^{d-1})$ satisfying relations 2.7 and 2.10 [8]. In fact, we will take these properties as our starting point for defining global symmetries in QFT.

Definition 2.1.1 (*Global symmetry*)**.** A quantum field theory is said to have a global G symmetry if there exist a set of operators $U_g(M^{d-1})$, supported on codimension-1 manifolds M^{d-1} and labelled by elements of G, which satisfy the following properties:

(1) They depend topologically on M^{d-1}.

(2) They satisfy the product rule 2.7, i.e. they constitute a unitary representation of G.

Later in Chapter 3, this new point of view will lead us to a direct generalization of the notion of symmetry in quantum field theory.

2.2 Gauge symmetries

We can consider *gauging* a global symmetry by promoting its action on the fields to be local, i.e. we consider "gauge transformations" $u(x) \in G$ of the form

$$\phi(x)^a \xrightarrow{u(x)} R(u(x))^a{}_b \phi(x)^b. \tag{2.12}$$

We usually require the map $u : X \to G$ to be continuous. This implies that we have some notion of topology on G, which in fact exists for continuous groups[4]. The set of all such maps u forms the group of all gauge transformations $\mathcal{G}$ on X.

It is well known that gauging a continuous symmetry introduces a gauge field or connection 1-form $A(x) = A_\mu(x) dx^\mu$ which takes values in the Lie algebra $\mathfrak{g}$ of G. Once we have a connection, we are able to compare values of the fields $\phi(x)$ at different points in spacetime by means of the covariant derivative $D_\mu = \nabla_\mu - iA_\mu$ (or, in the language of differential forms, the exterior covariant derivative $d_D = d - iA \wedge -$). Under gauge transformations, the gauge field transforms as

$$A_\mu(x) \xrightarrow{u(x)} A'_\mu(x) = u(x) A_\mu(x) u^{-1}(x) + iu(x)\partial_\mu u^{-1}(x). \tag{2.13}$$

The mathematical details of gauge theory are subtler when one works in general curved spacetimes, and are best described in the language of fiber bundles. The reader can find a quick exposition to this topic in [9], while more thorough reviews can be found in [10] and [11]. For the purposes of this work, it is worth noting that in general one can not define a connection over all of spacetime, but only locally by "patching" spacetime through an open cover. At intersections of the open cover, the distinct connections are related by gauge transformations. Roughly, we will refer to a particular collection of patches plus gauge transformations at the intersections as a *gauge bundle*.

2.2.1 Background vs. quantum gauge fields

Before considering the gauge field as a quantum entity, we could take A to be a fixed, classical *background* gauge field. In that case the partition function will depend on the choice of A, so that $\mathcal{Z} \equiv \mathcal{Z}[A]$. The background field A should couple to matter fields via the Noether currents of the G symmetry — this ensures, at least at the classical level, that the theory is invariant under gauge transformations[5]. However, it may happen that the partition function is *not* gauge invariant after the introduction of such couplings. If that is the case, we say that the symmetry has a *'t Hooft anomaly* — we can think about it as an obstruction to gauging the global symmetry at the quantum level.

We could also consider adding extra gauge invariant terms to the action which only depend on A. Such terms will define a "topological action" $S_{\text{top}}[A]$ if they only depend on the particular

[4] Remember that Lie groups are smooth manifolds, so in particular they are topological spaces.

[5] More generally one should make the replacement $d \to d_D$, which may also introduce coupling terms of higher order in A.

topology of the gauge bundle over which A is defined, but not on the specific connection chosen on it. Said action will not affect the equations of motion of the matter fields as it simply adds a global phase factor to the partition function. However, it will become relevant once we decide to gauge the global symmetry in the quantum theory. A well-known example is the *theta term* in 4d Yang-Mills theory:

$$S_{\text{top}}[A] = \frac{\theta}{8\pi^2} \int_X \text{Tr}(F \wedge F), \tag{2.14}$$

where $F = dA + iA \wedge A$ is the curvature 2-form defined by A. From the definition it is explicit that $S_{\text{top}}[A]$ does not depend on the particular choice of connection A, but only on the topology of the gauge bundle through the curvature 2-form. Furthermore, it can be proven that $\int_X \frac{1}{8\pi^2} \text{Tr}(F \wedge F)$ takes integer values — this quantity is known as the *instanton number* or the *2nd Chern class* of the gauge bundle. In order for the partition function to be single valued, we require θ to be defined modulo shifts of 2π.

The final step is promoting A to be a quantum field. For it to be dynamical, we introduce kinetic terms to the action such as $\int_X -\frac{1}{2}\text{Tr}(F \wedge *F)$. We are also allowed to add topological terms to the action as discussed above. Finally, we must integrate over all possible connections A in the path integral. More accurately, since we consider different configurations of the gauge field as physically equivalent if they are connected by a gauge transformation, we should only integrate over *equivalence classes* of A modulo $\mathcal{G}$. As an important remark, one should note that this also includes summing over all the different, inequivalent gauge bundles that can be built over X (also known as "topological sectors"). Thus, the full partition function becomes

$$\mathcal{Z} = \sum_{\substack{\text{sum over} \\ \text{gauge bundles}}} \int_{\mathcal{A}/\mathcal{G}} [\mathcal{D}A] \int [\mathcal{D}\phi] e^{iS[\phi_i, A, \ldots]}, \tag{2.15}$$

where we schematically represent the space of all connections modulo $\mathcal{G}$ as $\mathcal{A}/\mathcal{G}$. Given this expression, we can interpret $S_{\text{top}}[A]$ as giving different weights to different topological sectors inside the path integral.

2.2.2 Discrete gauge symmetries

The mathematical details of gauging continuous symmetries are well-known and were described in last section. We are now faced with the question of what gauging a discrete symmetry means. Essentially, we would like to introduce the analog concept of a connection A for the discrete symmetry.

Consider, for simplicity, that the symmetry group G is abelian. If the symmetry group was continuous, transporting the field ϕ around a closed loop $\gamma \subset X$ gives the field an extra phase $R(g)$, given by the *holonomy* of the gauge field around γ:

$$\text{Hol}(\gamma) = e^{i\int_\gamma A} = g. \tag{2.16}$$

We will take this as our starting point for defining A in the discrete case. To each loop γ in spacetime, assign a value $g \in G$ which corresponds to the holonomy of the gauge field around it. Moreover, this gauge field should satisfy the following properties:

(1) If we consider G as being equipped with the discrete topology[6], deforming γ continuously can not abruptly change the value of its holonomy. Therefore, the holonomy map should define a group homomorphism from equivalence classes of 1-cycles $[\gamma]$ to G.

(2) We consider two gauge fields as equivalent if they are related by a gauge transformation — these transformations do not change the values of the holonomies around loops.

Formally, the above two requirements amount to the statement that equivalence classes of gauge fields are described by elements in the *first cohomology group of X with coefficients in G*: $H^1(X, G)$. While item 1 means that we are only considering closed gauge fields which satisfy $dA = 0$ (i.e. cocycles), item 2 means that they are only defined up to exact terms (i.e. modulo coboundaries). The reader can find a short review of the basic notions of algebraic topology in [12], and a more thorough treatment in [13].

Another way to understand the above construction is through the language of symmetry operators introduced in Section 2.1. Instead of having a background gauge field, suppose that we are transporting a charged field ϕ along a 1-cycle γ which crosses a symmetry operator $U_g(M^{d-1})$ (see Fig. 2.4).

Figure 2.4: Transporting a charged field ϕ along a loop γ, crossing a symmetry defect $U_g(M^{d-1})$.

After ϕ traverses the whole loop, observe that this is analogous to having a background gauge field A with holonomy g around γ. In fact, the relation between these two concepts can be formalized by means of the Poincaré duality theorem [13]:

$$H^1(X, G) \simeq H_{d-1}(X, G). \tag{2.17}$$

Elements of the homology group $H_{d-1}(X, G)$ are equivalence classes of $(d-1)$-cycles modded out by boundaries — they can be interpreted as networks of $(d-1)$-dimensional submanifolds built out of non-trivial $(d-1)$-cycles in X, each one of them assigned an element $g \in G$. As the reader can recognize, these objects are nothing but networks of symmetry operators U_g, as shown in Figure 2.5.

[6]The discrete topology on a set X is defined by considering all of its subsets as open.

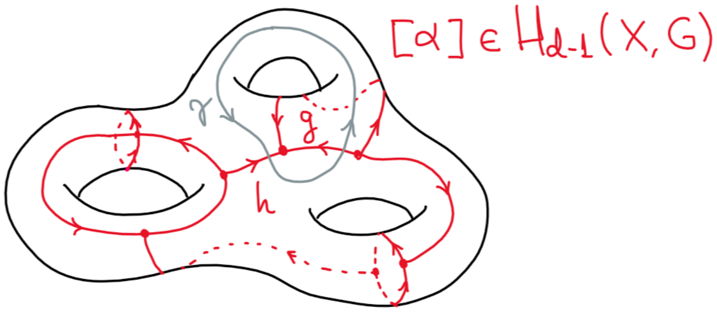

Figure 2.5: A background gauge field for a discrete symmetry group can be implemented by a network of defects $[\alpha] \in H_{d-1}(X, G)$. For example, the 1-cycle γ drawn in the picture has holonomy $= gh \in G$.

At each junction, the defects should satisfy a set of consistency conditions known as the *cocycle conditions*. If we trace a small loop around a junction between n symmetry operators U_{g_i}, the holonomy should be trivial (because the loop is contractible) — therefore,

$$g_1^{\pm 1} g_2^{\pm 1} \ldots g_n^{\pm 1} = 1 \tag{2.18}$$

at each junction. The ± 1 signs on each element will depend on the particular orientations of the defects (e.g. in-going versus out-going lines in 2 dimensions). Finally, note that a particular choice of connection A may define many topologically distinct networks of defects. Observe first that junctions can always be resolved as sets of trivalent vertices (see Fig. 2.6). However, there are in general multiple ways to resolve such junctions. Since all possible resolutions define the same physical configuration, we consider the resulting networks as equivalent under this "gauge freedom".

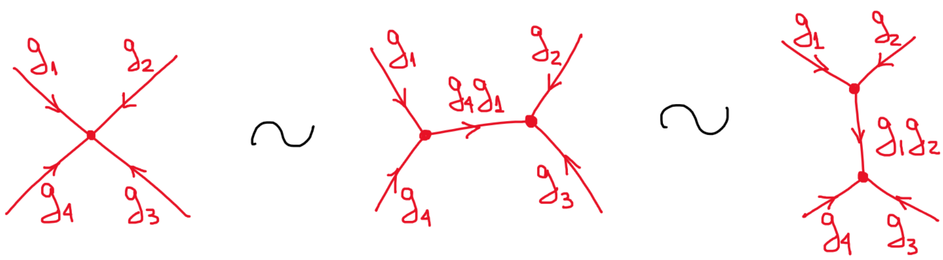

Figure 2.6: A quadruple junction of defects can be resolved as a set of trivalent vertices in two different ways.

The idea of gauging a discrete symmetry becomes clear now. In order to gauge a discrete global symmetry, we must sum over all inequivalent configurations of the background gauge field on X — these are given by elements in $H^1(X, G)$. By virtue of the Poincaré duality, this is equivalent to summing over all possible inequivalent networks of defects over X — these are given by elements in $H_{d-1}(X, G)$.

To further clarify the concepts described in this section, we shall give an example of this procedure in the context of 2d conformal field theory, following [6].

Example: Discrete (abelian) orbifolds in 2d CFT

Consider a 2d CFT defined on the torus $T^2 \simeq S^1 \times S^1$ with a global symmetry group G, which is discrete and abelian. In order to gauge it, we must some over all possible *twisted sectors*, i.e. over all possible choices for the background gauge field $A \in H^1(T^2, G)$. From what we discussed before, this is equivalent to summing over all possible networks of U_g defects $\in H_1(T^2, G)$ — such networks are specified by assigning a pair of elements $g_1, g_2 \in G$ to the two non-trivial cycles γ_1, γ_2 shown in Figure 2.7.

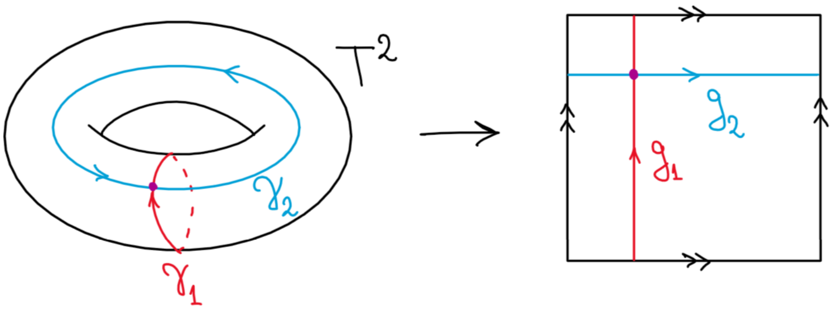

Figure 2.7: A general network of defects over a torus is defined by two elements $g_1, g_2 \in G$.

The partition function of the orbifolded (gauged) theory will then be the sum of partition functions for each twisted sector. More generally, we can weight each twisted sector by an extra phase factor, known as a choice of *discrete torsion* [14], which should only depend on the connection defined by g_1 and g_2:

$$\mathcal{Z}_G = \sum_{g_1, g_2} \epsilon(g_1, g_2) \mathcal{Z}_{g_1, g_2}. \tag{2.19}$$

Observe that a choice of discrete torsion is analogous to a choice of topological action $S_{\text{top}}[A]$ in the continuous case.

CHAPTER 3

Higher-form symmetries

3.1 q-form symmetries

As we saw in Chapter 2, a theory with an ordinary global symmetry G contains the following set of elements:

- Charged operators ϕ(x), which transform under representations of the symmetry group G. They are supported on 0-dimensional manifolds (points).
- Topological defects $U_g(M^{d-1})$, supported on codimension-1 manifolds M^{d-1}.
- If the symmetry is continuous, we also have a set of conserved 1-form currents j.

The latter point leads us to consider the following generalization. Suppose that we can find a conserved $(q+1)$-form "current" j in our theory, i.e.

$$\nabla_\alpha j^{\alpha\mu_1\mu_2\cdots\mu_q} = 0 \iff d(*j) = 0. \tag{3.1}$$

We can integrate $*j$ over codimension-$(q+1)$ manifolds M^{d-q-1} — the result will depend topologically on M^{d-q-1} as a consequence of Eq. 3.1. This suggests that, in analogy to the case of ordinary global symmetries, we could find a set of topological defects built out of these higher-dimensional currents. The corresponding defects should in turn depend on lower-dimensional manifolds.

Let us formalize this idea in the following definition, which arises as a direct generalization of Definition 2.1.1.

Definition 3.1.1 (*q-form symmetry*)**.** A quantum field theory is said to have a global G *q-form symmetry* if there exist a set of operators $U_g(M^{d-q-1})$, supported on codimension-$(q+1)$ manifolds M^{d-q-1} and labelled by elements of G, which satisfy the following properties:

(1) They depend topologically on M^{d-q-1}.

(2) They satisfy the product rule

$$U_g(M^{d-q-1})U_{g'}(M^{d-q-1}) = U_{gg'}(M^{d-q-1}). \tag{3.2}$$

Ordinary global symmetries follow as a particular case of this definition for $q = 0$. As such, we will refer to them as *0-form symmetries* from now on.

An important property which follows from the definition itself is that q-form symmetries can only be *abelian* for $q > 0$. Remember that the order in which two defects U_g, $U_{g'}$ are multiplied is determined by the usual operator time-ordering. For that matter, we can always deform a codimension-$(q+1)$ manifold so as to embed it inside a spatial slice $\mathcal{M}^{d-1}$. Now, consider each defect to be defined at slightly different times $t-\epsilon$ and $t+\epsilon$. The fact that the symmetry operators have codimension > 1 means that we can continuously deform them so as to exchange their positions (see Fig. 3.1). Hence, the order in which they act does not matter and the symmetry is abelian. This argument does not hold for 0-form symmetries — in that case defects will inevitably cross each other if we try to exchange their order.

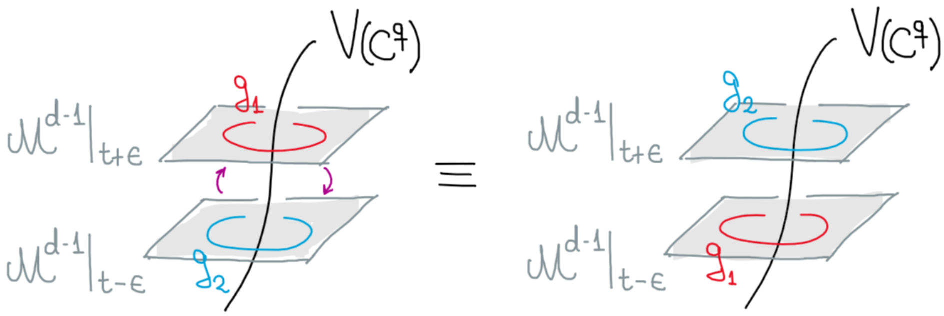

Figure 3.1: Higher-form symmetries are abelian for $q > 0$ since defects can be exchanged in time by a continuous deformation.

Let us talk about the charged objects for a higher-form symmetry. Charged operators should now be defined in higher-dimensional manifolds so that they are able to link around U_g defects. For a q-form symmetry, the charged objects are now extended operators supported on q-dimensional manifolds $\mathcal{C}^q$ — we will denote such operators by $V(\mathcal{C}^q)$. If the operator is in a representation $R(G)$ of the group, the analog of relation 2.10 is

$$U_g(S^{d-q-1})V(\mathcal{C}^q) = R(g)V(\mathcal{C}^q), \tag{3.3}$$

where now S^{d-q-1} is a $(d-q-1)$-dimensional sphere linking $\mathcal{C}^q$. The pictorial interpretation is similar to the one for the 0-form case discussed in Section 2.1. Deforming S^{d-q-1} leaves the result unaffected unless we cross $V(\mathcal{C}^q)$ (i.e. if we unlink the two operators). If that happens, the charged operator gets acted on by the corresponding group element, while the defect can be shrunk until it disappears (see Fig. 3.2).

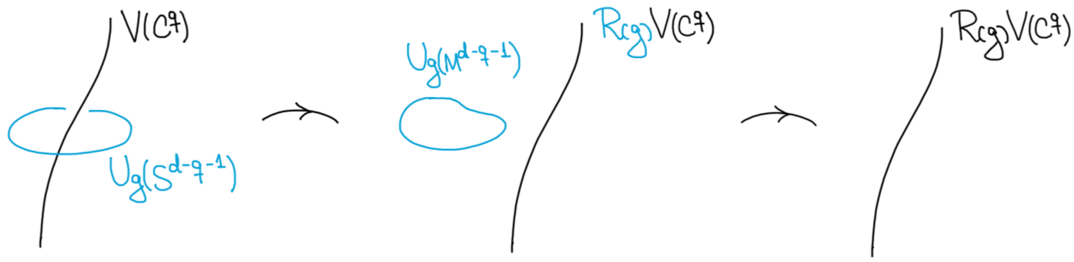

Figure 3.2: Graphical representation of the action of symmetry operators on charged fields, for a higher-form symmetry.

3.2 Gauging q-form symmetries

Just as in the case of 0-form symmetries, we can couple our theory to a classical background gauge field for a q-form symmetry. The gauge field will now be a $(q+1)$-form, and its holonomy is measured by integrating over $(q+1)$-dimensional manifolds:

$$\mathrm{Hol}(\Sigma^{q+1}) = \exp\left(i \oint_{\Sigma^{q+1}} B\right). \tag{3.4}$$

If the symmetry is discrete, by following the same reasoning as in the 0-form case we conclude that all possible background gauge field configurations are classified by $H^{q+1}(X, G)$. Furthermore, remember that in Section 2.2.2 we observed a duality between symmetry operators and curves measuring holonomy for a usual 1-form gauge field. Analogously, we can interpret $U_g(M^{d-q-1})$ as generating holonomy g along every $(q+1)$-dimensional manifold crossing M^{d-q-1}. Thus, at least for discrete q-form symmetries, coupling the theory to a background gauge field is equivalent to stretching a network of U_g defects along M. These networks should respect the same consistency conditions as in the 0-form case: cocycle conditions and invariance under changes of their topology. Once again, this is just a restatement of the Poincaré duality relating $H^{q+1}(X, G)$ to $H_{d-q-1}(X, G)$.

3.3 Examples of higher-form symmetries

3.3.1 $U(1)$ gauge theory in 4d

A well-known theory which possesses higher-form symmetries is Maxwell theory in 4 dimensions. In the absence of matter, it is defined by the following action:

$$S = \int_X -\frac{1}{2g^2} F \wedge *F, \tag{3.5}$$

where g is the gauge coupling constant. While $dF = 0$ by the Bianchi identity, $d * F = 0$ by the equations of motion. These relations are nothing else but a pair of conservation equations for two 2-form currents, known as the electric and magnetic currents:

$$j_e = \frac{2}{g^2}F, \quad j_m = \frac{1}{2\pi} * F. \tag{3.6}$$

They generate what are known as the $U(1)$ electric and magnetic 1-form symmetries of the theory, respectively. The corresponding symmetry operators are obtained by integrating j_e and j_m over 2-dimensional manifolds M:

$$U^E_{e^{i\alpha}}(M) = \exp\left(i\alpha \int_M j_e\right) = \exp\left(i\frac{2\alpha}{g^2}\int_M *F\right), \tag{3.7}$$

$$U^M_{e^{i\eta}}(M) = \exp\left(i\eta \int_M j_m\right) = \exp\left(i\frac{\eta}{2\pi}\int_M F\right). \tag{3.8}$$

These surface defects are referred to as Gukov-Witten operators [15, 16]. Following [6], we normalize j_e and j_m so that α and β are 2π-periodic.

Let us now discuss the charged objects for these pair of 1-form symmetries. For the electric symmetry, the charged objects are Wilson lines:

$$W_q(\gamma) = \exp\left(iq \oint_\gamma A\right). \tag{3.9}$$

In order to understand how the symmetry operators act on them, consider a electric defect supported on a sphere, which links a Wilson line extended along the time direction. At a fixed time, definition 3.7 tells us that $U^E_g(S^2)$ measures the electric charge enclosed by S^2. Thus, the corresponding action of the symmetry operators on Wilson lines is given by

$$U^E_{e^{i\alpha}}(S^2) W_q(\gamma) = e^{iq\alpha} W_q(\gamma). \tag{3.10}$$

We can further consider the action of the symmetry operators on the gauge field A. They act by shifting the gauge field by a flat (singular) connection with holonomy α around Σ, as can be seen by considering their action on small Wilson lines linking Σ.

Analogously, the charged operators of the magnetic 1-form symmetry are 't Hooft lines $T_m(\gamma)$, with $m \in 2\pi\mathbb{Z}$. This can be seen from the fact that the magnetic surface operators measure magnetic charges. The action of these defects on the 't Hooft lines is given by

$$U^M_{e^{i\eta}}(S^2) T_m(\gamma) = e^{i\frac{m}{2\pi}\eta} T_m(\gamma). \tag{3.11}$$

Consider now surface operators defined on manifolds N with boundary $\gamma = \partial N$. A magnetic operator U^M_g defined on N takes the form

$$U^M_{e^{i\eta}}(N) = \exp\left(i\frac{\eta}{2\pi}\int_N F\right) = \exp\left(i\frac{\eta}{2\pi}\oint_\gamma A\right). \tag{3.12}$$

This operator is not gauge invariant, since 2π shifts of the holonomy will change its value for general values of η. To compensate for this, we must place a Wilson loop $W_{-\eta/2\pi}(\gamma)$ in the boundary. Notice that this Wilson line is by itself not properly quantized — we thus refer to such lines as not being genuine line operators since they depend on a choice of surface. Analogously, electric operators supported on N will be bounded by improperly quantized 't Hooft loops. Since $U^E_{e^{i\alpha}}(N)$ introduces a singular connection with holonomy α around its support, we can interpret this operator as being the Dirac surface for a 't Hooft line with charge $m = \alpha$, which is in general not in $2\pi\mathbb{Z}$.

Let us now discuss gauging the 1-form symmetries of the theory. Notice that we can easily couple a pair of background 2-form gauge fields B_e and B_m to the theory via terms proportional to $B_e \wedge *j_e$ and $B_m \wedge *j_m$ in the action[1]. We could also consider gauging only discrete subgroups of the symmetries. Without loss of generality, consider gauging a $\mathbb{Z}_n$ subgroup of the magnetic $U(1)$. This will introduce a gauge field B which takes values in $H^2(X, \mathbb{Z}_n)$. In order to couple it to the theory, we first start with an ordinary 2-form field B and add the following terms to the action:

$$\Delta S = \int_X \frac{i}{2\pi} B \wedge F + \frac{in}{2\pi} B \wedge F_m. \tag{3.13}$$

Here $F_m = dA_m$ with A_m a usual $U(1)$ gauge field — this new variable will serve as a Lagrange multiplier which imposes the requirement that B takes values in $H^2(X, \mathbb{Z}_n)$ after being integrated over in the path integral.

Finally, consider adding charged matter to the theory. The magnetic symmetry stays unchanged, as it follows from the Bianchi identity for F which is always guaranteed. However, the equations of motion for F now include charged sources which break the electric 1-form symmetry. To be explicit, let us introduce matter fields with definite charge q — the electric $U(1)$ will then be broken to $\mathbb{Z}_q$. To prove this, notice that Wilson lines can now end on charged operators, forming a gauge invariant object $W_{-q}(\gamma)\phi_q(x)$ with $x = \partial\gamma$. If an electric surface operator $U^E_{e^{i\alpha}}(S^2)$ surrounding γ is to be topological, we should be able to unlink it from the Wilson line by moving it away from its endpoint (see Fig. 3.3). On the other hand, we can make the operator act on the Wilson line, giving it an extra phase which should be trivial. Thus,

$$e^{iq\alpha} = 1 \implies \alpha \in \frac{2\pi}{q}\mathbb{Z} \iff e^{i\alpha} \in \mathbb{Z}_q. \tag{3.14}$$

3.3.2 $SU(2)$ vs. $SO(3)$ Yang-Mills theory in 4d

We will now discuss the one-form symmetries of non-abelian gauge theories in 4 dimensions, focusing on the particular cases of $SU(2)$ and $SO(3)$ Yang-Mills theory.

Let us start with a free $SU(2)$ gauge theory, which admits Wilson lines in all representations of the group, while 't Hooft lines are constrained to have magnetic charges $m \in 4\pi\mathbb{Z}$. The theory has an electric 1-form symmetry valued in $\mathbb{Z}_2$, the center subgroup of $SU(2)$. It acts

[1] However, we must note that it is not possible to gauge both symmetries simultaneously, as they have a mixed 't Hooft anomaly [6].

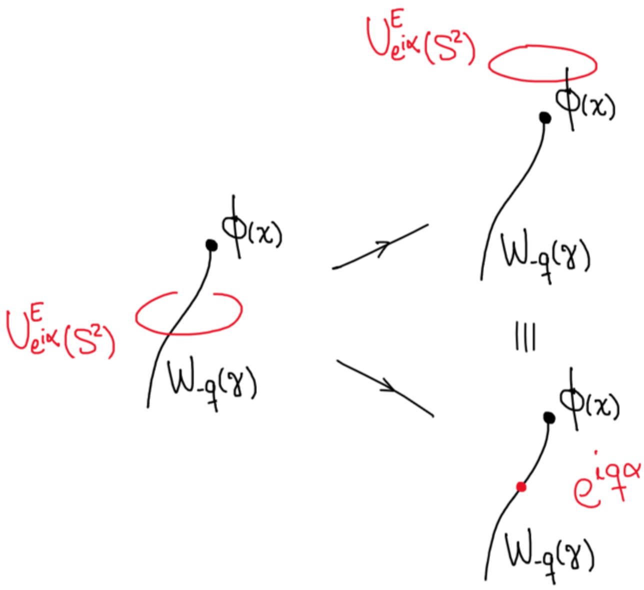

Figure 3.3: A defect linking a Wilson line which ends on a charged operator can be continuously deformed so as to act trivially, or can act on the line by giving it a non-trivial phase factor.

on the Wilson lines by measuring their "charge" under the action of $\mathbb{Z}_2$: while integer-spin representations are invariant under it, half-spin representations transform non-trivially (i.e. half-spin representations change sign under a 2π rotation). The symmetry operators are the analog of Gukov-Witten operators $U^E_n(M)$ associated to elements $n \in \mathbb{Z}_2$. they implement the center action on the lines as follows:

$$U^E_n(S^2)W_j(\gamma) = e^{2\pi i n j}W_j(\gamma). \tag{3.15}$$

In our particular case, the only non-trivial surface operator is the one associated to $n = 1 \in \mathbb{Z}_2$. We see that it acts trivially on Wilson lines with integer spin, while lines with half-integer spin obtain a factor of -1. Just as in the example of $U(1)$ gauge theory, we can interpret this action as shifting the gauge field A by a flat (singular) $\mathbb{Z}_2$-connection. In this way we explicitly verify that this is indeed a symmetry of the theory — the Lagrangian only depends on F, which is unaffected by such a shift.

We can also consider electric surface operators supported on manifolds with boundary. By the same arguments as in $U(1)$ gauge theory, an operator associated to $n \in \mathbb{Z}_2$ will be bounded by a 't Hooft line with magnetic charge $2\pi \cdot (n \bmod 2)$. For $n = 1$, one obtains 't Hooft lines with $m = 2\pi$, which we know are not allowed in $SU(2)$ Yang-Mills. In this context, said 't Hooft lines are interpreted as not being genuine line operators, and they will depend on the Dirac surface operator which they bound. On the other hand, Wilson lines are always genuine line operators since they are gauge invariant for any spin-j representation.

Let us proceed to discuss the consequences of gauging the $\mathbb{Z}_2$ 1-form symmetry. If we do that, the U^E_n operators become trivial as they now implement gauge transformations. As a consequence, the 't Hooft lines which previously lived at the boundary of said defects now become genuine line operators of the theory. On the other side, charged operators should be gauge invariant, i.e. have zero charge under the center symmetry. We conclude that only Wilson lines with integer spin survive the gauging procedure. Wilson lines with half-integer spin are well-defined only if they bound a *magnetic* surface defect, which as in the example of $U(1)$ gauge theory compensates for the gauge non-invariance of its boundary operator.

Thus, by gauging the electric 1-form symmetry we have gained a new symmetry — a magnetic 1-form symmetry valued in $\mathbb{Z}_2$. A magnetic surface operator $U_n^M(M)$ measures the flux of F around surfaces linking 't Hooft lines $T_m(\gamma)$, as in

$$U_n^M(S^2)T_m(\gamma) = e^{\frac{1}{2}inm}T_m(\gamma). \tag{3.16}$$

Once again, the only non-trivial operators are those with $n = 1 \in \mathbb{Z}_2$. 't Hooft lines with magnetic charges $m \in 4\pi\mathbb{Z}$ are uncharged under this symmetry, which is the reason why we did not consider the magnetic $\mathbb{Z}_2$ as a symmetry before. On the other side, 't Hooft lines with charges $m \in 2\pi\mathbb{Z}$ (and not in $4\pi\mathbb{Z}$) obtain a factor of -1.

The reader should have already noticed that, after gauging the electric 1-form symmetry of $SU(2)$ Yang-Mills, the set of allowed line operators has exactly reduced to the one allowed by $SO(3)$ Yang-Mills theory. Thus, we conclude that

$$\begin{gathered} SU(2) \text{ Yang-Mills theory with a gauge } \mathbb{Z}_2 \text{ 1-form symmetry} \\ \Longleftrightarrow \\ SU(2)/\mathbb{Z}_2 \simeq SO(3) \text{ Yang-Mills theory.} \end{gathered}$$

There is a small caveat in our logic, however. Remember that in general there exist many different ways to gauge a symmetry, given by assigning different weights to each topological sector. It can be shown that, by introducing the discrete analog of a theta therm for the electric 1-form symmetry, one can obtain two $SO(3)$ theories with different spectrums for their line operators [17]. In our analysis, we have implicitly assumed that all twisted sectors are weighted equally in the partition function.

3.4 Applications

We will now briefly discuss two important applications of higher-form symmetries: selection rules and their spontaneous symmetry breaking. The latter will allow us to rephrase the classification of phases in 4d gauge theory in terms of the electric and magnetic 1-form symmetries.

3.4.1 Selection rules

Higher-form symmetries will impose selection rules on correlation functions, just as in the 0-form case explained in the Introduction. Throughout this section we will assume that spacetime is a compact manifold.

Consider a charged operator $V(\mathcal{C}^q)$ defined on a non-trivial q-cycle in spacetime. We shall prove that its expectation value vanishes as a consequence of the higher-form symmetry. In fact, if we link the charged operator by a U_g defect, we can either make it act on the operator or collapse it by deforming it around spacetime, something which is always possible when the latter is compact (see Fig. 3.4). Therefore, we find that

$$R(g)\,\langle V(\mathcal{C}^q)\rangle = \langle V(\mathcal{C}^q)\rangle \implies \langle V(\mathcal{C}^q)\rangle = 0. \tag{3.17}$$

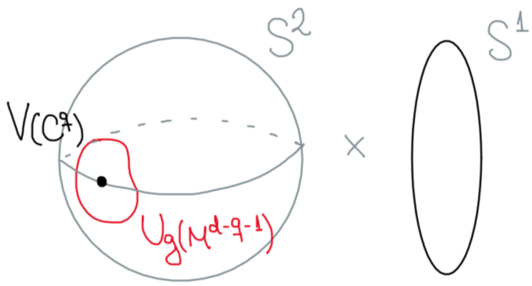

Figure 3.4: In this example, $X = S^2 \times S^1$ and the charged operator is supported on $\mathcal{C}^1 = \{\mathrm{pt}\} \times S^1$. The U_g defect can either act on $V(\mathcal{C}^1)$ or be collapsed on the other side of the sphere.

As a particular example, consider Wilson lines in either $U(1)$ or $SU(2)$ gauge theory, supported on non-trivial 1-cycles in a compact spacetime. Since they are charged under their respective electric 1-form symmetries, their vacuum expectation values will vanish. Similarly, if we consider 't Hooft lines in either $U(1)$ or $SO(3)$ gauge theory, their vacuum expectation values vanish as a consequence of the magnetic 1-form symmetry.

3.4.2 Spontaneous symmetry breaking

First, let us briefly review how large Wilson loops allow us to characterize the phases of 4d Yang-Mills theory. It can be proven that vacuum expectation values of such loops (in Euclidean signature) will typically fall into two different cases [18]:

- **Area law:** In this case one finds that $\langle W(\gamma)\rangle \sim e^{-\sigma A[\gamma]}$, where σ is a proportionality constant known as the *string tension* and $A[\gamma]$ is the area of the minimal surface bounded by γ.
- **Perimeter law**: In this case we have $\langle W(\gamma)\rangle \sim e^{-\mu L[\gamma]}$, with μ a proportionality constant and $L[\gamma]$ the perimeter of the Wilson loop.

Furthermore, these behaviors can be related to the effective potential between a pair of test "quarks" (for example, see [19]). In the first case, one shows that an area law for the Wilson line implies that the effective potential grows with distance. Thus, in that case we say that the theory is in the *confining phase* and that interactions are long-ranged. On the other hand, a perimeter law is a sign of *deconfinement* — the effective potential is a constant (at leading order) and thus there are no long-range interactions.

We can redefine our Wilson line by multiplying it with a term such that $\langle W(\gamma)\rangle$ becomes 1 when it follows a perimeter law. More concretely, set

$$W(\gamma) \longrightarrow W(\gamma)e^{\mu \oint_\gamma ds}. \tag{3.18}$$

Note that Wilson lines with area law still get a zero vacuum expectation value as $A[\gamma]$ grows faster than $L[\gamma]$. With this new definition, we can employ large Wilson loops as an *order parameter* for confinement:

$$\text{Area law: } \langle W(\gamma)\rangle = 0 \implies \text{Confining phase}$$
$$\text{Perimeter law: } \langle W(\gamma)\rangle \neq 0 \implies \text{Deconfining phase}$$

We now connect back to the description of these line operators in terms of their 1-form symmetries. We will interpret Wilson lines having non-zero vacuum expectation value as signaling the *spontaneous breaking* of their electric 1-form symmetry. This can be seen as a generalization of the 0-form case, where symmetry-broken phases can be characterized by non-zero correlators at large distances:

$$\lim_{|x|\to\infty} \langle \mathcal{O}^\dagger(x)\mathcal{O}(0)\rangle \sim \langle \mathcal{O}^\dagger(x)\rangle\langle \mathcal{O}(0)\rangle + ... \tag{3.19}$$

This is in contrast to the symmetry-preserving case where correlators decay exponentially with $|x|$. Analogously, in the 1-form case, Wilson lines following a perimeter law show long-range correlations which do not decay as γ grows larger, in contrast to those following an area law.

Finally, it is possible to extend the validity of Goldstone's theorem to the higher-form symmetry case[2]. As an example, let us focus on the particular case of 4d Maxwell theory. Wilson lines in this theory will satisfy perimeter laws, and therefore we conclude that the electric 1-form symmetry must be spontaneously broken in the IR. It turns out that the Nambu-Goldstone boson in this case is the *photon itself*. This can be understood by calculating the matrix element of the electric 2-form current (which is proportional to F) between the vacuum and a single photon state $|\epsilon, p\rangle$:

$$\langle 0|F_{\mu\nu}|\epsilon, p\rangle = (\epsilon_\mu p_\nu - \epsilon_\nu p_\mu)e^{ipx}. \tag{3.20}$$

In that way, we understand the masslessness of the photon as a consequence of the generalized Goldstone's theorem.

[2] A detailed proof can be found in [20].

APPENDIX A

Magnetic monopoles & 't Hooft lines

A.1 Monopoles in $U(1)$ gauge theory

It is interesting to consider, in usual electromagnetism, objects which act as sources of magnetic field, i.e. *magnetic monopoles*. At first sight, the existence of such objects is forbidden by Maxwell's equations. In fact, consider a magnetic monopole with charge m sitting at the origin. The magnetic flux around a small sphere surrounding it should equal the enclosed charge — however:

$$m = \int_{S^2} F = \int_{\partial S^2} A = 0, \tag{A.1}$$

using that $F = dA$ and Stokes' theorem. In other words, the existence of magnetic monopoles clashes with the exactness of F, which is guaranteed by the Bianchi identity.

We can evade this restriction by considering spacetime to have a non-trivial topology. For example, let us think of a magnetic monopole as "puncturing" spacetime so that we go from $M = \mathbb{R}^4$ to $M = (\mathbb{R}^3 - \{0\}) \times \mathbb{R}$. Mathematically, a $U(1)$ gauge theory is specified by choosing a $U(1)$ gauge bundle over M — in our example this is equivalent to choosing a gauge bundle over the sphere S^2. Let us now "patch" the sphere by its two hemispheres N and S. In each one of these patches we can consider gauge fields A_N and A_S, connected by a $U(1)$ gauge transformation at the overlap:

$$A_N = A_S + ie^{i\varphi(x)} de^{-i\varphi(x)} = A_S + d\varphi(x). \tag{A.2}$$

Note that our definition of A includes the gauge coupling constant e inside it. That said, let us calculate the magnetic flux in this particular configuration:

$$m = \int_{S^2} F = \int_N dA_N + \int_S dA_S = \int_\gamma A_N - \int_\gamma A_S = \int_\gamma d\varphi. \tag{A.3}$$

At this point, one might think about using Stokes' theorem once more to conclude that $m = 0$. However, we must take into account that $\varphi(x)$ is not necessarily a well-defined function over γ. Since we only require $e^{i\varphi(x)} \in U(1)$, $\varphi(x)$ can shift by integer multiples of 2π after going around a circle. Thus $d\varphi$ might not be the exterior derivative of a single-valued function, and Stokes' theorem fails to be valid. The correct conclusion is that

$$m = \int_{S^2} F = \int_{\gamma} d\varphi = \Delta\varphi \in 2\pi\mathbb{Z}. \tag{A.4}$$

This relation is known as the *Dirac quantization condition*, as it restricts the possible charges of the magnetic monopoles.

Just as the insertion of a Wilson line $W_q(\gamma)$ in the path integral can be interpreted as probing the theory with a test charge q tracing a worldline γ, we would like to have an analogous concept for a magnetic monopole. A *'t Hooft line $T_m(\gamma)$* is an operator defined via its insertion in the path integral — we restrict the integration over A to configurations which satisfy $\int_{S^2} F = m$ for any sphere surrounding γ at a fixed time [19]. Indeed, this corresponds to inserting a probe magnetic monopole with charge m and worldline γ.

There is an slightly different viewpoint on magnetic monopoles which the reader will find useful to understand. Instead of defining two gauge fields A_N and A_S in the two hemispheres of S^2, let us try defining a single gauge field A in a patch which covers the sphere as much as possible. In the limit where we cover all of S^2 but a single point $\{p\}$, we can calculate the magnetic flux around S^2 as follows:

$$\int_{S^2} F \to \int_{S^2-\{p\}} dA = \oint_{\gamma} A. \tag{A.5}$$

If we want this holonomy to be non-trivial, it must happen that the gauge field is singular at p. More generally, by foliating $\mathbb{R}^3 - \{0\}$ with spheres of all sizes, this procedure gives us a gauge field which is singular along a line, known as the *Dirac string*. If we demand the magnetic monopole to be a local entity, the Dirac string should be invisible to a small Wilson line wrapping around it — therefore,

$$W_q(\gamma) = e^{iq\oint_\gamma A} = e^{iqm} \overset{!}{=} 1 \implies m \in 2\pi\mathbb{Z}, \tag{A.6}$$

which is nothing else but the Dirac quantization condition.

A.2 Monopoles in $SU(2)$ vs. $SO(3)$ Yang-Mills theory

Let us now introduce magnetic monopoles (and their corresponding 't Hooft lines) in $SU(2)$ Yang-Mills theory[1]. We would like to find $U(1)$ monopoles embedded in the gauge group, so let us consider the $U(1)$ subgroup of $SU(2)$ generated by $h \in \mathfrak{su}(2)$, which is unique up to conjugation. Remember that the curvature 2-form is now valued in the Lie algebra — in particular, its restriction to the $U(1)$ subgroup will be of the form $F_{U(1)} = F \otimes h$. We define the magnetic charge m of an $SU(2)$ monopole by

$$m = \int_{S^2} F. \tag{A.7}$$

[1]The following discussion can be easily extended to Yang-Mills theory with general gauge groups.

We now show that m must follow a constraint similar to Eq. A.4. Consider a Wilson loop in the spin-j representation of $SU(2)$, measuring the holonomy of $A_{U(1)} = A \otimes h$ around a loop γ. Remember that the spin-j representation with $j \in \frac{1}{2}\mathbb{Z}$ is $(2j+1)$-dimensional — its (orthonormal) basis vectors are denoted as $|j,k\rangle$ where $k = -j, j+1, \dots, j$. Then:

$$W_j(\gamma) = \mathrm{Tr}_j \mathcal{P}\left\{\exp\left(i \oint_\gamma A_{U(1)}\right)\right\} = \sum_{k=-j}^{j} \langle j,k| \exp\left(i \oint_\gamma A \otimes h\right) |j,k\rangle$$
$$= \sum_{k=-j}^{j} \exp\left(ik \oint_\gamma A\right). \tag{A.8}$$

Again, we work in units where the coupling constant of the theory equals 1. In order for the Wilson loop to be well defined (i.e. properly quantized) we require that, under gauge transformations,

$$k \oint_\gamma A \to k \oint_\gamma A + 2\pi\mathbb{Z}, \quad \forall k \in \{-j, \dots, j\}. \tag{A.9}$$

Just as in the case of Maxwell theory, this requirement constrains the allowed fluxes of $F_{U(1)}$:

$$\int_{S^2} F_{U(1)} \in 2\pi\mathbb{Z} \cdot \mathbb{1} \implies e^{imh} = \mathbb{1} \in SU(2). \tag{A.10}$$

This result is generalized version of the Dirac quantization condition, which holds for all representations of $SU(2)$. To understand how this condition constraints the allowed magnetic monopoles, consider the Wilson line of a probe charge in the spin-$\frac{1}{2}$ representation. Condition A.10 tells us that $m \cdot \frac{1}{2} \in 2\pi\mathbb{Z}$. Thus, in contrast to the case of Maxwell theory, the minimum magnetic charge allowed for an $SU(2)$ monopole is 4π.

We now proceed to discuss magnetic monopoles in $SO(3) \simeq SU(2)/\mathbb{Z}_2$ Yang-Mills theory. Although $SO(3)$ and $SU(2)$ Yang-Mills are locally indistinguishable, the global properties of the gauge groups will allow for distinct sets of line operators. First of all, note that $SO(3)$ Yang-Mills does not admit Wilson lines in representations with half-integer spin. Condition A.10 will then only hold for integer spin representations, and the constraint on the allowed magnetic charges is relaxed to $m \in 2\pi\mathbb{Z}$. In other words, reducing the number of allowed Wilson lines has led to the apparition of new 't Hooft lines in the theory.

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