Conformal Field Theories of Generalized Parafermions

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CHAPTER 1

Introduction

The concept of symmetry is of utmost importance in the description of physical systems. In classical mechanics, Noether's theorem associates the existence of continuous symmetries of a theory to quantities conserved in time. For quantum mechanical systems, symmetries become a useful basis in which to organize the spectrum of a theory. Furthermore, in quantum field theory symmetries are one of the few non-perturbative tools available to study their dynamics, and allows one to obtain strong statements on fundamental quantities such as correlation functions.

A particularly interesting kind of symmetry is *conformal symmetry*, in which a field theory is invariant under coordinate transformations that leave the metric untouched up to a scaling factor. These theories are called *conformal field theories* (CFTs), and are ubiquitous whenever one tries to describe systems which do not depend on a length scale [1]. Such is the case for statistical physics systems at their critical points (which characterize a phase transition), for example [2].

In condensed-matter physics, conformal field theories also have an important role in the characterization of 2+1d topological phases of matter. Such phases allow for the existence of *anyonic* degrees of freedom — emergent excitations which have non-trivial braiding statistics beyond that of bosons and fermions [3]. The exchange of two anyons generally defines a unitary transformation on the space of ground states [4] — if these braiding transformations do not commute with each other, the anyons are said to satisfy *non-abelian statistics*. Non-abelian anyons are special in the sense that they may *fuse* into many possible different channels. For example, if an anyon of type a and an anyon of type b are brought together, the *fusion rule*

$$\phi_a \times \phi_b = \sum_c N_{ab}^{\ \ c} \phi_c.$$

tells us that this process may result in particle of type c whenever the coefficient $N_{ab}^{\ \ c}$ is non-zero, with multiplicity given by this same number.

It turns out that 2+1d topological phases of matter are dual to 1+1d conformal field theories through a bulk/boundary correspondence. Indeed, the computation of certain quantities of the bulk (such as ground-state wavefunctions) can be mapped to a corresponding computation in a conformal field theory (such as correlation functions) living at the boundary of the bulk [4]. Moreover, the anyonic excitations will directly correspond to fields in the associated CFT — in this sense, analyzing the field content of the boundary theory already gives us valuable information about the particle spectrum in the bulk. In fact, just as there is a notion of fusing anyons in the topological phase of matter, there is a sense in which fields of a conformal field theory can be fused to give rise to other fields. For certain types of CFTs in which the field spectrum is divided into finite sectors (called rational CFTs), the short-distance behavior of said fields define the *fusion rules of the theory*:

$$[\phi_i] \times [\phi_j] = \sum_k {N_{ij}}^k [\phi_k].$$

We are thus led to investigate, from the conformal field theory side of the correspondence, possible sets of fusion rules which may be used to describe non-abelian anyons in a topological phase of matter.

A large class of CFTs can be obtained by starting with theories associated to Lie algebras — the so-called Wess-Zumino-Witten models — and taking "quotients" between them. Through this construction, one can find a variety of examples of theories with rich fusion structures. An example of such "coset" theories are those containing degrees of freedom which are neither bosons or fermions — the so-called *parafermionic* conformal field theories. Although their usual construction is based on modding out a $\mathfrak{u}(1)$ algebra inside $\mathfrak{su}(2)$, one can generalize this procedure to arbitrary semisimple Lie algebras [5]. However, particular examples of such generalized parafermionic theories have not been explored in full detail in the literature. The main objective of this project is to amend this by analyzing the structure of such theories for $\mathfrak{g} = \mathfrak{su}(3)$.

The present document is organized as follows. In Chapter 2 we present some of the basic notions in the study of 2-dimensional conformal field theories, including important definitions such as those of fusion algebras. In Chapter 3 we introduce the Wess-Zumino-Witten models through a purely representation-theoretic perspective. Chapter 4 is devoted to explaining the coset construction and particular examples within, such as the usual $\mathbb{Z}_k$ parafermions. We also introduce the generalized parafermionic field theories (also known as Gepner parafermions) and present our results on the fusion rules of the $\mathfrak{su}(3)$ theories. Many tools from the representation theory of Lie and affine Lie algebras will be necessary to analyze these theories in detail — we have summarized the fundamental notions in Appendices A and B, focusing on their applications to our work rather than delving into their mathematical details.

CHAPTER 2

Notions of 2d Conformal Field Theory

A conformal field theory is a field theory which is invariant under conformal transformations, i.e. a theory with **conformal symmetry**. The requirement of conformal symmetry leads to powerful constraints, allowing one to greatly simplify computations and to obtain interesting, non-perturbative information about the theory. In the special case of dimension two, *local* conformal invariance allows one to completely determine the form of fundamental quantities such as correlation functions, thus rendering the theory "completely solvable".

However, the appropriate mathematical framework to describe conformal field theories is very different from the usual approach used to describe general quantum field theories. In fact, it is more adequate to first analyze the theory solely on the basis of its conformal symmetry, and then understand what consequences it has on the computation of physical observables after quantization. In this chapter we introduce the mathematical framework of conformal field theory, summarizing the basic aspects which will be necessary for the later parts of this project. The content of this chapter is mostly based in references [6] and [7], and its purpose is to give an overview of the basics instead of being a comprehensive review.

2.1 Conformal Symmetry in 2d

Consider M to be a 2-dimensional spacetime manifold, equipped with a background Riemannian metric $g_{\mu\nu}(x)$ (i.e. we do not consider the metric to be a dynamical field).

Definition 2.1.1. A **conformal transformation** is a coordinate transformation[1] which leaves the metric invariant up to a multiplicative factor:

$$g_{\mu\nu}(x) \longmapsto g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \stackrel{!}{=} \Lambda(x) g_{\mu\nu}(x). \tag{2.1}$$

[1]Here we take the "passive" point of view where a coordinate transformation φ corresponds to a change of charts, with the transition function given by φ.

Such transformations preserve angles between tangent vectors. For our purposes, we will always take g to be the flat Euclidean metric given by $g_{\mu\nu} = \mathrm{diag}(+1, +1)$.

Let us start by studying *infinitesimal* transformations of the form $x' = x + \epsilon(x)$. If such a transformation is to be conformal, condition 2.1 implies that

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = (\partial_\sigma \epsilon^\sigma) g_{\mu\nu} \implies \begin{cases} \partial_0 \epsilon_0 = +\partial_1 \epsilon_1 \\ \partial_0 \epsilon_1 = -\partial_1 \epsilon_0. \end{cases} \tag{2.2}$$

This set of equations can be recognized as the Cauchy-Riemann equations if we consider ϵ^0 and ϵ^1 as the real and imaginary parts of some complex function. This motivates us to introduce the following definitions:

$$\begin{cases} z = x^0 + ix^1, \quad \bar{z} = x^0 - ix^1, \\ \epsilon = \epsilon^0 + i\epsilon^1, \quad \bar{\epsilon} = \epsilon^0 - i\epsilon^1, \\ \partial = \dfrac{1}{2}(\partial_0 - i\partial_1), \quad \bar{\partial} = \dfrac{1}{2}(\partial_0 + i\partial_1). \end{cases} \tag{2.3}$$

In terms of these complex variables, Eq. 2.2 are expressed succintly as

$$\bar{\partial}\epsilon = 0, \tag{2.4}$$

which tells us that ϵ is a holomorphic function of z. Following common usage, we denote such functions as $\epsilon(z)$, while general smooth functions f are denoted as $f(z, \bar{z})$. Eq. 2.4 is also equivalent to $\partial\bar{\epsilon} = 0$, meaning that $\bar{\epsilon}$ is an antiholomorphic function of z. We denote such functions by $\bar{\epsilon}(\bar{z})$.

From the analysis above, we conclude that local conformal transformations are given by functions which are locally holomorphic. Assuming we can expand $\epsilon(z)$ around 0, we have

$$\begin{cases} z' = z + \epsilon(z) = z + \displaystyle\sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1}, \\ \bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \displaystyle\sum_{n\in\mathbb{Z}} \epsilon_n \bar{z}^{n+1}. \end{cases} \tag{2.5}$$

This leads us to introduce the generators

$$\ell_n = -z^{n+1}\partial, \quad \bar{\ell}_n = -\bar{z}^{n+1}\bar{\partial}, \tag{2.6}$$

which satisfy the following commutation relations:

$$\begin{cases} [\ell_m, \ell_n] = (m-n)\ell_{m+n}, \\ [\bar{\ell}_m, \bar{\ell}_n] = (m-n)\bar{\ell}_{m+n}, \\ [\ell_m, \bar{\ell}_n] = 0. \end{cases} \tag{2.7}$$

The $\{\ell_n\}_{n\in\mathbb{Z}}$ constitute a sub-algebra of 2.7 known as the Witt algebra — in this sense, the algebra of 2d conformal transformations is the direct sum of two copies of the Witt algebra, the other one generated by $\{\bar{\ell}_n\}_{n\in\mathbb{Z}}$.

The Virasoro Algebra

So far we have discussed conformal symmetry only at the classical level. It is important to know that, after quantization, the Hilbert space of physical states comprises a *projective* representation of the classical symmetry group G. At the level of the algebras, projective representations of $\mathfrak{g} = \mathrm{Lie}(G)$ are in one-to-one correspondence to usual representations of central extensions of $\mathfrak{g}$. The technical details are thoroughly explained in [8] — for our purposes, we simply note that the relevant algebra in our case is the central extension of the Witt algebra by $\mathbb{C}$, commonly known as the **Virasoro algebra**, which depends on a complex number c known as the **central charge**. If we label the quantum generators corresponding to the $\{\ell_n\}_{n\in\mathbb{Z}}$ as $\{L_n\}_{n\in\mathbb{Z}}$, the Virasoro algebra Vir_c is defined by the following commutation relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{2.8}$$

The quantum algebra of the 2d conformal transformations is then the direct sum of two copies of Vir_c, adding the generators $\{\bar{L}_n\}_{n\in\mathbb{Z}}$ corresponding to the antiholomorphic part of the algebra. Thus, the full quantum algebra is given by

$$\begin{cases} [L_m, L_n] = (m-n)L_{m+n} + \dfrac{c}{12}(m^3 - m)\delta_{m+n,0}, \\ [\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + \dfrac{c}{12}(m^3 - m)\delta_{m+n,0}, \\ [L_m, \bar{L}_n] = 0. \end{cases} \tag{2.9}$$

2.2 Primary Fields and the Energy-Momentum Tensor

We now turn our attention on fields which transform in particular ways under conformal transformations.

Definition 2.2.1. A field $\phi(z,\bar{z})$ has **conformal dimensions** $(h,\bar{h})$ if it transforms like

$$\phi(z,\bar{z}) \longmapsto \phi'(z',\bar{z}') = \lambda^{-h}\bar{\lambda}^{-\bar{h}}\phi(z,\bar{z}) \tag{2.10}$$

under scale transformations $z \mapsto z' = \lambda z$.

Definition 2.2.2. A field $\phi(z,\bar{z})$ with conformal dimensions $(h,\bar{h})$ is **primary** if it transforms like

$$\phi(z,\bar{z}) \longmapsto \phi'(z',\bar{z}') = \left(\frac{\partial z'}{\partial z}\right)^{-h}\left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}}\phi(z,\bar{z}) \tag{2.11}$$

under any *local* conformal transformation. If it only transforms in this way under *global* conformal transformations, it is called a **quasi-primary** field.

Definition 2.2.3. Given a quasi-primary field $\phi(z,\bar{z})$ with conformal dimensions $(h,\bar{h})$, we define its **scaling dimension** Δ and its **spin** s by

$$\Delta = h + \bar{h}, \qquad s = h - \bar{h}. \tag{2.12}$$

Remark 2.2.1. In the context of conformal field theory, holomorphic and antiholomorphic fields are also known as *chiral* and *anti-chiral* fields respectively. We make use of this terminology extensively throughout this work.

Since primary fields are defined in terms of their behavior under local conformal transformations, it will be useful to compute their infinitesimal variation under transformations of the form $z \mapsto z' = z + \epsilon(z)$. We have, at a point $(z, \bar{z})$:

$$\begin{aligned} \delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z}) &\equiv \phi'(z,\bar{z}) - \phi(z,\bar{z}) \\ &= -(h\phi\partial\epsilon + \epsilon\partial\phi)(z,\bar{z}) - (\bar{h}\phi\bar{\partial}\bar{\epsilon} + \bar{\epsilon}\bar{\partial}\phi)(z,\bar{z}). \end{aligned} \tag{2.13}$$

The above equation can thus be taken as an alternative definition of a primary field.

Noether's theorem states that every continuous symmetry of a theory leads to the existence of a conserved current j_μ which satisfies $\partial^\mu j_\mu = 0$. For the particular case of invariance under transformations of the form $x \mapsto x + \epsilon(z)$, the associated conserved current can be explicitly written as [7]

$$j_\mu(x) = T_{\mu\nu}(x)\epsilon^\nu(x), \tag{2.14}$$

where $T_{\mu\nu}(x)$ is known as the *energy-momentum tensor.* One can check that the conservation of j_μ implies that this tensor is both symmetric and traceless, i.e.

$$T_{\mu\nu} = T_{\nu\mu} \qquad \text{and} \qquad T_\mu{}^\mu = 0. \tag{2.15}$$

We can now switch to complex variables $z, \bar{z}$ as in (2.3). The components of the energy-momentum tensor in these new coordinates can be obtained through the usual transformation law of tensors under coordinate transformations. They read:

$$\begin{aligned} T_{zz} &= \frac{1}{2}(T_{00} - iT_{10}), \\ T_{\bar{z}\bar{z}} &= \frac{1}{2}(T_{00} + iT_{10}), \\ T_{z\bar{z}} &= T_{\bar{z}z} = 0. \end{aligned} \tag{2.16}$$

Together with the properties (2.15), the above relations imply that T_{zz} is *chiral*, while $T_{\bar{z}\bar{z}}$ is *anti-chiral.* We denote these components by $T(z)$ and $\bar{T}(\bar{z})$ respectively. Furthermore, we usually refer to $T(z)$ as *the* (chiral) energy-momentum tensor.

2.3 Radial Quantization

We know proceed to discuss the operator content of the theory through a special quantization formalism known as *radial quantization.* Radial quantization consists in applying the usual canonical quantization recipe with the time axis chosen to point outwards in the radial direction. Although this choice may seem strange at first, one of the main advantages of this particular quantization scheme is that it allows one to apply the machinery of contour integrals. Thus, we can make use of well-known results from complex analysis to obtain important information on the operator content of a conformal field theory.

Consider an Euclidean CFT defined on a cylinder of radius 1. It can be described by coordinates (x^0, x^1) where $x^0 \in \mathbb{R}$ parametrizes the infinite vertical direction, and $x^1 \in [0, 2\pi[$ the finite angular direction (cf. Fig. 2.1). As our notation suggests, we consider the infinite direction to be the time direction, while we consider a space direction that is compactified in a circle of radius 1. As usual, we identify $x^1 \sim x^1 + 2\pi$.

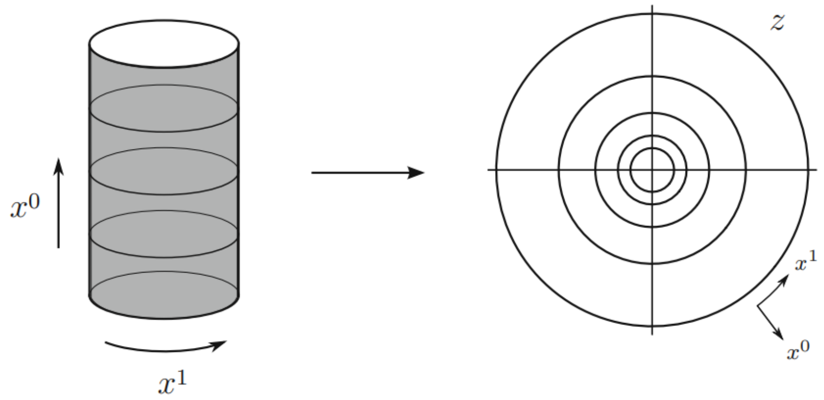

Figure 2.1: Map between the cylindrical coordinates (x^0, x^1) and the complex plane z. Source: [7].

We will map this cylinder to the usual complex plane by sending

$$(x^0, x^1) \longmapsto e^{x^0 + ix^1}. \tag{2.17}$$

Through this transformation, points at time $t \to -\infty$ are mapped to the origin, while points at time $t \to +\infty$ are all mapped to infinity (which can be regarded as a point in the Riemann sphere). Moreover, points at finite times are mapped to concentric circles in $\mathbb{C}$.

We can now apply the canonical quantization recipe for this particular choice of time axis. At each fixed time x^0, we associate a Hilbert space $\mathcal{H}$, the space of physical states of the theory. As usual, we assume the existence of a vacuum or *ground state* $|0\rangle \in \mathcal{H}$, which is invariant under the action of global conformal transformations. With this recipe, correlation functions can be obtained by calculating time-ordered vacuum expectation values of operators acting on $\mathcal{H}$. We also have a natural identification of the Hamiltonian and momentum operators in terms of the quantum conformal algebra operators. Indeed, since they correspond to the generators of time and space translations respectively, one finds that

$$H = L_0 + \bar{L}_0, \qquad P = i(L_0 - \bar{L}_0), \tag{2.18}$$

are the correct expressions for these operators in radial quantization.

By use of the map 2.17, a primary field on the cylinder can know be described through complex coordinates as $\phi(z, \bar{z})$. We can perform a Laurent expansion around $(z, \bar{z}) = (0, 0)$:

$$\phi(z, \bar{z}) = \sum_{n, \bar{m} \in \mathbb{Z}} z^{-n-h} \bar{z}^{-\bar{m}-\bar{h}} \phi_{n,\bar{m}}. \tag{2.19}$$

The factors of h and $\bar{h}$ in the exponents are introduced for later convenience (cf. Eq. 2.22). We quantize the field ϕ by promoting its Laurent coefficients (or "modes") to operators.

As noted previously, points at the infinite past $t \longrightarrow -\infty$ in the cylinder are mapped to a single point in $\mathbb{C}$, the origin. Knowing this, we can define an asymptotic *in-state* corresponding to ϕ via

$$|\phi\rangle = \lim_{(z,\bar{z}) \to (0,0)} \phi(z, \bar{z}) |0\rangle . \tag{2.20}$$

For this state to be well-defined, we require this limit to be non-singular. This leads to the following condition on the Laurent modes of ϕ:

$$\phi_{n,\bar{m}} |0\rangle = 0, \quad \text{for } n > -h \text{ or } \bar{m} > -\bar{h}. \tag{2.21}$$

This leads to an explicit form for the asymptotic in-state $|\phi_{\text{in}}\rangle$:

$$|\phi\rangle = \phi_{-h,-\bar{h}} |0\rangle . \tag{2.22}$$

We can also define a notion of hermitian conjugation for these operators. Notice that the usual notion of hermitian conjugation defined on Minkowski space does not affect the spacetime coordinates. However, in Euclidean space the "time" coordinate τ is related to the real time t by $\tau = it$. Thus, hermitian conjugation sends $x^0 \mapsto -x^0$, or equivalently sends

$$z = e^{x^0 + ix^1} \longmapsto e^{-x^0 + ix^1} = \frac{1}{z^*}. \tag{2.23}$$

The correct definition of $\phi^\dagger$ should then be

$$\phi^\dagger(z, \bar{z}) = \bar{z}^{-2h} z^{-2\bar{h}} \phi\left(\frac{1}{\bar{z}}, \frac{1}{z}\right), \tag{2.24}$$

with the prefactors chosen again for convenience. By performing a Laurent expension of $\phi^\dagger$ and comparing with the hermitian conjugate of Eq. 2.19, we see that the Laurent modes satisfy the condition

$$\left(\phi_{n,\bar{m}}\right)^\dagger = \phi_{-n,-\bar{m}}. \tag{2.25}$$

Using the hermitian conjugate of ϕ, we define the asymptotic *out-state* corresponding to it by

$$\langle\phi| = \lim_{(z,\bar{z}) \to (0,0)} \langle 0| \phi^\dagger(z, \bar{z}). \tag{2.26}$$

Using the mode expansion, we again arrive to a condition on the Laurent modes:

$$\langle 0| \phi_{n,\bar{m}} = 0, \quad \text{for } n < h \text{ and } \bar{m} < \bar{h}, \tag{2.27}$$

from which we can simplify the definition of the asymptotic out-state to

$$\langle\phi| = \langle 0| \phi_{+h,+\bar{h}}. \tag{2.28}$$

2.4 The Operator Product Expansion

As mentioned before, we can calculate correlation functions of products of operators by taking their time-ordered vacuum expectation value. In radial quantization, where time is parametrized by the radial coordinate, time-ordering now corresponds to a *radial* ordering $\mathcal{R}$. The latter is defined as

$$\mathcal{R}\{\phi_1(z)\phi_2(w)\} = \begin{cases} \phi_1(z)\phi_2(w), & \text{if } |z| > |w|, \\ \phi_2(w)\phi_1(z), & \text{if } |w| > |z|. \end{cases} \tag{2.29}$$

Later on we will often find expressions of the form

$$\oint_{C(w)} dz A(z)B(w), \tag{2.30}$$

with $C(w)$ a contour circling w counterclockwise. However, inside correlation functions, products of operators only make sense if they are radially ordered. Thus, we would like to compute the radially ordered expression of the above equation. In order to achieve this, we split the contour into two contours which go around the origin at radii larger and smaller than $|w|$, as shown in Fig. 2.2.

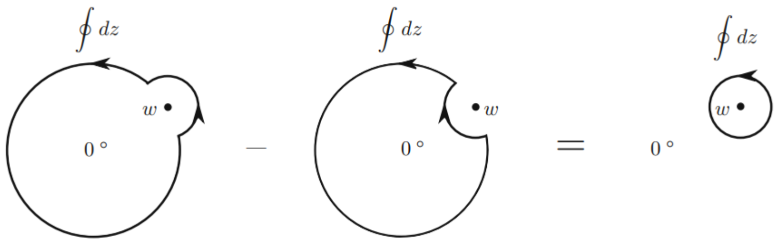

Figure 2.2: Splitting of the contour $C(w)$ into two contours centered around the origin. Source: [7].

With this, we can rewrite the previous expression using the new contours C_1, C_2 as

$$\begin{aligned} \int_{C(w)} dz\mathcal{R}\{A(z)B(w)\} &= \int_{C_1} dz A(z)B(w) - \int_{C_2} dz B(w)A(z) \\ &= \oint dz[A(z), B(w)]. \end{aligned} \tag{2.31}$$

This equation will allow us to relate commutation relation to radially-ordered operators of products, as we will see shortly.

Going back to the discussion on Noether's theorem in Sec. 2.2, we remember that conserved currents imply the existence of conserved charges given by integrating j^0 over all of space (at a fixed time). In radial quantization this means we are integrating j^0 over a circle of fixed radius, i.e. a contour integral. In complex coordinates, the charge associated to conformal symmetry is expressed as

$$Q = \frac{1}{2\pi i} \oint \left(dz T(z)\epsilon(z) + d\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z})\right). \tag{2.32}$$

Furthermore, at the quantum level the charge operator Q becomes a generator the corresponding symmetry transformations, i.e.

$$\delta_{\epsilon,\bar{\epsilon}}\phi(w,\bar{w}) = [Q,\phi(w,\bar{w})]. \tag{2.33}$$

Thus, replacing Eq. 2.32 above we have that

$$\begin{aligned}\delta_{\epsilon,\bar{\epsilon}}\phi(w,\bar{w}) &= \frac{1}{2\pi i}\oint dz[T(z)\epsilon(z),\phi(w,\bar{w})] + \text{a.c.}\\ &= \frac{1}{2\pi i}\oint_{C(w)} dz\epsilon(z)\mathcal{R}\{T(z)\phi(w,\bar{w})\} + \text{a.c.},\end{aligned} \tag{2.34}$$

where "a.c." denotes the anti-chiral part of the expression. We can now compare this to the explicit form for $\delta_{\epsilon,\bar{\epsilon}}\phi(w,\bar{w})$ (cf. Eq. 2.13). First, we rewrite the latter equation in terms of contour integrals by employing the following identities:

$$\begin{aligned}h\phi(w,\bar{w})\partial\epsilon(w) &= \frac{1}{2\pi i}\oint_{C(w)} dzh\frac{\epsilon(z)}{(z-w)^2}\phi(w,\bar{w}).\\ \epsilon(w)\partial\phi(w,\bar{w}) &= \frac{1}{2\pi i}\oint_{C(w)} dz\frac{\epsilon(z)}{z-w}\partial\phi(w,\bar{w}).\end{aligned} \tag{2.35}$$

Remark 2.4.1. Remember that for any holomorphic function $f(z)$, we can express its derivatives at a point w through Cauchy's differentiation formula:

$$\partial^n f(w) = \frac{n!}{2\pi i}\oint_\gamma dz\frac{f(z)}{(z-w)^{n+1}},\quad n\geq 0, \tag{2.36}$$

with γ a contour circling w counterclockwise.

We can now compare the two expressions inside the contour integrals, keeping in mind that $\oint_\gamma f = 0$ if f is a holomorphic function with no singularities inside the region bounded by γ. From this, we obtain (for the chiral part) that

$$\mathcal{R}\{T(z)\phi(w,\bar{w})\} = \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial\phi(w,\bar{w}) + \ldots, \tag{2.37}$$

where we omit writing any non-singular terms. Eq. 2.37 is known as an **operator product expansion** (or OPE), in which a product of two nearby fields is expressed as a series expansion on $(z-w)$. Furthermore, the coefficients are given in terms of other fields and their derivatives. We will usually denote OPEs in the following way:

$$T(z)\phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial\phi(w,\bar{w}), \tag{2.38}$$

where we leave the radial ordering implicit and $\sim$ denotes equality up to non-singular terms. The anti-chiral part gives an analogous relation for the OPE between $\bar{T}$ and ϕ:

$$\bar{T}(\bar{z})\phi(w,\bar{w}) \sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\bar{\partial}\phi(w,\bar{w}). \tag{2.39}$$

Eqs. 2.38 and 2.39 can be considered as an alternate way to define a primary field through its OPE with the energy-momentum tensor.

An important OPE is that between the energy-momentum tensor and itself. Here we simply state the result without proof, which reads [6]:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{2.40}$$

with c a constant which we identify as the central charge as follows. First, we perform a mode expansion for the energy-momentum tensor:

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n, \tag{2.41}$$

so that the L_n are the Laurent modes of T. The factors of 2 in the exponent will be explained later in this Section.

> **Remark 2.4.2.** Remember that for any holomorphic function $f(z)$ which admits a Laurent expansion around a point w, we can calculate its Laurent coefficients through the following contour integral:
>
> $$a_n = \frac{1}{2\pi i}\oint_\gamma dz \frac{f(z)}{(z-w)^{n+1}}, \tag{2.42}$$
>
> with γ circling w counterclockwise, such that $f(z) = \sum_n a_n (z-w)^n$.

Let us compute the charge (2.32) for the particular choice of $\epsilon(z) = \epsilon_n z^{n+1}$, corresponding to the generator ℓ_n of the conformal algebra. The chiral part gives

$$\begin{aligned} Q_n &= \frac{1}{2\pi i}\oint dz T(z)(\epsilon_n z^{n+1}) \\ &= \epsilon_n \sum_{m\in\mathbb{Z}} \frac{1}{2\pi i}\oint dz L_m z^{n-m-1} \\ &= \epsilon_n L_n. \end{aligned} \tag{2.43}$$

In this sense, we can identify the Laurent modes of the energy-momentum tensor with the generators of conformal transformations shown in Sec. 2.1. At the quantum level, their algebra should then satisfy the Virasoro algebra with some central charge. Indeed, we can compute the commutator $[L_m, L_n]$ with the same strategy as before, by expressing the Laurent modes of T in terms of contour integrals. A long calculation then gives that

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{2.44}$$

Thus, the constant c in the OPE is interpreted as the central charge of the theory. In this way, the energy-momentum tensor characterizes a conformal field theory by specifying a particular realization of the Virasoro algebra.

From the previous calculations, we can observe the following remarks:

(1) Knowing the singular part of the OPE of T with itself is equivalent to knowing the commutation relations of its Laurent modes. Indeed, this is true for general fields: knowing their OPE immediately gives us the commutation relations between their Laurent modes and viceversa. For example, using the OPE (2.38) we can compute the commutation relations of L_k and ϕ_n (omitting the anti-chiral indices, which are left unaffected):

$$[L_m, \phi_n] = ((h-1)m - n)\phi_{m+n}. \tag{2.45}$$

(2) The energy-momentum tensor is *not* a primary field, as can be seen by comparing Eq. 2.40 with the general OPE between T and a primary field ϕ. However, it can be shown that the energy-momentum tensor $T(z)$ is *quasi-primary* with conformal dimensions $(h, \bar{h}) = (2, 0)$, which explains the choice of exponents for the Laurent expansion of T (cf. Eq. 2.41).

Normal Ordering

As can be seen from the OPEs computed so far, the leading terms in the series expansion are divergent as the positions of the two fields get closer and closer. This is a general phenomenon in quantum field theory — products of fields at the same point may diverge, and hence must be regularized for them to make sense. In free field theories this is achieved by imposing a *normal ordering* on products of fields, which is equivalent to "subtracting" the divergent vacuum expectation value of said product. In conformal field theory, the analogous prescription would be to subtract the singular part of the OPE of two fields.

More specifically, if two fields $\phi(z), \chi(w)$ have OPE

$$\phi(z)\chi(w) = \sum_{n=-\infty}^{N} \frac{1}{(z-w)^n} \{\phi\chi\}(w) \tag{2.46}$$

(with N a positive integer), then we define the **normal ordered product** $N(\phi\chi)(z)$ as

$$N(\phi\chi)(z) = \{\phi\chi\}_0(z), \tag{2.47}$$

i.e. as the regular term of order zero (in $z-w$) in the OPE. With this definition, one can find that the regular part of the OPE of $\phi(z)$ and $\chi(w)$ can be naturally expressed as a sum of normal-ordered products:

$$\phi(z)\chi(w) = (\text{singular terms}) + \sum_{n=0}^{\infty} \frac{(z-w)^n}{n!} N(\partial^n\phi\chi)(w). \tag{2.48}$$

2.5 The State-Operator correspondence

We now proceed to study the structure of the Hilbert space in a conformal field theory, while a more systematic view will be given through the lens of representation theory in Section 2.6.1

As mentioned in Sec. 2.2, the Hamiltonian of a conformal field theory is given in terms of the Virasoro algebra operators as $H = L_0 + \bar{L}_0$. Therefore, energy eigenstates correspond to simultaneous eigenstates of L_0 and $\bar{L}_0$. In fact, we will now prove that an asymptotic in-state corresponding to a primary field $\phi(w, \bar{w})$ is such an energy eigenstate.

First, we derive some properties of the modes L_n when acting on the ground state. For the asymptotic in-state corresponding to $T(z)$ to be well-defined, we find that

$$L_i \left|0\right\rangle = 0, \quad i > -2. \tag{2.49}$$

Now, by making use of the commutation relations of L_0 and ϕ_n (c.f. Eq. 2.45) and Eq. 2.22 for the state $\left|\phi\right\rangle$, we find that

$$L_0 \left|\phi\right\rangle [L_0, \phi_{-h}] \left|0\right\rangle = h\phi_{-h} \left|0\right\rangle = h \left|\phi\right\rangle . \tag{2.50}$$

As stated above, we find that the state ϕ is an eigenstate for L_0 with eigenvalue h. Following a similar procedure for the anti-chiral modes, ϕ is also an eigenstate for $\bar{L}_0$ with eigenvalue $\bar{h}$. What about the action of the modes L_n with $n \neq 0$? First, observe that for $n > 0$:

$$L_n \left|\phi\right\rangle = [L_n, \phi_{-h}] \left|0\right\rangle = ((h-1)n + h)\phi_{-h+n} \left|0\right\rangle = 0. \tag{2.51}$$

However, for L_n with $n < 0$ we find a non-trivial result. First, notice from the commutation relations of the L_n modes that

$$[L_0, L_n] = -nL_n. \tag{2.52}$$

Thus, the modes L_{-n} with $n > 0$ increase the eigenvalue of an L_0 eigenstate by n. We can then construct a tower of excited states by applying these modes successively to $\left|\phi\right\rangle$:

$$L_{-k_1} L_{-k_2} \cdots L_{-k_n} \left|\phi\right\rangle . \tag{2.53}$$

Such states are called the *descendants* of $\left|\phi\right\rangle$, and have an L_0 eigenvalue equal to $h + k_1 + \cdots + k_n \equiv h + N$, where N is called the *level* of the descendant.

It turns out that each of these descendant states can be obtained as an asymptotic in-state of an associated *descendant field*. Without showing the explicit calculations, we summarize in Table 2.1 the first few descendant states and their corresponding descendant fields for levels $N = 1, 2$ and 3 [7].

State	Field	Level
$\left\|\phi\right\rangle$	$\phi(z)$	0
$L_{-1} \left\|\phi\right\rangle$	$\partial\phi(z)$	1
$L_{-1}L_{-1} \left\|\phi\right\rangle$	$\partial^2\phi(z)$	2
$L_{-2} \left\|\phi\right\rangle$	$N(T\phi)(z)$	2
$L_{-1}L_{-1}L_{-1} \left\|\phi\right\rangle$	$\partial^3\phi(z)$	3
$L_{-2}L_{-1} \left\|\phi\right\rangle$	$N(T\partial\phi)(z)$	3
$L_{-3} \left\|\phi\right\rangle$	$N(\partial T\phi)(z)$	3

Table 2.1: Descendant states and fields associated to a primary field $\phi(z)$, for levels $N \geq 3$.

In fact, for conformal field theories there is a one-to-one correspondence between the states in the Hilbert space and the operators of the theory, known as the *state-operator correspondence*. The set of all the descendants of a field $\phi(z)$ is known as its *conformal family* $[\phi]$. It can be obtained by taking normal ordered products of derivatives of ϕ and T, as seen explicitly in Table 2.1 for the first few levels.

2.6 Unitary Minimal Models

2.6.1 Highest Weight Representations of the Virasoro Algebra

It is well-known that symmetries are a convenient way to organize the spectrum of a quantum mechanical theory. Since the action of a symmetry on the Hilbert space commutes with the Hamiltonian, the energy eigenstates of the theory can be organized into various representations of the corresponding symmetry group. For this reason, we expect that the energy eigenstates of a conformal field theory will split into representations of the Virasoro algebra. Since the chiral sector of the algebra does not interact with the anti-chiral sector, we can study them separately and then take tensor products to obtain a proper representation of the full Virasoro algebra.

First, we choose the generator L_0 to be the only diagonal generator (as it doesn't commute with the rest of L_n generators) in the representation space, which is commonly known as a *Verma module*. We denote an eigenstate of L_0 through its eigenvalue h by $|h\rangle$, such that

$$L_0 |h\rangle = h |h\rangle . \tag{2.54}$$

As we saw in Section 2.5, the L_n generators act as ladder operators for L_0, where they are raising operators for $n < 0$ and lowering operators for $n > 0$. We say that $|h\rangle$ is a highest-weight state in the Verma module if it is annihilated by all lowering operators, i.e.

$$L_n |h\rangle = 0, \quad n > 0. \tag{2.55}$$

We can then generate other states in the module by successive application of the raising operators L_{-n} (with $n > 0$).

We can also define an inner product in the Verma module through our definition of hermitian conjugation of modes: $L_n^\dagger = L_{-n}$ (c.f. Eq. 2.25). The inner product of $L_{-r_1} \cdots L_{-r_n} |h\rangle$ and $L_{-s_1} \cdots L_{-s_m} |h\rangle$ is then defined as

$$\langle h| L_{r_1} \cdots L_{r_n} L_{-s_1} \cdots L_{-s_m} |h\rangle , \tag{2.56}$$

with $\langle h| = |h\rangle^\dagger$ the dual state to $|h\rangle$. The hermiticity of L_0 then forces h to be real.

The same procedure follows for the anti-chiral sector. For a particular choice of the central charge in Vir_c, we denote the Verma module generated by a highest weight vector $|h\rangle$ as $V_{h,c}$ (while for the anti-chiral sector we denote the module generated by $|\bar{h}\rangle$ as $\bar{V}_{\bar{h},c}$). In general, the Hilbert space of a CFT will have the form

$$\bigoplus_{h,\bar{h}} V_{h,c} \otimes \bar{V}_{\bar{h},c}. \tag{2.57}$$

The reader may have already noticed the similarity of this process with the analysis of the CFT Hilbert space in Sec. 2.5. Indeed, we can recognize the asymptotic in-state of a primary field $\phi(z)$ with conformal dimensions $(h, \bar{h})$ as a highest-weight vector with L_0 eigenvalue h and $\bar{L}_0$ eigenvalue $\bar{h}$. Its corresponding Verma module will be generated by its descendant states. Finally, the Hilbert space of the theory will be the direct sum of the Verma modules corresponding to each of the fields present in the theory.

It may happen that for certain values of h and c, the Verma module generated by a state $|h\rangle$ turns out to be *reducible*. This means that it contains subspaces which are by themselves Verma (sub)modules. Such submodules are generated by a highest-weight state $|\chi\rangle \in V_{h,c}$. As it is annihilated by all the L_n with $n > 0$, this state is found to be a *null vector*, i.e. it has zero norm. In fact, its inner product with any other state of the Verma module turns out to be zero:

$$\langle h| L_{k_1} \cdots L_{k_n} |\chi\rangle = 0. \tag{2.58}$$

Hence in particular, $\langle \chi|\chi\rangle = \||\chi\rangle\|^2 = 0$. We can easily obtain an irreducible Verma module from $V_{h,c}$ by quotienting it out by the submodule generated by χ.

However, it may also happen that a Verma module contains states with negative norm, which are not allowed in unitary theories. Unitarity then imposes a constraint on the allowed values of h. In other words, there are only certain values allowed for the conformal dimensions of primary fields. Without giving the explicit calculations, we simply present the allowed values of h depending on the central charge of the theory [6]:

(1) For $c \geq 1$, all representations with $h \geq 0$ are unitary.

(2) For $0 < c < 1$, there is only a discrete set of points (c, h) such that $V_{h,c}$ corresponds to a unitary representation of the Virasoro algebra. The central charge is constrained to be of the form

$$c = 1 - \frac{6}{m(m+1)}, \tag{2.59}$$

with $m \geq 2$ an integer. For a particular value of c, the allowed values for h are given by:

$$h_{r,s}(m) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}, \tag{2.60}$$

with r, s integers such that $1 \leq r \leq m-1$ and $1 \leq s \leq r-1$.

Conformal field theories with central charges given by 2.59 are known as *unitary minimal models*. As noted above, such theories only allow for a finite number of highest-weight representations, i.e. a finite number of primary fields. More generally, it is known that theories with a finite number of primaries must necessarily have a *rational* central charge, hence they are referred to as *rational conformal field theories* (RCFTs). More generally, an RCFT is a theory where the primary fields can be organized in a finite set of irreducible representations of an extended symmetry algebra (which enhances the Virasoro algebra of the theory) [6]. We will see examples of such extended symmetries in Section 3.

To conclude, we must specify the way in which the chiral and anti-chiral sectors of the theory combine to form the full-fledged Hilbert space. A simple solution is to choose the combination

$$\mathcal{H} = \bigoplus_h V_{h,c} \otimes \bar{V}_{h,c}, \tag{2.61}$$

where the sum goes exactly once through all the allowed values of h for the given central charge c. Such theories are called *diagonal*. Although other choices can be made, we will restrict ourselves to diagonal theories throughout this work.

2.6.2 Fusion Rules

For unitary minimal models, the existence of null vectors highly constrains the form of the OPEs between primary fields. We will not present the detailed calculations, but simply state the final result [7]. Denote by $\phi_{(p,q)}$ the primary field which has conformal dimension $h_{p,q}$. Then:

$$[\phi_{(p_1,q_1)}] \times [\phi_{(p_2,q_2)}] = \sum_{\substack{k=1+|p_1-p_2| \\ k+p_1+p_2 \text{ odd}}}^{p_1+p_2-1} \sum_{\substack{l=1+|q_1-q_2| \\ l+q_1+q_2 \text{ odd}}}^{q_1+q_2-1} [\phi_{(k,l)}]. \tag{2.62}$$

The above formula is interpreted as follows: the OPE between a field in the conformal family of $\phi_{(p_1,q_1)}$ and a field in the conformal family of $\phi_{(p_2,q_2)}$ only contains fields which are in the conformal families of the allowed $\phi_{(k,l)}$. Thus for unitary minimal models, the OPEs define an algebraic structure on the space of fields, known as the *fusion rules* of the theory.

A similar phenomenon happens for arbitrary rational conformal field theories. In general, the OPEs of primary fields define a *fusion algebra* of the form

$$[\phi_i] \times [\phi_j] = \sum_k N_{ij}{}^k [\phi_k], \tag{2.63}$$

where the indices label all the possible primary fields of the theory[2]. It can be shown that the *fusion coefficients* $N_{ij}{}^k$ must be positive integers, and that they are only zero if and only if the three point function $\langle \phi_i \phi_j \phi_k \rangle$ vanishes. Finally, we note that it is usual convention to drop the brackets on the conformal families when writing out their fusion rules.

Let us make a few extra comments on the fusion algebra 2.63. It is both a commutative and associative algebra, which follows from the commutativity and associativity of the OPE. The unit element corresponds to the conformal family of the identity operator $[\mathbb{1}]$. At the level of the fusion coefficients, these three conditions imply that

$$\begin{cases} N_{i1}{}^k = \delta_{ik} \\ N_{ij}{}^k = N_{ji}{}^k \\ \sum_l N_{kj}{}^l N_{il}{}^m = \sum_l N_{ij}{}^l N_{lk}{}^m \end{cases} \tag{2.64}$$

[2] More accurately, in this expression the indices label the chiral parts of the fields. A similar relation holds for the anti-chiral parts.

Example: The Ising CFT

One of the simplest (diagonal) minimal models is obtained for $m = 3$, where $c = 1/2$ and there are only 3 allowed values for h: $0, \frac{1}{2}$ and $\frac{1}{16}$. This theory is known as the *Ising CFT*, for its relation to the Ising model at the critical point [9]. By comparison with the statistical model, the primary fields of the theory are interpreted to be the spin operator σ and the energy density operator ε. Their conformal dimensions are given by

$$(h, \bar{h})_\sigma = \left(\frac{1}{16}, \frac{1}{16}\right), \qquad (h, \bar{h})_\varepsilon = \left(\frac{1}{2}, \frac{1}{2}\right). \tag{2.65}$$

Of course, we also have the identity operator $\mathbb{1}$ with conformal dimensions $(0, 0)$. The fusion rules of the theory can be easily obtained from Eq. 2.62. We organize them in a table as follows, omitting the trivial fusion rules for the identity operator:

$\times$	σ	ε
σ	$\mathbb{1} + \varepsilon$	
ε	σ	$\mathbb{1}$

Table 2.2: Fusion rules for the Ising CFT.

CHAPTER 3

Wess-Zumino-Witten Models

In Chapter 2, we reviewed some of the basic properties of conformal field theories in two dimensions. By exploiting the infinite-dimensional nature of the local conformal algebra $\mathrm{Vir}_c \oplus \overline{\mathrm{Vir}}_c$, we managed to find a variety of results (such as the existence of the unitary minimal models) by studying the representation theory of the Virasoro algebra.

We now turn our attention to conformal field theories where the local conformal symmetry is *enhanced* by extra symmetries, in the sense that the theory has a larger symmetry algebra compatible with the Virasoro algebra. In fact, there is a special set of CFTs known as *Wess-Zumino-Witten models* (or WZW models for short), whose symmetries are associated to general Lie algebras [10]. Although WZW models have an explicit Lagrangian description, we will instead investigate the structure of these models through the lens of representation theory, just as we did for the Virasoro algebra in the previous chapter.

3.1 Current Algebras

Wess-Zumino-Witten models are characterized by the existence of a set of fields called *currents* — chiral primary fields $j^a(z)$ with conformal dimension $h = 1$. The OPEs between these currents are found to be of the following form [7]:

$$j^a(z) j^b(w) \sim \frac{k\delta_{ab}}{(z-w)^2} + \sum_c \frac{i f_{abc}}{z-w} j^c(w), \tag{3.1}$$

with k and f_{abc} some set of scalar coefficients. These relations define what is known as the *current algebra* of the model. The associativity of the OPE between three such currents implies that the f_{abc} satisfy the Jacobi identity — thus, one can interpret them as the structure constants of some Lie algebra $\mathfrak{g}$. We will limit ourselves to consider either semisimple Lie algebras or the abelian $\mathfrak{u}(1)$ algebra and direct sums between them.

As we already know, the OPE between fields is equivalent to the set of commutation relations between their Laurent modes. If we introduce the Laurent expansion of the $j^a(z)$,

$$j^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} j_n^a, \tag{3.2}$$

then Eq. 3.1 traduces into the following commutation relations for the j_n^a:

$$[j_n^a, j_m^b] = \sum_c i f_{abc} j_{n+m}^c + kn\delta_{ab}\delta_{n+m,0}. \tag{3.3}$$

If the f_{abc} are the structure constants of a Lie algebra $\mathfrak{g}$, the above relations define what is known as the *affine Lie algebra* $\hat{\mathfrak{g}}_k$ associated to $\mathfrak{g}$. Mathematically, this algebra can be constructed as a particular central extension of $\mathfrak{g}$ characterized by a constant k, known as the *level* of the affine algebra. Moreover, notice that the zero modes of the currents satisfy the commutation relations of $\mathfrak{g}$:

$$[j_0^a, j_0^b] = i\sum_c f_{abc} j_0^c. \tag{3.4}$$

Thus, the Lie algebra $\mathfrak{g}$ is naturally a finite subalgebra of its affinization $\hat{\mathfrak{g}}_k$.

In what follows, we will need to understand some of the basic properties and definitions within the theory of Lie and affine Lie algebras, specially those related to their representation theory. We summarize the necessary technical notions in Appendices A and B.

3.2 The Sugawara Construction

As mentioned in the introduction to this section, this current algebra is compatible with the Virasoro algebra in some natural way. In fact, there exists an intrinsic definition of the energy-momentum tensor in terms of the currents of the WZW model. This particular definition is known as the *Sugawara energy-momentum tensor* [11], and we will study it in what follows.

For the moment, assume that the Lie algebra $\mathfrak{g}$ associated to the model is simple. The Sugawara construction starts by making the following ansatz for the energy-momentum tensor:

$$T(z) = \gamma \sum_a N(j^a j^a)(z). \tag{3.5}$$

Here γ is an arbitrary constant which one can fix by demanding the currents to have conformal dimension $h = 1$ with respect to $T(z)$ field, i.e. that their OPEs with $T(z)$ are of the form (2.38) for $h = 1$. The necessary calculations yield the following value for γ:

$$\gamma = \frac{1}{2(k + C_{\mathfrak{g}})}, \tag{3.6}$$

where $C_{\mathfrak{g}}$ is known as the dual Coxeter number of $\mathfrak{g}$ (c.f. Appendix A). Furthermore, by calculating the OPE of $T(z)$ with itself we can obtain the central charge of the theory, which is found to be

$$c = \frac{k \dim \mathfrak{g}}{k + C_{\mathfrak{g}}}. \tag{3.7}$$

One can further verify the central charge is bounded from below by the rank of $\mathfrak{g}$, and from above by the dimension of $\mathfrak{g}$. This implies that WZW models always have central charge $c \geq 1$.

We can now obtain the Laurent modes L_n of the Sugawara energy-momentum tensor by writing out its Laurent expansion. They can be expressed in terms of the Laurent modes of the currents as

$$L_n = \frac{1}{2(k + C_{\mathfrak{g}})} \sum_a \left(\sum_{m \leq -1} j^a_m j^a_{n-m} + \sum_{m \geq 0} j^a_{n-m} j^a_m \right). \tag{3.8}$$

The fact that $T(z)$ is an energy-momentum tensor implies that these modes satisfy the Virasoro algebra (2.8). Moreover, we can also find the commutation relations between the modes L_n and j^a_m:

$$[L_n, j^a_m] = -m j^a_{n+m}. \tag{3.9}$$

In particular, the zero modes of the currents commute with the generators of the Virasoro algebra, implying that the corresponding WZW model has a local G symmetry, where G is a Lie group with $\mathfrak{g}$ as its Lie algebra[1].

Remark 3.2.1. We have chosen a particular "basis" for the currents $j^a(z)$, in which the modes j^a_n are orthonormal with respect to the Killing form of the affine Lie algebra:

$$K(j^a_n, j^b_m) = \delta^{ab} \delta_{n+m,0}. \tag{3.10}$$

In an arbitrary basis J^a, the commutation relations read as in Eq. B.2:

$$[J^a_n, J^b_m] = \sum_c i \tilde{f}^{ab}{}_c J^c_{n+m} + knK(J^a_0, J^b_0)\delta_{n+m,0}, \tag{3.11}$$

with $\tilde{f}^{ab}{}_c$ the structure constants in the new basis. The Sugawara energy-momentum tensor is then given by

$$T(z) = \frac{1}{2(k + C_{\mathfrak{g}})} \sum_{a,b} \frac{1}{K(J^a_0, J^b_0)} N(J^a J^b)(z). \tag{3.12}$$

We can now extend the Sugawara construction to the case of semisimple Lie algebras. Since a semisimple Lie algebra can be expressed as a direct sum of simple Lie algebras, the Sugawara energy-momentum tensor is simply the sum of energy-momentum tensors of each factor. All these energy-momentum tensors commute with each other (since the currents corresponding to each factor commute), and thus the central charge of the theory is also the sum of the central charges.

Example 3.2.1. Consider the Lie algebra $\mathfrak{su}(2)$. Its dimension and dual Coxeter number are given by $\dim \mathfrak{su}(2) = 3$ and $C_{\mathfrak{su}(2)} = 2$ respectively. Thus, the $\widehat{\mathfrak{su}}(2)_k$-WZW model has central charge

$$c = \frac{3k}{2 + k}. \tag{3.13}$$

[1] To be more accurate, we should also take into account the anti-chiral sector. Aside from the currents j^a, WZW models have anti-chiral currents (i.e. primary fields with $\bar{h} = 1$) which do not interact with the chiral ones, but also generate an affine $\hat{\mathfrak{g}}_k$ algebra. Thus, the theory actually enjoys a local $G \times \bar{G}$ symmetry.

We can express the current algebra in terms of the Cartan-Weyl basis of $\widehat{\mathfrak{su}}(2)_k$. If we label the currents as H, E^+ and E^-, then the commutation relations of their modes are given by (cf. Appendix B)

$$\begin{aligned}
[H_n, H_m] &= kn\delta_{n+m,0}, \\
[H_n, E^{\pm}_m] &= \pm\sqrt{2}E^{\pm}_{n+m}, \\
[E^+_n, E^-_m] &= \sqrt{2}H_{n+m} + kn\delta_{n+m,0}.
\end{aligned} \tag{3.14}$$

As usual, the above relations translate into the following OPEs for the currents:

$$\begin{aligned}
H(z)H(w) &\sim \frac{k}{(z-w)^2}, \\
H(z)E^{\pm}(w) &\sim \frac{\pm\sqrt{2}}{z-w}E^{\pm}(w), \\
E^+(z)E^-(w) &\sim \frac{k}{(z-w)^2} + \frac{\sqrt{2}}{z-w}H.
\end{aligned} \tag{3.15}$$

3.3 WZW Primary Fields

In Section 2.2 we defined the concept of a primary field, which transforms in a distinguished way with respect to the Virasoro algebra. For WZW models, it will also be useful to define fields which transform in a "covariant" way with respect to the local G symmetry.

Definition 3.3.1. A chiral field $\phi_\lambda(z)$ is **WZW primary** if its OPE with the currents $j^a(z)$ is of the form

$$j^a(z)\phi_\lambda(w) \sim -\frac{t^a_\lambda \phi_\lambda(w)}{z-w}, \tag{3.16}$$

where λ labels a representation of $\mathfrak{g}$ and t^a_λ are the generators of $\mathfrak{g}$ in that particular representation[2].

The above OPE will result in a certain action of the current modes on the asymptotic in-state $|\phi_\lambda\rangle = \lim_{z\to 0}\phi_\lambda(z)\,|0\rangle$, which is found to be as follows:

$$\begin{aligned}
j^a_0\,|\phi_\lambda\rangle &= -t^a_\lambda\,|\phi_\lambda\rangle, \\
j^a_n\,|\phi_\lambda\rangle &= 0, \quad \text{for } n>0.
\end{aligned} \tag{3.17}$$

We now prove that these relations imply that WZW primary fields are also primary fields in the usual sense (i.e. Virasoro primary fields). Indeed, by employing the explicit form of the L_n modes in terms of the current modes (3.8), one finds that

$$\begin{aligned}
L_0\,|\phi_\lambda\rangle &= \frac{1}{2(k+C_{\mathfrak{g}})}\sum_a j^a_0 j^a_0\,|\phi_\lambda\rangle, \\
L_n\,|\phi_\lambda\rangle &= 0, \quad \text{for } n>0.
\end{aligned} \tag{3.18}$$

[2]We use the notation $t^a_\lambda\phi_\lambda$ as a compact way to write $\sum_j (t^a_\lambda)_{ij}(\phi_\lambda)_j$, where the matrix indices run from 1 to the dimension of the representation $\dim\lambda$.

We recognize the action of L_0 to be proportional to the action of the Casimir operator of $\mathfrak{g}$, given by $\sum_a t^a_\lambda t^a_\lambda$. Since the vector space generated by the $|\phi_\lambda\rangle$ naturally has a $\mathfrak{g}$-module structure for the representation λ, the Casimir operator acts on them simply through multiplication by a constant equal to $(\lambda, \lambda+2\rho)$ [6]. Thus, we conclude that the field $\phi_\lambda(z)$ is a Virasoro primary with conformal dimension

$$h_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2(k + C_\mathfrak{g})}. \tag{3.19}$$

Note that the action of the t^a_λ on the vector $|\phi_\lambda\rangle$ has not been fully determined yet. However, we would ideally want to have a definition of primary fields where all states in the $\mathfrak{g}$-module can be somehow obtained from the state $|\phi_\lambda\rangle$. Indeed, this will happen if this vector is the highest-weight state of the representation. Denote then the state as $|\hat{\lambda}\rangle$, where $\hat{\lambda}$ stands for the extension of the highest weight λ to an affine weight. In the same fashion, we denote the corresponding WZW primary field as $\hat{\lambda}(z)$. The state $|\hat{\lambda}\rangle$ then satisfies the following conditions:

$$E^{\pm\alpha}_n |\hat{\lambda}\rangle = H^i_n |\hat{\lambda}\rangle = E^\alpha_0 |\hat{\lambda}\rangle = 0, \tag{3.20}$$

for $n > 0$ and $\alpha \in \Delta_+$. Moreover, $|\hat{\lambda}\rangle$ is an eigenvector of H^i_0 with eigenvalue λ^i — in the Chevalley basis, this implies that

$$h^i_0 |\hat{\lambda}\rangle = \lambda_i |\hat{\lambda}\rangle, \tag{3.21}$$

with λ_i the corresponding Dynkin labels of λ. From now on, we will refer to states with the above properties as *WZW primary states*.

It can be shown [6] that the relevant highest-weight states one should consider are those corresponding to integrable representations $\hat{\lambda} \in P^k_+$. In fact, any correlation function in the WZW model involving a field associated to a non-integrable representation vanishes. Furthermore, since the set of integrable representations at any level k is finite, we conclude that there is only a finite number of WZW primary fields in any WZW model. However, in general there may be an infinite number of Virasoro primary fields, which are reorganized according to the integrable representations of $\hat{\mathfrak{g}}_k$. In this sense, WZW models satisfy our definition of a rational conformal field theory.

Finally, we observe that WZW models are also unitary as a consequence of the unitarity of the integrable representations of $\hat{\mathfrak{g}}_k$ [6]. Moreover, as in the case of the unitary minimal models, we must choose a way in which the chiral and anti-chiral sectors mix together to form the full Hilbert space of the theory. We will again limit ourselves to the study of diagonal theories, in which fields transform in the same representation in both the chiral and anti-chiral sectors.

3.4 Fusion Rules of WZW Models

We now turn our attention to the study of the fusion algebra between WZW primary fields. Given that the primaries of a $\hat{\mathfrak{g}}_k$-WZW model are in one-to-one correspondence to the integrable representations of its associated affine Lie algebra, the fusion rules take the form

$$\hat{\lambda} \times \hat{\mu} = \sum_{\hat{\nu} \in P^k_+} N^{(k)\,\hat{\nu}}_{\hat{\lambda}\hat{\mu}} \hat{\nu}. \tag{3.22}$$

We will not delve into the general properties of the fusion coefficients and their computation, which is a lengthy process. For the purposes of our work, it will suffice to know that there are explicit expressions for the fusion coefficients of the $\widehat{\mathfrak{su}}(2)_k$ and $\widehat{\mathfrak{su}}(3)_k$-WZW models, which we now present.

$\widehat{\mathfrak{su}}(2)_k$ Fusion Coefficients

The explicit formula for the fusion coefficients in $\widehat{\mathfrak{su}}(2)_k$ was first found in [12]. It reads:

$$N_{\hat{\lambda}\hat{\mu}}^{(k)\,\hat{\nu}} = \begin{cases} 1, & \begin{array}{c}\text{if } |\lambda_1 - \mu_1| \leq \nu_1 \leq \min\{\lambda_1 + \mu_1, 2k - \lambda_1 - \mu_1\} \\ \text{and } \lambda_1 + \mu_1 + \nu_1 = 0 \bmod 2;\end{array} \\ 0, & \text{otherwise.} \end{cases} \tag{3.23}$$

$\widehat{\mathfrak{su}}(3)_k$ Fusion Coefficients

The case of $\widehat{\mathfrak{su}}(3)_k$ is a bit more involved. Here we present the explicit expression for the coefficients

$$N_{\hat{\lambda}\hat{\mu}\hat{\nu}}^{(k)} = N_{\hat{\lambda}\hat{\mu}}^{(k)\,\hat{\nu}^*}, \tag{3.24}$$

where $\hat{\nu}^*$ denotes the affine weight associated to the weight conjugate to ν. For the particular case of $\mathfrak{su}(N)$, conjugating a weight simply reverses the order of its Dynkin labels, i.e. $\lambda_i^* = \lambda_{N-1-i}$. Thus, the coefficients $N_{\hat{\lambda}\hat{\mu}}^{(k)\,\hat{\nu}}$ can be easily obtained from the ones mentioned above. The explicit formula was found in [13], and requires some previous definitions.

First we define the coefficients

$$\mathcal{N}_{\lambda\mu\nu} = \delta_{k_0}(k_0^{\max} - k_0^{\min} + 1), \tag{3.25}$$

where

$$\begin{aligned} k_0^{\min} &= \max\{\lambda_1 + \lambda_2, \mu_1 + \mu_2, \nu_1 + \nu_2, L_1 - \min\{\lambda_1, \mu_1, \nu_1\}, L_2 - \min\{\lambda_2, \mu_2, \nu_2\}\}, \\ k_0^{\max} &= \min\{L_1, L_2\}, \end{aligned} \tag{3.26}$$

and

$$\delta_{k_0} = \begin{cases} 1, & \text{if } k_0^{\max} \geq k_0^{\min} \text{ and } L_1, L_2 \in \mathbb{Z}_{\geq 0}; \\ 0, & \text{otherwise.} \end{cases} \tag{3.27}$$

Furthermore, the L_i are given by

$$L_i = (\lambda + \mu + \nu, \omega_i), \tag{3.28}$$

where ω_1, ω_2 are the fundamental weights of $\mathfrak{su}(3)$. With this definitions, the fusion coefficients (3.24) are given by

$$N_{\hat{\lambda}\hat{\mu}\hat{\nu}}^{(k)} = \delta_{k_0} \times \begin{cases} 0, & \text{if } k < k_0^{\min}; \\ \mathcal{N}_{\lambda\mu\nu} - (k_0^{\max} - k), & \text{if } k_0^{\min} \leq k \leq k_0^{\max}; \\ \mathcal{N}_{\lambda\mu\nu}, & \text{if } k > k_0^{\max}. \end{cases} \tag{3.29}$$

CHAPTER 4

Coset Theories

Up to now we have found a variety of examples of conformal field theories, such as the unitary minimal models and the $\hat{\mathfrak{g}}_k$-WZW models. In this chapter, we will find an even larger family of conformal field theories known as *coset theories*, which are defined through "quotients" of WZW models. In fact, coset theories include as particular cases the models we have previously studied.

A special kind of coset theories are those which contain *parafermionic* degrees of freedom, i.e. fields with fractional spin. The usual construction of such theories involve quotienting out a $\mathfrak{u}(1)$ algebra inside $\mathfrak{su}(2)$. We will study these theories in some detail, discussing their field content and how to represent the latter in terms of vertex operators.

However, one could construct even more general parafermionic theories by modding out a $\mathfrak{u}(1)^{\mathrm{rk}\mathfrak{g}}$ algebra inside an arbitrary semisimple Lie algebra $\mathfrak{g}$. This construction was proposed by Gepner in 1987 [5], but the properties of such theories have not been fully explored yet. This leads us to investigate the $\widehat{\mathfrak{su}}(3)_k/\widehat{\mathfrak{u}}(1)^2$ parafermion theory in more detail, computing the field content of the theory as well as its fusion rules.

4.1 The Coset Construction

Consider the $\hat{\mathfrak{g}}_k$-WZW model associated to a semisimple Lie algebra $\mathfrak{g}$, and consider a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. This embedding can be generalized to an embedding of affine algebras $\hat{\mathfrak{h}}_{k'} \subset \hat{\mathfrak{g}}_k$ — thus we can also consider the $\hat{\mathfrak{h}}_{k'}$-WZW model contained inside the original one. Of course, there must be a relationship between the levels k and k', constrained by the structure of the two Lie algebras. This relation is given by

$$k' = kx_e, \tag{4.1}$$

where x_e is the embedding index corresponding to $\mathfrak{h} \hookrightarrow \mathfrak{g}$ (c.f. Appendix A).

We can now write out the Sugawara energy-momentum tensors for each WZW model:

$$T_{\mathfrak{g}}(z) = \frac{1}{2(k + C_{\mathfrak{g}})} \sum_a N(j^a_{\mathfrak{g}} j^a_{\mathfrak{g}})(z), \tag{4.2}$$

$$T_{\mathfrak{h}}(z) = \frac{1}{2(k' + C_{\mathfrak{h})}} \sum_b N(j^b_{\mathfrak{h}} j^b_{\mathfrak{h}})(z). \tag{4.3}$$

Of course, the currents which generate the $\hat{\mathfrak{h}}_{k'}$ algebra are particular linear combinations of those which generate the full $\hat{\mathfrak{g}}_k$ algebra. In particular, they are primary fields with conformal dimension $h = 1$ with respect to both $T_{\mathfrak{g}}(z)$ and $T_{\mathfrak{h}}(z)$. Thus, the OPEs of the energy-momentum tensors with the $j^b_{\mathfrak{h}}$ are as follows:

$$\begin{aligned} T_{\mathfrak{g}}(z) j^b_{\mathfrak{h}}(w) &\sim \frac{1}{(z-w)^2} j^b_{\mathfrak{h}}(w) + \frac{1}{z-w} \partial j^b_{\mathfrak{h}}(w), \\ T_{\mathfrak{h}}(z) j^b_{\mathfrak{h}}(w) &\sim \frac{1}{(z-w)^2} j^b_{\mathfrak{h}}(w) + \frac{1}{z-w} \partial j^b_{\mathfrak{h}}(w). \end{aligned} \tag{4.4}$$

Since both expressions are equal, we can substract them to get an OPE $\big(T_{\mathfrak{g}}(z) - T_{\mathfrak{h}}(z)\big) j^b_{\mathfrak{h}}(w)$ that only consists of regular terms. Furthermore, since $T_{\hat{\mathfrak{h}}}(z)$ is constructed from the currents $j^b_{\hat{\mathfrak{h}}}$, this implies that the OPE $(T_{\hat{\mathfrak{g}}}(z) - T_{\hat{\mathfrak{h}}}(z)) T_{\hat{\mathfrak{h}}}(z)$ is regular as well. At the level of commutation relations, a regular OPE is equivalent to the fact that the modes of each field commute with each other. In our particular case, this means that

$$[L^{\mathfrak{g}}_n - L^{\mathfrak{h}}_n, L^{\mathfrak{h}}_m] = 0. \tag{4.5}$$

We will now verify that the modes $L^{\mathfrak{g}}_n - L^{\mathfrak{h}}_n \equiv L^{\mathfrak{g}/\mathfrak{h}}_n$ satisfy the Virasoro algebra commutation relations. Indeed,

$$\begin{aligned} [L^{\mathfrak{g}/\mathfrak{h}}_n, L^{\mathfrak{g}/\mathfrak{h}}_m] &= [L^{\mathfrak{g}}_n, L^{\mathfrak{g}}_m] - [L^{\mathfrak{h}}_n, L^{\mathfrak{h}}_m] \\ &= (m-n) L^{\mathfrak{g}/\mathfrak{h}}_{m+n} + (c_{\hat{\mathfrak{g}}_k} - c_{\hat{\mathfrak{h}}_{k'}}) \frac{m^3 - m}{12} \delta_{m+n,0}. \end{aligned} \tag{4.6}$$

We conclude that the theory associated to the energy-momentum tensor $(T_{\mathfrak{g}} - T_{\mathfrak{h}})(z) \equiv T_{\mathfrak{g}/\mathfrak{h}}(z)$ is a consistent conformal field theory with central charge

$$c_{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{k'}} \equiv c_{\hat{\mathfrak{g}}_k} - c_{\hat{\mathfrak{h}}_{k'}}. \tag{4.7}$$

Hence, through this procedure (known in the literature as the Goddard-Kent-Olive construction [14]) we have built a new theory out of a pair of WZW models, known as the *coset theory* $\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{k'}$.

Example 4.1.1. Consider the family of coset theories given by

$$\frac{\widehat{\mathfrak{su}}(2)_k \oplus \widehat{\mathfrak{su}}(2)_1}{\widehat{\mathfrak{su}}(2)_{k'}}. \tag{4.8}$$

These theories are known as *diagonal* since we construct them through the affinization of the natural embedding $\mathfrak{su}(2) \hookrightarrow \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, where the embedded $\mathfrak{su}(2)$ is generated by the sum of generators of each parent $\mathfrak{su}(2)$:

$$j^a_{\text{diag}}(z) = j^a_{(1)} + j^a_{(2)}. \tag{4.9}$$

Since the generators j^a_1 and j^b_1 commute by construction, one can easily derive from the commutation relations of the modes of j^a_{diag} that the level k' must be equal to the sum of the parent levels $k' = k + 1$. Finally, if we compute the central charge of the theory with help of Eqs. 3.13 and 4.7, we obtain:

$$c = \frac{3k}{k+2} + 1 - \frac{3(k+1)}{k+3} = 1 - \frac{6}{(k+2)(k+3)}. \tag{4.10}$$

After switching to a new variable $m \equiv k+2$, we identify the central charges of these theories to be exactly those of the unitary minimal models (cf. Eq. 2.59). Indeed, it can be rigorously proved that these coset theories are indeed the unitary minimal models we have studied before [6].

4.2 Field Identifications

We can now discuss the field content of the coset theory, starting from the field content of the $\hat{\mathfrak{g}}_k$ and $\hat{\mathfrak{h}}_{k'}$ WZW models. As seen in Sec. 3.3, (WZW) primary fields of a WZW model are in one-to-one correspondence to the integrable representations of its associated affine Lie algebra. Furthermore, corresponding to an embedding $\hat{\mathfrak{h}}_{k'} \subset \hat{\mathfrak{g}}_k$ of affine Lie algebras there is an associated set of *branching rules*, which tell us how to "split" an integrable representation $\hat{\lambda}$ of $\hat{\mathfrak{g}}_k$ into different integrable representations $\hat{\mu}$ of $\hat{\mathfrak{h}}_{k'}$:

$$\hat{\lambda} \longmapsto \bigoplus_{\hat{\mu}} b_{\hat{\lambda},\hat{\mu}} \hat{\mu}. \tag{4.11}$$

A grosso modo, the branching functions $b_{\hat{\lambda},\hat{\mu}}$ will in some sense correspond to the primary fields $\Phi^{\hat{\lambda}}_{\hat{\mu}}$ of the coset theory. A more accurate version of this statement involves a discussion of character theory, which we do not cover throughout this work. We will simply summarize the main results which follow from the analysis of the characters involved in the branching (4.11) — the reader can find a detailed account of character theory and its relation to conformal field theory in [6].

It turns out that not all the primary fields $\Phi^{\hat{\lambda}}_{\hat{\mu}}$ are different for different values of the weights $\hat{\lambda}$ and $\hat{\mu}$ — in other words, we need to identify some fields with each other. The issue of *field identifications* was solved in the general case by Gepner [15], who obtained the relations

$$\Phi^{\hat{\lambda}}_{\hat{\mu}} \sim \Phi^{A\hat{\lambda}}_{\tilde{A}\hat{\mu}}. \tag{4.12}$$

Here A denotes a transformation on the weights known as an *outer automorphism*, and $\tilde{A}$ is such that $\mathcal{P}A = \tilde{A}\mathcal{P}$. Without delving into the mathematical details of such transformations, it will suffice to know how outer automorphisms act on affine weights for algebras of the type $A_r = \mathfrak{su}(r+1)$. For these algebras, the outer automorphism group $O(\hat{A}_{r+1})$ is isomorphic to $\mathbb{Z}_{r+1}$, and is generated by an element a which acts as [6]

$$a[\lambda_0, \lambda_1, \ldots, \lambda_r] = [\lambda_r, \lambda_0, \ldots, \lambda_{r-1}]. \tag{4.13}$$

Together with the branching condition (A.39), these two relations completely fix the field spectrum of the coset theory. We will investigate concrete examples of these field identifications in what follows. The analysis of the characters also imply that the conformal dimension of $\Phi^{\hat{\lambda}}_{\hat{\mu}}$ is $h_\lambda - h_\mu + n^\lambda_\mu$, for n^λ_μ an integer known as the *grade* of $\hat{\mu}$ in $\hat{\lambda}$. Finally, one also has the following relation between the primary fields G_λ of $\hat{\mathfrak{g}}_k$, H_μ of $\hat{\mathfrak{h}}_{k'}$, and Φ^λ_μ of the coset theory [15]:

$$G_\lambda(z) = \Phi^\lambda_\mu(z) H_\mu(z). \tag{4.14}$$

4.3 The $\mathbb{Z}_k$ Parafermions

We now proceed to study the coset theory

$$\frac{\widehat{\mathfrak{su}}(2)_k}{\widehat{\mathfrak{u}}(1)}, \tag{4.15}$$

known as the $\mathbb{Z}_k$ **parafermion CFT**. It was first studied by Zamolodchikov and Fateev as the CFT arising from a lattice statistical system with $\mathbb{Z}_k$ symmetry [16]. The name "parafermion" will become clearer in what follows — for now, we point out that the theory has central charge

$$c = \frac{3k}{k+2} - 1 = \frac{2(k-1)}{k+2}. \tag{4.16}$$

For $k = 2$, we have $c = 1/2$ and the $\mathbb{Z}_2$ parafermions turn out to be equivalent to the Ising CFT.

In order to study the field spectrum of the theory, we start by determining the branching rules of $\widehat{\mathfrak{su}}(2)_k$ representations into $\widehat{\mathfrak{u}}(1)$ representations. A highest weight representation (λ) of $\mathfrak{su}(2)$ naturally splits into $\mathfrak{u}(1)$ representations by the branching

$$(\lambda) \longmapsto (\lambda)_1 \oplus (\lambda-2)_1 \oplus \cdots \oplus (-\lambda)_1 = \bigoplus_{\substack{m=-\lambda \\ \lambda-m=0 \bmod 2}}^{\lambda} (m)_1, \tag{4.17}$$

where each m corresponds to a weight in the weight system of λ (c.f. Example A.6.1), and the subscript denotes the fact that the $\mathfrak{u}(1)$ representations are 1-dimensional. We label said representations through the value of their generator, known as the *charge* of the representation. The above splitting is equivalent to the branching condition A.39, which in this particular case reads $\lambda - m = 0 \bmod 2$.

We now consider the affinization of the branching (4.17). At level k, we know all $\mathfrak{su}(2)$ representations with $0 \le \lambda \le k$ are allowed (since they correspond to integrable representations in P_+^k). However, we are not yet familiar with the affinization of $\mathfrak{u}(1)$, which requires some extra discussion. The details again need the theory of characters and modular invariants, and so we will limit ourselves to describe the main facts necessary for this work.

The $\widehat{\mathfrak{u}}(1)_k$ Theory

We start by listing some of the basic properties of the (chiral) *free boson CFT*. This theory has central charge $c = 1$, and its basic building block is a bosonic field $\phi(z)$ (with $h = 0$) which satisfies

$$\phi(z)\phi(w) \sim -\ln(z-w). \tag{4.18}$$

The free boson theory naturally has a current $j(z) = i\partial\phi(z)$. More interestingly, one can construct the so-called *vertex operators* by taking normal-ordered exponentials of the boson $\phi(z)$:

$$V_\alpha(z) =: e^{i\alpha\phi(z)} : . \tag{4.19}$$

We will usually drop the normal-ordering symbol for convenience. The vertex operator $V_\alpha(z)$ has conformal dimension $h = \alpha^2/2$.
The affinization of $\mathfrak{u}(1)$ is characterized by the commutation relations

$$[j_n, j_m] = kn\delta_{n+m,0} \tag{4.20}$$

between the Laurent modes of a single current j. For $k = 1$, these are the commutation relations between the modes of the free boson $\phi(z)$ [6]. However, notice that by an appropiate rescaling of the current we can absorb the factor of k in the right hand side of the equation. Thus, this tells us that the notion of level is immaterial for $\widehat{\mathfrak{u}}(1)$.
Consider now the theory of a boson compactified on a circle of radius $\sqrt{2k}$. By analyzing the characters of the theory, one can find that it has an extended symmetry algebra generated by the fields

$$j = i\partial\phi(z), \qquad \Gamma^\pm = e^{i\sqrt{2k}\phi(z)}. \tag{4.21}$$

The allowed primary fields of the theory are then found to be vertex operators of the form [6]

$$V_{m/\sqrt{2k}} = e^{i\frac{m}{\sqrt{2k}}\phi(z)}, \tag{4.22}$$

with $-k+1 \leq m \leq k$. Conventionally, this theory is denoted as $\widehat{\mathfrak{u}}(1)_k$.

With this said, we notice that the embedding $\mathfrak{u}(1) \hookrightarrow \mathfrak{su}(2)$ gives rise to an embedding $\widehat{\mathfrak{u}}(1)_k \hookrightarrow \widehat{\mathfrak{su}}(2)_k$ where the vertex operators in $\widehat{\mathfrak{u}}(1)_k$ correspond to each weight $m \in \Omega_\lambda$. As for the field identifications, the only non-trivial outer automorphism of $\widehat{\mathfrak{su}}(2)_k$ acts on $\lambda = [\lambda_0, \lambda_1] = [k-\lambda, \lambda]$ by switching the labels. On the charge m, the corresponding automorphism acts by sending $m \mapsto m+k$ [15]. Thus, we have the identifications

$$\Phi^\lambda_m \sim \Phi^{k-\lambda}_{m+k}, \tag{4.23}$$

where the charge is of course defined modulo $2k$.The conformal dimensions of these fields are

$$h^\lambda_m \equiv h_\lambda - h_m = \frac{\lambda(\lambda+2)}{4(k+2)} - \frac{m^2}{4k}, \tag{4.24}$$

i.e. n^λ_m vanishes for this particular choice [17]. After the corresponding identifications, the theory has $\frac{1}{2}k(k+1)$ different (WZW) primary fields.

4.3.1 Vertex representations

In this Section we will show how to represent the $\widehat{\mathfrak{su}}(2)_k$ currents in terms of vertex operators. This procedure will make evident the parafermionic nature of the degrees of freedom in the $\mathbb{Z}_k$ coset theory.

First, let us first obtain the vertex operator representation of $\widehat{\mathfrak{su}}(2)_1$ as follows. Starting from the theory of a free boson, we can check that the currents

$$H(z) = i\partial\phi, \qquad E^\pm = e^{\pm i\sqrt{2}\phi} \tag{4.25}$$

satisfy the same commutation relations as the Cartan-Weyl generators of $\widehat{\mathfrak{su}}(2)_1$. In fact, their OPEs can be easily computed by using known results for the free boson CFT (see e.g. Section 6.3 in [6]):

$$\begin{aligned} H(z)H(w) &\sim \frac{1}{(z-w)^2}, \\ H(z)E^{\pm}(w) &\sim \frac{\pm\sqrt{2}}{z-w}E^{\pm}, \\ E^{+}(z)E^{-}(w) &\sim \frac{1}{(z-w)^2} + \frac{\sqrt{2}}{z-w}H. \end{aligned} \tag{4.26}$$

We see this agrees with the OPEs found previously in Eq. 3.15.

This description has its roots in the decomposition of the $\widehat{\mathfrak{su}}(2)_k$ theory into a $\widehat{\mathfrak{u}}(1)$ theory and the $\mathbb{Z}_1$ coset as follows:

$$\widehat{\mathfrak{su}}(2)_1 = \frac{\widehat{\mathfrak{su}}(2)_1}{\widehat{\mathfrak{u}}(1)} \times \widehat{\mathfrak{u}}(1) \simeq \widehat{\mathfrak{u}}(1). \tag{4.27}$$

Indeed, the $\mathbb{Z}_1$ theory has central charge $c = 0$ and thus equals the trivial theory with only the identity operator. Generalizing this idea to $\widehat{\mathfrak{su}}(2)_k$, we could think about decomposing its currents into products of a field in the $\mathbb{Z}_k$ theory and a vertex operator as follows:

$$\begin{aligned} H(z) &= i\sqrt{k}\partial\phi(z), \\ E^{+}(z) &= \sqrt{k}\psi(z)e^{i\sqrt{2/k}\phi(z)}, \\ E^{-}(z) &= \sqrt{k}\psi^{\dagger}(z)e^{-i\sqrt{2/k}\phi(z)}. \end{aligned} \tag{4.28}$$

Their OPEs will give the correct expressions (3.15) given that

$$\psi(z)\psi^{\dagger}(w) \sim \frac{1}{(z-w)^{2-2/k}}, \tag{4.29}$$

a relation which was found in [16] through the lattice construction of the $\mathbb{Z}_k$ parafermions. Furthermore, we can easily read the conformal dimensions of $\psi(z)$ from the above definitions:

$$h_{E^+} = 1, \; h_{\mathcal{V}_{\sqrt{2/k}}} = \frac{1}{k} \implies h_{\psi} = 1 - \frac{1}{k}. \tag{4.30}$$

From this we conclude that the spin of ψ is also $1 - \frac{1}{k}$ (as it is a chiral field), and thus it is neither a fermion or a boson. We say that ψ is a *parafermionic* field since it can have arbitrary rational values for its spin.

4.4 The Gepner Parafermions

Gepner [5] proposed the idea of generalizing the $\mathbb{Z}_k$ theory to the coset

$$\frac{\widehat{\mathfrak{g}}_k}{\widehat{\mathfrak{u}}(1)^{\mathrm{rk}\mathfrak{g}}}, \tag{4.31}$$

with $\mathrm{rk}\mathfrak{g}$ the rank of the Lie algebra $\mathfrak{g}$. In his paper, Gepner mimicked the vertex operator representation technique for a general semisimple Lie algebra, starting from the theory of $\mathrm{rk}\mathfrak{g}$ non-interacting bosons and constructing the currents as in Eq. 4.28. We will not show the detail construction of these vertex operator representations (as they require some discussion on the free boson zero-modes [6]), but instead will focus on doing some computations for the particular case of $\widehat{\mathfrak{g}}_k = \widehat{\mathfrak{su}}(3)_k$.

We first establish the field content of the theory. An arbitrary representation then splits into $\mathfrak{u}(1)^2$ representations as in Eq. 4.17, which reads:

$$(\Lambda_1, \Lambda_2) \longmapsto \bigoplus_{(\lambda_1,\lambda_2)\in\Omega_\Lambda} (\lambda_1, \lambda_2). \tag{4.32}$$

The branching condition corresponding to this branching reads $\Lambda_1 + 2\Lambda_2 = \lambda_1 + 2\lambda_2 \pmod 3$ [17]. After affinization, the allowed representations for $\widehat{\mathfrak{su}}(3)_k$ are those with Dynkin labels

$$(\Lambda_1, \Lambda_2) \text{ s.t. } \Lambda_1 + \Lambda_2 \leq k. \tag{4.33}$$

Furthermore, the embedding $\mathfrak{u}(1)^2 \hookrightarrow \mathfrak{su}(3)$ is sent to an embedding $\widehat{\mathfrak{u}}(1)^2_k \hookrightarrow \widehat{\mathfrak{su}}(3)_k$ where the vertex operators correspond to weights $\lambda \in \Omega_\Lambda$, and are of the form

$$e^{\frac{i}{\sqrt{k}}\lambda\cdot\phi(z)}. \tag{4.34}$$

Here $\phi(z) = (\phi_1(z), \phi_2(z))$ is a pair of two (non-interacting) free bosons and $\lambda \cdot \phi(z) \equiv \sum_i \lambda^i \phi^i(z)$. Since the action of a raising operator is to shift λ by a simple root (either α_1 or α_2), the primary fields of the $\widehat{\mathfrak{u}}(1)^2$ theory are restricted to have $\lambda \sim \lambda + k\Lambda_{\text{root}}$. Finally, there are now two types of field identifications, corresponding to the two non-trivial outer automorphisms of $\widehat{\mathfrak{su}}(3)_k$:

$$\Phi^{(\Lambda_1,\Lambda_2)}_{(\lambda_1,\lambda_2)} \sim \Phi^{(k-\Lambda_1-\Lambda_2,\Lambda_1)}_{(\lambda_1+k,\lambda_2)} \sim \Phi^{(\Lambda_2,k-\Lambda_1-\Lambda_2)}_{(\lambda_1,\lambda_2+k)}. \tag{4.35}$$

Once we are done with determining the primary fields of the coset theory, the fusion rules can be derived by employing formula (3.29) and the fact that the $\mathfrak{u}(1)^2$ representations simply fuse by summing their charges.

For $k = 2$, the fusion table was obtained in [17]. In this project, we obtained the fusion tables for $k = 3, 4$ and 5 by means of a computer program which implements the above field identifications and fusion rules. However, the field spectrum of these theories is very large (30, 80, and 175 primary fields respectively after identifications), and thus the fusion tables become quite complicated. For this reason, we follow a similar approach to Ardonne and Schoutens [17] in which fields are *grouped* in a way consistent with their fusion rules as follows. We partition the primary fields into various groups $A, B, \dots$ and write, for example

$$A \times B = C + D \tag{4.36}$$

if the fusion of any field in group A with any other field in group B gives a field in group C plus a field in group D. A systematic grouping of the fields can be done for any k by means of the field identifications (4.35). Indeed, we can partition the fields into groups based on their $\mathfrak{su}(3)$ highest weight, such that two fields Φ^Λ_λ and $\Phi^{\Lambda'}_{\lambda'}$ are in the same group if $\Lambda \sim \Lambda'$ in the sense that

$$(\Lambda_1, \Lambda_2) \sim (k - \Lambda_1 - \Lambda_2, \Lambda_1) \sim (\Lambda_2, k - \Lambda_1 - \Lambda_2). \tag{4.37}$$

Following this recipe, we can reduce the complete fusion table into *group fusion rules* for each k between 2 and 5. The case $k = 2$ was also shown in [17], and turns out to be equivalent to the fusion table of the Fibonacci anyons [3]:

$\times$	Ψ	A
Ψ	Ψ	
A	A	$\Psi + A$

Table 4.1: Group fusion rules for the $\widehat{\mathfrak{su}}(3)_2/\widehat{\mathfrak{u}}(1)^2$ coset theory.

We now present the group fusion tables for $k = 3, 4, 5$, which in the best of our knowledge have not been yet studied in the literature.

$\times$	Ψ	X	Y	A
Ψ	Ψ			
X	X	$2Y$		
Y	Y	$\Psi + A$	$2X$	
A	A	$3X$	$3Y$	$3\Psi + 2A$

Table 4.2: Group fusion rules for the $\widehat{\mathfrak{su}}(3)_3/\widehat{\mathfrak{u}}(1)^2$ coset theory.

$\times$	Ψ	X	Y	A	B
Ψ	Ψ				
X	X	$Y + A$			
Y	Y	$\Psi + B$	$X + A$		
A	A	$Y + B$	$X + B$	$\Psi + A + B$	
B	B	$X + A + B$	$Y + A + B$	$X + Y + A + B$	$\Psi + X + Y + A + 2B$

Table 4.3: Group fusion rules for the $\widehat{\mathfrak{su}}(3)_4/\widehat{\mathfrak{u}}(1)^2$ coset theory.

$\times$	Ψ	X	Y	Z	W	A	B
Ψ	Ψ						
X	X	$Y+Z$					
Y	Y	$\Psi+A$	$X+W$				
Z	Z	$W+A$	$X+B$	$Y+W+B$			
W	W	$Y+B$	$Z+A$	$\Psi+A+B$	$X+Z+B$		
A	A	$X+W+B$	$Y+Z+B$	$Y+Z+A+B$	$X+W+A+B$	$\Psi+Z+W+2A+B$	
B	B	$Z+A+B$	$W+A+B$	$X+Z+W+A+B$	$Y+Z+W+A+B$	$X+Y+Z+W+A+2B$	$\Psi+X+Y+Z+W+2A+2B$

Table 4.4: Group fusion rules for the $\widehat{\mathfrak{su}}(3)_5/\widehat{\mathfrak{u}}(1)^2$ coset theory.

CHAPTER 5

Conclusions

Throughout this document we introduced the framework of 2d conformal field theory, which exploits the infinite-dimensional symmetry algebra of local conformal transformations to obtain important results on the structure of a theory. Moreover, we introduced the Wess-Zumino-Witten models — important examples of 2d CFTs characterized by affine Lie algebras $\widehat{\mathfrak{g}}_k$ — and understood their field content through the lens of representation theory. Finally, with the help of well-known results on the structure of the $\widehat{\mathfrak{su}}(3)_k$ WZW model, we managed to compute the field spectrum and the associated fusion rules in the $\widehat{\mathfrak{su}}(3)_k/\widehat{\mathfrak{u}}(1)$ parafermionic coset theory. We saw that as k gets larger and larger the number of allowed primary fields increases tremendously — this led us to introduce a grouping scheme that reduces the whole set of fusion rules into smaller, more manageable fusion tables. We obtained the full results for levels $k = 3, 4$ and 5.

The group fusion tables deserve further exploration from the physical point of view. For example, it is unclear if the grouping scheme represents some physical process or if it is a simple mathematical gadget. It would also be interesting to further understand the properties of these fusion tables — for example, they may possess symmetries where exchanging certain groups with each other leave the fusion rules invariant. Furthermore, motivated by the bulk/boundary correspondence mentioned in the Introduction, it would be useful to investigate what kind of anyonic systems are described by the fusion rules obtained in this work. Other properties which have a fundamental role in the study of such systems, such as the F and R-matrices [3], deserve further study as well. In particular, the fusion table obtained for $k = 3$ may represent a generalization of the Fibonacci anyons, although a more extensive exploration of their properties will be needed to support this claim.

APPENDIX A

Representation Theory of Simple Lie Algebras

In this Appendix, we present some of the basic definitions in the representation theory of simple Lie algebras. The content of this chapter is largely based in reference [6], which gives a practical account of the topic at hand. We also make use of references [18] and [19] for some of the more mathematical details. Furthermore, we assume the reader to be familiar with the fundamental notions in the theory Lie algebras. However, we try to make this and the following Appendices as self-contained as possible.

We remind the reader that a Lie algebra $\mathfrak{g}$ is a vector space equipped with a Lie bracket, i.e. a bilinear binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which

(1) Is alternating, i.e. $[x, x] = 0,\ \forall x \in \mathfrak{g}$.

(2) Satisfies the Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0,\ \forall x, y, z \in \mathfrak{g}$.

If we choose a basis $\{J^a\}$ $(a = 1, \dots, \dim \mathfrak{g})$ of the Lie algebra, then the Lie bracket operation is completely determined by a set of *commutation relations*:

$$[J^a, J^b] = i \sum_c f^{ab}_{\ \ c} J^c. \tag{A.1}$$

The coefficients $f^{ab}_{\ \ c}$ are called the *structure constants* of $\mathfrak{g}$. Furthermore, a Lie algebra is called *simple* if it does not contain any non-trivial ideal, i.e. if there is no linear subspace $\mathfrak{a}$ of $\mathfrak{g}$ such that $[a, x] \in \mathfrak{a}$ for all $a \in \mathfrak{a}, x \in \mathfrak{g}$. A Lie algebra is *semisimple* if it is the direct sum of simple Lie algebras.

We now consider actions of a Lie algebra on an arbitrary vector space V. The space $\mathrm{End}(V)$ of endomorphisms on V has a natural Lie algebra structure, given by the usual commutator bracket:

$$[A, B] = A \circ B - B \circ A,\quad \forall A, B \in \mathrm{End}(V). \tag{A.2}$$

A representation of $\mathfrak{g}$ on V is then a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, i.e. a linear map which preserves the Lie bracket structure:

$$\rho([x, y]_{\mathfrak{g}}) = [\rho(x), \rho(y)]_{\mathfrak{gl}(V)}. \tag{A.3}$$

In this context, V is called a representation space for $\mathfrak{g}$ or a $\mathfrak{g}$-module. Furthermore, a representation is called *irreducible* if there is no proper subspace $W \subset V$ that is invariant under the action of all $\rho(x)$.

A.1 Generators in the Cartan-Weyl basis

It is useful to consider a distinguished basis of $\mathfrak{g}$ known as the **Cartan-Weyl basis**, which is constructed as follows. We consider a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, generated by a maximal set of elements $\{H^i\}$ ($i = 1, \dots, r$) such that $[H^i, H^j] = 0$. The dimension of $\mathfrak{h}$ is called the *rank* of $\mathfrak{g}$. By definition of the Cartan subalgebra the $\{\mathrm{ad}_{H^i}\}$ are simultaneously diagonalizable, and hence we can complete the set $\{H^i\}$ to a basis $\{H^i\} \cup \{E^\alpha\}$ of $\mathfrak{g}$ such that

$$\mathrm{ad}_{H^i}(E^\alpha) = [H^i, E^\alpha] = \alpha^i E^\alpha. \tag{A.4}$$

The vector $\alpha = (\alpha^1, \dots, \alpha^r)$ is known as a *root*, and we denote the set of $\dim(\mathfrak{g}) - r$ roots by Δ. The roots are naturally elements of the dual space $\mathfrak{h}^*$ by the relation

$$\alpha(H^i) = \alpha^i.$$

We can obtain the commutation relations between the generators as follows. The Jacobi identity implies that $[H^i, [E^\alpha, E^\beta]] = (\alpha^i + \beta^i)[E^\alpha, E^\beta]$, from which we conclude that:

(1) If $\beta = -\alpha$, then $[E^\alpha, E^\beta]$ commutes with all the H^i. This implies that $[E^\alpha, E^\beta]$ itself is an element of the Cartan subalgebra, thus a linear combination of the H^i. In fact, it happens that one can find a normalization in which

$$[E^\alpha, E^{-\alpha}] = 2\frac{\alpha \cdot H}{|\alpha|^2}. \tag{A.5}$$

(2) If $0 \neq \alpha + \beta \notin \Delta$, then $[E^\alpha, E^\beta] = 0$.

(3) Finally, if $\alpha + \beta \in \Delta$ then $[E^\alpha, E^\beta]$ must be proportional to $E^{\alpha+\beta}$.

Therefore, the Cartan-Weyl basis is characterized by the following commutation relations:

$$\begin{aligned} &[H^i, H^j] = 0, \\ &[H^i, E^\alpha] = \alpha^i E^\alpha, \\ &[E^\alpha, E^\beta] = \begin{cases} N_{\alpha,\beta} E^{\alpha+\beta}, & \alpha + \beta \in \Delta, \\ \frac{2}{|\alpha|^2} \alpha \cdot H, & \alpha = -\beta, \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{A.6}$$

with the $N_{\alpha,\beta}$ a set of constants determined by the particular structure of the Lie algebra $\mathfrak{g}$.

A.2 The Killing Form

On any Lie algebra $\mathfrak{g}$ we can always define the **Killing form**, a symmetric bilinear form given by

$$K(X,Y) = \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y). \tag{A.7}$$

The Killing form is *invariant*, i.e. it satisfies the property:

$$K([X,Y],Z) = K(X,[Y,Z]), \quad \forall X,Y,Z \in \mathfrak{g}. \tag{A.8}$$

Throughout this work we will prefer to use a slightly different normalization of the Killing form, given by

$$K(X,Y) = \frac{1}{2C_{\mathfrak{g}}} \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y), \tag{A.9}$$

where $C_{\mathfrak{g}}$ is a constant called the *dual Coxeter number* of $\mathfrak{g}$ (c.f. Eq. A.19). For semisimple Lie algebras it can be proved that this form is always nondegenerate, and thus defines an scalar product on $\mathfrak{g}$.

When considering a generic basis $\{J^a\}$ of $\mathfrak{g}$, we will assume it is orthonormal with respect to K:

$$K(J^a, J^b) = \delta^{ab}. \tag{A.10}$$

In the Cartan-Weyl basis, we can assume the same holds for the generators of the Cartan subalgebra: $K(H^i, H^j) = \delta_{ij}$. One can find that the Killing form evaluated on every other combination of generators vanishes, except for

$$K(E^\alpha, E^{-\alpha}) = \frac{2}{|\alpha|^2}. \tag{A.11}$$

As the Killing form is an scalar product on $\mathfrak{g}$, in particular it defines an isomorphism between the Cartan subalgebra $\mathfrak{h}$ and its dual. In fact, for $H \in \mathfrak{h}$ we have that $K(H,\cdot) \in \mathfrak{h}^*$ — conversely, for every linear form $\gamma \in \mathfrak{h}$ we obtain a corresponding element $H^\gamma \in \mathfrak{h}$ such that

$$\gamma(H^i) = K(H^i, H^\gamma). \tag{A.12}$$

For the case of a root $\alpha \in \Delta$, we obtain the element $\alpha \cdot H \in \mathfrak{h}$. Through this isomorphism we can moreover define a scalar product on the dual space (and thus in particular on the roots) through

$$(\gamma, \beta) = K(H^\gamma, H^\beta). \tag{A.13}$$

Indeed, $(\alpha,\alpha) = |\alpha|^2$ coincides with the usual definition used before.

A.3 Weights

As seen in Eq. A.4, the roots α are the eigenvalues of H in the adjoint representation, with the E^α being the corresponding basis eigenvectors. Furthermore, the zero eigenvalue has degeneracy r, corresponding to the Cartan generators H^i acting on themselves. For an arbitrary representation, consider a basis $\{|\lambda\rangle\}$ of eigenvectors of H, i.e.

$$H^i |\lambda\rangle = \lambda^i |\lambda\rangle. \tag{A.14}$$

The vector $\lambda = (\lambda^1, \dots, \lambda^r)$ is called a *weight*, and it naturally lives in the space $\mathfrak{h}^*$ through the action $\lambda(H^i) = \lambda^i$. Notice that for the adjoint representation, the weights are exactly the roots $\alpha \in \Delta$.

The commutator between H^i and E^α shows that the action of the latter change the eigenvalue of a state by α:

$$H^i E^\alpha |\lambda\rangle = [H^i, E^\alpha] |\lambda\rangle + E^\alpha H^i |\lambda\rangle \tag{A.15}$$

$$= (\lambda^i + \alpha^i)(E^\alpha |\lambda\rangle). \tag{A.16}$$

So $E^\alpha |\lambda\rangle$ is proportional to the state $|\lambda + \alpha\rangle$. For this reason, the operators E^α are also known as *ladder operators*. It can be shown that any weight in a finite-dimensional representation is such that $(\alpha, \lambda)/|\alpha|^2 \in \mathbb{Z}$. In particular, it is true for the roots $\beta \in \Delta$, an important point which we will use later on.

A.4 Simple Roots and the Cartan Matrix

A base $B = \{\alpha_1, \dots, \alpha_r\}$ for the root space is a set of linearly independent roots which span $\mathfrak{h}^*$, such that any root $\alpha \in \Delta$ can be expanded as

$$\alpha = \sum_{i=1}^{r} k_i \alpha_i$$

with all $k_i \neq 0$ of the same sign. If they are positive (negative), α is called a positive (negative) root, and we denote the set of such roots as Δ_+ (Δ_-). The α_i are called *simple roots*, and it is always possible to find a base for the root system.

We define the Cartan matrix by

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{\alpha_j^2},$$

where all the A_{ij} are integers (as pointed out in the last section). One can prove that A_{ij} satisfies the following properties:

(1) $A_{ij} \leq 0$ for $i \neq j$.

(2) $A_{ij} = 0 \iff A_{ji} = 0$.

(3) $A_{ij} A_{ji} = 0, 1, 2$ or 3, for $i \neq j$.

Furthermore, these properties imply that the roots of a simple Lie algebra have at most two different lengths (called the short roots and long roots), and the ratio between their lengths has to be either 2 or 3. If all roots are of the same length, we say that the Lie algebra is *simply laced*.

We also introduce the *coroots* $\alpha^\vee$, defined by

$$\alpha^\vee = \frac{2\alpha_i}{|\alpha_i|^2}. \tag{A.17}$$

Then the Cartan matrix can be expressed as $A_{ij} = (\alpha_i, \alpha_j^\vee)$. Finally, we define the *highest root* θ, the unique root such that when expressed as $\theta = \sum_i m_i \alpha_i$ it gives the maximal value of the sum $\sum_i m_i$.

When expressed in terms of α_i, the coefficients are called the *marks* a_i; when expressed in terms of $\alpha_i^\vee$, they are called the *comarks* $a_i^\vee$:

$$\theta = \sum_{i=1}^{r} a_i \alpha_i = \sum_{i=1}^{r} a_i^\vee \alpha_i^\vee; \quad a_i, a_i^\vee \in \mathbb{N}. \tag{A.18}$$

With this said, we can define the *dual Coxeter number* as

$$C_{\mathfrak{g}} = \sum_i a_i^\vee + 1. \tag{A.19}$$

Remark A.4.1. In order to fully fix the normalization of the roots, we must choose a specific length for them. The standard convention is to set $|\theta|^2 = 2$ (or equivalently, the long roots have all length $\sqrt{2}$).

A.5 Generators in the Chevalley Basis

Another distinguished basis of the Lie algebra $\mathfrak{g}$ is the so called *Chevalley basis*, which makes manifest the fact that the structure of $\mathfrak{g}$ is completely determined by its corresponding Cartan matrix. We construct it as follows: for each simple root α_i, we define three generators

$$e^i = E^{\alpha_i}, \quad f^i = E^{-\alpha_i}, \quad h^i = \frac{2}{|\alpha|^2}\alpha_i \cdot H, \tag{A.20}$$

which satisfy the following commutation relations:

$$\begin{aligned} [h^i, h^j] &= 0, \\ [h^i, e^j] &= A_{ji} e^j, \\ [h^i, f^j] &= -A_{ji} f^j, \\ [e^i, f^j] &= \delta_{ij} h^j. \end{aligned} \tag{A.21}$$

However, in general these generators will not form a basis of $\mathfrak{g}$. The remaining generators are obtained by repeated commutation of the e^i and f^j, and are subject to the constraints

$$\begin{aligned} (ad_{e^i})^{1-A_{ji}} e^j &= 0, \\ (ad_{f^i})^{1-A_{ji}} f^j &= 0. \end{aligned} \tag{A.22}$$

Finally, we can easily compute the Killing form between the $\{h^i\}$ generators to be $K(h^i, h^j) = (\alpha_i^\vee, \alpha_j^\vee)$.

A.6 Fundamental Weights

Since the weights live in the same space as the roots (the dual space $\mathfrak{h}^*$), we can also expand them in terms of the simple roots. However, it will be more convenient to define an alternative basis $\{\omega_i\}$ dual to the simple coroots:

$$(\omega_i, \alpha_j^\vee) = \delta_{ij}. \tag{A.23}$$

The ω_i are called the *fundamental weights*. A general weight λ is then expressed as a linear combination of them as

$$\lambda = \sum_{i=1}^{r} \lambda_i \omega_i, \quad \lambda_i = (\lambda, \alpha_i^\vee). \tag{A.24}$$

The λ_i are called the *Dynkin labels* of λ, and they are all integers for finite-dimensional representations of $\mathfrak{g}$ (following the discussion in Section A.3). In particular, note that for $\lambda = \alpha_i$ a simple root the Dynkin labels coincide with the elements of the Cartan matrix, $A_{ij} = (\alpha_i, \alpha_j^\vee)$. Thus, we have the relation:

$$\alpha_i = \sum_{j=1}^{r} A_{ij} \omega_j. \tag{A.25}$$

Moreover, note that the Dynkin labels are the eigenvalues of the Cartan subalgebra generators in the Chevalley basis:

$$h^i \left|\lambda\right\rangle = \alpha_i^\vee \cdot H \left|\lambda\right\rangle = (\alpha^\vee, \lambda) \left|\lambda\right\rangle = \lambda_i \left|\lambda\right\rangle . \tag{A.26}$$

There is a distinguished weight ρ called the *Weyl vector*, which has all of its Dynkin labels equal to 1. Equivalently, it can be written as

$$\rho = \sum_i \omega_i = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha. \tag{A.27}$$

We can construct a matrix $F_{ij} = (\omega_i, \omega_j)$ by taking scalar products between the fundamental weights, known as the *quadratic form matrix*. From the definition of the fundamental weights, we see that this matrix represents the change of basis between $\{\omega_i\}$ and $\left\{\alpha_i^\vee\right\}$:

$$\omega_i = \sum_j F_{ij} \alpha_j^\vee. \tag{A.28}$$

Furthermore, the above remark implies that F_{ij} is related to the Cartan matrix by

$$F_{ij} = (A^{-1})_{ij} \frac{\left|\alpha_j\right|^2}{2}. \tag{A.29}$$

The weight and root lattices

It will be useful to consider the lattices generated by the fundamental weights and the simple roots, known as the *weight lattice* and the *root lattice* respectively. These lattices are simply the span over the integers of $\{\omega_i\}$ and $\{\alpha_i\}$:

$$\Lambda_{\text{weight}} = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_r, \tag{A.30}$$

$$\Lambda_{\text{root}} = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_r. \tag{A.31}$$

Since highest weights of finite-dimensional irreducible representations have integer Dynkin labels, they are in particular elements of Λ_{weight}. On the other side, the action of a ladder operator E^{α_i} is to shift a weight by α_i — thus the repeated action of ladder operators is characterized by a vector $\nu \in \Lambda_{\text{root}}$. As noted previously, the roots are in particular weights for the adjoint representation, and so $\Lambda_{\text{root}} \subset \Lambda_{\text{weight}}$.

Example A.6.1 ($\mathfrak{su}(2)$)**.** The most basic example of a simple Lie algebra is $\mathfrak{su}(2)$, whose Cartan matrix is given by $A = (2)$ (and thus there is only a single simple root $\alpha_1 \equiv \alpha$ of length $\sqrt{2}$). Following Eqs. A.21, one can easily obtain the $\mathfrak{su}(2)$ commutation relations in the Chevalley basis:

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f. \tag{A.32}$$

These three generators will suffice to obtain a basis of the algebra, and thus the latter has dimension 3. In the Cartan-Weyl basis, the commutation relations read

$$[E^+, E^-] = H, \qquad [H, E^+] = \sqrt{2}E^+, \qquad [H, E^-] = -\sqrt{2}E^-, \tag{A.33}$$

where we denote $E^{\pm\alpha} \equiv E^\pm$. From Eq. A.25, we see that the fundamental weight ω is related to the simple root by $\alpha = 2\omega$. We present the weight and root lattices of $\mathfrak{su}(2)$ in Figure A.1.

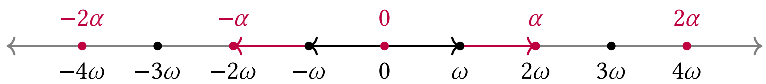

Figure A.1: Weight and root lattices of $\mathfrak{su}(2)$. Points in Λ_{weight} (Λ_{root}) are shown in black (purple).

Example A.6.2 ($\mathfrak{su}(3)$)**.** The next non-trivial example of a simple Lie algebra is that of $\mathfrak{su}(3)$, whose Cartan matrix is given by

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{A.34}$$

There are two simple roots α_1 and α_2 which have the same length. Again, the Cartan matrix gives the relation between the simple roots and the fundamental weights:

$$\begin{aligned} \alpha_1 &= 2\omega_1 - \omega_2, \\ \alpha_1 &= -\omega_1 + 2\omega_2. \end{aligned} \tag{A.35}$$

One can again construct the Chevalley basis of the algebra, although there are a larger number of commutation relations as the algebra will turn out to be 8-dimensional. We limit ourselves to present the root system of $\mathfrak{su}(3)$ and its fundamental weights in Figure A.2.

A.7 Highest-weight Representations

Without delving too deep into the mathematical details, we point out that any finite-dimensional irreducible representation of a semisimple Lie algebra can be characterized by a *highest-weight state* $|\lambda\rangle$, where $\lambda = (\lambda_1, \dots, \lambda_r)$ denotes the set of its Dynkin labels. The highest weight λ is defined as the one that, when expanded in terms of simple roots, gives the maximal sum for its coefficients in that basis. This implies that

$$E^\alpha |\lambda\rangle = 0, \quad \forall \alpha \in \Delta_+.$$

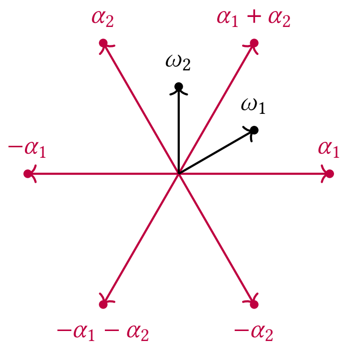

Figure A.2: Root system and fundamental weights of $\mathfrak{su}(3)$.

Furthermore, one can also prove that a highest weight must have non-negative Dynkin labels. Such a weight is in general said to be *dominant*. Conversely, every dominant weight characterizes a unique finite-dimensional irreducible representation of $\mathfrak{g}$. Following this discussion, we will usually denote a representation by its highest weight λ.

All states in the representation space of λ can be obtained through the action of lowering operators:

$$E^{-\beta_1} E^{-\beta_2} \cdots E^{-\beta_n} \left|\lambda\right\rangle, \quad \beta_i \in \Delta_+.$$

The set of eigenvalues of all states in the representation space forms the *weight system* of the representation, denoted by Ω_λ. There is a simple algorithm to compute all the allowed weights, starting from the highest weight $\lambda = (\lambda_1, \ldots, \lambda_r)$. For each positive Dynkin label λ_k, we compute the set $\{\lambda - \alpha_k, \lambda - 2\alpha_k, \ldots, \lambda - \lambda_k \alpha_k\}$. We then repeat this process for each of the weights obtained in this way, until there are no more weights with positive Dynkin labels.

Example A.7.1. Consider the adjoint representation $(1,1)$ of $\mathfrak{su}(3)$. We can apply the above algorithm to compute all the weights in $\Omega_{(1,1)}$, which we depict graphically in Figure A.3.

$$\begin{tikzcd}
(1,1) \arrow[r, "-\alpha_2"] \arrow[d, "-\alpha_1"'] & (2,-1) \arrow[d, "-\alpha_1"'] & \\
(-1,2) \arrow[r, "-\alpha_2"] & (0,0) \arrow[r, "-\alpha_2"] \arrow[d, "-\alpha_1"'] & (1,-2) \arrow[d, "-\alpha_1"'] \\
& (-2,1) \arrow[r, "-\alpha_2"] & (-1,-1)
\end{tikzcd}$$

Figure A.3: Weight system of the adjoint representation $(1,1)$ of $\mathfrak{su}(3)$.

A.8 Lie Algebra Embeddings

Consider a Lie subalgebra $\mathfrak{p} \subset \mathfrak{g}$. Then we refer to the specific inclusion $\mathfrak{p} \hookrightarrow \mathfrak{g}$ as an *embedding* of Lie algebras (or simply an embedding). A (finite-dimensional) representation λ of $\mathfrak{g}$, when restricted to the action of $\mathfrak{p}$, generally splits into a direct sum of irreducible representations of the latter as

$$\lambda \longmapsto \bigoplus_{\mu \in P_+} b_{\lambda\mu}\mu. \tag{A.36}$$

Here P_+ denotes the set of dominant weights for $\mathfrak{p}$, which we known are in one-to-one correspondence with the irreducible representations of $\mathfrak{p}$. The numbers $b_{\lambda\mu}$ are called the *branching coefficients*, and they specify the multiplicity of a particular representation μ in the above decomposition.

We can obtain the branching coefficients corresponding to the splitting A.36 as follows. One can see that the inclusion map $\mathfrak{p} \hookrightarrow \mathfrak{g}$ necessarily restricts to an embedding of Cartan subalgebras as $\mathfrak{h}_\mathfrak{p} \hookrightarrow \mathfrak{h}_\mathfrak{g}$. By taking the dual of this map, this induces a *projection map* (or projection matrix) $\mathcal{P}$ between the dual spaces $\mathfrak{h}_\mathfrak{g}^* \to \mathfrak{h}_\mathfrak{p}^*$, where the weights naturally live. We then start by projecting all the weights in Ω_λ into $\mathfrak{h}$-weights, and then reorganize the latter into weight systems of irreducible representations of $\mathfrak{h}$.

Example A.8.1. We can give a simple example of the above process for the embedding $\mathfrak{su}(2) \hookrightarrow \mathfrak{su}(3)$. In fact, there are two inequivalent embeddings of $\mathfrak{su}(2)$ into $\mathfrak{su}(3)$ — without giving the mathematical details, we present the corresponding projection maps schematically in Figure A.4, and point out that a more detailed discussion can be found in ref. [18].

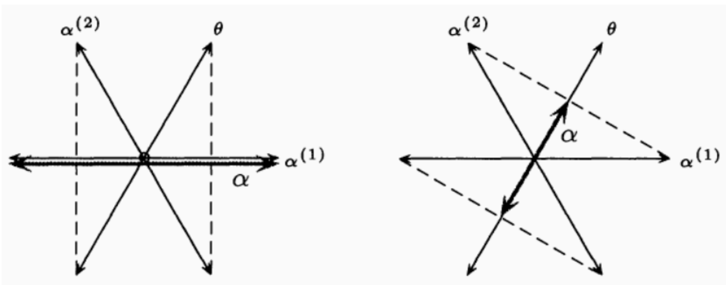

Figure A.4: Projection maps for the two inequivalent embeddings $\mathfrak{su}(2) \hookrightarrow \mathfrak{su}(3)$. The dashed lines follow the direction in which the root system of $\mathfrak{su}(3)$ is projected. Source: [18]

From the above figure we can infer the branching coefficients for the adjoint representation $(1,1)$ of $\mathfrak{su}(3)$. In the first case, we observe that the projected weights can be grouped as

$$(1,1) \longmapsto (4) \oplus (2), \tag{A.37}$$

where we denote representations through their Dynkin labels (if we label the $\mathfrak{su}(2)$ representations through their spin, the right hand side reads $2 \oplus 1$). On the other hand, the second case is seen to give the different grouping

$$(1,1) \longmapsto (2) \oplus 2(1) \oplus (0) \tag{A.38}$$

(or $1 \oplus 2\frac{1}{2} \oplus 0$ in physics notation).

There is a necessary (but not sufficient) condition for the branching coefficient $b_{\lambda\mu}$ to not vanish. Namely, μ must lie somewhere in the projection $\mathcal{P}\Omega_\lambda$ of the weight system of λ. Since any weight in this weight system can be written as $\lambda - \nu$ for some $\nu \in \Lambda_{\text{root}}$, the aforementioned necessary condition can be also written as

$$\mathcal{P}\lambda - \mu \in \mathcal{P}\Lambda_{\text{root}}. \tag{A.39}$$

The above relation is known as the *branching condition.*

Finally, it will be useful to define the *embedding index* x_e of an embedding $\mathfrak{p} \hookrightarrow \mathfrak{g}$, given by the following ratio:

$$x_e = \frac{|\mathcal{P}\theta_{\mathfrak{g}}|^2}{|\theta_{\mathfrak{p}}|^2}. \tag{A.40}$$

APPENDIX B

Representation Theory of Affine Lie Algebras

In this Appendix we present the definition of an affine Lie algebra $\hat{\mathfrak{g}}_k$ corresponding to a (semisimple) Lie algebra $\mathfrak{g}$, and summarize some of the basic definitions in their representation theory (although in much less detail than for the case of usual simple Lie algebras). Once again, the content here is largely based in reference [6], with some of the mathematical details taken from reference [18].

B.1 Generators in the Cartan-Weyl basis

In order to arrive to the affine Lie algebra $\mathfrak{g}_k$, we start by adjoining to it the algebra $\mathbb{C}[t, t^{-1}]$ of Laurent polynomials in a variable t:

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]. \tag{B.1}$$

Its generators are given by $J_n^a \equiv J^a \otimes t^n$. We now consider the central extension of $\tilde{\mathfrak{g}}$ obtained by adjoining a central element $\hat{k}$ which commutes with all the J_n^a:

$$[J_n^a, J_m^b] = if^{ab}_{\ \ c} J_{n+m}^c + \hat{k} n K(J^a, J^b)\delta_{n+m,0}. \tag{B.2}$$

For a basis $\{J^a\}$ orthonormal with respect to the Killing form, the above commutation relations read

$$[J_n^a, J_m^b] = if^{ab}_{\ \ c} J_{n+m}^c + \hat{k} n \delta_{ab}\delta_{n+m,0}. \tag{B.3}$$

We can rewrite the commutation relations of this new algebra in terms of an *affine Cartan-Weyl basis*. We define the affine generators as $H_n^i \equiv H^i \otimes t^n$ and $E_n^\alpha \equiv H^i \otimes t^n$ — one can then find their commutation relations to be

$$\begin{aligned}
&[H_n^i, H_m^j] = \hat{k} n \delta^{ij} \delta_{n+m,0}, \\
&[H_n^i, E_m^\alpha] = \alpha^i E_{n+m}^\alpha, \\
&[E_n^\alpha, E_m^\beta] = \begin{cases} N_{\alpha,\beta} E_{n+m}^{\alpha+\beta}, & \alpha+\beta \in \Delta, \\ \frac{2}{|\alpha|^2}\left(\alpha \cdot H_{n+m} + \hat{k} n \delta_{n+m,0}\right), & \alpha = -\beta, \\ 0, & \text{otherwise}, \end{cases}
\end{aligned} \tag{B.4}$$

We further add to $\tilde{\mathfrak{g}}' = \tilde{\mathfrak{g}} \oplus \mathbb{C}\hat{k}$ an extra generator L_0 which, together with the generators H_0^i and $\hat{k}$, form a Cartan subalgebra of $\tilde{\mathfrak{g}}'$. Its commutation relations with the J_n^a are given by

$$[L_0, J_n^a] = -n J_n^a.$$

The rest of generators $\left\{E_n^\alpha\right\}$ and $\left\{H_n^i\right\}_{n\neq 0}$ will correspond to the ladder operators of this algebra. Thus, we finally arrive to the *affine Lie algebra*

$$\hat{\mathfrak{g}}_k = \mathfrak{g} \oplus \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\hat{k} \oplus \mathbb{C}L_0. \tag{B.5}$$

As in the case of the Virasoro algebra Vir_c, the subscript k labels the particular central extension of the Lie algebra, and its called the *level.* In terms of the algebra, k denotes the eigenvalue of $\hat{k}$ — the latter commutes with the rest of generators of $\hat{\mathfrak{g}}_k$, and thus simply acts as multiplication by k.

B.2 The affine Killing form and Chevalley basis

We can extend the Killing form of $\mathfrak{g}$ to $\hat{\mathfrak{g}}_k$ by setting

$$K(J_n^a, J_m^b) = \delta^{ab} \delta_{n+m,0}.$$

We supplement this with the conditions

$$K(J_n^a, \hat{k}) = 0, \qquad K(\hat{k}, \hat{k}) = 0, \qquad K(J_n^a, L_0) = 0, \qquad K(L_0, \hat{k}) = -1, \tag{B.6}$$

which can be obtained by applying the invariance property of the Killing form. The value of $K(L_0, L_0)$ must be specified by hand — by convention, it is chosen to be

$$K(L_0, L_0) = 0. \tag{B.7}$$

There is a slight arbitrariness in the definition of L_0, where we can always choose to shift it by $a\hat{k}$ without affecting the commutation relations. Such a shift changes the value of $K(L_0, L_0)$ to $-2a$.

As in the case of usual Lie algebras, the above defined Killing form is an scalar product on $\hat{\mathfrak{g}}_k$. Hence it defines an isomorphism between the Cartan subalgebra and its dual, and induces an scalar product on the latter space. We again define a weight as a vector in the dual of the Cartan subalgebra (or equivalently, as the eigenvalues of the Cartan generators on a simultaneous eigenvector of them):

$$\hat{\lambda} = (\hat{\lambda}(H_0^1), \ldots, \hat{\lambda}(H_0^r); \hat{\lambda}(\hat{k}); \hat{\lambda}(-L_0)) \equiv (\lambda; k_\lambda, n_\lambda). \tag{B.8}$$

We call $\hat{\lambda}$ an *affine weight*. The *finite part* λ corresponds to a usual weight in the simple Lie algebra by the inclusion $\mathfrak{g} \hookrightarrow \hat{\mathfrak{g}}_k$. The scalar product induced by the Killing form on the weight space is given by

$$(\hat{\lambda}, \hat{\mu}) = (\lambda, \mu) + k_\lambda n_\mu + k_\mu n_\lambda, \tag{B.9}$$

where in the right hand side (λ, μ) represents the usual scalar products of weights in $\mathfrak{g}$.

The affine weights corresponding to the adjoint representations are called the *affine roots*. In that case k_λ vanishes, since the adjoint action of $\hat{k}$ is always zero. Therefore, the adjoint roots have the general form

$$\hat{\beta} = (\beta; 0; n), \tag{B.10}$$

and thus the scalar product of affine roots is simply given by the scalar product of their finite parts: $(\hat{\alpha}, \hat{\beta}) = (\alpha, \beta)$. The affine root corresponding to E_n^α is

$$\hat{\alpha} = (\alpha; 0; n), \quad \alpha \in \Delta, n \in \mathbb{Z}, \tag{B.11}$$

while the one associated to H_n^i (for $n \neq 0$) is $n\delta$, with $\delta \equiv (0; 0; 1)$. From here onwards we denote $\alpha \equiv (\alpha; 0; 0)$, so that the affine roots are associated to E_n^α are written as $\hat{\alpha} = \alpha + n\delta$. The set of all affine roots then reads

$$\hat{\Delta} = \{\alpha + n\delta, \alpha \in \Delta\} \cup \{n\delta\}. \tag{B.12}$$

Furthermore, observe that the roots $n\delta$ have zero length, since $(\delta, \delta) = 0$. We say that such roots are *imaginary*, while the rest are said to be *real*.

B.3 Simple roots and the Cartan matrix

An affine Lie algebra has $r + 1$ simple roots — the first r of them correspond to the simple roots of $\mathfrak{g}$:

$$\alpha_i = (\alpha_i; 0; 0). \tag{B.13}$$

The last one (which should include δ) is conveniently chosen as

$$\alpha_0 \equiv (-\theta; 0; 1) \equiv -\theta + \delta, \tag{B.14}$$

with θ the highest root of $\mathfrak{g}$. With this convention, the set of positive roots is given by

$$\hat{\Delta}_+ = \{\alpha + n\delta; n > 0, \alpha \in \Delta\} \cup \{\alpha; \alpha \in \Delta_+\} \cup \{n\delta; n > 0\}. \tag{B.15}$$

With this said, we can now define the *extended Cartan matrix* to be

$$\widehat{A}_{ij} = (\alpha_i, \alpha_j^\vee), \tag{B.16}$$

where the indices run from 0 to r, and the affine coroots are given by

$$\hat{\alpha}^\vee \equiv \frac{2}{|\hat{\alpha}|^2}(\alpha; 0; n) = (\alpha^\vee; 0; \frac{2}{|\alpha|^2} n). \tag{B.17}$$

We will denote the simple coroots by $\alpha_0^\vee \equiv \alpha_0$ and $\alpha_i^\vee \equiv (\alpha_i^\vee; 0; 0)$. The entries in the extra row/column of the extended Cartan matrix can be easily computed:

$$(\alpha_0, \alpha_j^\vee) = -(\theta, \alpha_j^\vee) = -\sum_{i=1}^{r} a_i (\alpha_i, \alpha_j^\vee) = -\sum_{i=1}^{r} a_i A_{ij}. \tag{B.18}$$

The extended Cartan matrix shares some of the properties of the usual Cartan matrix, which will not be necessary for the purposes of our work.

We further define the zeroth mark a_0 to be 1 (the marks corresponding to the finite parts are left untouched). This implies the zeroth comark is also 1 since the finite part of α_0 is a long root:

$$a_0^\vee = a_0 \frac{|\alpha_0|^2}{2} = 1. \tag{B.19}$$

The extended Cartan matrix has one zero eigenvalue, reflecting the fact that the affine scalar product is *semi*positive definite. We can express the imaginary root $\delta = \alpha_0 + \theta$ as

$$\delta = \sum_{i=0}^{r} a_i \alpha_i = \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee .$$

With this definition, the dual Coxeter number then reads $C_\mathfrak{g} = \sum_{i=0}^{r} a_i^\vee$.

B.4 Fundamental weights

We define the fundamental weights as the basis dual to the simple coroots, similarly to usual Lie algebra case. We further take them to be L_0 eigenstates with 0 eigenvalue. For $i \neq 0$, they are given by

$$\hat{\omega}_i = (\omega_i; a_i^\vee; 0) \tag{B.20}$$

(the condition $(\hat{\omega}_i, \alpha_0^\vee) = 0$ fixes the $\hat{k}$ eigenvalue). On the other hand, the zeroth fundamental weight is fixed to be

$$\hat{\omega}_0 = (0; 1; 0)$$

and it is called the *basic fundamental weight*. With $\omega_i \equiv (\omega_i; 0; 0)$ as usual, we have $\hat{\omega}_i = a_i^\vee \hat{\omega}_0 + \omega_i$ for $i \neq 0$. The scalar product between fundamental weights is found to be

$$\begin{aligned} (\hat{\omega}_i, \hat{\omega}_j) &= (\omega_i, \omega_j) = F_{ij}, \quad i, j \neq 0; \\ (\hat{\omega}_0, \hat{\omega}_i) &= (\hat{\omega}_0, \hat{\omega}_0) = 0, \quad i \neq 0. \end{aligned} \tag{B.21}$$

An arbitrary affine weight is expanded in terms of the fundamental weights and δ as

$$\hat{\lambda} = \sum_{i=0}^{r} \lambda_i \hat{\omega}_i + \ell \delta, \qquad \ell \in \mathbb{R}. \tag{B.22}$$

We can read off the $\hat{k}$ eigenvalue k (the level) from this expression:

$$k = \hat{\lambda}(\hat{k}) = \sum_{i=0}^{r} a_i^\vee \lambda_i, \tag{B.23}$$

which implies that the zeroth Dynkin label of $\hat{\lambda}$ is

$$\lambda_0 = k - \sum_{i=1}^{r} a_i^\vee \lambda_i = k - (\lambda, \theta). \tag{B.24}$$

In what follows, we denote affine weights through their Dynkin labels as $\hat{\lambda} = [\lambda_0, \ldots, \lambda_r]$, omitting the L_0 eigenvalue.

Affine weights with non-negative integer Dynkin labels are also called *dominant*. However, note that this characteristic depends on the level k (as seen from the above equation). We denote by P_k^+ the set of all dominant weights at a fixed level k. The finite part of an affine dominant weight is itself a dominant weight: $\hat{\lambda} \in P_k^+ \implies \lambda \in P^+$ (but not viceversa).

B.5 Integrable highest-weight representations

A highest-weight representation is characterized by a unique highest state $|\hat{\lambda}\rangle$ annihilated by all ladder operators associated to positive roots (c.f. Eq. B.15):

$$E_0^\alpha |\hat{\lambda}\rangle = E_n^{\pm\alpha} |\hat{\lambda}\rangle = H_n^i |\hat{\lambda}\rangle = 0; \qquad n > 0, \alpha \in \Delta_+. \tag{B.25}$$

The eigenvalues of the state are given by the highest weight $\hat{\lambda}$:

$$H_0^i |\hat{\lambda}\rangle = \lambda^i |\hat{\lambda}\rangle \quad (\text{for } i \neq 0), \quad \hat{k} |\hat{\lambda}\rangle = k |\hat{\lambda}\rangle, \tag{B.26}$$

and we set $L_0 |\hat{\lambda}\rangle = 0$ by convention (again, this fixes the arbitrariness in the definition of L_0). In the Chevalley basis (which we have not described in detail), we equivalently have

$$h^i |\hat{\lambda}\rangle = \lambda_i |\hat{\lambda}\rangle, \quad i = 0, 1, \ldots, r. \tag{B.27}$$

All other states in the representation are generated by acting on $|\hat{\lambda}\rangle$ by lowering operators. Furthermore, since $\hat{k}$ comutes with all operators, all states share the same level k.

For affine Lie algebras, the analogues of irreducible finite-dimensional representations of $\mathfrak{g}$ are those highest-weight representations whose projections onto the $\mathfrak{su}(2)$ algebra associated to any real root are finite [6, 18]. It turns out that all weights $\hat{\lambda}'$ in the weight system $\Omega_{\hat{\lambda}}$ (again defined as the set of all weights in $\hat{\lambda}$) of such a representation satisfy

$$\lambda_i' \in \mathbb{Z}; \quad i = 0, 1, \ldots, r, \tag{B.28}$$

and $\lambda_i \in \mathbb{Z}_+$ for the highest weight. In particular this implies that $\lambda_0 = k - (\theta, \lambda) \in \mathbb{Z}_+$, and given that $(\lambda, \theta) \in \mathbb{Z}_+$, we obtain the condition

$$k \in \mathbb{Z}_+, \qquad k \geq (\lambda, \theta). \tag{B.29}$$

This means that for a fixed value of the level k, there are only a finite number of integrable highest-weight representations. In fact, integrable highest-weight representations will be in one-to-one correspondence with the dominant affine weights $\hat{\lambda} \in P_+^k$.

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