Math 245

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Week 1

September 8 - September 13

1.1 Inner Product

Definition 1.1.1 (Inner Product Space)**.** *An* inner product space *(over* $\mathbb{C}$*) is a* vector space V *and a function:*

$$\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$$

satisfying:

i. $\langle \vec{x}+\vec{y}, \vec{z}\rangle = \langle \vec{x}, \vec{z}\rangle + \langle \vec{y}, \vec{z}\rangle$ *for all* $\vec{x}, \vec{y}, \vec{z} \in V$

ii. $\langle c\vec{x}, \vec{y}\rangle = c\,\langle \vec{x}, \vec{y}\rangle$ *for all* $\vec{x}, \vec{y} \in V$ *and* $c \in \mathbb{C}$

iii. $\langle \vec{x}, \vec{y}\rangle = \overline{\langle \vec{y}, \vec{x}\rangle}$ *for all* $\vec{x}, \vec{y} \in V$

iv. $\langle \vec{x}, \vec{x}, \in\rangle\, \mathbb{R}_{>0}$ *if* $\vec{x} \neq \vec{0}$, $\langle \vec{x}, \vec{x}\rangle = 0$ *otherwise*

Example 1.1.1.1 (Standard Complex Inner Product)**.**

$$V = \mathbb{C}^n \qquad\qquad \langle (x_1,\ldots,x_n),(y_1,\ldots,y_n)\rangle = x_1\overline{y_1} + \ldots + x_n\overline{y_n}$$

Properties i, ii, *and* iii *clearly hold. For* iv, *for any* $\vec{x} = (a_1 + b_1 i, \ldots, a_n + b_n i)$

$$\langle \vec{x}, \vec{x}\rangle = (a_1+b_1 i)\overline{(a_1+b_1 i)} + \ldots + (a_n+b_n i)\overline{(a_n+b_n i)} \tag{1.1}$$
$$= a_1^2 + b_1^2 + \ldots + a_n^2 + b_n^2 \tag{1.2}$$

This is the standard complex inner product*. If we replace* $\mathbb{C}^n$ *with* $\mathbb{R}^n$ *then we get the* standard real inner product *(dot product).*

Example 1.1.1.2 (L^2 Inner Product)**.**

$$V = C([0,1]) \qquad\qquad \langle f, g\rangle = \int_0^1 f\overline{g}$$

This is called the L^2 *inner product on* V

Example 1.1.1.3 (Frobenius Inner Product)**.**

$$V = M_n(\mathbb{C}) \qquad\qquad \langle A, B\rangle = tr\left(A\overline{B^\intercal}\right)$$

This is called the Frobenius inner product *on* V*. It satisfies* iv *because, for* $A = (a_{ij})$*,* $B = (b_{ij})$ *we have:*

$$tr\left(A\overline{B^\intercal}\right) = \sum a_{ij}\overline{b_i j}$$

which is equivalent to the standard complex inner product*.*

1.2 Cauchy-Bunyakovsky-Schwarz Inequality

Definition 1.2.1 (Length)**.** *If* $\vec{v}$ *is a vector in an* inner product space*, the* length *of* $\vec{v}$ *is:*

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v}\rangle}$$

Theorem 1.2.1 (Cauchy-Schwarz)**.** *Let* $\vec{x}, \vec{y} \in V$ *be vectors in an* inner product space*, then:*

$$|\langle \vec{x}, \vec{y}\rangle| \leq \|\vec{x}\| \, \|\vec{y}\|$$

Proof. If $\vec{y} = \vec{0}$, this is trivial. Otherwise, for any $c \in \mathbb{C}$

$$\begin{aligned} 0 &\leq \|\vec{x} - c\vec{y}\|^2 && (1.3)\\ &\leq \|\vec{x}\|^2 - \overline{c}\,\langle \vec{x}, \vec{y}\rangle - c\,\langle \vec{y}, \vec{x}\rangle + c\overline{c}\,\|y\|^2 && (1.4)\end{aligned}$$

So, let $c = \dfrac{\langle \vec{x}, \vec{y}\rangle}{\|y\|^2}$:

$$\begin{aligned} 0 &\leq \|\vec{x}\|^2 - \frac{\overline{\langle \vec{x}, \vec{y}\rangle}}{\|y\|^2}\langle \vec{x}, \vec{y}\rangle - \frac{\langle \vec{x}, \vec{y}\rangle}{\|y\|^2}\langle \vec{y}, \vec{x}\rangle + \frac{\langle \vec{x}, \vec{y}\rangle\,\overline{\langle \vec{x}, \vec{y}\rangle}}{\|y\|^4}\|y\|^2 && (1.5)\\ &\leq \|\vec{x}\|^2 - \frac{|\langle \vec{x}, \vec{y}\rangle|^2}{\|y\|^2} - \frac{|\langle \vec{x}, \vec{y}\rangle|^2}{\|y\|^2} + \frac{|\langle \vec{x}, \vec{y}\rangle|^2}{\|y\|^2} && (1.6)\\ |\langle \vec{x}, \vec{y}\rangle|^2 &\leq \|\vec{x}\|^2\,\|\vec{y}\|^2 && (1.7)\\ |\langle \vec{x}, \vec{y}\rangle| &\leq \|\vec{x}\|\,\|\vec{y}\| && (1.8)\end{aligned}$$

□

Remark. *We can define the angle between* $\vec{x}, \vec{y}$ *as* $\cos^{-1}\dfrac{|\langle \vec{x}, \vec{y}\rangle|}{\|\vec{x}\|\,\|\vec{y}\|}$.

1.3 Orthogonality

Definition 1.3.1 (Orthogonality)**.** *Two vectors $\vec{x}, \vec{y}$ are* orthogonal *if and only if $\langle \vec{x}, \vec{y} \rangle = 0$.*

Definition 1.3.2 (Unit vector)**.** *A* unit vector *is a vector of length 1.*

Definition 1.3.3 (Orthogonal Set)**.** *An* orthogonal set *is a set S where $\langle \vec{x}, \vec{y} \rangle = 0$ for all $\vec{x}, \vec{y} \in S$, $\vec{x} \neq \vec{y}$.*

Definition 1.3.4 (Orthonormal Set)**.** *An* orthonormal set *is an* orthogonal set *in which each vector is a* unit vector*.*

Example 1.3.4.1 (Standard Basis)**.** *The standard basis in $\mathbb{R}^n$ is* orthonormal

Theorem 1.3.1 (Orthonormal Coordinates)**.** *Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be an* orthogonal basis *of an* inner product space *V. Then for any $\vec{x} \in V$ we have:*

$$\vec{x} = \sum_{i=1}^{n} \frac{\langle \vec{x}_i, \vec{v}_i \rangle}{\|\vec{v}_i\|^2} \vec{v}_i$$

Proof. Write $\vec{x} = a_1\vec{v}_1 + \ldots + a_n\vec{v}_n$. Then, for any i:

$$\begin{aligned}
\langle \vec{x}, \vec{v}_i \rangle &= \langle a_1\vec{v}_1 + \ldots + a_n\vec{v}_n, \vec{v}_i \rangle && (1.9)\\
&= a_1 \langle \vec{v}_1, \vec{v}_i \rangle + \ldots + a_n \langle \vec{v}_n, \vec{v}_i \rangle && (1.10)\\
&= a_i \langle \vec{v}_i, \vec{v}_i \rangle && (1.11)\\
a_i &= \frac{\langle \vec{x}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2} && (1.12)
\end{aligned}$$

□

Remark. *If $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is* orthonormal*, then $\vec{x} = \sum_{i=1}^{n} \langle \vec{x}_i, \vec{v}_i \rangle \vec{v}_i$*

Remark. *The $\vec{v}_i$ coordinate of $\vec{x}$ depends only $\vec{x}$ and $\vec{v}_i$. It does not depend on any other vectors in the basis.*

Remark. *In finite dimensions,* inner product spaces *always have* orthnormal bases*.*

Theorem 1.3.2 (Orthogonal $\implies$ Linear Independence)**.** *Let S be an* orthogonal set *of non-zero vectors, then S is linearly independent.*

Proof. For any $\vec{v}_1, \ldots, \vec{v}_n \in S$, set $a_1\vec{v}_1 + \ldots + a_n\vec{v}_n = \vec{0}$. By similar construction as 1.3.1, we can show for any i that:

$$a_i \langle \vec{v}_i, \vec{v}_i \rangle = 0$$

Since $\vec{v}_i \neq 0$ by assumption, $a_i = 0$ for all i. □

1.4 Gram-Schmidt Procedure

Given a basis $\{\vec{w}_1, \dots \vec{w}_n\}$ for a (finite dimensional) *inner product space* V, the Gram-Schmidt gives an *orthogonal basis* for V as follows:

Step 1 Set $\vec{v}_1 = \vec{w}_1$

Step 2 Set $\vec{v}_2 = \vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$

...

Step i Set $\vec{v}_i = \vec{w}_i - \sum_{j=1}^{i-1} \frac{\langle \vec{w}_2, \vec{v}_i \rangle}{\|\vec{v}_i\|^2}\vec{v}_i$

Claim 1.4.1. $\{\vec{v}_1, \dots, \vec{v}_n\}$ *is an* orthogonal basis *of* V

Proof. We first check that $\{\vec{v}_1, \dots, \vec{v}_n\}$ is *orthogonal*.

We proceed by induction on i. If $n = 1$, we are vacuously done.

Otherwise, assume that $\{\vec{v}_1, \dots, \vec{v}_i\}$ is *orthogonal*. For any $1 \le j \le i$ we have:

$$\langle \vec{v}_{i+1}, \vec{v}_j \rangle = \left\langle \vec{w}_{i+1} - \sum_{k=1}^{i} \frac{\langle \vec{w}_{i+1}, \vec{v}_k \rangle}{\|\vec{v}_k\|^2}\vec{v}_k, \vec{v}_j \right\rangle \tag{1.13}$$

$$= \left\langle \vec{w}_{i+1} - \frac{\langle \vec{w}_{i+1}, \vec{v}_j \rangle}{\|\vec{v}_j\|^2}\vec{v}_j, \vec{v}_j \right\rangle \tag{1.14}$$

$$= \langle \vec{w}_{i+1}, \vec{v}_j \rangle - \frac{\langle \vec{w}_{i+1}, \vec{v}_j \rangle}{\|\vec{v}_j\|^2} \|\vec{v}_j\|^2 \tag{1.15}$$

$$= \langle \vec{w}_{i+1}, \vec{v}_j \rangle - \langle \vec{w}_{i+1}, \vec{v}_j \rangle \tag{1.16}$$

$$= 0 \tag{1.17}$$

Furthermore, $\vec{v}_i \neq \vec{0}$. For $i = 1$ we have $\vec{v}_1 = \vec{w}_1 \neq \vec{0}$ by assumption. Otherwise, we have $\vec{v}_i = \vec{w}_i - \vec{x}$ for $x \in \text{span}\{\vec{w}_1, \dots, \vec{w}_{i-1}\}$. Thus $\vec{v}_i$ is a nonzero linear combination of $\{\vec{w}_1, \dots, \vec{w}_i\}$ and is therefore non-zero.

Thus $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a set of n vectors in an n-dimensional space that are orthogonal and nonzero. By 1.3.2 they are linearly independent, and thus a basis of V. $\square$

Remark. *To obtain an* orthonormal basis *of* V*, simply divide each* $\vec{v}_i$ *by its length. This is called* normalizing.

1.5 Orthogonal Complement

Definition 1.5.1 (Orthogonal Complement)**.** *Let V be an* inner product space *and $W \subset V$ a subspace. The orthogonal complement of W is:*

$$W^{\perp} = \{\vec{v} \in V : \langle \vec{v}, \vec{w} \rangle = 0 \textit{ for all } \vec{w} \in W\}$$

Example 1.5.1.1 (Orthogonal complement of entire space is empty)**.**

$$W = V \qquad\qquad W^{\perp} = \{\vec{v} \in V : \langle \vec{v}, \vec{w} \rangle = 0 \textit{ for all } \vec{w} \in V\} = \left\{\vec{0}\right\}$$

$$\textit{because } \langle \vec{v}, \vec{v} \rangle = 0 \implies \vec{v} = \vec{0}$$

Example 1.5.1.2 (Orthogonal complement of empty subspace is the entire space)**.**

$$W = \left\{\vec{0}\right\} \qquad\qquad W^{\perp} = V$$

Example 1.5.1.3 (Orthogonal complelment of a line is a plane)**.**

$$V = \mathbb{R}^3 \qquad W = \{(0,0,z) : z \in \mathbb{R}\} \qquad W^{\perp} = \{(x,y,0) : x, y \in \mathbb{R}\}$$