1 Problem

In this assignment we will be obtaining the least squares solution to the system

$$\begin{array}{rcrcl} 3x & - & 2y & = & 3 \\ & & 2y & = & 5 \\ 4x & + & 4y & = & 4 \end{array} \tag{1}$$

by using QR decomposition. The assignment will be carried out in four parts. In the first and second parts, we will use the Gram-Schmidt process and Householder transformations, respectively, to find the QR decomposition of the coefficient matrix of the system above. In the third part, we will use a matlab command to generate the QR decomposition. The fourth part will contain the solution of (1).

1.1 Part 1: Gram-Schmidt

Recall that a QR decomposition of an $n \times k$ matrix A is a factorization of A into the product QR, where Q is an $n \times k$ orthogonal matrix ($Q^T Q = I$) and R is a $k \times\ k$ upper triangular matrix. Using the Gram-Schmidt process allows us to determine the columns of the matrix Q by using the columns of A as the set of vectors to orthonormalize. After obtaining Q, we can use the identity

$$R = Q^T A \tag{2}$$

to compute the individual components of R.

Proceeding with the Gram-Schmidt process using vectors $a_1 = (3, 0, 4)^t$, $a_2 = (-2, 3, 4)^t$ from

$$A = \begin{bmatrix} 3 & -2 \\ 0 & 3 \\ 4 & 4 \end{bmatrix}$$

we obtain the following orthonormal vectors

$$u_1 = \frac{a_1}{|a_1|} = (\frac{3}{5}, 0, \frac{4}{5})^t$$

$$u_2 = \frac{w_2}{|w_2|} = (\frac{-16}{25}, \frac{3}{5}, \frac{12}{25})^t$$

where $w_2 = a_1 - (a_2 u_1) u_1 = (-16/5, 3, 12/5)^t$. Using these vectors as the columns of Q, we have

$$Q = \begin{bmatrix} 3/5 & -16/25 \\ 0 & 3/5 \\ 4/5 & 12/25 \end{bmatrix}.$$

Then, since A is 3×2, we know that R must be 2×2. So by using (2), we obtain

$$R = \begin{bmatrix} 5 & 2 \\ 0 & 5 \end{bmatrix}$$

.

We conclude that

$$A = \begin{bmatrix} 3/5 & -16/25 \\ 0 & 3/5 \\ 4/5 & 12/25 \end{bmatrix} \cdot \begin{bmatrix} 5 & 2 \\ 0 & 5 \end{bmatrix} = QR$$

1.2 Part 2: Householder Transformations

Using A as defined in part 1, we wish to apply orthogonal transformations H_1, H_2 to A in order to zero the entries below the main diagonal. If we take $Q = (H_2 H_1)^T$, then we see that the resultant product $H_2 H_1 A = R$, as $A = QR$, hence $Q^T A = Q^T Q R = R$.

Using the definition of the Householder reflector, we take $H_1 = I - 2u_1 u_1^t$ where $u_1 = a_1 + sgn(a_1)\|a_1\|e$ and $e = (1, 0, 0)^t$. Thus,

$$u_1 = \frac{(3,0,4)^t + 5(1,0,0)^t}{\|(3,0,2)^t + 5(1,0,0)^t\|} = \frac{1}{\sqrt{5}}(2,0,1)^t$$

And

$$H_1 = \frac{1}{5} \begin{bmatrix} -3 & 0 & -4 \\ 0 & 0 & 0 \\ -4 & 0 & 3 \end{bmatrix}$$

Multiplying by A, we get

$$H_1 A = \begin{bmatrix} -5 & -2 \\ 0 & 3 \\ 0 & 4 \end{bmatrix}$$

Proceeding in a similar manner using the vector $a_2 = (3, 4)^t$ from the second column of $H_1 A$, we construct $u_2 = \frac{1}{\sqrt{5}}(2, 1)^t$, from which we find H_2 to be

$$H_2 = \frac{1}{5}\begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix}$$

Multiplying H_2 by the block (vector, really) containing a_2, we obtain

$$H_2 A_{2:3,2} = \frac{1}{5}\begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$$

Putting this all together, we may conclude that the upper triangular matrix R is given by

$$R = \begin{bmatrix} -5 & -2 \\ 0 & -5 \\ 0 & 0 \end{bmatrix}.$$

We may then construct Q as the product of H_1 and $\tilde{H}_2$, where

$$\tilde{H}_2 = \begin{bmatrix} 1 & 0 \\ 0 & H_2 \end{bmatrix}$$

Thus, Q is given by

$$Q = \frac{1}{5}\begin{bmatrix} -3 & 0 & -4 \\ 0 & 0 & 0 \\ -4 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & -4/5 & 3/5 \end{bmatrix} = \begin{bmatrix} -3/5 & 16/25 & -12/25 \\ 0 & 0 & 0 \\ -4/5 & -12/25 & 9/25 \end{bmatrix}.$$

Part 3: MATLAB

Using the qr function in MATLAB, the following results are generated.

```
>> A = [3 -2; 0 3; 4 4]

A =

3     -2
0      3
4      4

>> [Q, R] = qr(A)

Q =

-0.6000     0.6400    -0.4800
0    -0.6000    -0.8000
-0.8000    -0.4800     0.3600

R =

-5     -2
0     -5
0      0
```

1.3 Part 4: Least Squares Solution

Using the Gram-Schmidt QR decomposition, the solution to the system can be found by solving

$$\begin{bmatrix} 5 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3/5 & -16/25 \\ 0 & 3/5 \\ 4/5 & 12/25 \end{bmatrix}^T \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$$

By using any conventional method, we find the solution to be

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19/25 \\ 3/5 \end{bmatrix}$$