1 Problem

In this assignment we will being utilizing the Gauss-Seidel method to approximate the solution of a linear system within 10^{-5} in the l_∞ norm. The system in question is

$$\begin{array}{rrrcr} 2x_1 & -x_2 & +x_3 & = & -1 \\ 2x_1 & +2x_2 & +2x_3 & = & 4 \\ -x_1 & -x_2 & +2x_3 & = & -5. \end{array} \tag{1}$$

To obtain the iterative method we intend to use, we isolate the diagonal and lower triangular terms on one side of the equality signs, which yields

$$\begin{array}{rrrcrrr} 2x_1 & & & = & -1 & +x_2 & -x_3 \\ 2x_1 & +2x_2 & & = & 4 & -2x_3 & \\ -x_1 & -x_2 & +2x_3 & = & -5. & & \end{array}$$

Converting the system into matrix form, we have

$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \\ -5 \end{bmatrix} \tag{2}$$

Let the coefficient matrices on the L.H.S. and R.H.S. be denoted by $D+L$ and U, respectively, and let C be the constant vector $(-1, 4, -5)^t$. Furthermore, let the vector $(x_1, x_2, x_3)^t$ on the R.H.S. be the kth term of the sequence $\left\{\mathbf{x}^{(n)}\right\}$, and let the same vector on the L.H.S. be the $(k-1)th$ term of the sequence. Then, we can express (2) as

$$(D+L)\mathbf{x}^k = U\mathbf{x}^{k-1} + C.$$

Then, assuming that $D+L$ is non-singular, we can isolate $\mathbf{x}^k$ and obtain

$$\begin{aligned} \mathbf{x}^k &= (D+L)^{-1}U\mathbf{x}^{k-1} + (D+L)^{-1}C \\ &= \begin{bmatrix} 0 & 1/2 & -1/2 \\ 0 & -1/2 & -1/2 \\ 0 & 0 & -1/2 \end{bmatrix} \mathbf{x}^{k-1} + \begin{bmatrix} -1/2 \\ 5/2 \\ -3/2 \end{bmatrix} \end{aligned}$$

which is exactly the Gauss-Seidel iterative method.

The matrix $(D+L)^{-1}U$ is called the Gauss-Seidel matrix, and if its Spectral Radius lies within the unit disk ($|z| < 1$), the sequence $\left\{\mathbf{x}^{(n)}\right\}$ is guaranteed to converge to the solution of (1). For this particular problem, it is easily verifiable that $|\rho((D+L)^{-1}U)| = 1/2$, hence we may proceed without any problems.

2 Source Code

```
% File Name: GS_hw4.m
% Assignment: Project 4
% Student: Joseph Free
% Course: MATH3261
%
% Purpose: This function takes an nxn coefficient matrix and constant
% n-dimensional constant vector corresponding to a linear system and
% attempts to find an approximate solution x within a user specified
% tolerance through the use of the Gauss-Seidel iterative method.
%
% Required input:
%
%   A         -- A square coefficent matrix.
%   C         -- An n-dimensional constant vector.
%   n         -- Dimension of the matrix A.
%   x0        -- Initial guess vector.
%   TOL       -- User specified tolerance.
%   N         -- Maximum number of iterations to attempt.
%

function [ x, iter ] = GS_hw4( A, C, n, x0, TOL, N )

U = zeros(n);
for i = 1:n
    for j = i:n-1
        U(i,j+1) = A(i,j+1);
    end
end
```

```
% Initialize variables.
DL = A - U;
invDL = DL^-1;
MG = invDL*-U;
MC = invDL*C;

% If the Gauss-Seidel matrix is convergent, continue with the method.
if ( max(eig(MG)) < 1 )
    iter = 0;
    x = x0;
    while( abs( max( abs(x) ) - max( abs( MG*x + MC ) ) ) > TOL )
        iter = iter + 1;
        x = vpa(MG*x + MC);
        if ( iter > N )
            clear x
            fprintf('Max iterations exceeded!\n')
            break
        end
    end
% If the Gauss-Seidel matrix is not convergent, get the hell out of Dodge.
else
    fprintf('Non-convergent matrix.\n');
end
end
```

```
% File Name: main_hw4.m
% Assignment: Project 4
% Student: Joseph Free
% Course: MATH3261
%
% Purpose: This script is the driver for thee GS_hw4 function.
% All results for this assignment are derived from these instructions.
%

A = [ 2 -1 1; 2 2 2; -1 -1 2; ] %#ok<*NOPTS>
C = [-1; 4; -5;]
x0 = [0;0;0;];

[x, iter] = GS_hw4( A, C, size(A,1), x0, 10^-5, 100)
```

3 Results

As seen in the Source Code section above, in the main_hw4 script we supply the function GS_hw4 with the coefficient matrix and constant vector from (1), the dimension of the matrix, an initial guess vector of $(0, 0, 0)^t$, tolerance of $10^{-}5$, and a maximum number of iterations. Using this data the GS_hw4 function constructs the Gauss-Seidel matrix, checks the criterion for convergence, and then proceeds to iterate through the sequence. Upon completion, it will yield the approximate solution $\mathbf{x}$ in addition to the number of iterations required to reach within $10^{-}5$ of the actual solution. The results proper can be seen below:

```
Command Window

>> main_hw4

A =

     2    -1     1
     2     2     2
    -1    -1     2


C =

    -1
     4
    -5


x =

   0.99999570846557617187
    2.00000476837158203125
 -0.99999976158142089844


iter =

    22
```

Thus, we see that after 22 iterations, we have a sufficiently accurate approximation to the solution of (1).