Solutions

1. Three matrices, **A**, **B** and **C**, are shown in the R output, below.

```
> ## Recall that R fills matrices by column, by default
> (A<- matrix(c(2,3,5,4,1,5,7,8),nrow=4,ncol=2))

     [,1] [,2]
[1,]    2    1
[2,]    3    5
[3,]    5    7
[4,]    4    8

> (B<- matrix(c(6,9,3,1),nrow=4,ncol=1))

     [,1]
[1,]    6
[2,]    9
[3,]    3
[4,]    1

> (C<- matrix(c(3,8,5,2,8,6,1,4),nrow=4,ncol=2))

     [,1] [,2]
[1,]    3    8
[2,]    8    6
[3,]    5    1
[4,]    2    4
```

Complete the following items.

(a) $\mathbf{A} + \mathbf{C}$ and give the size of the result.

```
> (result<-A+C)

     [,1] [,2]
[1,]    5    9
[2,]   11   11
[3,]   10    8
[4,]    6   12
```

```
> dim(result)

[1] 4 2
```

(b) $\mathbf{A} - \mathbf{C}$ and give the size of the result.

```
> (result<- A-C)

     [,1] [,2]
[1,]   -1   -7
[2,]   -5   -1
[3,]    0    6
[4,]    2    4

> dim(result)

[1] 4 2
```

(c) $\mathbf{B}'\mathbf{A}$ and give the size of the result.

```
> (result<- t(B)%*%A)

     [,1] [,2]
[1,]   58   80

> dim(result)

[1] 1 2
```

(d) $\mathbf{A}\mathbf{C}'$

```
> (result<- A%*%t(C))

     [,1] [,2] [,3] [,4]
[1,]   14   22   11    8
[2,]   49   54   20   26
[3,]   71   82   32   38
[4,]   76   80   28   40

> dim(result)

[1] 4 4
```

(e) $\mathbf{C}'\mathbf{A}$

```
> (result<- t(C)%*%A)

     [,1] [,2]
[1,]   63   94
[2,]   55   77

> dim(result)

[1] 2 2
```

2. Two matrices, **X** and **Y**, for a simple linear regression problem are shown in the `R` output, below.

```
> (X<- matrix(c(1,1,1,1,1,1,4,1,2,3,3,4),ncol=2))


     [,1] [,2]
[1,]    1    4
[2,]    1    1
[3,]    1    2
[4,]    1    3
[5,]    1    3
[6,]    1    4


> (Y<- matrix(c(16,5,10,15,13,22),ncol=1))


     [,1]
[1,]   16
[2,]    5
[3,]   10
[4,]   15
[5,]   13
[6,]   22
```

Complete the following items.

(a) $\mathbf{Y}'\mathbf{Y}$

```
> t(Y)%*%Y

     [,1]
[1,] 1259
```

(b) $\mathbf{X'Y}$

```
> (XtY<- t(X)%*%Y)

     [,1]
[1,]   81
[2,]  261
```

(c) $\mathbf{X'X}$

```
> (XtX<- t(X)%*%X)

     [,1] [,2]
[1,]    6   17
[2,]   17   55
```

(d) $(\mathbf{X'X})^{-1}$

```
> (XtXinv<- solve(XtX))

           [,1]       [,2]
[1,]  1.3414634 -0.4146341
[2,] -0.4146341  0.1463415

> MASS::fractions(XtXinv)

     [,1]   [,2]
[1,]  55/41 -17/41
[2,] -17/41   6/41
```

(e) $(\mathbf{X'X})^{-1}\mathbf{X'Y}$

```
> XtXinv%*%XtY

          [,1]
[1,] 0.4390244
[2,] 4.6097561
```

(f) Use `lm` in `R` to perform the implied simple linear regression. Report the estimates of the regression model parameters.

```
> coefficients(lm(Y ~ 0 + X))

       X1        X2
0.4390244 4.6097561
```

3. Consider the following linear combinations of the (not necessarily independent or uncorrelated) random variables, Y_1, Y_2, Y_3 and Y_4

$$\begin{aligned} W_1 &= \frac{1}{4}(Y_1 + Y_2 + Y_3 + Y_4) \\ W_2 &= \frac{1}{2}(Y_1 + Y_2) - \frac{1}{2}(Y_3 + Y_4) \end{aligned}$$

   (a) State the above linear combinations in matrix notation.

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix}$$

$$\mathbf{W} = \mathbf{AY}$$

   where $\mathbf{A}$ is the 2×4 matrix of numbers, etc.

   (b) Find the expectation of the random vector $\mathbf{W} = [W_1, W_2]^T$.
   $\mathrm{E}(\mathbf{W}) = \mathrm{E}(\mathbf{AY}) = \mathbf{A}\mathrm{E}(\mathbf{Y})$, which is

$$\begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} \mathrm{E}(Y_1) \\ \mathrm{E}(Y_2) \\ \mathrm{E}(Y_3) \\ \mathrm{E}(Y_4) \end{bmatrix} = \begin{bmatrix} 1/4(\mathrm{E}(Y_1) + \mathrm{E}(Y_2) + \mathrm{E}(Y_3) + \mathrm{E}(Y_4)) \\ 1/2(\mathrm{E}(Y_1) + \mathrm{E}(Y_2)) - 1/2(\mathrm{E}(Y_3) + \mathrm{E}(Y_4)) \end{bmatrix}$$

   (c) Find the variance of $\mathbf{W}$. In your solution, please use the shorthand notation, $\sigma_{ij} = \mathrm{Cov}(Y_i, Y_j)$. This should help to make things less messy. You may use pencil, if you like, rather than typesetting your solution.

   $\mathrm{Var}(\mathbf{W}) = \mathrm{Var}(\mathbf{AY}) = \mathbf{A}\mathrm{Var}(\mathbf{Y})\mathbf{A}^T$, which is

$$\begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{bmatrix} \begin{bmatrix} 1/4 & 1/2 \\ 1/4 & 1/2 \\ 1/4 & -1/2 \\ 1/4 & -1/2 \end{bmatrix}$$

   Thus, after some matrix multiplication and using the symmetry of variance matrices, i.e., $\mathrm{Cov}(Y_i, Y_j) = \mathrm{Cov}(Y_j, Y_i)$, i.e., $\sigma_{ij} = \sigma_{ji}$ (similar for that of the W_i), we have $\mathrm{Var}(\mathbf{W})$ with entries
   $\mathrm{Var}(W_1) = 1/16(\sigma_{11} + \sigma_{22} + \sigma_{33} + \sigma_{44} + 2(\sigma_{12} + \sigma_{13} + \sigma_{14} + \sigma_{23} + \sigma_{24} + \sigma_{34}))$
   $\mathrm{Var}(W_2) = 1/4(\sigma_{11} + \sigma_{22} + \sigma_{33} + \sigma_{44} + 2(\sigma_{12} + \sigma_{34} - \sigma_{13} - \sigma_{14} - \sigma_{23} - \sigma_{24}))$
   $\mathrm{Cov}(W_1, W_2) = 1/8(\sigma_{11} + \sigma_{22} - \sigma_{33} - \sigma_{44} + 2(\sigma_{12} - \sigma_{34}))$
   You may have written σ_i^2 for σ_{ii} or given similar symbolic notation for the W variance–covariance terms. Fine.