Homework #4

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1.

Suppose that the amount of fluid in a small can of soda can be well approximated by a Normal distribution. Let X be the amount of soda (in milliliters) in a single can and $X \sim N(\mu = 222,\ \sigma = 5)$.

(a) $P(X > 230) =$

$$\begin{aligned} P(X > 230) &= P\left(\frac{X - \mu_X}{\sigma_X} > \frac{230 - 222}{5}\right) \\ &= P\left(Z > \frac{8}{5}\right) \\ &= P(Z > 1.6) \end{aligned}$$

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1-pnorm( 1.6 )

## [1] 0.05479929
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(b) Suppose we take a random sample of 6 cans such that the six cans are independent. What is the expected value of the mean of those six cans? In other words, what is $E(\bar{X})$?

$$E(\bar{X}) = E(X) = \mu$$

(c) What is $Var(\bar{X})$? (Recall we denote this as $\sigma^2_{\bar{X}}$)

$$\sigma^2_{\bar{X}} = \frac{\sigma^2}{n} = \frac{25}{6} = 4.166\bar{6}$$

(d) What is the standard deviation of $\bar{X}$? (Recall we denote this as $\sigma_{\bar{X}}$)

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{5}{\sqrt{6}} = 2.041$$

(e) What is the probability that the sample mean will be greater than 230 ml? That is, find $P(\bar{X} > 230)$.

$$\begin{aligned} P(\bar{X} > 230) &= P\left(\frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}} > \frac{230 - 222}{5/\sqrt{6}}\right) \\ &= P\left(Z > \frac{8}{5/\sqrt{6}}\right) \\ &= P(Z > 3.919) \end{aligned}$$

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1-pnorm( 3.919 )

## [1] 4.445856e-05
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2.

Suppose that the number of minutes that I spend waiting for my order at Big Foot BBQ can be well approximated by a Normal distribution with mean $\mu = 10$ minutes and standard deviation $\sigma = 1.5$ minutes.

(a) Tonight I am planning on going to Big Foot BBQ. What is the probability I have to wait less than 9 minutes?

$$\begin{aligned} P(X < 9) &= P\left(\frac{X-\mu}{\sigma} < \frac{9-10}{1.5}\right) \\ &= P\left(Z < -0.666\bar{6}\right) \end{aligned}$$

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pnorm( -0.66667 )
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## [1] 0.2524915
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(b) Over the next month, I'll visit Big Foot BBQ 5 times. What is the probability that the mean waiting time of those 5 visits is less than 9 minutes? (This assumes independence of visits but because I don't hit the same restaurant the same night each week, this assumption is probably ok.)

$$\begin{aligned} P(\bar{X} < 9) &= P\left(\frac{\bar{X}-\mu_{\bar{X}}}{\sigma_{\bar{X}}} < \frac{9-10}{1.5/\sqrt{5}}\right) \\ &= P\left(Z < -1.49\right) \end{aligned}$$

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pnorm( -1.49 )
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## [1] 0.06811212
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3.

A bottling company uses a machine to fill bottles with a tasty beverage. The bottles are advertised to contain 300 milliliters (ml), but in reality the amount varies according to a normal distribution with mean $\mu = 298$ ml and standard deviation $\sigma = 3$ ml. (For this problem, we'll assume σ is known and carry out the calculations accordingly).

(a) What is the probability that a randomly chosen bottle contains less than 296 ml?

$$\begin{aligned} P(X < 296) &= P\left(\frac{X-\mu}{\sigma} < \frac{296-298}{3}\right) \\ &= P\left(Z < \frac{-2}{3}\right) \end{aligned}$$

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pnorm( -2/3 )
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## [1] 0.2524925
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(b) Given a simple random sample of size$n = 6$ bottles, what is the probability that the sample mean is less than 296 ml?

$$\begin{aligned} P(\bar{X} < 296) &= P\left(\frac{\bar{X}-\mu}{\sigma_{\bar{X}}} < \frac{296-298}{3/\sqrt{6}}\right) \\ &= P\left(Z < -1.633\right) \end{aligned}$$

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pnorm( -1.633 )
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## [1] 0.0512345
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(c) What is the probability that a single bottle is filled within 1 ml of the true mean $\mu = 298$ ml? *Hint: Draw the distribution and shade in what probability you want... then convert that to a question about standard normals. To find the answer using a table or R, you need to look up two values and perform a subtraction.*

$$
\begin{aligned}
P(297 < X < 299) &= P\left(\frac{297-298}{3} < \frac{X-\mu}{\sigma} < \frac{299-298}{3}\right) \\
&= P\left(-\frac{1}{3} < Z < \frac{1}{3}\right) \\
&= P\left(Z < \frac{1}{3}\right) - P\left(Z < -\frac{1}{3}\right)
\end{aligned}
$$

```
pnorm(1/3) - pnorm(-1/3)
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## [1] 0.2611173
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(d) What is the probability that the mean of 10 randomly selected bottles is within 1 ml of the mean? What about a sample of 100?

$$
\begin{aligned}
P(297 < \bar{X}_{10} < 299) &= P\left(\frac{297-298}{3/\sqrt{10}} < \frac{\bar{X}-\mu}{\sigma/\sqrt{10}} < \frac{299-298}{3/\sqrt{10}}\right) \\
&= P(-1.054 < Z < 1.054) \\
&= P(Z < 1.054) - P(Z < -1.054)
\end{aligned}
$$

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pnorm(1.054) - pnorm(-1.054)
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## [1] 0.7081171
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$$
\begin{aligned}
P(297 < \bar{X}_{100} < 299) &= P\left(\frac{297-298}{3/\sqrt{100}} < \frac{\bar{X}-\mu}{\sigma/\sqrt{100}} < \frac{299-298}{3/\sqrt{100}}\right) \\
&= P(Z < 3.333) - P(Z < -3.333)
\end{aligned}
$$

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pnorm(3.333) - pnorm(-3.333)
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## [1] 0.9991
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(e) If a sample of size $n = 50$ has a sample mean of $\bar{x} = 298$, should this be evidence that the filling machine is out of calibration? i.e., assuming the machine has a mean fill amount of $\mu = 300$ and $\sigma = 3$, what is $P\left(\bar{X} \leq 298\right)$?

Critically this calculation is assuming that $\mu = 300$.

$$
\begin{aligned}
P(\bar{X} < 298) &= P\left(\frac{\bar{X}-\mu}{\sigma_{\bar{X}}} < \frac{298-300}{3/\sqrt{50}}\right) \\
&= P(Z < -4.714)
\end{aligned}
$$

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pnorm( -4.714 )
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## [1] 1.214503e-06
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The sample mean we saw is very far into the tail of the distribution of values of $\bar{x}$ that we should see if $\mu = 300$. So either we got really really unlucky with our sample, or the machine has drifted out of calibration. Considering we know that the machine periodically needs to be re-calibrated, Occam's razor would suggest the more likely event is that the machine has drifted out of calibration.