Chapter 1

The Symplectic Group

A natural starting place is the study of the symplectic group. The symplectic group is one of the classical matrix groups that arises in the study of bilinear forms.

Let V be a vector space over $F = \mathbb{R}$ or $\mathbb{C}$.

Definition 1.1. We define a bilinear form on V as a function $B : V \times V \to F$ such that $B_x : y \mapsto B(x,y)$ and $B_y : x \mapsto B(x,y)$ are linear. If $(x_i)_{i=1}^n$ is a choice of basis for V, then B has a matrix representation

$$(B(x_i, x_j))_{i,j\in[n]} \text{ where } B(x,y) = B(\sum_{i=1}^n a_i x_i, \sum_{i=1}^n b_i x_i) = x^\top B y = \sum_{i=1}^n \sum_{j=1}^n a_i b_j B(x_i, x_j).$$

Definition 1.2. We say that bilinear form B is nondegenerate if the matrix form of B is of full rank.

It makes sense to assume our bilinear form is nondegerate. If it weren't, we could produce a linear subspace V' of V such that B restricted to V' was of full rank.

Definition 1.3. We say that B is symmetric if $\forall x, y \in V, B(x,y) = B(y,x)$ and that B is alternating if $\forall x \in V, B(x,x) = 0$.

When the characteristic of the underlying field is not 2, other words for alternating are anti-symmetric and skew-symmetric. These definitions are particularly important for Proposition 1.4.

Proposition 1.4. *Let B be a bilinear form on V and say $x \perp_B y$ iff $B(x,y) = 0$. Then $(x \perp_B y \implies y \perp_B x)$ iff B is symmetric or alternating.*

Proof. $\Leftarrow$ Suppose B is symmetric. If $B(x,y) = 0$, then $B(y,x) = B(x,y) = 0$. Suppose B is alternating. From

$$B(x+y, x+y) = B(x,x) + B(x,y) + B(y,x) + B(y,y) = B(x,y) + B(y,x) = 0,$$

we have that $B(x,y) = -B(y,x)$. If $B(x,y) = 0$, then $B(y,x) = -B(x,y) = 0$.
$\Rightarrow$ Since B is bilinear

$$B(x, B(x,y)z - B(x,z)y) = B(x,y)B(x,z) - B(x,z)B(x,y) = 0. \tag{1.1}$$

Now suppose that $x \perp_B y \implies y \perp_B x$. Then by 1.1

$$B(B(x,y)z - B(x,z)y, x) = B(x,y)B(z,x) - B(x,z)B(y,x) = 0. \tag{1.2}$$

If we set $z = x$ in the above equation, we get $B(x,x)(B(x,y) - B(y,x) = 0$. So

$$\forall x \in V, \text{ either } B(x,x) = 0 \text{ or } \forall y \in V\, B(x,y) = B(y,x). \tag{1.3}$$

Now suppose that B is not symmetric. Then there exists $x, y \in V$ such that $B(x,y) \neq B(y,x)$. Then $B(x,x) = B(y,y) = 0$ by 1.3. We wish to prove that B is alternating, so suppose that $\exists z \in V$ such that $B(z,z) \neq 0$. By 1.3, $B(z,x) = B(x,z)$ and $B(z,y) = B(y,z)$. By 1.2, we have that

$$B(x,y)B(z,x) - B(x,z)B(y,x) = B(x,z)(B(x,y) - B(y,x)) = 0$$

and

$$B(y,x)B(z,y) - B(y,z)B(x,y) = B(z,y)(B(y,x) - B(x,y)) = 0.$$

Since $B(x,y) \neq B(y,x)$, $B(x,z) = B(x,z) = B(z,y) = B(y,z) = 0$. But then

$$B(x, z+y) = B(x,y) \neq B(y,x) = B(z+y, x).$$

By 1.3, $B(z+y, z+y) = B(z,z) + B(z,y) + B(y,z) + B(y,y) = B(z,z) = 0$, a contradiction. Therefore B is alternating, completing the proof. □

$\perp_B$ should be an obvious generalization of the notion of perpendicularity.

Proposition 1.5. *Suppose B is a nondegenerate alternating bilinear form on a vector space V over F. Then V is of even dimension.*

The most direct way to show this is to construct a basis so that the matrix representation of your alternating bilinear form is of the form of 1.4.

Definition 1.6. Let B be a nondegenerate, alternating bilinear form over V. Then we say that (V, B) is a symplectic vector space.

Here we are continuing with our generalization of inner product spaces.

Proposition 1.7. *Let (V, B) be a symplectic vector space of dimension $2n$. Then V has a basis in which B is of the form*

$$J_n = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}. \tag{1.4}$$

Such a basis is called a symplectic basis of (V, B).

Proof. We choose an ordered basis $(w_i)_{i=1}^{2n}$ by a greedy algorithm. Set $W_i = \text{span}\{(w_j)_{j=1}^{i}\} \cup \text{span}\{(w_j)_{j=n+1}^{n+i}\}$, defined at the appropriate time as we specify our basis. For any subspace $W \subset V$, define the $W^\perp$ by

$$W^\perp = \{v \in V \mid B(v,w) = 0 \text{ for } w \in W\}.$$

Suppose $w \in W_i$. Since B is nondegenerate, so is B restricted to W_i. Let $(w'_k)_{k=1}^i$ be an ordered basis for W_i, regardless of our prior choices of w_i. Let $B(w'_i, w'_j) = b_{i,j}$ be the matrix elements of the restriction of B to W_i in basis $(w'_k)_{k=1}^i$. Write $w = \sum_{j=1}^i c_j w'_j$.

We wish to prove that $w \notin W_i^\perp$. If $B(w, w'_k) = 0$ for all $k \in [i]$, then $B(w, w') = 0$ for all $w' \in W$. Hence $w \in W^\perp$ iff $B(w, w'_k) = 0$ for all $k \in [i]$. Finally

$$w \in W_i^\perp \text{ iff } B(w, w'_k) = \sum_{j=1}^{i} c_j B(w'_j, w'_k) = \sum_{j=1}^{i} c_j b_{j,k} = 0.$$

But since B restricted to W_i is nondegenerate, $\sum_{j=1}^i c_j b_{j,k} = 0 \implies c_j = 0 \forall j \in [i]$. Hence $w = 0$ and we can conclude that $W_i \cap W_i^\perp = 0$.

Since $W^\perp$ is a linear subspace of V and B is nondegenerate, B restricted to $W^\perp$ is nondegenerate. Thus $\exists w_{i+1},\ w'_{n+i+1} \in W_i^\perp$ such that $B(w_{i+1}, w'_{n+i+1}) \neq 0$. Suppose $B(w_{i+1}, w'_{n+i+1}) = b$. Then set $w_{n+i+1} = -\frac{w_{n+i+1}}{b}$ so that $B(w_{i+1}, w_{n+i+1}) = -1 = -B(w_{n+i+1}, w_{i+1})$. Let

$$y = x - \sum_{k=1}^{i} B(x, w_{n+i+1}) w_i + \sum_{k=1}^{i} B(x, w_i) w_{n+i+1}.$$

Then $y \in W_i^\perp$

If $W_i \neq V$, then $\exists w_{i+1},\ w'_{n+i+1} \in W_i^\perp$ such that $B(w_{i+1}, w'_{n+i+1}) \neq 0$. Suppose $B(w_{i+1}, w'_{n+i+1}) = b$. Then set $w_{n+i+1} = -\frac{w_{n+i+1}}{b}$ so that $B(w_{i+1}, w_{n+i+1}) = -1 = -B(w_{n+i+1}, w_{i+1})$. The algorithm terminates when $W_{2n} = V$ and $(w_i)_{i=1}^{2n}$ is an order basis in which B has matrix form J. □

This particular J is chosen because it is "nice." We can easily construct such a basis for proposition 1.7 using a greedy algorithm.

Definition 1.8. Let (V_1, B_1) and (V_2, B_2) be two symplectic vector spaces. A linear transformation $L : V_1 \to V_2$ is a symplectomorphism of (V_1, B_1), (V_2, B_2) if

$$B_2(L(x), L(y)) = B_1(x, y), \forall x, y \in V_1.$$

Most classical matrix are defined in this way as isogenies.

Definition 1.9. The group of symplectic automorphisms of (V, B), (V, B) is called the symplectic group $Sp((V, B))$.

We have a collection of morphisms and we want to find linear representations of those morphisms. That will give us $\mathrm{Sp}_n(F)$.

Proposition 1.10. *Let (V, B) be a symplectic vector space of dimension $2n$ with a fixed symplectic basis. Then M is a matrix representation of some symplectic automorphism of (V, B) iff*

$$M^\top J_n M = J_n.$$

Proof. Fix a symplectic basis of V. Let $[u]$ represent the coordinate vector for $u \in V$ in the symplectic basis. Then

$$B(v,w) = [w]^T J[v].$$

Let M be a matrix of a linear transformation $\phi : V \to V$ in the symplectic basis. Then ϕ is symplectic if and only if

$$B(\phi(v),\phi(w)) = \phi(v,w), \text{ for all } v,w \in V.$$

In the symplectic basis this condition says

$$[\phi(w)]^T J[\phi(v)] = [w]^T J[v]$$

or

$$[w]^T M^T JM[v] = [w]^T J[v], \text{ for all } v,w \in V.$$

Since v,w are arbitrary this is equivalent to the statement. □

This characterization follows directly from the fact that any symplectomorphism can be represented by J under an appropriate choice of basis.

Definition 1.11. The symplectic matrix group is defined by

$$Sp_{2n}(S) = \{M \in M_{2n}(S) \mid M^\intercal JM = J\}$$

where $S = \mathbb{Z},\mathbb{Q},\mathbb{R}$, or $\mathbb{C}$.

The following properties come in handy when working with the symplectic group.

Proposition 1.12. *For $A,B,C,D \in M_n(S)$, the following are equivalent:*

1. $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp_{2n}(S)$
2. $A^\intercal C = C^\intercal A,\ B^\intercal D = D^\intercal B,\ A^\intercal D - C^\intercal B = I_n.$

Proof. Follows from the block representation of M on the condition $M^T JM = J$. □

Proposition 1.13. *For all $M \in Sp_{2n}(S)$, $det(M) = 1$.*

Chapter 2

Complex Structures of Real Symplectic Spaces

As vector spaces over $\mathbb{R}$, $\mathbb{R}^{2n} \cong \mathbb{C}$ by the map

$$v = \begin{bmatrix} x \\ y \end{bmatrix} \leftrightarrow z = x + iy.$$

The operation $z \mapsto iz$ corresponds to $v \mapsto -J_n v$, giving $\mathbb{R}^{2n}$ a complex structure. It is now natural to ask the ways in which we can equip $\mathbb{R}^{2n}$ a complex structure.

Definition 2.1. Let V be a vector space over $\mathbb{R}$. We say that $J \in \operatorname{Aut} V$ is a complex structure on V if

$$J^2 = -\operatorname{id}_V .$$

If V is a symplectic vector space, we may ask that the complex structure J be compatible with the symplectic structure on V.

Definition 2.2. Let B be a symplectic form on V. We say that complex structure J is compatible with B if $\forall v, w \in V$, $B(Jx, Jy) = B(x, y)$.

Now suppose that J is a complex structure on V compatible with B. Define

$$B'(x, y) \coloneqq B(v, Jy) \text{ for } x, y \in V.$$

Since J and B are compatible,

$$g(Jx, y) = B(x, y),$$

and since $J^2 = -1$ and B is alternating,

$$g(x, y) = g(y, x)$$

and

$$g(Jx, Jy) = g(x, y).$$

Therefore g is symmetric and non-degenerate. If g is positive semi-definite, then it is also Hermitian. In that case, we say that J is a positive compatible complex structure and the triple $(V, B, J$ a Kähler vector space. As before, a natural question to ask is what are the positive compatible complex structures on a real vector space.

Chapter 3

Lagrangian Subspaces and Polarizations

Definition 3.1. Let (V, B) be a symplectic vector space and W a linear subspace of V. Then the B-orthogonal complement of W is

$$W^{\perp} = \{v \in V \mid B(v, w) = 0 \text{ for all } w \in W\}.$$

W is Lagrangian iff $W^{\perp} = W$.

Definition 3.2. Suppose W_1 and W_2 are Lagrangian subspaces of V. Then $W_1 \diamond_V W_2$ iff $V = W_1 \oplus W_2$. We denote $T(W_1) = \{W_2 \subset V \mid W_1 \diamond_V W_2\}$.

Proposition 3.3. *For any Lagrangian subspace W_1 of V, there exists Lagrangian subspace W_2 of V such that $W_1 \diamond_W W_2$.*

Proposition 3.4. *Let $L(V)$ ddenote the Lagrangian subspaces $L \subset V$. Then $L(\mathbb{R}^{2n}) \cong U(n)/O(n)$.*

Proof. test

Proposition 3.5. *test*

test $\square$

Proposition 3.6. *Suppose $W_1 \diamond W_2$ in V. Then there exists a symplectic basis $\{x_i\}_{i=1}^{2n}$ of V such that $\{x_i\}_{i=1}^{n}$ is a basis of W_1 and $\{x_i\}_{i=n+1}^{2n}$ is a basis of W_2. Such a basis is called adapted to the decomposition.*

Proposition 3.7. *The set of symplectomorphisms that preserve W act transitively on $T(W)$.*

Proposition 3.8. *Suppose W_1, W_2, and W_3 and Lagrangian subspaces of V and that $W_1 \diamond_V W_2$ and $W_1 \diamond_V W_3$. Then there is a symplectomorphism $\phi : V \to V$ such that $\phi(W_1) = W_1$ and $\phi(W_2) = W_3$.*

For a real symplectic vector space $(V_{\mathbb{R}}, B_{\mathbb{R}})$, denote its complexification by $(V_{\mathbb{C}}, B_{\mathbb{C}})$ where

$$V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V_{\mathbb{R}}.$$

and $B_{\mathbb{R}}$ is extended to $V_{\mathbb{C}}$ by linearity over $\mathbb{C}$. We define conjugation in $V_{\mathbb{C}}$ by

$$\overline{cv} = \overline{c}v \text{ for } v \in \mathbb{R}$$

real-linearly extended to $V_{\mathbb{C}}$.

Definition 3.9. A Lagrangian subspace is real if it is a complexification of a Lagrangian subspace in $V_{\mathbb{R}}$. We say that a Lagrangian subspace W is positive iff

$$-iB(x, \overline{x}) > 0, \forall x \in W.$$

Theorem 3.10. *Let W_1 be a real Lagrangian subspace and W_2 be a positive Lagrangian subspace of of $V_{\mathbb{C}}$. Then $W_1 \diamond_V W_2$.*

Let $\overline{W} = \{\overline{x} \in V \mid x \in W\}$.

Definition 3.11. If $W \diamond_V \overline{W}$ and W is positive, then $W \oplus \overline{W}$ is a positive polarization of $V_{\mathbb{C}}$.

Theorem 3.12. *There is a natural bijection between the collection of positive complex structures on V_R and the collection of positive polarizations of $V_{\mathbb{C}}$.*

Theorem 3.13. *Let $V_{\mathbb{R}}$ be a symplectic vector space of dimension $2n$. The set of positive polarizations $V_{\mathbb{C}} = W \oplus \overline{W}$ is parameterized by the Siegel upper half plane*

$$\mathbb{H}_n = \{M \in M_n(\mathbb{C}) \mid M \textit{ symmetric and } \operatorname{Im} M \textit{ positive definite}\} \cong Sp_n(\mathbb{R})/U(g).$$

Chapter 4

Moduli Space of Polarized Tori

Definition 4.1. Let $b = \{b_1, b_2, ..., b_{2g}\}$ be $\mathbb{R}$-linearly independent vectors in $\mathbb{C}^n$. Then

$$L = \mathbb{Z}b_1 + \mathbb{Z}b_2 + \ldots + \mathbb{Z}b_{2n}$$

is a lattice subgroup of $\mathbb{C}^n$.

Definition 4.2. $X = \mathbb{C}^n/L$ is an n-dimensional complex torus.

Theorem 4.3. *Any connected compact complex Lie group X of dimension n is a complex torus.*

Definition 4.4. Given a lattice L of $\mathbb{C}^n$ with basis $b = \{b_1, b_2, ..., b_{2n}\}$, the matrix

$$\Pi_b = \begin{bmatrix} b_1 & b_2 & \ldots & b_{2n} \end{bmatrix}$$

is called a period matrix of lattice L with respect to basis b.

Proposition 4.5. *Two bases $a = \{a_1, a_2, ..., a_{2n}\}$ and $b = \{b_1, b_2, ..., b_{2n}\}$ generate the same lattice iff*

$$\Pi_a = M\Pi_b$$

with $M \in M_{n,n}(\mathbb{Z})$ and $det(M) = \pm 1$.

Definition 4.6. An algebraic set is the locus of zeros of a finite collection of polynomials.

Definition 4.7. A complex manifold X is called projective algebraic if there is a holomorphic embedding $\phi : X \to \mathbb{P}^N$ such that $\phi(X)$ is a regular algebraic set.

Definition 4.8. A torus X is an abelian variety if it is projective algebraic.

Theorem 4.9. *All complex tori are compact Kähler manifolds.*

Theorem 4.10. *A compact Kähler manifold endowed with a positive line bundle admits a projective embedding.*

Definition 4.11. A positive polarization on $X = \mathbb{C}^n/L$ is a positive definite hermitian form H on $\mathbb{C}^n$ that $\operatorname{Im} H$ is integer valued and alternating on L.

Theorem 4.12. *If torus X is equipped with a positive definite hermitian form H, then the corresponding line bundle is positive.*

Proposition 4.13. *Let L_1 and L_2 be lattices of $\mathbb{C}^n$ and let ϕ be a linear isomorphism of $\mathbb{C}^n$ that takes L_1 to L_2. Then ϕ induces a homeomorphism ϕ' of $\mathbb{C}^n/L_1$ and $\mathbb{C}^n/L_2$.*

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "\phi"] \arrow[d] & \mathbb{C}^n \arrow[d] \\ \mathbb{C}^n/L_1 \arrow[r, "\phi'"] & \mathbb{C}^n/L_2 \end{tikzcd}$$

Definition 4.14. We say that two polarized abelian varieties (X_1, H_1) and (X_2, H_2) are isomorphic if there exists and isomorphism $\phi' : X_1 \to X_2$ such that

$$H_1(x, y) = H_2(\phi'(x), \phi'(y)).$$

Proposition 4.15. *Fix a complex vector space V. There is a bijection f between hermitian forms H on V and real-valued alternating forms E on V that satisfy $E(ix, iy) = E(x, y)$ given by*

$$E(x, y) = \operatorname{Im} H(x, y) \text{ and } H(x, y) = E(ix, y) + iE(x, y)$$

Proposition 4.16. *A real-valued alternating form E provides a polarization on torus X via bijection f iff the following Riemann Relations are satisfied:*

1. $\Pi_e[E]_e^{-1}\Pi_e^{\intercal}$
2. $i\Pi_e[E]_e^{-1}\overline{\Pi_e^{\intercal}}$

Proposition 4.17. *Let $X = \mathbb{C}^n/L$ be a torus with polarization H. There exists a basis e of L such that $E = \operatorname{Im} H$ has the matrix representation*

$$[E]_e = \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix}$$

where $D = diag(d_1, ..., d_n)$ with integers $d_i \geq 0$ and $d_i \mid d_{i+1}$.

Definition 4.18. Such a basis e is called a symplectic basis of L. $D = \operatorname{diag}(d_1, ..., d_n)$ is called the type of the polarization of the abelian variety. A polarization is called principal if it is of type $(1, 1, ..., 1)$.

Lemma 4.19. *Given n relatively prime integers $(m_i)_{i=1}^n$, there exists*

$$M = \begin{bmatrix} m_1 & m_2 & \dots & m_n \\ * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * \end{bmatrix}$$

such that $M \in \mathrm{GL}_n(\mathbb{Z})$.

Proposition 4.20. *Let L be a polarized lattice in $\mathbb{C}^n$ with polarization of type $D = diag(d_i)_{i=1}^n$ and symplectic basis $e = \{e_i\}_{i=1}^{2n}$. Let $e' = \{\frac{1}{d_i}e_i\}_{i=1}^n$ be a basis for $\mathbb{C}^n$. Then*

$$\Pi_{e'} = \begin{bmatrix} D & Z \end{bmatrix},$$

where $Z \in \mathbb{H}_n$. Let this map from abelian varieties of specified type to $\mathbb{H}_n$ be called g_D.

Proposition 4.21. *To type D and $Z \in \mathbb{H}_n$, we can construct a polarized abelian variety A such that $g_D(A) = Z$.*

Proposition 4.22. *Given a type D, the Siegel upper half space $\mathbb{H}_n$ is a moduli space for polarized abelian varieties of type D with symplectic basis.*

Definition 4.23. Let

$$G_D := \{M \in \mathrm{Sp}(\mathbb{Q}) \mid M^\intercal L_D \subset L_D\}$$

and for $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G_D$ and $Z \in \mathbb{H}_n$, define the fractional linear transformation action by

$$M(Z) = (AZ + B)(CZ + D)^{-1}.$$

Theorem 4.24. *For Z_1, $Z_2 \in \mathbb{H}$ with polarization type D fixed, the following are equivalent:*

- *polarized abelian varieties $g_D(Z_1)$ and $g_D(Z_2)$ are isomorphic*
- *$Z_2 = M(Z)$ for some $M \in G_D$*

Theorem 4.25. *When $n = 2$, $G_{\mathrm{diag}(1,N)}$ is the paramodular group*

$$K(N) = Sp_4(\mathbb{Q}) \cap \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N\mathbb{Z} \\ \frac{1}{N}\mathbb{Z} & \frac{1}{N}\mathbb{Z} & \frac{1}{N}\mathbb{Z} & \mathbb{Z} \end{bmatrix}$$

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