Quantum optics
Standing Wave quantization

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Homework of Lesson 1

This homework is complemented with introductory and concluding video material. Have a look at the video before starting the homework. Once you have completed the problem, go to the corresponding quiz for evaluating your answers. Once you complete the quiz you will have access to the solution to the problem (pdf file).

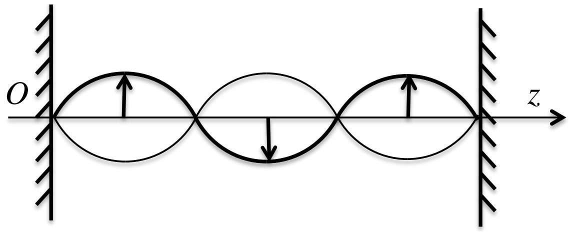

FIGURE 1 – *Planar Fabry-Pérot cavity.*

The goal of the problem is to apply the field quantization method presented in lesson 1 directly to a standing wave field instead of a free plane wave mode. We consider a one dimensional mode defined by two planar mirrors separated by a distance L_{cav} along the Oz direction (see figure 1). We consider a mode with linear polarization $\boldsymbol{\varepsilon}$.

1 Classical standing wave

The expression of the classical vector potential (using Coulomb gauge), electric field an magnetic field of a standing wave is

$$\boldsymbol{A}(z,t) = A(t)\,\boldsymbol{\varepsilon}\sin(k\cdot z) \tag{1}$$
$$\boldsymbol{E}(z,t) = E(t)\,\boldsymbol{\varepsilon}\sin(k\cdot z) \tag{2}$$
$$\boldsymbol{B}(z,t) = A(t)\,\boldsymbol{k}\times\boldsymbol{\varepsilon}\cos(k\cdot z) \tag{3}$$

where $A(t)$ and $E(t)$ are real field amplitudes and the wave-vector $\boldsymbol{k}$ is oriented along the z axis.

1. For which values of $k = |\boldsymbol{k}|$ does this mode match a cavity of length L_{cav} ? From now on, k takes one of these values.
2. From the well known equation $E(t) = -\dot{A}(t)$ and from Maxwell equations, show that $\dot{E}(t) = \omega^2 A(t)$ with $\omega = ck$.
3. We define the dimensionless parameter $\alpha(t) = 1/(2\,\mathscr{E}_{sw}^{(1)})\left(\omega A(t) - iE(t)\right)$ where $\mathscr{E}_{sw}^{(1)}$ is a normalization factor with the dimention of an electric field. What is the differential equation determining the time evolution of α ?
4. The energy of the mode is defined by

$$H = \frac{\varepsilon_0}{2}\int_{V_{cav}}\left(\boldsymbol{E}(\boldsymbol{r},t)^2 + c^2\boldsymbol{B}(\boldsymbol{r},t)^2\right)d^3r \tag{4}$$

where $V_{cav} = L_{cav}\, L^2$ is the quantization volume for a planar cavity of length L_{cav}. The length L is the size of the quantization volume in the transverse direction with respect to the mode axis. Show that for a proper choice of $\mathscr{E}_{sw}^{(1)}$ one has $H = \hbar\omega|\alpha(t)|^2 = \hbar\omega|\alpha(0)|^2$.

5. By analogy with the case of a plane wave, how can one define a pair of canonically conjugated variables P and Q ? Give the expression of P and Q as a function of $E(t)$ and $A(t)$. What is the dimension of these variables P and Q.
6. Express $\boldsymbol{A}(z,t)$ and $\boldsymbol{E}(z,t)$ as a function of $\alpha(t)$ and $\alpha^*(t)$.

2 Quantization

1. Following the same procedure as presented in lesson 1 for quantizing a plane wave, how can you define quantum field operators $\hat{\boldsymbol{A}}(z)$ and $\hat{\boldsymbol{E}}(z)$ as a function of a non-hermitian operator $\hat{b}$ verifying the commutation relation $[\hat{b}, \hat{b}^\dagger] = 1$.
2. The travelling wave electric field operators with wavevector $\boldsymbol{k}$ and polarization $\boldsymbol{\varepsilon}$ is defined by
$$\hat{\boldsymbol{E}}_{\boldsymbol{k},\boldsymbol{\varepsilon}}(z) = i\ \mathscr{E}_k^{(1)}\, \boldsymbol{\varepsilon} \left(\hat{a}_{\boldsymbol{k},\boldsymbol{\varepsilon}} e^{i\boldsymbol{k}\cdot z} - \hat{a}^\dagger_{\boldsymbol{k},\boldsymbol{\varepsilon}} e^{-i\boldsymbol{k}\cdot z}\right) \tag{5}$$
with $\mathscr{E}_k^{(1)} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V_{cav}}}$. By expressing the superposition of two counter-propagating travelling waves modes $\hat{\boldsymbol{E}}_{\boldsymbol{k},\boldsymbol{\varepsilon}}(z) + \hat{\boldsymbol{E}}_{-\boldsymbol{k},\boldsymbol{\varepsilon}}(z)$ as a function of $\sin(kz)$ and $\cos(kz)$, express $\hat{b}$ as a function of $\hat{a}_{\boldsymbol{k},\boldsymbol{\varepsilon}}$ and $\hat{a}_{-\boldsymbol{k},\boldsymbol{\varepsilon}}$. Give the expression of an operator $\hat{c}$ creating one photon in a mode varying spatially as $\cos(kz)$.
Using the commutation relations $[\hat{a}_{\boldsymbol{k},\boldsymbol{\varepsilon}}, a^\dagger_{\boldsymbol{k},\boldsymbol{\varepsilon}}] = 1$ check that these expressions are consistent with the expected commutation relations $[\hat{b}, \hat{b}^\dagger] = 1$ and $[\hat{c}, \hat{c}^\dagger] = 1$. What is the value of $[\hat{b}, \hat{c}^\dagger]$?
3. Remember that one photon in the travelling wave mode $(\boldsymbol{k}, \boldsymbol{\varepsilon})$ has a momentum $\boldsymbol{p}_{\boldsymbol{k}} = \hbar\boldsymbol{k}$. Give, without calculations, the average value of the momentum for a one photon state stored in the cavity mode. What are the possible results of a measurement of the momentum for this field ?

Concluding remark

The quantization method of a standing wave presented here applies to any mode of the classical electromagnetic field, like the gaussian mode of a laser cavity or the dipole radiation mode of an antenna. Indeed the method applies to any classical phenomena corresponding to periodic evolution of independent classical variables.

The concluding video illustrates this point by introducing the quantization of an oscillating LC electric circuit. You will learn that superconducting circuits cooled down to 10 mK temperature effectively behave quantum mechanically although they are involving a huge number of electrons. This demonstrated the generality of the canonical quantization procedure, which applies to "macroscopic variables" like a current carried by a macroscopic number of electrons. The take home message is : any "macroscopic variables" of classical harmonic oscillators can be quantized by introducing creation and destruction operators $\hat{b}^\dagger$ and $\hat{b}$ with commutation relation $[\hat{b}, \hat{b}^\dagger] = 1$. This is easy. The not so easy part is the realization of situations where quantum behaviour of oscillators manifests experimentally. You will learn more about this in the next lessons and homework.