Quantum optics
Multimode radiation field states: localized single photon state

M. Brune, A. Aspect

Homework of Lesson 4

Introduction

In the lesson you learned how to quantize the multimode radiation field. The method is based on the decomposition of an arbitrary classical electromagnetic field into plane waves modes by using Fourier transform. Each of these modes is then quantized independently of all the others as a single harmonic oscillator. In this homework, we will use this multimode field description formalism in order to explicitly describe the state of a single photon wavepacket and its propagation. We will introduce the concept of a localized mode of the radiation field.

For the sake of simplicity, we will consider here a one-dimensional wavepacket with a well defined linear polarization $\vec{\varepsilon}$. Generalizing this approach to 3D would provide a full description of the propagation of any quantum state of the radiation field. In particular it would include diffraction effects that are obviously not describes by our simplified 1D approach.

1 Single photon wavepacket

One considers the most general, normalized one-photon state : $|1\rangle_{\text{wp}} = \sum_\ell c_\ell |1_\ell\rangle$. The index ℓ represents a plane wave propagating in the Ox direction with the wave vector $\mathbf{k}_\ell = k_\ell \mathbf{e}_x$, where $\mathbf{e}_x$ is a unit vector defining the propagation direction, and a fixed polarization $\vec{\varepsilon}$. The state $|1_\ell\rangle$ represents a one-photon state in mode ℓ and 0 photons in any other mode.

We recall the expression of the electric field operator for a 1D radiation field :

$$\hat{\boldsymbol{E}}(x) = i \sum_\ell \mathscr{E}_\ell^{(1)} \, \vec{\varepsilon} \left(\hat{a}_\ell e^{i k_\ell \cdot x} - \hat{a}_\ell^\dagger e^{-i k_\ell \cdot x} \right) \qquad \text{with} \qquad \mathscr{E}_\ell^{(1)} = \sqrt{\frac{\hbar \omega_\ell}{2 \varepsilon_0 L S}} \qquad \omega_\ell = c k_\ell \tag{1}$$

where S is the transverse cross section of the beam and L the length of the 1D quantization volume. In our simplified 1D model, the field amplitude is assumed to be constant over the surface S and to vanish outside S, as seen in video 2.2. Introducing $\hat{\boldsymbol{E}}^{(+)}(x) = i \sum_\ell \mathscr{E}_\ell^{(1)} \, \vec{\varepsilon} \, \hat{a}_\ell \, e^{i k_\ell \cdot x}$ and $\hat{\boldsymbol{E}}^{(-)}(x) = \left(\hat{\boldsymbol{E}}^{(+)}(x) \right)^\dagger$ one has

$$\hat{\boldsymbol{E}}(x) = \hat{\boldsymbol{E}}^{(+)}(x) + \hat{\boldsymbol{E}}^{(-)}(x). \tag{2}$$

1. Express the field state $|\psi_1(t)\rangle_{\text{wp}}$ at time t for the initial state $|\psi_1(0)\rangle_{\text{wp}} = |1\rangle_{\text{wp}}$. Be careful while taking into account the contribution of vacuum whose energy is $E_V = \sum_\ell 1/2 \, \hbar \omega_\ell$.

   Solution : Using the Hamiltonian $\hat{H}_R = \sum_\ell \hbar \omega_\ell (\hat{a}_\ell^\dagger \hat{a}_\ell + 1/2)$ the energy of state $|1_\ell\rangle$ is $E_{1_\ell} = \sum_{\ell'} \hbar \omega_{\ell'} (\delta_{\ell,\ell'} + 1/2) = E_V + \hbar \omega_\ell$, where $\delta_{\ell,\ell'}$ is the Kronecker symbol. One thus has

$$|\psi_1(t)\rangle_{\text{wp}} = \sum_\ell c_\ell e^{-i E_{1_\ell} t/\hbar} |1_\ell\rangle = e^{-i E_V t/\hbar} \sum_\ell c_\ell e^{-i \omega_\ell t} |1_\ell\rangle \tag{3}$$

In all the following parts of the problem, we will use the Hamiltonian

$$\hat{H}'_R = \hat{H}_R - E_V = \sum_\ell \hbar\omega_\ell \hat{a}^\dagger_\ell \hat{a}_\ell \tag{4}$$

instead of $\hat{H}_R$. These two Hamiltonian only differ by the constant energy E_V. You can check that the amplitude of electric field vacuum fluctuations are not affected by this redefinition of the vacuum energy. Most physical effects related to vacuum fluctuations are well described by $\hat{H}'_R$.

2. Rewrite $|\psi_1(t)\rangle_{\text{wp}}$ by using the Hamiltonian $\hat{H}'_R$ instead of $\hat{H}_R$.

   Solution :

$$|\psi_1(t)\rangle = \sum_\ell c_\ell e^{-i\omega_\ell t}|1_\ell\rangle \tag{5}$$

   As usual in quantum mechanics, removing a constant energy from the Hamiltonian amounts to remove a global phase factor from the evolution of the state. This does not change the predictions for observables quantities.

3. Give the explicit expression of the vectors $\hat{a}_\ell\, e^{i\boldsymbol{k}_\ell\cdot\boldsymbol{x}}|\psi_1(t)\rangle_{\text{wp}}$.

   Solution :

$$\hat{a}_\ell\, e^{ik_\ell\cdot x}|\psi_1(t)\rangle_{\text{wp}} = e^{ik_\ell\cdot x}\sum_{\ell'} c_{\ell'} e^{-i\omega_{\ell'} t}\ \hat{a}_\ell|1_{\ell'}\rangle = c_\ell\, e^{ik_\ell(x-ct)}\ |0\rangle. \tag{6}$$

   We used $\hat{a}_\ell|1_{\ell'}\rangle = \delta_{\ell,\ell'}|0\rangle$. Note that, for $\ell \neq \ell'$, it is the null vector with norm 0, not to be confused with the vector $|0\rangle$ representing the vacuum state, whose norm is 1.

4. Show that $\hat{\boldsymbol{E}}^{(+)}(x)|\psi_1(t)\rangle_{\text{wp}} = \boldsymbol{E}^{(+)}_{\text{wp}}(x,t)|\psi_R\rangle$. Give the expression of the function $\boldsymbol{E}^{(+)}_{\text{wp}}(x,t)$ and of the state $|\psi_R\rangle$.

   Solution :

$$\hat{\boldsymbol{E}}^{(+)}(x)|\psi_1(t)\rangle_{\text{wp}} = i\Big(\sum_\ell\ \mathscr{E}^{(1)}_\ell\ \vec{\boldsymbol{\varepsilon}}\,\hat{a}_\ell\, e^{ik_\ell\cdot x}\Big)|\psi_1(t)\rangle_{\text{wp}} \tag{7}$$

   By using (5) one gets.

$$\hat{\boldsymbol{E}}^{(+)}(x)|\psi_1(t)\rangle_{\text{wp}} = i\,\vec{\boldsymbol{\varepsilon}}\Big(\sum_\ell\ \mathscr{E}^{(1)}_\ell\ c_\ell\, e^{ik_\ell(x-ct)}\Big)\ |0\rangle. \tag{8}$$

   By identification we have $|\psi_R\rangle = |0\rangle$ and

$$\boldsymbol{E}^{(+)}_{\text{wp}}(x,t) = i\,\vec{\boldsymbol{\varepsilon}}\sum_\ell\ \mathscr{E}^{(1)}_\ell\ c_\ell\, e^{ik_\ell(x-ct)} \tag{9}$$

   This expression can be interpreted as the spatial variation of the electric field complex amplitude of the field mode in which the $|1\rangle_{\text{wp}}$ state is prepared.

5. Give the average value of the electric field operator $\langle\boldsymbol{E}(x,t)\rangle = {}_{\text{wp}}\langle\psi_1(t)|\hat{\boldsymbol{E}}(x)|\psi_1(t)\rangle_{\text{wp}}$.

   Solution : One has $\langle\boldsymbol{E}(x,t)\rangle = {}_{\text{wp}}\langle\psi_1(t)|\hat{\boldsymbol{E}}^{(+)}(x)|\psi_1(t)\rangle_{\text{wp}} + \text{c.c.}$ where c.c. corresponds to complex conjugate. Using (5) and (8) one easily finds

$$\langle\boldsymbol{E}(x,t)\rangle = {}_{\text{wp}}\langle\psi_1(t)|\ \cdot\ \big(\hat{\boldsymbol{E}}^{(+)}(x)|\psi_1(t)\rangle_{\text{wp}}\big) + \text{c.c.} = 0 \tag{10}$$

6. We introduce the operator $\hat{A}_{\rm wp}(t) = \sum_\ell c_\ell^*(t)\,\hat{a}_\ell$. Calculate the commutator $[\hat{A}_{\rm wp}(t), \hat{A}^\dagger_{\rm wp}(t)]$. Express $|1\rangle_{\rm wp}$ as a function of the operator $\hat{A}^\dagger_{\rm wp}(t)$ and of the vacuum state $|0\rangle$. Give the physical interpretation of the operator $\hat{A}_{\rm wp}(t)$.

   Solution : From $[\hat{a}_\ell, \hat{a}^\dagger_\ell] = 1$ one gets $[\hat{A}_{\rm wp}(t), \hat{A}^\dagger_{\rm wp}(t)] = 1$.
   One has $|\psi_1(t)\rangle_{\rm wp} = \hat{A}^\dagger_{\rm wp}(t)|0\rangle$. In particular, the initial wavepacket is $|1\rangle_{\rm wp} = \hat{A}^\dagger_{\rm wp}(0)|0\rangle$. At any time, the one-photon wavepacket results from the action of a single creation operator on vacuum. The operators $\hat{A}_{\rm wp}$ and $\hat{A}^\dagger_{\rm wp}$ can be interpreted as single localized photon annihilation and creation operators respectively.
   Note that the time dependence of these creation operators corresponds to the propagation of the wavepacket.

2 Gaussian single photon wavepacket

For all the following parts of the problem, we now define

$$c_\ell = \mathscr{N}\frac{1}{(2\pi)^{1/4}\sqrt{\Delta k}}\,e^{-\frac{(k_\ell-k_0)^2}{4\Delta k^2}}\,, \tag{11}$$

corresponding to a gaussian wavepacket centered around k_0. The variance of the probability distribution of the wavevector k_ℓ is Δk^2.

We assume that $\Delta k \ll k_0$ so that one will consider that $\mathscr{E}^{(1)}_\ell = \sqrt{\frac{\hbar\omega_\ell}{2\varepsilon_0 LS}} \simeq \sqrt{\frac{\hbar\omega_0}{2\varepsilon_0 LS}} = \mathscr{E}^{(1)}_{\omega_0}$ with $\omega_0 = ck_0$.

1. Express the norm of $|1\rangle_{\rm wp}$ as a discrete sum over the modes. By transforming the discrete sum into an integral (see note below), give the explicit expression of $\mathscr{N}$ as a function of the size of the quantization volume L.

   Note :
   - The rule for transforming a discrete sum over k_ℓ in one dimension into an integral over the continuous variable k is

   $$\sum_{k_\ell}[f(k_\ell)] \to \frac{L}{2\pi}\int_{-\infty}^{+\infty} dk\,[f(k)]\,, \tag{12}$$

   where $f(k)$ is a function of k.

   - We remind the expression of a normalized Gaussian probability distribution of a continuous variable y centered at y_0

   $$P_G(y) = \frac{1}{\sqrt{2\pi}\sigma}\,e^{-\frac{(y-y_0)^2}{2\sigma^2}} \tag{13}$$

   where σ is the standard deviation of the variable y.

   Solution : We first transform the discrete sum involved in the norm of $|1\rangle_{\rm wp}$ into an integral

   $$\|\,|1\rangle_{\rm wp}\,\|^2 = \sum_{k_\ell}|c_\ell|^2 \to \frac{L}{2\pi}\int_{-\infty}^{+\infty} dk\,\mathscr{N}^2\frac{1}{\sqrt{2\pi}\Delta k}\,e^{-\frac{(k-k_0)^2}{2\Delta k^2}}\,. \tag{14}$$

   By performing the integral, normalization leads to $\mathscr{N} = \sqrt{\frac{2\pi}{L}}$.

2. Calculate the explicit expression of $\boldsymbol{E}^{(+)}_{\rm wp}(x,t)$ for the considered gaussian wave packet.

   Note : Using the definition $g(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, f(K)\, e^{iKX}$, the Fourier transform of the gaussian $f(K) = e^{-\frac{K^2}{2\Delta K^2}}$ is the function $g(X) = \frac{1}{\Delta X}\, e^{-\frac{X^2}{2\Delta X^2}}$ with $\Delta X = 1/\Delta K$.

   Solution : Transforming the discrete sum (9) into an integral, using the definition (11) of c_ℓ, and defining $K = k - k_0$, one gets

$$\begin{aligned}\boldsymbol{E}^{(+)}_{\rm wp}(x,t) &= i\,\vec{\boldsymbol{\varepsilon}}\ \mathscr{E}^{(1)}_{\omega_0}\,\frac{1}{(2\pi)^{1/4}\sqrt{\Delta k}}\sqrt{\frac{L}{2\pi}}\int_{-\infty}^{+\infty} dk\, e^{-\frac{(k-k_0)^2}{4\Delta k^2}}\ e^{ikx} \\ &= i\,\vec{\boldsymbol{\varepsilon}}\ \mathscr{E}^{(1)}_{\omega_0}\,\frac{1}{(2\pi)^{1/4}\sqrt{\Delta k}}\sqrt{\frac{L}{2\pi}}\int_{-\infty}^{+\infty} dK\, e^{-\frac{(K)^2}{4\Delta k^2}}\ e^{iKX} e^{ik_0x}.\end{aligned}$$

   Defining $\Delta K^2 = 2\Delta k^2$ and using the above expression of the Fourier transform of a gaussian, one finally gets

$$\boldsymbol{E}^{(+)}_{\rm wp}(x,t) = i\,\vec{\boldsymbol{\varepsilon}}\ \mathscr{E}^{(1)}_{\omega_0}\,\frac{1}{(2\pi)^{1/4}\sqrt{\Delta k}}\sqrt{\frac{L}{2\pi}}\,\sqrt{2}\,\Delta k\, e^{-\Delta k^2\,X^2}\ e^{ik_0x} \tag{15}$$

3. Give the explicit expression of the one photon detection rate $w^{(1)}(x,t) = s \parallel \hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r})|\psi_1(t)\rangle_{\rm wp} \parallel^2$ as a function of $X = x - ct$, $\mathscr{E}^{(1)}_{\omega_0}$, L and Δk. The parameter s is a coefficient characterizing the detector efficiency. Does this expression depend on L ?

   Solution : Using (8) one has $w^{(1)}(x,t) = s\left|\boldsymbol{E}^{(+)}_{\rm wp}(x,t)\right|^2$. From the expression (14) of $\boldsymbol{E}^{(+)}_{\rm wp}(x,t)$ one gets

$$w^{(1)}(x,t) = s\ {\mathscr{E}^{(1)}_{\omega_0}}^2\,\frac{L}{\pi}\,\frac{1}{\sqrt{2\pi}}\,\Delta k\, e^{-2\Delta k^2\,(x-ct)^2}. \tag{16}$$

   The wavepacket has a gaussian spatial variation along the propagation axis x. From the definition (1), the product ${\mathscr{E}^{(1)}_\ell}^2 L$ is independant of L. This is consistent with the fact that all measurable physical quantities should not depend on the (arbitrarily large) dimension of the quantization volume. Note that our expression of $w^{(1)}(x,t)$ still depends on the transverse cross section S, which is a real physical parameter of the wavepacket.

4. What is the position of the wavepacket at time t ? What is the spatial width Δx of the wavepacket. Does it depends on time ?

   Solution : Equ. (16) is a gaussian function of X centered at $x = ct$. It propagates at the velocity of light, without deformation. The spatial width of the single photon pulse is $\Delta x = \frac{1}{2\Delta k}$. It is independent of time. The wavepacket thus propagates without dispersion, as expected for light propagation in vacuum. This is in contrast to the wavepacket spreading occurring in the case of massive particles propagation.

5. Using creation and annihilation operators defined in question (1.6), we define $\hat{A}_{\rm wp} = \hat{A}_{\rm wp}(0)$ and $= \hat{A}^\dagger_{\rm wp} = \hat{A}^\dagger_{\rm wp}(0)$. We also define a dimensionless function $F(x)$ describing the spatial mode structure of the wavepacket by $\boldsymbol{E}^{(+)}_{\rm wp}(x,0) = i\,\vec{\boldsymbol{\varepsilon}}\ \mathscr{E}^{(1)}_{\omega_0}\,F(x)$. We introduce the electric field operator

$$\hat{\boldsymbol{E}}_{\rm wp}(x) = i\ \mathscr{E}^{(1)}_{\omega_0}\ \vec{\boldsymbol{\varepsilon}}\left(\hat{A}_{\rm wp}F(x) - \hat{A}^\dagger_{\rm wp}F^*(x)\right). \tag{17}$$

   We now use this single mode expression for describing the electric field corresponding to the gaussian mode wavepacket considered above. Give the expression of $F(x)$.

Using directly the properties of the operators $\hat{A}_{\text{wp}}$ and $\hat{A}^{\dagger}_{\text{wp}}$, calculate $\langle \hat{\boldsymbol{E}}_{\text{wp}}(x)\rangle_{\text{vac}} = {}_{\text{wp}}\langle 0|\hat{\boldsymbol{E}}_{\text{wp}}(x)|0\rangle_{\text{wp}}$ and $\Delta \boldsymbol{E}^{\text{vac}}_{\text{wp}}(x)^2$ in the vacuum state $|0\rangle_{\text{wp}}$ as a function of $\mathscr{E}^{(1)}_{\omega_0}$ and $F(x)$. Do these result depends on L?

Solution :

From the expression (15), one has

$$F(x) = \frac{1}{(2\pi)^{1/4}}\sqrt{\frac{L\Delta k}{\pi}}\; e^{-\frac{x^2}{4\Delta x^2}}\, e^{ik_0 x}, \tag{18}$$

which is obviously dimensionless.

As we have ${}_{\text{wp}}\langle 0|\hat{A}_{\text{wp}}|0\rangle_{\text{wp}} = 0$ one finds $\langle \hat{\boldsymbol{E}}_{\text{wp}}(x)\rangle_{\text{vac}} = 0$.

The variance of $\hat{\boldsymbol{E}}_{\text{wp}}(x)$ in the vacuum is

$$\begin{aligned}\Delta \boldsymbol{E}^{\text{vac}}_{\text{wp}}(x)^2 &= {}_{\text{wp}}\langle 0|\big(\hat{\boldsymbol{E}}_{\text{wp}}(x)\big)^2|0\rangle_{\text{wp}} = {}_{\text{wp}}\langle 0| - \mathscr{E}^{(1)\,2}_{\omega_0}\Big(\hat{A}_{\text{wp}}f(x) - \hat{A}^{\dagger}_{\text{wp}}F^*(x)\Big)^2|0\rangle_{\text{wp}} \\ &= -\,\mathscr{E}^{(1)\,2}_{\omega_0}\;{}_{\text{wp}}\langle 0|\hat{A}^2_{\text{wp}}F(x)^2 + \hat{A}^{\dagger 2}_{\text{wp}}F^*(x)^2 - \big(\hat{A}_{\text{wp}}\hat{A}^{\dagger}_{\text{wp}} + \hat{A}^{\dagger}_{\text{wp}}\hat{A}_{\text{wp}}\big)\big|F(x)\big|^2|0\rangle_{\text{wp}}\end{aligned}$$

Using the commutator $[\hat{A}_{\text{wp}}, \hat{A}^{\dagger}_{\text{wp}}] = 1$, one finds

$$\Delta \boldsymbol{E}^{\text{vac}}_{\text{wp}}(x)^2 = \mathscr{E}^{(1)\,2}_{\omega_0}\big|F(x)\big|^2 = \mathscr{E}^{(1)\,2}_{\omega_0}\, L\,\frac{\Delta k}{\pi}\,\frac{1}{\sqrt{2\pi}}\; e^{-\frac{x^2}{2\Delta x^2}}. \tag{19}$$

In this expression, the L dependance of $\mathscr{E}^{(1)\,2}_{\omega_0}$ is compensated by that of $\big|F(x)\big|^2$ and the result is independent of L. For a single localized mode, we thus find finite vacuum fluctuation. Note that the "intensity" of vacuum fluctuations associated to this mode have the same spatial dependance as the intensity of a one photons state prepared in the same mode.

3 Advanced complementary question

The first order spatial correlation function of the radiation field in the state $|\psi\rangle$ is defined by $g^{(1)}(x,x') = \langle\psi|\hat{\boldsymbol{E}}^{(-)}(x)\,\hat{\boldsymbol{E}}^{(+)}(x')|\psi\rangle$.

This correlation function can be measured with a Michelson interferometer. The single photon detection rate at the output of the interferometer has an interference term which is proportional to the real part of $\langle\psi|\hat{\boldsymbol{E}}^{(-)}(x,t)\hat{\boldsymbol{E}}^{(+)}(x,t-\Delta L/c)|\psi\rangle \propto g^{(1)}(x, x-\Delta L/c)$, where ΔL is the optical path difference between the two arms of the interferometer.

For the one-photon radiation field considered above, we define

$$g^{(1)}_{\text{wp}}(x,x') = {}_{\text{wp}}\langle 1|\hat{\boldsymbol{E}}^{(-)}(x)\,\hat{\boldsymbol{E}}^{(+)}(x')|1\rangle_{\text{wp}} \tag{20}$$

Show that $g^{(1)}_{\text{wp}}(x,x') = \mathcal{C}\;\mathscr{E}^{(1)\,2}_{\omega_0}\, f(x,x')$, where $\mathcal{C}$ is a normalization constant that one will not calculate. Give the explicit expression of $f(x,x')$. What is its value for $x = x' = 0$.

Solution : Using (8), and (9) one has

$$\hat{\boldsymbol{E}}^{(+)}(x')|1\rangle_{\rm wp} = i\vec{\varepsilon}\, E^{(+)}_{\rm wp}(x')|0\rangle$$
$$_{\rm wp}\langle 1|\hat{\boldsymbol{E}}^{(-)}(x) = -i\vec{\varepsilon}\, E^{(+)*}_{\rm wp}(x)|0\rangle.$$

Using (14) leads to

$$g^{(1)}_{\rm wp}(x,x') = \mathcal{C}\ \ \mathscr{E}^{(1)^2}_{\omega_0}\, e^{-\frac{x^2}{4\Delta x^2}}\ e^{-\frac{x'^2}{4\Delta x^2}}\ \cos k_0(x'-x) \tag{21}$$

where $\mathcal{C}$ is a normalization constant. We thus have

$$f(x,x') = e^{-\frac{x^2}{4\Delta x^2}}\, e^{-\frac{x'^2}{4\Delta x^2}}\ \cos k_0(x'-x). \tag{22}$$

The first order correlation function $g^{(1)}_{\rm wp}(x,x')$ is an oscillating function of $x - x'$ with a spatial period equal to the optical wavelength $\lambda_0 = 2\pi/k_0$. It is modulated by an envelope, which is the product of two gaussian. For $x = x' = 0$ the envelope reaches t its maximal value of 1, meaning interferences with a visibility of 1.

The fast spatial oscillation of $g^{(1)}_{\rm wp}(x,x')$ over a spatial length of about Δx are the signature of the spatial coherence of the considered wavepacket. The width Δx characterizes the coherence length of this wavepacket. You can find it amazing : we have constructed a quantum state of radiation whose average electric field is null, corresponding to a random field amplitude if one considers an ensemble average over many realizations. This field does not have a well-defined phase, but it leads to high contrast interferences reflecting its spatial correlation length. This shows that, strictly speaking, interferences are not sensitive to the absolute phase of a wave but only to the relative phase between interfering paths.

Concluding remarks

We have seen here how to use the quantum formalism describing a multimode radiation field in order to explicitly describe the propagation of a single photon wavepacket. Indeed, the final expression of $|\psi(t)\rangle_{\rm wp}$ generalizes to the description of an arbitrary n photon state in the considered gaussian mode, which can be defined as

$$|n\rangle_{\rm wp} = \frac{1}{\sqrt{n!}}\left(\hat{A}^{\dagger}_{\rm wp}\right)^n|0\rangle. \tag{23}$$

Note that this formalism automatically describes the spatial shape of the wavepacket as well as its propagation with time. To get a real feeling of the power of this formalism, you should try to express the same state in the number state basis of plane wave modes! You find it too difficult, it is normal!