Quantum optics
Shot-noise limit of interferometry

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Homework of Lesson 6

Introduction

Lesson 6 was devoted to the description of single photon interference experiments. In the lesson we used an explicit multimode description of a realistic single-photon wavepacket. You have learnt that a single photon wavepacket leads to the same interference pattern as one would get by sending a classical field in the same interferometer.

In the homework, we will address the more practical issue of the limit of an interferometric measurement. In a fully classical field framework, the precision of a measurement is only limited by technical noise, such as the electric noise of detectors and amplifiers, or residual vibrations of the interferometer. We will study here how the granularity of photons as well as vacuum fluctuations affects the precision of a phase measurement performed with a Mach-Zehnder interferometer. The goal of the homework is to compare the ultimate quantum limit one can reach in a phase-shift measurement by using either number states or quasi-classical, coherent, states at the input of the interferometer.

We will establish that the quantum limit of a phase shift measurement is intimately related to vacuum fluctuations entering by the unused input of the interferometer. Thus, in contrast to the approach used in the lesson for describing single photon detection rates, we will need here to consider explicitly the apparently empty input channel of the interferometer in order to determine the ultimate precision of phase measurements. We will do it either for a N_1 photon radiation field or for a coherent radiation field, with the same average photon number.

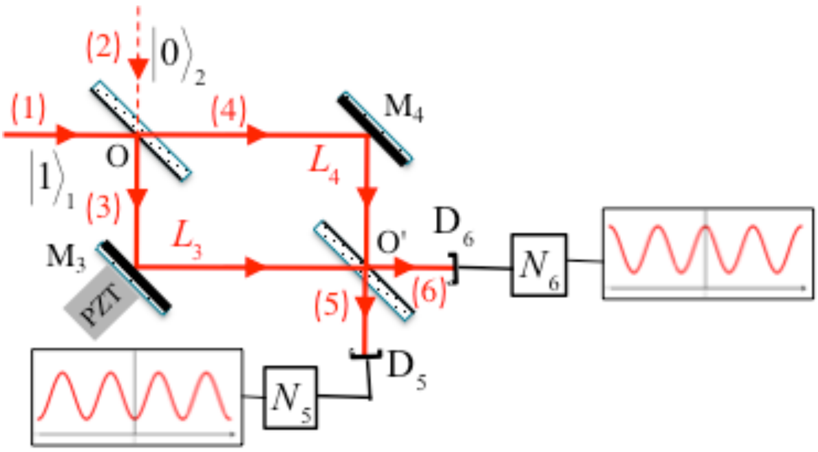

FIGURE 1 – **Mach-Zehnder interferometer**

We consider here the same Mach-Zehnder interferometer as introduced in lesson 6 (figure 1). We will adopt the same simplified approach as in lesson 2 for describing an N photon wavepacket as a truncated, single-mode plane wave of transverse cross section S and of length L. The multimode approach used in lesson 6 could as well be used and would lead to similar results.

In our simplified model, we define six single-mode field operators that we express as time dependent operators using the Heisenberg formalism. We define the field operators of modes $j =$ 1 to 4 by

$$\hat{\boldsymbol{E}}_j^{(+)}(\boldsymbol{r}, t) = i\ \mathscr{E}^{(1)}\ \vec{\boldsymbol{\varepsilon}}\, \hat{a}_j e^{i(\boldsymbol{k}_j\cdot\boldsymbol{r}-\omega t)} \qquad \text{with} \qquad \mathscr{E}_\ell^{(1)} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 L^3}}, \tag{1}$$

where the origin of $\boldsymbol{r}$ is at O. The field operators describing output modes $j = 5$ and 6 are defined by

$$\hat{\boldsymbol{E}}_j^{(+)}(\boldsymbol{r}', t) = i\ \mathscr{E}^{(1)}\ \vec{\boldsymbol{\varepsilon}}\, \hat{a}_j e^{i(\boldsymbol{k}_j\cdot\boldsymbol{r}'-\omega t)}\,, \tag{2}$$

where the position $\boldsymbol{r}'$ has its origin at O'. All modes have the same angular frequency ω. Note that in these expressions, the annihilation operators $\hat{a}_j$ are not time-dependent. The two input modes of the interferometer correspond to $j = 1$ and 2. They are coupled to modes $j = 3$ and 4 by the first beamsplitter. After propagation, these modes are the input modes of the second beamsplitter. Modes 5 an 6 are output modes of the interferometer. We denote by R and T the intensity reflexion and transmission coefficients of the two mirrors, with $R + T = 1$. We also define photon number operators $\hat{N}_j = \hat{a}_j^\dagger \hat{a}_j$. Note that, in the Heisenberg representation, photon numbers operators are unchanged : they remain time-independent as they do individually commute with the radiation field Hamiltonian H_R.

Note : **The result (15-16) of question 1.4 can be admitted in order to directly solve parts 2 and 3**.

But doing the calculation of section 1 by yourself is the best way to get a deep understanding of the formalism of quantum optics.

1 Field operators transformation

1. We recall here the relation linking field operators $\hat{\boldsymbol{E}}_j^{(+)}(\mathrm{O}, t)$ with $j = 1$, 2, 3 and 4 at position O (corresponding to $\boldsymbol{r} = 0$) as given in the lesson

$$\hat{\boldsymbol{E}}_3^{(+)}(\mathrm{O}, t) = R^{1/2}\hat{\boldsymbol{E}}_1^{(+)}(\mathrm{O}, t) + T^{1/2}\hat{\boldsymbol{E}}_2^{(+)}(\mathrm{O}, t) \tag{3}$$

$$\hat{\boldsymbol{E}}_4^{(+)}(\mathrm{O}, t) = T^{1/2}\hat{\boldsymbol{E}}_1^{(+)}(\mathrm{O}, t) - R^{1/2}\hat{\boldsymbol{E}}_2^{(+)}(\mathrm{O}, t)\,. \tag{4}$$

Express the corresponding relations for $\hat{a}_j$ operators.

Solution :

Introducing expression (1) of field operators at $\boldsymbol{r} = 0$, one finds

$$\hat{a}_3 = R^{1/2}\,\hat{a}_1 + T^{1/2}\,\hat{a}_2 \tag{5}$$

$$\hat{a}_4 = T^{1/2}\,\hat{a}_1 - R^{1/2}\,\hat{a}_2\,. \tag{6}$$

Note that these relations are the same as the one directly introduced in lesson 2.

2. Express $\hat{\boldsymbol{E}}_j^{(+)}(\mathrm{O}', t)$ with $j = 3$ and 4 as a function of $\hat{\boldsymbol{E}}_j^{(+)}(\mathrm{O}, t)$ with $j = 3$ and 4. One will introduce the propagation phase factors $\varphi_3 = \omega L_3/c$ and $\varphi_4 = \omega L_4/c$. We assume that reflection at mirors M_3 and M_4 do not introduce additional phase shifts.

Solution :

$$\hat{\boldsymbol{E}}_3^{(+)}(\mathrm{O}', t) = \hat{\boldsymbol{E}}_3^{(+)}(\mathrm{O}, t - L_3/c) = \hat{\boldsymbol{E}}_3^{(+)}(\mathrm{O}, t)\, e^{i\varphi_3} \tag{7}$$

$$\hat{\boldsymbol{E}}_4^{(+)}(\mathrm{O}', t) = \hat{\boldsymbol{E}}_4^{(+)}(\mathrm{O}, t - L_4/c) = \hat{\boldsymbol{E}}_4^{(+)}(\mathrm{O}, t)\, e^{i\varphi_4}\,. \tag{8}$$

3. We recall the relation linking field operators $\hat{\boldsymbol{E}}_j^{(+)}(\mathrm{O},t)$ with $j = 3$, 4, 5 and 6 at position O' (corresponding to $\boldsymbol{r}' = 0$) as given in the lesson

$$\begin{aligned}\hat{\boldsymbol{E}}_5^{(+)}(\mathrm{O}',t) &= -R^{1/2}\hat{\boldsymbol{E}}_3^{(+)}(\mathrm{O}',t) + T^{1/2}\hat{\boldsymbol{E}}_4^{(+)}(\mathrm{O}',t) &(9)\\ \hat{\boldsymbol{E}}_6^{(+)}(\mathrm{O}',t) &= T^{1/2}\hat{\boldsymbol{E}}_3^{(+)}(\mathrm{O}',t) + R^{1/2}\hat{\boldsymbol{E}}_4^{(+)}(\mathrm{O}',t)\,. &(10)\end{aligned}$$

Express the corresponding relations for $\hat{a}_j$ operators.

Solution :
Using expression (1)(2) of field operators and equations (7-10), one finds

$$\begin{aligned}\hat{a}_5 &= -R^{1/2}\hat{a}_3\, e^{i\varphi_3} + T^{1/2}\hat{a}_4\, e^{i\varphi_4} &(11)\\ \hat{a}_6 &= T^{1/2}\hat{a}_3\, e^{i\varphi_3} + R^{1/2}\hat{a}_4\, e^{i\varphi_4}\,. &(12)\end{aligned}$$

In all the following parts, we consider the case of balanced beamsplitters such that $R = T = 1/2$.

4. We define the phases $\Phi = \omega(L_3 + L_4)/c$ and $\varphi = \omega(L_3 - L_4)/c$. Show that the output mode operators $\hat{a}_5$ and $\hat{a}_6$ can be expressed as

$$\begin{aligned}\hat{a}_5 &= e^{i\Phi/2}\big[-i\sin(\varphi/2)\,\hat{a}_1 - \cos(\varphi/2)\,\hat{a}_2\big] &(13)\\ \hat{a}_6 &= e^{i\Phi/2}\big[\cos(\varphi/2)\,\hat{a}_1 + i\sin(\varphi/2)\,\hat{a}_2\big]\,. &(14)\end{aligned}$$

The global phase factor $e^{i\Phi/2}$ is irrelevant in all the following parts. We will drop it and use the simplified expressions

$$\begin{aligned}\hat{a}_5 &= -i\sin(\varphi/2)\,\hat{a}_1 - \cos(\varphi/2)\,\hat{a}_2 &(15)\\ \hat{a}_6 &= \cos(\varphi/2)\,\hat{a}_1 + i\sin(\varphi/2)\,\hat{a}_2\,. &(16)\end{aligned}$$

Solution : One gets the result by combining (5-6) and (11-12).

2 Number state input

1. We consider the input state $|\Psi_{12}\rangle = |N_1\rangle|N_2\rangle$ with N_1 photons in mode 1 and N_2 photons in mode 2. Express $N_5 = \langle\hat{N_5}\rangle$ and $N_6 = \langle\hat{N_6}\rangle$ as a function of N_1 and N_2. Is there a crossed interference term between the two input radiation fields ?

Solution : Using (15) one has

$$N_5 = \langle N_1, N_2|\big[i\sin(\varphi/2)\,\hat{a}_1^\dagger - \cos(\varphi/2)\,\hat{a}_2^\dagger\big]\big[-i\sin(\varphi/2)\,\hat{a}_1 - \cos(\varphi/2)\,\hat{a}_2\big]|N_1, N_2\rangle. \quad (17)$$

Developing the product of operators, all cross-term involving the two modes vanish when taking the average value and one gets

$$N_5 = \sin^2(\varphi/2)\,N_1 + \cos^2(\varphi/2)\,N_2. \quad (18)$$

Similarly

$$N_6 = \cos^2(\varphi/2)\,N_1 + \sin^2(\varphi/2)\,N_2. \quad (19)$$

One finds the sum of the two interference pattern one would get with photons send either in input 1 or in input 2.

In all the following questions of this section, we now consider the input state $|\Psi_{12}\rangle = |N_1\rangle|0_2\rangle$ with vacuum in input 2. For measuring the phase shift φ with the best possible precision, we consider the observable $\hat{N}_d = \hat{N}_6 - \hat{N}_5$. We assume that the two detectors have unit efficiency so that during the measurement time $T = L/c$ all output photons are finally detected.

2. Calculate the average value $N_d = \langle \hat{N}_d \rangle$ as a function of N_1 and φ.

Solution : Using (18) and (19) with $N_2 = 0$ one gets

$$N_d = \langle \hat{N}_d \rangle = N_1 \cos \varphi \, . \tag{20}$$

3. Calculation of the average value of $\hat{N}_d^2$.

(a) Express $\hat{N}_5$ and $\hat{N}_6$ and then $\hat{N}_d$ as a function of φ, $\hat{N}_1$, $\hat{N}_2$ and of $(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1)$.

Solution : Developping the operator product in (17) one gets

$$\hat{N}_5 = \sin^2(\varphi/2)\hat{N}_1 + \cos^2(\varphi/2)\hat{N}_2 - i \sin(\varphi/2)\cos(\varphi/2)(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1). \tag{21}$$

In a similar way

$$\hat{N}_6 = \cos^2(\varphi/2)\hat{N}_1 + \sin^2(\varphi/2)\hat{N}_2 + i \sin(\varphi/2)\cos(\varphi/2)(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1). \tag{22}$$

We thus have

$$\hat{N}_d = \cos\varphi\,(\hat{N}_1 - \hat{N}_2) + i \sin\varphi\,(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1) \tag{23}$$

(b) Calculate $\langle \hat{N}_d^2 \rangle$, the average value of $\hat{N}_d^2$ as a function of N_1 and φ.

Solution : From expression (23) of $\hat{N}_d$ the $\hat{N}_2$ term will not contribute in any way to the average value $\langle \hat{N}_d^2 \rangle$ as mode 2 is in the vacuum. In the development of $\hat{N}_d^2$, the cross-term $\hat{N}_1(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1)$ also has a vanishing average value because it involves average values of $\hat{a}_2$ or $\hat{a}_2^\dagger$ in the vacuum. We thus have

$$\langle \hat{N}_d^2 \rangle = \cos^2\varphi\, N_1^2 - \sin^2\varphi\, \langle N_1, 0_2|(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1)^2|N_1, 0_2\rangle. \tag{24}$$

Developing the product

$$(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1)^2 = \hat{a}_1^{\dagger 2}\hat{a}_2^2 + \hat{a}_2^{\dagger 2}\hat{a}_1^2 - \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1^\dagger \hat{a}_2. \tag{25}$$

The two first terms vanish when taking the average value on number states as well as the last term, which is proportional to $\hat{N}_2$. Using the commutator $[\hat{a}_2, \hat{a}_2^\dagger] = 1$, the remaining term writes $\hat{N}_1(\hat{N}_2 + 1)$. Due to the factor $(N_2 + 1)$ the second input mode contributes to $\langle \hat{N}_d^2 \rangle$ (and thus to the variance ΔN_d^2) even if $N_2 = 0$. Finally

$$\langle \hat{N}_d^2 \rangle = \cos^2\varphi\, N_1^2 + \sin^2\varphi\; N_1. \tag{26}$$

The first term in the sum is simply the square of $\langle \hat{N}_d \rangle$ given by equation (20). The second term is the contribution of vacuum in mode 2.

(c) Calculate the variance $\Delta N_d^2 = \langle \hat{N}_d^2 \rangle - \langle \hat{N}_d \rangle^2$ of $\hat{N}_d$ as a function of N_1 and φ.

Solution : From (20) and (26) one gets

$$\Delta N_d^2 = \sin^2\varphi\; N_1. \tag{27}$$

Note that this variance would be zero if mode 2 was neglected. This shows that vacuum fluctuations entering unused input of the interferometer have an essential contribution the fluctuations of $\hat{N}_d$.

4. The measurement error $\Delta\varphi$ of an estimate of φ based on the measurement of N_d (cf. question 2.2) is given by

$$\frac{dN_d}{d\varphi}\,\Delta\varphi = \Delta N_d. \tag{28}$$

Express $\Delta\varphi$ as a function of N. Does the sensitivity depends on φ? Comment this result.

Solution :

$$\Delta\varphi = \frac{\Delta N_d}{\left|\frac{dN_d}{d\varphi}\right|} = \frac{1}{\sqrt{N_1}} \tag{29}$$

This result does not depend on φ. In a usual situation when using an interferometer for measuring a phase shift, one would intuitively expect a better sensitivity by adjusting the interferometer close to mid-fringe ($\varphi \simeq \pi/2 \bmod \pi$), thus maximizing the measurement sensitivity $\left|\frac{dN_d}{d\varphi}\right| = \left|N_1 \sin\varphi\right|$, i.e. maximizing the dependency of the signal N_d with respect to the parameter φ. This is indeed the best adjustment if the measurement noise ΔN_d is a constant. This is the case if it corresponds to technical noise in the detectors, which is independent of φ. However, if detectors are noiseless, ΔN_d as given by (27) has a variation with φ, which just compensates for the variation of the sensitivity. As you will see when performing the same calculation in the case of a quasi-classical field state (next section) this is a purely quantum result, which does not survive if the input state has any intensity fluctuation.

The $1/\sqrt{N_1}$ factor is the same as the statistical factor one would get by repeating N_1 times the measurement with N_1 independent photons. This noise is intimately linked to the fact that a given amount of radiation field energy corresponds to a finite number of photons. Individual photodetection events can be seen as independent random events. The statistical noise associated to this fundamentaly discretized stochastic process is named the “shot noise”. You will see in the next section that it coincides with the “standard quantum limit” (SQL) defined as the quantum limit of measurement precision when using coherent (quasi-classical) states at the input of the interferometer.

3 Coherent state input

We will now resume question 2.2 to 2.5 for the input state $|\Psi_{12}\rangle = |\alpha\rangle_1|0\rangle_2$, where $|\alpha\rangle_1$ corresponds to a coherent state in mode 1 with average photon number $N = |\alpha|^2$. Coherent states describe light emitted by a laser, or any classical monochromatic source.

We recall here the definition of a coherent state as introduced in homework 2.

The coherent state $|\alpha\rangle$ is defined by

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{30}$$

where $\alpha = |\alpha|e^{i\theta}$ is a complex number. This number can be seen as the dimensionless complex amplitude of the average radiation field associated to the coherent state $|\alpha\rangle$. The average photon number in the state $|\alpha\rangle$ is $|\alpha|^2$.

As you can check easily, coherent states are eigenvectors of $\hat{a}$ with the eigenvalue α

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{31}$$

Another important relation is obtained by conjugation

$$\langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha|. \tag{32}$$

You are encouraged to solve this section of the problem by only using the two previous relations, but not the explicit expression of $|\alpha\rangle$.

1. Calculate the average value $N_d = \langle \hat{N}_d \rangle$ as a function of N_1 and φ. One will use the result of question 2.3.a.

   Solution : Using the expression (23) of $\hat{N}_d$ and using (31)(32) one gets

$$N_d = N_1 \cos\varphi\,. \tag{33}$$

2. Calculation of the average value of $\hat{N}_d^2$.

   (a) Calculate $\langle \hat{N}_d^2 \rangle$, the average value of $\hat{N}_d^2$ as a function of N_1 and φ.

   Solution : From expression (23) of $\hat{N}_d$ the $\hat{N}_2$ term will not contribute in any way to the average value $\langle \hat{N}_d^2 \rangle$ as mode 2 is in the vacuum. In the development of $\hat{N}_d^2$, the cross-term $\hat{N}_1(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1)$ also has a vanishing average value because they involve averages values of $\hat{a}_2$ or $\hat{a}_2^\dagger$ in the vacuum. This is the same argument as in the case of number state input in section 2. We thus have

$$\langle \hat{N}_d^2 \rangle = \cos^2\varphi\, \langle \alpha_1 | \hat{N}_1^2 | \alpha_1 \rangle - \sin^2\varphi\, \langle \alpha_1, 0_2 | (\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1)^2 | \alpha_1, 0_2 \rangle. \tag{34}$$

   One has $\hat{N}_1^2 = \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1^\dagger \hat{a}_1 = \hat{a}_1^\dagger (\hat{a}_1^\dagger \hat{a}_1 + 1)\hat{a}_1$, which is the expression of $\hat{N}_1$ in the "normal order", i.e. with all creation operators on the left-hand side of products and all the annihilation operators on the right. This order is well adapted for calculating average values in coherent states by using (31) and (32). One thus gets

$$\langle \alpha_1 | \hat{N}_1^2 | \alpha_1 \rangle = N_1^2 + N_1. \tag{35}$$

   Developing the product

$$(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1)^2 = \hat{a}_1^{\dagger 2} \hat{a}_2^2 + \hat{a}_2^{\dagger 2} \hat{a}_1^2 - \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1^\dagger \hat{a}_2. \tag{36}$$

   As in the case of number states, the two first terms vanish when taking the average value on the vacuum in mode 2, as well as the last term, which is proportional to $\hat{N}_2$. Using the commutator $[\hat{a}_2, \hat{a}_2^\dagger] = 1$, the remaining term writes $\hat{N}_1(\hat{N}_2 + 1)$. Finally

$$\langle \hat{N}_d^2 \rangle = \cos^2\varphi\,(N_1^2 + N_1) + \sin^2\varphi\; N_1 = \cos^2\varphi\, N_1^2 + N_1. \tag{37}$$

   Here again, as in the case of a number state input, the term $\sin^2\varphi\; N_1$ originates from vacuum fluctuations of input mode 2.

   (b) Calculate the variance $\Delta N_d^2 = \langle \hat{N}_d^2 \rangle - \langle \hat{N}_d \rangle^2$ of $\hat{N}_d$.

   Solution : From (33) and (37) one gets

$$\Delta N_d^2 = N_1. \tag{38}$$

   This is different from the case of a number sate input as seem on (27). In addition to the contribution $\sin^2\varphi\; N_1$ of the empty mode 2, we have a $\cos^2\varphi\; N_1$ term, which originates from photon number fluctuations of the coherent input state.

3. Express the phase measurement error $\Delta\varphi$ as a function of N and φ. Compare with the case of number state input in mode 1.

Solution :

$$\Delta\varphi = \frac{\Delta N_d}{\left|\frac{dN_d}{d\varphi}\right|} = \frac{1}{|\sin\varphi|\sqrt{N_1}} \tag{39}$$

In contrast to the case of number state input this result does depend on φ. The measurement error is minimized when the measurement sensitivity $\left|\frac{dN_d}{d\varphi}\right| = \left|N_1 \sin\varphi\right|$ is maximum, at $\varphi \simeq \pi/2 \mod \pi$. These are the operating points that maximize the variation of the signal $\langle N_d \rangle$ with respect to the parameter φ. The additional noise with respect to the case of number state input, corresponds to fluctuations in the input coherent sate $\Delta N_1^2 = N_1$ as compared to $\Delta N_1^2 = 0$ for a number state, and which manifests in the first term of equation (34).
For φ set at optimal measurement phase, the phase measurement noise is $1/\sqrt{N_1}$. It defines the "standard quantum limit" (SQL) as the ultimate measurement precision one can reach when using coherent states, also called quasi-classical states. Note that for number states, and in the absence of technical noise, the standard quantum limit is reached at any value of the phase difference φ, but the limit is not passed. Note also that that number states with $N_1 > 1$ are extremely difficult to prepare in optics, whereas coherent states with huge photons numbers are easily produced by lasers. You will see in our second course of quantum optics, that one can improve the performance of coherent state input, and go bellow the SQL, by injecting another kind of quantum states of light, the so-called "squeezed states", into the unused input channel of the interferometer.