Quantum optics
The coherent states of light

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Homework of Lesson 2

Introduction

In lesson 2 you have encountered photon number states (or "Fock states") of light as the elementary energy excitation of a quantized field. Amazingly, these state have energy but the average value of the electric field vanishes. They have a well-defined frequency but a totally undetermined phase! In the last practice quiz, you have seen that a non-zero average electric field emerges from a coherent superposition of $|n\rangle$ and $|n+1\rangle$ number states. The phase of the average field directly reflects the relative phase of the two probability amplitudes of this superposition. However, you can check that the average amplitude of this field is much less than what you could expect from a classical field with the same energy. This is related to the fact that in this superposition a significant part of the energy is still contained in a component with undetermined phase.

The goal of the problem is to introduce a class of superposition of number states whose average field has minimal quantum fluctuations and whose amplitude is as close as possible to that of a classical field with the same energy.

Problem

We consider a single mode plane wave radiation described so that the field operator is

$$\hat{\boldsymbol{E}}_\ell(\boldsymbol{r}) = i\ \mathscr{E}_\ell^{(1)}\ \vec{\boldsymbol{\varepsilon}}_\ell \left(\hat{a}_\ell e^{i\boldsymbol{k}_\ell\cdot\boldsymbol{r}} - \hat{a}_\ell^\dagger e^{-i\boldsymbol{k}_\ell\cdot\boldsymbol{r}}\right) \qquad \text{with} \qquad \mathscr{E}_\ell^{(1)} = \sqrt{\frac{\hbar\omega_\ell}{2\varepsilon_0 L^3}} \qquad \omega_\ell = ck_\ell \tag{1}$$

where $\hat{a}_\ell$ and $\hat{a}_\ell^\dagger$ are photon annihilation and creation operators. The field hamiltonian is $\hat{H}_\ell = \hbar\omega_\ell\ (\hat{a}_\ell^\dagger\hat{a}_\ell + 1/2)$. Remember the important relations

$$[\hat{a}_\ell, \hat{a}_\ell^\dagger] = 1 \tag{2}$$
$$\hat{a}_\ell|n_\ell\rangle = \sqrt{n_\ell}|n_\ell - 1\rangle \tag{3}$$
$$\hat{a}_\ell^\dagger|n_\ell\rangle = \sqrt{n_\ell+1}|n_\ell + 1\rangle \tag{4}$$

where $|n_\ell\rangle$ represents the radiation state containing n photons in the mode defined by $\ell = (\vec{\boldsymbol{\varepsilon}}_\ell, \boldsymbol{k}_\ell)$. As we consider a single mode, we will drop the index ℓ in all the following.

1 Single-mode field amplitude evolution

1. Let us consider $|\psi(t)\rangle$ an arbitrary quantum state of radiation at time t in a single mode ℓ. Express $\boldsymbol{E}(\boldsymbol{r},t) = \langle\psi(t)|\hat{\boldsymbol{E}}(\boldsymbol{r})|\psi(t)\rangle$ as a function of $\alpha(t) = \langle\psi(t)|\hat{a}|\psi(t)\rangle$ and of $\alpha^*(t) = \langle\psi(t)|\hat{a}^\dagger|\psi(t)\rangle$.

2. The general form of the state at time $t=0$ reads $|\psi(0)\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$. Express $|\psi(t)\rangle$ in the photon number states basis. Express $\alpha(t)$ as a function of $\alpha_0 = \alpha(0)$ and ω.
3. Give the general expression of $\boldsymbol{E}(\boldsymbol{r},t)$ as a function of $\mathscr{E}^{(1)}$, α_0 and α_0^*.

2 Coherent states of radiation : quasi-classical Glauber states

A single mode coherent state (or "quasi-classical state" or "Glauber state") is defined by the state vector :

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \tag{5}$$

where $\alpha = |\alpha| e^{i\phi}$ is a complex number.

1. Show that $|\alpha\rangle$ is an eigenvector of $\hat{a}$. What is the corresponding eigenvalue ?
   These rules make calculation of average values of field operators very simple if the operators are in *normal order* i.e. the $\hat{a}^\dagger$ operators on the left and the $\hat{a}$ operators on the right.
2. **Average value of electric field.** Write the expression of the average electric field amplitude and of the average energy as a function of α. Compare these values to the case of a number state.
3. **Photon number distribution**. Show that the probability $P(n)$ for having n photons is a Poisson law. Give the value of the average photon number $\bar{n} = \langle \hat{N} \rangle$ and of its variance $\Delta N^2 = \langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2$ as a function of α. How does the relative photon number fluctuation $\Delta n/\bar{n}$ depends on $\bar{n}$?
4. **Time evolution.** Let us consider the state $|\alpha_0\rangle$ at time $t=0$. Show that at any time t, the field state $|\psi(t)\rangle$ remains a coherent state characterized by a complex amplitude $\alpha(t)$. Give the expression of $\alpha(t)$.
5. **Field amplitude fluctuations.** Calculate the variance $\Delta \mathbf{E}^2$ of the electric field operator $\hat{\mathbf{E}}(\mathbf{r})$ as a function of $\mathscr{E}^{(1)}$. Does this variance depend on α ?
6. **Quadrature fluctuations.** As seen in video 1.7 (at time 3'08"), the field quadrature operators are defined by

$$\hat{Q} = \sqrt{\frac{\hbar}{2}} \left(\hat{a} + \hat{a}^\dagger \right) \tag{6}$$

$$\hat{P} = -i\sqrt{\frac{\hbar}{2}} \left(\hat{a} - \hat{a}^\dagger \right) \tag{7}$$

   We remind the expression of the electric field amplitude as a function of quadrature operators

$$\hat{\boldsymbol{E}}_\ell(\boldsymbol{r}) = \mathscr{E}^{(1)} \, \vec{\boldsymbol{\varepsilon}} \sqrt{\frac{2}{\hbar}} \left(\hat{P} \sin \boldsymbol{k}\cdot\boldsymbol{r} - \hat{Q} \cos \boldsymbol{k}\cdot\boldsymbol{r} \right). \tag{8}$$

   The operator $1/\sqrt{\hbar}\,\hat{P}$ (resp. $1/\sqrt{\hbar}\,\hat{Q}$) can be seen as the dimensionless electric field amplitude at positions $\boldsymbol{r}$ such that $\boldsymbol{k}\cdot\boldsymbol{r} = \pi/2 \; mod[\pi]$ (resp. $\boldsymbol{k}\cdot\boldsymbol{r} = 0 \; mod[\pi]$).
   Calculate ΔP^2 and ΔQ^2 for the coherent state $|\alpha\rangle$. Check that it minimizes the Heisenberg uncertainty relation $\Delta P.\Delta Q \geq \hbar/2$.

Take home message of homework 2

In the homework you have discovered the properties of remarkable quantum sates of single-mode light denoted as $|\alpha\rangle$ (α is a complex number), which can be defined as the eigenstates of the non-hermitian operator $\hat{a}$. You were given an explicit expression of this states in the photon number basis. You have shown that the amplitude fluctuation of the corresponding electric field has the same magnitude as that of vacuum. Indeed these states are minimal with respect to the canonically conjugated operators $\hat{P}$ and $\hat{Q}$ (seen video 1.7, at time 3'08"). In this sense, for a given energy, they are the best possible compromise maximizing the average electric field amplitude while keeping quantum fluctuations at the same minimal level as in the vacuum field.

Another important feature of coherent states is that they are much easier to prepare than number states. A coherent sates $|\alpha\rangle$ is a very good quantum description of the field radiated by a laser.

To be more specific, let us consider a section of a laser beam of length L corresponding to the mode one detects during an observation time $T = L/c$ (see video 2.2, at 6' 35" for more information about this important definition of a "localized mode"). If one detects on average $\bar{n}$ photons during the observation time T, the field state contained in a cylinder of length L is well represented by a coherent state $|\alpha\rangle$ with $|\alpha|^2 = \bar{n}$. This important result is far from being trivial and its demonstration is well beyond the scope of this course. A related question is the determination of the phase ϕ of the complex amplitude $\alpha = |\alpha|e^{i\phi}$, defining the physical phase of the electric field of a given laser field. This phase exists and can be revealed by sending two independent lasers on the same photodiode. If their frequency difference $\delta\nu$ is within the bandwidth of the detector, one will observe directly on an oscilloscope a beat-note at frequency $\delta\nu$ between the two lasers. One will see on the screen a sinusoidal function with a well defined phase revealing the phase difference between the two lasers. This rises an interesting but somewhat puzzling question : why is there a well defined phase ? There is a priori no reason that the phase takes a preferred value between 0 and 2π. Indeed, this problem is related to the question of "symmetry breaking" which occurs in other fields of physics like magnetism : an homogeneous ferromagnetic material has a well defined magnetization whereas it has a priori no reason to be magnetized in a preferred direction.

Important results

Notation : when a ket representing a field state contains a Latin letter like in $|n\rangle$ one usually have in mind a number state. When one uses a Greek letter like in $|\alpha\rangle$ one usually intend to represent a coherent state.

$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

$\langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha|$

$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$

The photon number distribution corresponding to the state $|\alpha\rangle$ is a Poisson law with $\bar{n} = |\alpha|^2$ and $\Delta n = |\alpha| = \sqrt{\bar{n}}$. If $\bar{n} \gg 1$ it is very well approximated by the Gaussian distribution $P(n) = \frac{1}{\sqrt{2\pi}\Delta n} e^{-\frac{(n-\bar{n})^2}{2\Delta n^2}}$.