Quantum optics
Generation of photon pairs by parametric down conversion

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Homework of Lesson 5

Introduction

The lesson 5 focused on the description of real photon wavepackets. As a first example, you have seen how to apply the multimode quantum optics formalism in order to describe a single photon wavepacket emitted by a single excited atom decaying by spontaneous emission (video 5.3). You were shown that this process leads to an exponentially decaying radiation field as a function of time. The corresponding wavepacket is a superposition of one-photon states in many different modes, with a Loretzian energy distribution.

You also saw (video 5.5) that single photon wavepacket can be produced using pairs of photons emitted in a radiative cascade. Detection of the first photon emitted in a cascade efficiently announces the emission of a second, "heralded" photon emitted after a very short time interval.

We will consider here the generation of photon pairs based on a non-linear optics process named "parametric down-conversion", which was also introduced in video 5.5. This is another source of heralded photons, with the advantage of emission of photon pairs in well-defined directions as opposed to photons generated by spontaneous emission. The goal of the problem is to show that parametric down conversion generates extremely well-isolated wavepacket in well-defined direction and with a duration in the range of 1 ps.

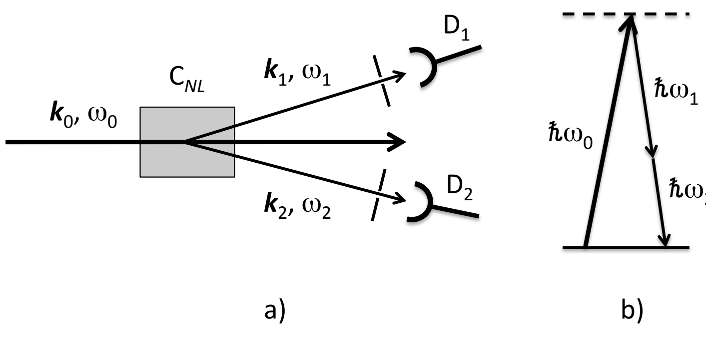

FIGURE 1 – **Parametric down-conversion**. **a)** *A plane wave with wavevector $\boldsymbol{k}_0$ is incident on the non-linear crystal C_{NL}. Two plane waves with wavevectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are generated in two different directions and detected on photon counters D_1 and D_2 after two narrow diaphragms defining the propagation direction of detected modes.* **b)** *Symbolic scheme representing parametric down-conversion. A photon with energy $\hbar\omega_0$ is destroyed and two photons with energy $\hbar\omega_1$ and $\hbar\omega_2$ are created.*

Parametric down-conversion (figure 1) is a non linear process, in which a single photon with angular frequency ω_0 is divided into two photons with angular frequencies ω_1 and ω_2. Energy conservation implies that $\hbar\omega_1 + \hbar\omega_2 = \hbar\omega_0$. We consider the conversion of a plane wave input "pump" mode with a wavevector $\boldsymbol{k}_0$ into pairs of modes with wavevectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. Photons emitted in mode 1 (resp. mode 2) are detected using the detector D_1 (resp. D_2) at position $\boldsymbol{r}_1$ (resp. $\boldsymbol{r}_2$).

In a birefringent crystal, the so called phase-matching condition determines the possible directions of emission of a pair of photons produced by parametric down-conversion. We select a pair of such directions with small diaphragms. For the sake of simplicity, we assume that all relevant field modes have a linear polarization $\vec{\varepsilon}$, which is perpendicular to the plane defined by vectors $\boldsymbol{k}_0$, $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. These propagation modes respectively correspond to photon annihilation operators $\hat{a}_0$, $\hat{a}_1$ and $\hat{a}_2$.

We describe here the action of the non-linear crystal by generalizing the approach that we adopted for describing the action of a beam-slitter. Non-absorbing optical devices, including non-linear crystals can be described formally by a unitary transformation of input modes into output modes. This unitary transform can also be seen as the evolution operator transforming an input radiation state into an output radiation state. It corresponds to a unitary evolution operator generated by some Hamiltonian. We now admit that in the case of parametric down-conversion, the relevant hamiltonian reads

$$\hat{H}_{NL} = \hbar\Omega\,(\hat{a}_0\,\hat{a}_1^\dagger\,\hat{a}_2^\dagger + \hat{a}_0^\dagger\,\hat{a}_1\,\hat{a}_2) \tag{1}$$

were Ω is a non-linear coupling strength and depends on the efficiency of the crystal.

The first term in $\hat{H}_{NL}$ has a simple physical interpretation, it couples the input state $|N_0, 0, 0\rangle$ to the state $|N_0 - 1, 1_1, 1_2\rangle$. It thus destroys one input photon in mode $\boldsymbol{k}_0$ and generates one photon pair in modes $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. Note that in order to have an hermitian Hamitonian we have also to include a second term in $\hat{H}_{NL}$, which corresponds to the opposite non-linear process : the destruction of one photon in mode $\boldsymbol{k}_1$ and one photon in mode $\boldsymbol{k}_2$ for creating one photon with a larger energy in mode $\boldsymbol{k}_0$. This process is named "sum-frequency generation".

We consider here parametric down-conversion in the case of a continuous pump field. For manipulating well-defined finite photon numbers, we follow the approach introduced in lesson 2 (video 2.2). We consider that each plane wave mode is truncated transversally with a cross section S and longitudinally with a length $L = c/T_\mathrm{m}$ along the propagation direction. Here, T_m is a wavepacket duration equal to the total measurement time of the continuous radiation fields. Numbers of photons are defined as the number of photons crossing the area S during the time T_m. The photon flux in mode j is defined by $I_j = N_j/T_\mathrm{m}$. The field per photon for a mode with angular frequency ω_j is thus

$$\mathscr{E}^{(1)}_{\omega_j} = \sqrt{\frac{\hbar\omega_j}{2\varepsilon_0 L S}}\ . \tag{2}$$

1 Photon pair generation : three coupled mode model

We first consider a three mode model. The free radiation field hamiltonian thus reduces to

$$\hat{H}_R = \sum_{i=0}^{2} \hbar\omega_i\,\hat{a}_i^\dagger\hat{a}_i. \tag{3}$$

We chose here to remove from $\hat{H}_R$ the constant vacuum energy.

1. The initial radiation state at time 0 is $|\psi_i\rangle = |\psi(0)\rangle = |N_0, 0, 0\rangle$. Show that the matrix element $\langle\psi_f|\hat{H}_{NL}|\psi_i\rangle$ is different from 0 for a single state $|\psi_f\rangle$. Give the expression of $|\psi_f\rangle$ and of the corresponding matrix element $\langle\psi_f|\hat{H}_{NL}|\psi_i\rangle$. Show that these two states have the same energy E_0. What is the value of E_0 ?

Solution : We have $H_{NL}|\psi_i\rangle = \hbar\Omega\sqrt{N_0}|N_0-1,1_1,1_2\rangle$. As $\hat{a}_1\,\hat{a}_2|N_0,0,0\rangle = 0$, the second term of H_{NL} does not contributes. We thus find that the initial state only couples to $|\psi_f\rangle = |N_0-1,1_1,1_2\rangle$ and the corresponding matrix element is $\langle\psi_f|\hat{H}_{NL}|\psi_i\rangle = \hbar\Omega\sqrt{N_0}$. From the relation $\hbar\omega_1 + \hbar\omega_2 = \hbar\omega_0$, the two coupled states have the same energy $E_0 = N_0\hbar\omega_0$.

2. By applying time dependent perturbation theory to H_{NL} with respect to the unperturbed Hamiltonian H_R, give the expression of the state $|\psi(T)\rangle$ at time T to first order in ΩT. Check that $|\psi(T)\rangle$ is normalized to first order in $\Omega T\sqrt{N_0}$. What is the probability P_{pair} to generate a pair of photons in mode 1 and 2 ? Show that the rate of pair generation $\phi_{\mathrm{pair}} = P_{\mathrm{pair}}/T_{\mathrm{m}}$ can be written $\phi_{\mathrm{pair}} = \eta_{\mathrm{pair}}\phi_0$. What is the value of the down-conversion efficiency η_{pair} ?

Reminder on time-dependent perturbation theory.

We define $|\varphi_j\rangle$ as eigenvectors of the Hamiltonian $\hat{H}_0$ with eigenvalue E_j. Let us consider the total Hamitonian $\hat{H} = \hat{H}_0 + \hat{V}_{\mathrm{pert}}$. If the system is prepared in state $|\varphi_n\rangle$ at time $t = 0$, the first order perturbative expression of the state $|\psi(T)\rangle$ at time T is

$$|\psi(T)\rangle = e^{-iE_nT/\hbar}|\varphi_n\rangle + \frac{1}{i\hbar}\sum_{j\neq n} e^{-iE_jT/\hbar}\int_0^T dt\, e^{-i(E_j-E_n)t/\hbar}\,\langle\varphi_j|\hat{V}_{\mathrm{pert}}|\varphi_n\rangle\,|\varphi_j\rangle\,. \qquad (4)$$

This expression is valid as far as the sum of probabilities of having any state $|\varphi_j\rangle$, with $j \neq n$ is small as compared to 1.

Solution : Application of (4) in the case of only two coupled degenerate states leeds to

$$|\psi(T)\rangle = e^{-iE_0T/\hbar}\big(|N_0,0,0\rangle - i\Omega T\sqrt{N_0}\,|N_0-1,1_1,1_2\rangle\big), \qquad (5)$$

whose norm is 1 to first order in $\Omega T\sqrt{N_0}$ (and neglecting second order terms). The factor $e^{-iE_0T/\hbar}$ is here a global phase factor, which can be omitted. The probability to generate a pair is thus

$$P_{\mathrm{pair}} = N_0\big(\Omega T\big)^2\,. \qquad (6)$$

This perturbative approach is valid provided $P_{\mathrm{pair}} \ll 1$. The probability P_{pair} represents the probability to generate a pair with an input wavepacket of duration T_{m} and containing N_0 photons in the pump mode. We thus have

$$\phi_{\mathrm{pair}} = P_{\mathrm{pair}}/T_{\mathrm{m}} = \big(\Omega T\big)^2\,\frac{N_0}{T_{\mathrm{m}}} = \eta_{\mathrm{pair}}\phi_0, \qquad (7)$$

with the definition $\eta_{\mathrm{pair}} = \big(\Omega T\big)^2$. Note that in our effective non-linear hamiltonian model, neither the coupling strength Ω nor the interaction time T have separately a well defined physical meaning. The only physically relevant parameter of the model is the product ΩT, which determines the actual down-conversion efficiency η_{pair}.

For all following questions in this section, we shift the time origin by T and we consider as a new initial state at time 0 the output state

$$|\psi_{out}\rangle = |\psi(T)\rangle = |N_0,0,0\rangle - i\sqrt{\eta_{\mathrm{pair}}N_0}\,|N_0-1,1_1,1_2\rangle\,. \qquad (8)$$

Following the approach of lesson 5, we now apply the Heisenberg formalism for evaluating single and double photodetection rates $w^{(1)}(\boldsymbol{r},t)$ and $w^{(2)}(\boldsymbol{r}_1,t_1,\boldsymbol{r}_1,t_1)$.

We recall that, in the Heisenberg formalism, the statevector does not evolve with time, whereas operators become time-dependent. If you are not familiar with the Heisenberg formalism, you should carefully view once again video 5.1.

3. Give the expression of the single photodetection rate $w^{(1)}(\boldsymbol{r}_1, t_1) = s \parallel \hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}_1, t_1)|\psi_{out}\rangle \parallel^2$ and $w^{(1)}(\boldsymbol{r}_2, t_2)$ at detectors D_1 and D_2 for a radiation field in the state $|\psi_{out}\rangle$. Express the result as a function of the detector efficiency $\eta_j = s\frac{\hbar\omega_j}{2\varepsilon_0 c}$, η_{pair}, S and the pump intensity $\phi_0 = N_0/T_{\text{m}}$. Note that the detector D_j is located at position $\boldsymbol{r}_j$ and only detects photons in mode j.

Solution : Using $w^{(1)}(\boldsymbol{r}_1, t_1) = s \parallel \hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}_1, t_1)|\psi_{out}\rangle \parallel^2$. One has

$$\hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}_1, t_1)|\psi_{out}\rangle = i\,\vec{\boldsymbol{\varepsilon}} \sum_{j=0}^{3} \mathscr{E}_{\omega_j}^{(1)}\, \hat{a}_j\, e^{i(\boldsymbol{k}_j\cdot\boldsymbol{r}_1 - \omega_j t_1)}|\psi_{out}\rangle. \tag{9}$$

As detector D_1 only detects mode 1, only the $j=1$ term contributes to $w^{(1)}(x_1, t_1)$ and one has

$$\hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}_1, t_1)|\psi_{out}\rangle = i\,\vec{\boldsymbol{\varepsilon}}\ \mathscr{E}_{\omega_1}^{(1)}\, \hat{a}_1\, e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}_1 - \omega_1 t_1)}\big(|N_0, 0, 0\rangle + \alpha\sqrt{\eta_{\text{pair}} N_0}\,|N_0 - 1, 1_1, 1_2\rangle\big) \tag{10}$$

$$= i\,\vec{\boldsymbol{\varepsilon}}\ \mathscr{E}_{\omega_1}^{(1)}\, e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}_1 - \omega_1 t_1)}\,\alpha\sqrt{\eta_{\text{pair}} N_0}\,|N_0 - 1, 0_1, 1_2\rangle. \tag{11}$$

The single photon detection rate is thus

$$w^{(1)}(x_1, t_1) = s\Big(\ \mathscr{E}_{\omega_1}^{(1)}\Big)^2\ \eta_{\text{pair}} N_0\ = s\frac{\hbar\omega_1}{2\varepsilon_0 LS}\eta_{\text{pair}} N_0. \tag{12}$$

Introducing the detector quantum efficiency $\eta_1 = s\frac{\hbar\omega_1}{2\varepsilon_0 c}$ and using the definition of the pump field intensity $\phi_0 = N_0/T_{\text{m}} = N_0\, c/L$ one gets

$$w^{(1)}(x_1, t_1) = \eta_1 \frac{c}{LS}\eta_{\text{pair}} N_0 = \frac{1}{S}\,\eta_1\,\eta_{\text{pair}}\,\phi_0\,. \tag{13}$$

Similarly

$$w^{(1)}(x_2, t_2) = \frac{1}{S}\,\eta_2\,\eta_{\text{pair}}\,\phi_0\,. \tag{14}$$

Note that $w^{(1)}(x_1, t_1) = w^{(1)}(x_2, t_2)$ if detection efficiencies at ω_1 and ω_2 are equal. In the continuous pump regime, single photon detection rates are independent of time and of position.

4. Express of the double photodetection rate $w^{(2)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2)$ as a function of η_j, η_{pair}, S, T_{m} and $\phi_0 = N_0/T_{\text{m}}$.

Solution : By definition $w^{(2)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2) = s^2 \parallel \hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}_2, t_2)\,\hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}_1, t_1)|\psi_{out}\rangle \parallel^2$. Using (11) one finds

$$\hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}_2, t_2)\,\hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}_1, t_1)|\psi_{out}\rangle$$

$$= i\vec{\boldsymbol{\varepsilon}}\ \mathscr{E}_{\omega_1}^{(1)}\,\alpha\sqrt{\eta_{\text{pair}} N_0}\, e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}_1 - \omega_1 t_1)}\,\hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}_2, t_2)|N_0 - 1, 0_1, 1_2\rangle \tag{15}$$

$$= -\ \mathscr{E}_{\omega_1}^{(1)}\ \mathscr{E}_{\omega_2}^{(1)}\,\alpha\sqrt{\eta_{\text{pair}} N_0}\, e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}_1 - \omega_1 t_1)}\ e^{i(\boldsymbol{k}_2\cdot\boldsymbol{r}_2 - \omega_1 t_2)}\,|N_0 - 1, 0_1, 0_2\rangle, \tag{16}$$

which leads to

$$w^{(2)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2) = s^2\Big(\ \mathscr{E}_{\omega_1}^{(1)}\ \mathscr{E}_{\omega_2}^{(1)}\Big)^2\ \eta_{\text{pair}} N_0 = \frac{1}{S^2}\,\eta_1\,\eta_2\,\eta_{\text{pair}}\,\phi_0 \frac{1}{T_{\text{m}}}\,. \tag{17}$$

We have implicitly assumed that the position $\boldsymbol{r}_j$ of detector D_j at time t_j is inside the moving volume LS of the wavepacket. Within these limits, $w^{(2)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2)$ is constant, whereas it is null as far as one detector is outside the wavepacket.

5. Express $g^{(2)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2) = w^{(2)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2)/\big(w^{(1)}(\boldsymbol{r}_1, t_1) \cdot w^{(1)}(\boldsymbol{r}_2, t_2)\big)$ as a function of P_{pair}. Comment this result.

Solution :

$$g^{(2)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2) = \frac{w^{(2)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2)}{w^{(1)}(\boldsymbol{r}_1, t_1) \cdot w^{(1)}(\boldsymbol{r}_2, t_2)} = \frac{1}{\eta_{\text{pair}} \phi_0 T_{\text{m}}} = \frac{1}{\eta_{\text{pair}} N_0} = \frac{1}{P_{\text{pair}}}. \tag{18}$$

As $P_{\text{pair}} \ll 1$, one has $g^{(2)} \gg 1$. This is in contrast with the case of a classical field state, which correspond to $g^{(2)} = 1$. In this last case, the second order photodetection rate equals the product of single photodetection rates. It means that there are no correlations between two photodetection events, which are thus independent stochastic events. In a classical radiation field, photons can thus be seen as randomly distributed independant particles. Here, photons being generated by pairs, the double photodetection rate is much higher that the product of single photodetection signals.

2 Photon pair generation : multimode model

In fact, there are many modes pairs, which satisfy the conditions

$$\omega_0 = \omega_1 + \omega_2 \tag{19}$$

$$\boldsymbol{k}_0 \simeq \boldsymbol{k}_1 + \boldsymbol{k}_2, \tag{20}$$

even if the directions of $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are fixed. We assume that the two diaphragms placed after the non-linear crystal define two directions, corresponding to unit vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ such that condition (20) is exactly fulfilled for a particular frequency pair $(\omega_1^{(0)}, \omega_2^{(0)})$. Note that due to dispersion and birefringence in anisotropic crystals, these two direction can be well separated with respect to the pump field. There are many pairs of modes with angular frequency ω_1 and ω_2 propagating along $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and approximately fulfilling (20). In contrast, (19) must be rigorously fulfilled so we consider pairs of frequencies such that δ by $\omega_1 = \omega_1^{(0)} + \delta$ and we have $\omega_2 = \omega_2^{(0)} - \delta$.

We now consider that the interaction with the non-linear crystal prepares the initial state

$$|\psi'_{out}\rangle = |N_0, 0, 0\rangle + |N_0 - 1\rangle \otimes \sqrt{\eta_{\text{pair}} N_0} |\psi_{\text{pair}}\rangle, \tag{21}$$

where η_{pair} represents the total pair-generation efficiency and where

$$|\psi_{\text{pair}}\rangle = \sum_{\delta} C(\delta)\, |1_{\boldsymbol{u}_1,\delta}, 1_{\boldsymbol{u}_2,-\delta}\rangle \tag{22}$$

is a multimode, normalized, pair state. In (23), the state $|1_{\boldsymbol{u}_1,\delta}\rangle$ (resp. $|1_{\boldsymbol{u}_2,-\delta}\rangle$) is a one photon state in the mode propagating along $\boldsymbol{u}_1$ (resp. $\boldsymbol{u}_2$) and with angular frequency $\omega_1 = \omega_1^{(0)} + \delta$ (resp. $\omega_2 = \omega_2^{(0)} - \delta$). The perturbative approach adopted here is valid for $\eta_{\text{pair}} N_0 \ll 1$.

1. What is the total probability P_{pair}^{tot} of getting a down-converted pair of photons ?

Solution : As all terms in the sum over δ in (23) correspond to orthogonal photon pair states, P_{pair}^{tot} is the sum of the probabilities of having a given pair state $|1_{\boldsymbol{u}_1,\delta}, 1_{\boldsymbol{u}_2,-\delta}\rangle$. By summing these probabilities and using the normalisation of $|\psi_{\text{pair}}\rangle$, one gets

$$P_{\text{pair}}^{\text{tot}} = \eta_{\text{pair}} N_0. \tag{23}$$

2. Give the expression of the single photodetection rate $w^{(1)}(\boldsymbol{r}_1, t_1)$ and $w^{(1)}(\boldsymbol{r}_2, t_2)$ at detectors D_1 and D_2 for a radiation field in the state $|\psi'_{out}\rangle$. Express the result as a function of s, η_{pair}, $\mathscr{E}^{(1)}_{\omega_j}$, $C(\delta)$ and N_0.

   Solution : Using (11) and (22) one has

$$\hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}_1, t_1)|\psi'_{out}\rangle \quad = \quad i\,\vec{\varepsilon} \sum_{\delta} \mathscr{E}^{(1)}_{\omega_1}\, e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}_1 - \omega_1 t_1)} \sqrt{\eta_{\text{pair}} N_0}\, C(\delta)\, |N_0 - 1, 0_1, 1_2\rangle \quad (24)$$

   and finally

$$w^{(1)}(x_1, t_1) = s\,\eta_{\text{pair}} N_0 \sum_{\delta} \left(\mathscr{E}^{(1)}_{\omega_1} \right)^2 \left| C(\delta) \right|^2 . \qquad (25)$$

   Similarly

$$w^{(1)}(x_2, t_2) = s\,\eta_{\text{pair}} N_0 \sum_{\delta} \left(\mathscr{E}^{(1)}_{\omega_2} \right)^2 \left| C(\delta) \right|^2 . \qquad (26)$$

3. In the limit $|\delta| \ll \omega_1^{(0)}$, $\omega_2^{(0)}$ express $w^{(1)}(x_j, t_j)$ as a function of η_j, η_{pair}, S and ϕ_0.

   Solution : Using the condition $|\delta| \ll \omega_1^{(0)}$, $\omega_2^{(0)}$, one has $\mathscr{E}^{(1)}_{\omega_j} \simeq \mathscr{E}^{(1)}_{\omega_j^{(0)}}$. Using the normalization of $C(\delta)$, $\eta_j = s\frac{\hbar\omega_j}{2\varepsilon_0 c}$ and $\phi_0 = N_0 c/L$ one gets

$$w^{(1)}(x_j, t_j) \simeq s\,\eta_{\text{pair}} N_0 \left(\mathscr{E}^{(1)}_{\omega_1^{(0)}} \right)^2 \left| C(\delta) \right|^2 = \frac{1}{S}\,\eta_j\,\eta_{\text{pair}}\,\phi_0 \,. \qquad (27)$$

4. Give the expression of the double photodetection rate $w^{(2)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2)$ as a function of propagation times $\tau_1 = \frac{\boldsymbol{u}_1\cdot\boldsymbol{r}_1}{c}$, $\tau_2 = \frac{\boldsymbol{u}_2\cdot\boldsymbol{r}_2}{c}$, s, δ, $\mathscr{E}^{(1)}_{\omega_1}$, $\mathscr{E}^{(1)}_{\omega_2}$ and $C(\delta)$.

   Solution : Using (10,11) one finds

$$\hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}_2, t_2)\, \hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}_1, t_1)|\psi_{out}\rangle$$

$$= -\sqrt{\eta_{\text{pair}} N_0} \sum_{\delta} C(\delta)\; \mathscr{E}^{(1)}_{\omega_1}\; \mathscr{E}^{(1)}_{\omega_2}\, e^{i\omega_1(\tau_1 - t_1)}\, e^{i\omega_2(\tau_2 - t_2)}\, |N_0 - 1, 0_1, 0_2\rangle, \qquad (28)$$

   which leads to

$$w^{(2)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2) = s^2\,\eta_{\text{pair}} N_0 \left| \sum_{\delta} C(\delta)\; \mathscr{E}^{(1)}_{\omega_1}\; \mathscr{E}^{(1)}_{\omega_2}\, e^{i\delta(\tau_1 - \tau_2 - t_1 + t_2)} \right|^2 . \qquad (29)$$

   In the following parts, we assume $|\delta| \ll \omega_1^{(0)}$, $\omega_2^{(0)}$. We also take $\tau_1 = \tau_2$ without loss of generality and we define $\tau = t_1 - t_2$.

5. Give an approximate value of $w^{(2)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2)$ as a function of η_j, η_{pair}, S, ϕ_0, τ, δ and $C(\delta)$. We will denote this approximate value by $w^{(2)}(\tau)$.

   Solution :

$$w^{(2)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2) \simeq \frac{1}{S^2}\,\eta_1\,\eta_2\,\eta_{\text{pair}}\,\phi_0 \frac{1}{T_{\text{m}}} \left| \sum_{\delta} C(\delta)\, e^{i\delta\tau} \right|^2 = w^{(2)}(\tau)\,. \qquad (30)$$

   Due to correlations between the radiation modes propagating along $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$,the double photodetection rate is not proportional to the product $w^{(1)}(\boldsymbol{r}_1, t_1) \cdot w^{(1)}(\boldsymbol{r}_2, t_2)$.

6. We assume that $C(\delta)$ is the gaussian function

$$C(\delta) = \mathcal{N}(L)\, e^{-\frac{\delta 2}{4\sigma_\delta^2}} \tag{31}$$

Where the normalization constant $\mathcal{N}(L) = \sqrt{\frac{2\pi c}{L}}\,\frac{1}{(2\pi)^{1/4}\sqrt{\sigma_\delta}}$. This normalization factor is obtained by using

$$\|\, |\psi_{\text{pair}}\rangle\, \|^2 = \sum_\delta \left|C(\delta)\right|^2 = \frac{L}{2\pi c}\int d\delta \left|C(\delta)\right|^2 = 1\,, \tag{32}$$

where the factor $\frac{L}{2\pi c} = dn/d\omega$ is the 1D density of mode.

For real non-linear crystals, the bandwidth of parametric down conversion is typically $\sigma_\delta \simeq 10^{13} s^{-1}$.

(a) What is the probability $P_{\text{pair}}(\delta)$ of producing a pair (ω_1, ω_2) with $\omega_1 = \omega_1^{(0)} + \delta$ and $\omega_2 = \omega_2^{(0)} - \delta$. What is the width at $1/\sqrt{e}$ of this probability distribution ?

Solution : One has $P_{\text{pair}}(\delta) = |C(\delta)|^2 = \mathcal{N}^2\, e^{-\frac{\delta 2}{2\sigma_\delta^2}}$. The variance of this gaussian distribution is σ_δ^2.

(b) Transforming the discrete sum into an integral, and performing the integration give the explicit expression of $w^{(2)}(\tau)$ as a function of η_j, η_{pair}, S, ϕ_0 and $\Delta\tau = \frac{1}{2\sigma_\delta}$.

Note : We recall the expression of the Fourier transform of a gaussian

$$F(x) = e^{-\alpha x^2} \xrightarrow{\text{F.T.}} G(k) = \frac{1}{\sqrt{2\alpha}} e^{-\frac{k^2}{2\alpha}} \tag{33}$$

Solution : Transforming the sum into an integral and performing the Fourrier transform we first have

$$\sum_\delta C(\delta)\, e^{i\delta\tau} = \mathcal{N}(L)\frac{L}{2\pi c}\int d\delta\, C(\delta) e^{i\delta\tau} = \mathcal{N}(L)\frac{L}{2\pi c}\, 2\sigma_\delta\, e^{-\sigma_\delta^2\,\tau^2} \tag{34}$$

Using (31), one finds

$$w^{(2)}(\tau) = \frac{1}{S^2}\,\eta_1\,\eta_2\,\eta_{\text{pair}}\,\phi_0\,\frac{2\sigma_\delta}{\pi\sqrt{2\pi}}\, e^{-2\sigma_\delta^2\,\tau^2}\,. \tag{35}$$

Defining $\Delta\tau = \frac{1}{2\sigma_\delta}$, one finally gets

$$w^{(2)}(\tau) = \frac{1}{S^2}\,\eta_1\,\eta_2\,\eta_{\text{pair}}\,\frac{\phi_0}{\Delta\tau}\,\frac{1}{\pi\sqrt{2\pi}}\, e^{-\frac{\tau^2}{2\Delta\tau^2}}\,. \tag{36}$$

(c) Give the expression of the normalized second order correlation function $g^{(2)}(\tau)$. Express the result as a gaussian function times a function of $N_{\text{pair}}(\Delta\tau)$ defined as the number of pairs generated during the time duration $\Delta\tau$.

Solution : Using (28) and (37) one finds

$$g^{(2)}(\tau) = \frac{1}{\eta_{\text{pair}}\,\phi_0 \Delta\tau}\,\frac{1}{\pi\sqrt{2\pi}}\, e^{-\frac{\tau^2}{2\Delta\tau^2}}\,. \tag{37}$$

The number of pairs generated during $\Delta\tau$ is $N_{\text{pair}}(\Delta\tau) = \eta_{\text{pair}}\, N_0 \Delta\tau$. On finally has

$$g^{(2)}(\tau) = \frac{1}{N_{\text{pair}}(\Delta\tau)}\,\frac{1}{\pi\sqrt{2\pi}}\, e^{-\frac{\tau^2}{2\Delta\tau^2}} \tag{38}$$

The second order correlation function is gaussian and peaked around $\tau = 0$, with a width $\Delta\tau \simeq 1$ ps. In addition, even if one generates millions of pairs per second, one has $N_{\text{pair}}(\Delta\tau) \simeq 10^{-6}$ and $g^{(2)}(0)$ is a huge number, much larger than 1. As a consequence of the large bandwidth of parametric down conversion, the two photons of a pair are strongly correlated in time. That's why one considers that the two photons of a pair generated by down-conversion are created "at the same time". This is another purely quantum feature : there is no classical field with the $g^{(2)}(\tau)$ that you calculated here.

As far as one only performs photon counting experiments, one can consider that parametric down-conversion directly generates pairs of photons emitted at the same, random, time. Note however that it is a highly non-trivial picture : as the input pump field considered here is continuous one expects that it generates continuous output fields propagating in the $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ directions. The peaked shape of the function $w^{(2)}(\tau)$ can however be interpreted in the following way : Once one photon is detected along direction $\boldsymbol{u}_1$ the radiation field propagating in the direction $\boldsymbol{u}_2$ is "projected" into a narrow single photon wavepacket. That is a source of heralded photons in the same sense as in video 5.5 of the lesson. But one could also perform other kind of measurements that will be presented in future lessons, more precisely homodyne measurements, for which the pairs of photons image does not render an account of the observations.