# Quantum optics
# Field state transformation on a beam splitter

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Homework of Lesson 3

## Introduction

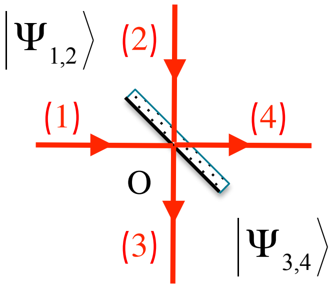


FIGURE 1 – *Coupling of modes 1 and 3 into modes 3 and 4 on a beamsplitter.*

In the lesson you learned how to transform field operators on a beam splitter. We introduced the transformation of input field operators $\hat{\mathbf{E}}_1^{(+)}$ and $\hat{\mathbf{E}}_2^{(+)}$ into output field operators $\hat{\mathbf{E}}_3^{(+)}$ and $\hat{\mathbf{E}}_1^{(+)}$, which reads

$$\hat{\mathbf{E}}_3^{(+)} = r\hat{\mathbf{E}}_1^{(+)} + t\hat{\mathbf{E}}_2^{(+)} \tag{1}$$

$$\hat{\mathbf{E}}_4^{(+)} = t\hat{\mathbf{E}}_1^{(+)} - r\hat{\mathbf{E}}_2^{(+)}, \tag{2}$$

where $r$ and $t$ are the amplitude reflection and transmission coefficients characterizing the beam splitter. In these expressions, these coefficients are assumed to be real. The formalism applies to lossless beam splitters for which $r^2 + t^2 = 1$. We assume that all four modes have the same frequency and polarization and that the input field wave-vectors are matched so that there are only two output modes. These equations are the quantum counterparts of the rules linking the classical fields emerging from a beam splitter as a function of the classical input fields. Remember the minus sign on the second line, which, as shown in video 3.1, is needed for energy conservation. Getting rid of normalization constant and the polarization one has

$$\hat{a}_3 = r\hat{a}_1 + t\hat{a}_2 \tag{3}$$

$$\hat{a}_4 = t\hat{a}_1 - r\hat{a}_2, \tag{4}$$

where $\hat{a}_i$ is the annihilation operator of one photon in mode $i$. By conjugation, one gets similar equations for $\hat{a}_i^\dagger$ creation operators.

As seen in lesson 2, $|\Psi_{1,2}\rangle$ and $|\Psi_{3,4}\rangle = \hat{U}|\Psi_{1,2}\rangle$ both represent the same global state of the field in two different basis. They are connected by a unitary transform $\hat{U}$. Note that, although field operators are not the same on both side of the beamsplitter, the state $|\Psi\rangle$ of the system is defined in a single Hilbert space and allows one to calculate the field amplitude at any point of space.

In the lesson, it was claimed that the most efficient way to calculate the value of an observable in the output state is to express it in the input mode basis, rather than expressing the state in the output basis. It is nevertheless possible to find a formally compact expression of the state in the output basis. The goal of the problem is to show how to do it and to use this expression for simple specific physical situations.

## Problem

## 1 Number state transformation

1. Let us consider the state $|n\rangle_1$ with $n$ photon in mode 1. Express $|n\rangle_1$ as a function of $\hat{a}_1^\dagger$ and of the vacuum state $|0\rangle_1$.

   ***Solution :*** Using $\hat{a}_1^\dagger|n\rangle_1 = \sqrt{n+1}|n+1\rangle_1$, one gets $|n\rangle_1 = \frac{1}{\sqrt{n!}}\,\hat{a}_1^{\dagger n}\,|0\rangle_1$.

2. Express $|\Psi\rangle = |n\rangle_1 \otimes |m\rangle_2 = |n,m\rangle_{1,2}$ as a function of $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$ acting on the two mode vacuum $|0,0\rangle$. Note that the vacuum state does not depends on the choice of the basis.

   ***Solution :*** $|\Psi\rangle = \frac{1}{\sqrt{n!m!}}\,\hat{a}_1^{\dagger n}\hat{a}_2^{\dagger m}\,|0,0\rangle$

3. Express the same state $|\Psi\rangle$ as a function of $\hat{a}_3^\dagger$ and $\hat{a}_4^\dagger$ output field creation operators.

   ***Solution :*** Inverting the h.c. of equ. (3) and (4), one expresses $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$ as a function of $\hat{a}_3^\dagger$ and $\hat{a}_4^\dagger$

$$\hat{a}_1^\dagger = r\hat{a}_3^\dagger + t\hat{a}_4^\dagger \tag{5}$$
$$\hat{a}_2^\dagger = t\hat{a}_3^\dagger - r\hat{a}_4^\dagger. \tag{6}$$

   By substitution in the expression of $|n,m\rangle_{1,2}$ one gets

$$|\Psi\rangle = \frac{1}{\sqrt{n!m!}}\,\hat{a}_1^{\dagger n}\hat{a}_2^{\dagger m}\,|0,0\rangle = \frac{1}{\sqrt{n!m!}}\,(r\hat{a}_3^\dagger + t\hat{a}_4^\dagger)^n(t\hat{a}_3^\dagger - r\hat{a}_4^\dagger)^m\,|0,0\rangle. \tag{7}$$

   In these two representations of the same vector, the first one leads to a simple expression of $|\Psi\rangle$ in the (1) and (2) modes number state basis. Equivalently, the second expression leads to a similar expression of $|\Psi\rangle$ in the (3) and (4) modes number state basis. However, going one step further, if one develops explicitly the product of operators, the state in the output basis is very complex, with a huge number of terms. That's why, except in a few simple situation (see below), one usually prefers to work with the input basis by expressing the output mode operators as a function of the input one.

4. Let us consider the state $|\Psi\rangle = |1,0\rangle_{1,2}$. Give the expression of this state in the output mode basis.

   ***Solution :*** $|\Psi\rangle = |1,0\rangle_{1,2} = r|1,0\rangle_{3,4} + t|0,1\rangle_{3,4}$.

   As expected, the initial one photon state is reflected in mode (3) with a probability $r^2$ or transmitted in mode (4) with a probability $t^2$.

5. Same question as above for the state $|\Psi\rangle = |1,1\rangle_{1,2}$ corresponding to exactly one photon in each input mode. We now consider a '50/50' beamsplitter with $r^2 = t^2 = 1/2$. What is the probability to finally detect ons photon in each output mode ? Interpret this result in term of interference between two undistinguishable quantum processes.

***Solution :***

$$\begin{aligned}|\Psi\rangle &= |1,1\rangle_{1,2} = \hat{a}_1^\dagger \hat{a}_2^\dagger\,|0,0\rangle = (r\hat{a}_3^\dagger + t\hat{a}_4^\dagger)(t\hat{a}_3^\dagger - r\hat{a}_4^\dagger)\,|0,0\rangle \\ &= \left[r\,t\,\hat{a}_3^{\dagger 2} + (t^2-r^2)\,\hat{a}_3^\dagger\hat{a}_4^\dagger - t\,r\,\hat{a}_4^{\dagger 2}\right]\,|0,0\rangle \\ &= \left[\sqrt{2}\,r\,t|2,0\rangle_{3,4} + (t^2-r^2)|1,1\rangle_{3,4} - \sqrt{2}\,t\,r|0,2\rangle_{3,4}\right],\end{aligned}$$

where we have used $[\hat{a}_3^\dagger, \hat{a}_4^\dagger] = 0$.

For a 50/50 beamsplitter one has $|\Psi\rangle = 1/\sqrt{2}\,[|2,0\rangle_{3,4} - |0,2\rangle_{3,4}]$. The $|1,1\rangle_{3,4}$ term vanishes due to destructive interference between two undistinguishable processes : input photons in mode (1) and (2) both transmitted or both reflected. This is a highly non-trivial quantum effect : there is a quantum interference involving two initially independent photons eventually emitted by two independent sources arriving at the same time on a beamsplitter. This effect was first observed by Hong Ou and Mandel in 1987 by placing a single photon counters on each output mode. For matched photons in time, they effectively observed a suppression of coincidence in the two photon detection signal.

6. We consider here $|\Psi\rangle = |n,0\rangle_{1,2}$. What is the expression of $|\Psi\rangle$ in the output basis. What is the probability $P(k, n-k)$ of detecting $k$ photons in mode (3) and (n-k) photons in mode (4) ?

***Solution :***

$$\begin{aligned}|\Psi\rangle &= \frac{1}{\sqrt{n!}}\,\hat{a}_1^{\dagger n}\,|0,0\rangle = \frac{1}{\sqrt{n!}}\,(r\hat{a}_3^\dagger + t\hat{a}_4^\dagger)^n\,|0,0\rangle \\ &= \frac{1}{\sqrt{n!}}\sum_{k=0}^{n}\binom{n}{k}\,(r\,\hat{a}_3^\dagger)^k\,(t\,\hat{a}_4^\dagger)^{n-k}|0,0\rangle \\ &= \sum_{k=0}^{n}\binom{n}{k}^{1/2} r^k\,t^{n-k}|k,n-k\rangle_{3,4}\end{aligned}$$

Where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. One thus have

$$P(k,n-k) = \left|\binom{n}{k}^{1/2} r^k\,t^{n-k}\right|^2 = \binom{n}{k} r^{2k}\,t^{2(n-k)} \tag{8}$$

This is a binomial distribution. It is the classical law of probability for the distribution of $n$ variables that can take only two values with probabilities $p = r^2$ and $1-p = t^2$ respectively. The $n$ photons state thus behaves as a collection of $n$ independent photons, each of them being individually reflected or transmitted.

As a simple illustration, consider the case of a pair of photons impinging at the same time on the same input of a 50/50 beamsplitter. According to the general result (8), we expect P(0,2) = P(2,0) = 1/4 and P(1,1) = 1/2. We can interpret this result assuming that we have two independent photons in channel 1. Each photon has a probability 1/2 to be reflected or transmitted. For independent scattering of photons, the possible outputs are : two photons in channel 3 ; two photons on channel 4 ; the first photon in 3 and the second in 4, and vice versa. This is 4 possibilities, hence the probabilities 1/4, 1/4, and $2 \times 1/4$ . This is in strong contrast with the behavior of two indistinguishable photons impinging simultaneously on the two input of a beamsplitter, which interfere in such a way that both end up in the same output, and one never has one photon in each output, as shown in question 1.5.

## 2 Coherent state transformation

We now consider the state $|\Psi\rangle = |\alpha\rangle_1 \otimes |0\rangle_2 = |\alpha,0\rangle_{1,2}$ where $|\alpha\rangle_1$ represents a coherent state in mode (1).

As seen in homework 2, the coherent state $|\alpha\rangle_1$ is defined by

$$|\alpha\rangle_1 = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle_1, \tag{9}$$

where $\alpha = |\alpha| e^{i\phi}$ is a complex number. We remind that the coherent state $|\alpha\rangle_1$ is an eigenvectors of the annihilation operator $\hat{a}_1$ with the eigenvalue $\alpha$. Reciprocally, we admit that any quantum state $|\psi\rangle_1$ such that $\hat{a}_1|\psi\rangle_1 = \alpha|\psi\rangle_1$ is the coherent state $|\alpha\rangle_1$. You can easily show this property by converting the eigenvalue equation into a recurrence relation on the coefficients of the eigenvector $|\Psi\rangle$. We have shown in homework 2 that the average photon number in state $|\alpha\rangle_1$ is $\bar{n}_1 = |\alpha_1|^2$.

1. By considering $\hat{a}_3|\Psi\rangle$ and $\hat{a}_4|\Psi\rangle$ show that $|\Psi\rangle = |\alpha'\rangle_3 \otimes |\alpha''\rangle_4 = |\alpha', \alpha''\rangle_{3,4}$ is the tensor product of coherent states in modes (3) and (4). Express $\alpha'$ and $\alpha''$ as a function of $\alpha$, $r$ and $t$.

   ***Solution :*** $\hat{a}_3|\Psi\rangle = (r\,\hat{a}_1 + t\,\hat{a}_2)|\alpha, 0\rangle_{1,2} = r\,\alpha|\alpha, 0\rangle_{1,2} = r\,\alpha\,|\Psi\rangle$. We have used $\hat{a}_1|\alpha\rangle_1 = \alpha|\alpha\rangle_1$ and $\hat{a}_2|0\rangle_2 = 0$.

   As a result, $|\Psi\rangle$ is an eigenvector of $\hat{a}_3$ with eigenvalue $r\,\alpha$. Using the property given above, it implies that $|\Psi\rangle$ is in a coherent state with respect to mode (3). In a similar way one has $\hat{a}_4|\Psi\rangle = (t\,\hat{a}_1 - r\,\hat{a}_2)|\alpha, 0\rangle_{1,2} = t\,\alpha\,|\Psi\rangle$ and $|\Psi\rangle$ is in a coherent state of complex amplitude $t\,\alpha$ with respect to mode (4).

   Finaly $|\Psi\rangle = |\alpha'\rangle_3 \otimes |\alpha''\rangle_4 = |\alpha', \alpha''\rangle_{3,4}$ with $\alpha' = r\,\alpha$ and $\alpha'' = t\,\alpha$.

2. What is the transmitted photon number in mode (3) and (4) ?

   ***Solution :*** One has $\bar{n}_3 = r^2\,\bar{n}_1$ and $\bar{n}_4 = t^2\bar{n}_1$ respectively. The coefficients $r^2$ and $t^2$ being the energy transmission and reflection coefficients of the beamsplitter, the result effectively corresponds to the expected result for an input classical field whose amplitude corresponds to $\bar{n}_1$ photons.

## Concluding remarks

In lesson 3, you were given a simple recipe describing one of the most important element of quantum optics : the beamsplitter. The method consists in calculating average values of observables defined in the output modes by first expressing these operators as a function of input mode operators. Then one takes this expression and calculates average values using the input state. This approach is usually much simpler as compared to the direct calculation of output average values by first expressing the field state in the output mode basis.

In this homework, you have seen that for some particular simple input states, the output state can anyway be easily expressed, leading interesting insight into the physics of a beamsplitter. Of course, these simple situations can as well be described using the formalism developed in the lesson, in which the input basis is preferred.

You may feel uncomfortable with this formalism, where a single state vector is expressed either in the input or output mode basis. In this formalism the radiation field is in a steady state and a one photon state is delocalized in the full quantisation volume. If you feel uncomfortable with this picture, there is a simple way to solve the problem by using the wave-packet approach introduced in lecture 2 (video 2.2). It consists in describing a real single photon state in mode $\ell$ as a propagating localized wavepacket of length $L$ and transverse section $S_l$.

In the framework of this wavepacket formalism, the unitary operator $U$ defined in the lesson 3 by $|\Psi\rangle_{3,4} = \hat{U}|\Psi\rangle_{1,2}$, can be interpreted as the unitary evolution operator describing the time evolution of an input wavepacket localized before the beamsplitter into output wavepackets scattered in modes (3) and (4) after some interaction time $t$. Explicit calculation of wavepacket propagation is heavy and you nearly never need to perform it. But it is useful to keep in mind the following picture : the relation $|\Psi\rangle_{3,4} = \hat{U}|\Psi\rangle_{3,4}$, which expresses a basis change for a single delocalized state vector can be as well interpreted as describing the evolution of an input wavepacket $|\Psi\rangle_{1,2}$ into an output wavepacket $|\Psi\rangle_{3,4}$ by a unitary evolution.