2.) Proove that the result is true for Backward error Analysis of Bordered LU factorization

$$A + \Delta A = \check{L}\check{U} \text{ with } |\Delta A| \le \gamma_n |\check{L}||\check{U}|$$

Base case:

Let $n = 1$. A and ΔA are both 1×1 and $\check{L}$ and $\check{U}$ are scalars. therefore $A = LU = a_{11}$, and $\Delta A = \Delta a_{11}$. Thus $(A + \Delta A)$ with $\check{L}$ and $\check{U} = a_{11}$ and $|\Delta A| \le \gamma_1 |a_{11}|^2$ therefore the base case is true

induction hypothesis:

Assume that the statement is true for all matrices of size $< n$

induction step:

consider an $n \times n$ matrix with factorization $A = LU$

Let ΔA be a matrix of the same size of A such that $|\Delta A| \le \gamma_n |\check{L}||\check{U}|$, where $\check{L}$ an $\check{U}$ are the factors of $A + \Delta A$

We want to show that the algorithm computes the correct factorization for $A + \Delta A$ with $|\Delta A| \le \gamma_n |\check{L}||\check{U}|$.

to do this expand the equation

$$\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) + \left(\begin{array}{c|c} \Delta A_{00} & \delta a_{01} \\ \hline \delta a_{10}^T & \delta \alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} \check{L}_{00} & 0 \\ \hline \check{l}_{10}^T & 1 \end{array}\right) \left(\begin{array}{c|c} \check{U}_{00} & \check{u}_{01} \\ \hline 0 & \upsilon_{11} \end{array}\right)$$

$$= \begin{pmatrix} A_{00} \Delta A_{00} & a_{01} \delta a_{01} \\ a_{10}^T \delta a_{10}^T & \alpha_{11} \delta \alpha_{11} \end{pmatrix} = \check{L} \check{U}$$

$\check{L}$ can be written as $\check{L} = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10} & 1 \end{array} \right)$

where L_{00} is a submatrix and L_{10} a column vector

$\check{U}$ can be written as $\check{U} = \left(\begin{array}{c|c} U_{00} & u_{01} \\ \hline 0 & \upsilon_{11} \end{array} \right)$

we can derive

$A_{00} = L_{00} U_{00}$

$a_{01} = A_{00} L_{10} + L_{00} u_{01}$

$a_{10}^T = U_{00} (a_{10}^T + \delta a_{10})$

$\alpha_{11} = u_{11}$

So we get

$$\begin{bmatrix} A_{00} + \Delta A_{00} + L_{00} U_{00} & A_{00} l_{10} + \delta a_{01} + L_{00} u_{01} \\ U_{00}(a_{10} + \delta a_{10}) & u_{11} (\alpha_{11} + \delta \alpha_{11}) \end{bmatrix}$$

since $\check{L}$ is lower triangular with 1's on the diagonal, L_{00} and U_{00} are lower and upper triangular matrices we can write

$$L_{00}U_{00} = A_{00} - \Delta A_{00} - L_{00}U_{01} - l_{10}\delta a_{10}$$

substituting into the expression

$$A_{00}A_{00}l_{10} + \delta a_{01} - L_{00}U_{01} - l_{10}\delta a_{10}U_{00}(a_{10} + \delta a_{10})u_{11}(\alpha_{11} + \Delta\alpha_{11}) =$$

$$\begin{pmatrix} A_{00} - \Delta A_{00} - L_{00}U_{01} - l_{10}\delta a_{10} & L_{00}U_{01} + L_{00}U_{00}l_{10} + l_{10}\delta\alpha_{11} \\ U_{00}L_{00} & U_{00}u_{11} \end{pmatrix}$$

from the equation we see

$$\Delta A = -L_{00}U_{01} - l_{10}\delta a_{10} + L_{00}U_{00}l_{10} + l_{10}\delta\alpha_{11}$$

we take the absolute value on both sides

$$|\Delta A| \leq |L_{00}U_{01}| + |l_{10}\delta a_{10}| + |L_{00}U_{00}l_{10}| + |l_{10}\delta\alpha_{11}|$$

we use the triangle inequality

$$|L_{00}U_{00}l_{10}| \leq |L_{00}||U_{00}||l_{10}|$$

L_{00} and U_{00} are lower and upper triangular

$|L_{00}| = 1 \quad |U_{00}| \leq \gamma_n$

γ_n is a bound on the magnitude of the elements of U_{00}

we also have $|L_{10}| \leq \gamma_n$ and $|U_{01}| \leq \gamma_n$

using the inequalities we can write

$$|\Delta A| \leq \gamma_n (|L_{00}||U_{01}| + |L_{10}||U_{00}| + |L_{00}||U_{00}||L_{10}| + |L_{10}||S_{211}|)$$

simplifying

$$|\Delta A| \leq \gamma_n |\check{L}||\check{U}|$$

therefore we have shown

$(A + \Delta A) = \check{L}\check{U}$ with $|\Delta A| \leq \gamma_n |\check{L}||\check{U}|$

for the bordered LU factorization

by Mathematical induction the ~~statement~~ holds