1)

a) Bordered cholesky algorithm

$$A = \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) \quad L = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)$$

substitute matrices into $A = LL^T$

$$\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)\left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)^T$$

$$= \left(\begin{array}{c|c} L_{00}L_{00}^T & a_{01} \\ \hline l_{10}^T L_{00}^T & l_{10}^T l_{10} + \lambda_{11}^2 \end{array}\right)$$

we conclude

$$\begin{array}{c|c} L_{00} = \text{Chol}(A)_{00} & a_{01} \\ \hline l_{10}^T = a_{10}^T L_{00}^{-T} & \lambda_{11} = \sqrt{\alpha_{11} - l_{10}^T l_{10}} \end{array}$$

which gives us the algorithm

1. partition $A \rightarrow \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right)$
2. assume $A_{00} := L_{00} = \text{chol}(A)$ has been computed by prev iterations.
3. overwrite $a_{10}^T := l_{10}^T = a_{10}^T L_{00}^{-T}$
4. overwrite $\alpha_{11} := \sqrt{\alpha_{11} - l_{10}^T l_{10}}$

b) $A = \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right)$ and $L = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline L_{10}^T & \lambda_{11} \end{array}\right)$

Proof

Base case: $n=1$.

A is 1×1 matrix and spd with a positive diagonal element α_{11}. Therefore the unique cholesky factorization of A is given by $\lambda_{11} = \sqrt{\alpha_{11}}$, λ_{11} being positive.

Inductive step:
Assume the theorem holds for an $n\times n$ matrix A and consider an $(n+1)\times(n+1)$ spd matrix A,

$$A = \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{01}^T & \alpha_{11} \end{array}\right)$$

Let L be the bordered cholesky factorization of A.

$$L = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)$$

we want to show L is well defined and unique.

Since A is SPD, we have $\alpha_{11} > 0$, and so $\lambda_{11} = \sqrt{\alpha_{11}}$ and column vector $l_{10} = a_{01}/\lambda_{11}$

we let L_{00} be the lower triangular matrix with positive diagonal entries so that $A_{00} - l_{10}l_{10}^T = L_{00}L_{00}^T$, which exist by lemma 5.4.4.2

we compute $LL^T = \begin{bmatrix} L_{00} & 0 \\ l_{10} & \lambda_{11} \end{bmatrix} \begin{bmatrix} L_{00} & l_{10}^T \\ 0 & \lambda_{11} \end{bmatrix}$

$$= \begin{bmatrix} L_{00}L_{00} + 0 & L_{00}l_{10}^T + 0 \\ l_{10}L_{00} + 0 & l_{10}l_{10}^T + \lambda_{11}^2 \end{bmatrix}$$

$$= \begin{bmatrix} A_{00} & a_{01}^T \\ a_{01} & \alpha_{11} \end{bmatrix}$$

thus we have found a lower triangular matrix L with positive diagonal entries such that $A = LL^T$

By induction the theorem holds.

- l_{10} is well defined because $l_{10} = 1/\lambda_{11} \times a_{01}$ and we defined $\lambda_{11} = \sqrt{\alpha_{11}}$. so it exists and is positive. it is also uniquely defined because λ_{11} is the only value that satisfies $\sqrt{\alpha_{11}}$
- α_{11} has to be positive