Week 3:
Probability and distributions

Phase 2

Key Ideas

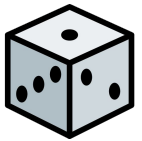
- Statistics is particularly useful when making inferences about data.

- We can never be 100% certain about our inferences. It's all about quantifying the certainty of an observation using **probability**.

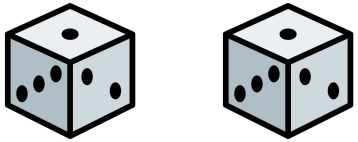
- In general, observations follow a **probability distribution**, i.e. they exhibit different probabilities.

Examples

Image you have one dice:



- Chance of getting 1 when rolling it? 1/6
- Chance of getting a 1 or 2 in the next roll? 2/6
- Chance of getting either 1, 2, 3, 4, 5, or 6 on the next roll? 1 (100%)
- Chance of not rolling a 2? 5/6



Now you have two dice:

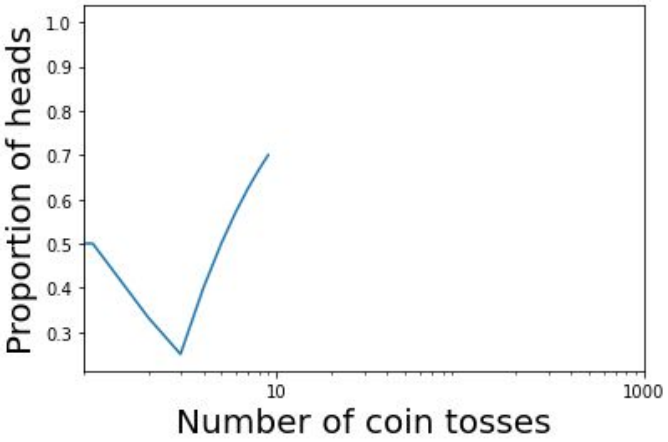
- What is the chance of getting two 1s? 1/6 x 1/6 = 1/36

Probability: a formal definition

The probability of an outcome is the **proportion** of times the outcome would occur if we observed the **random** process an **infinite** number of times.

Probability

Example: Probability of heads after N coin tosses…



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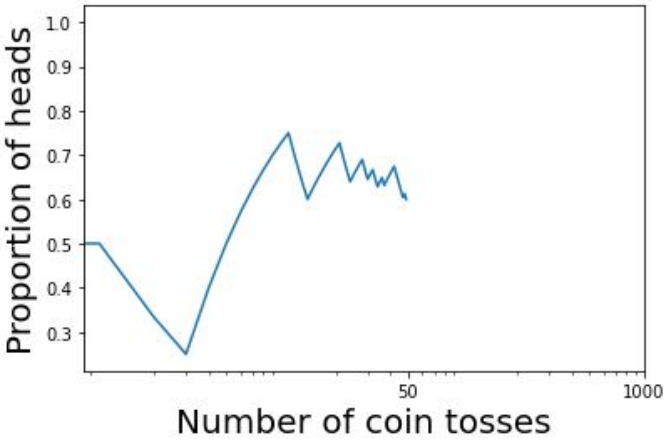

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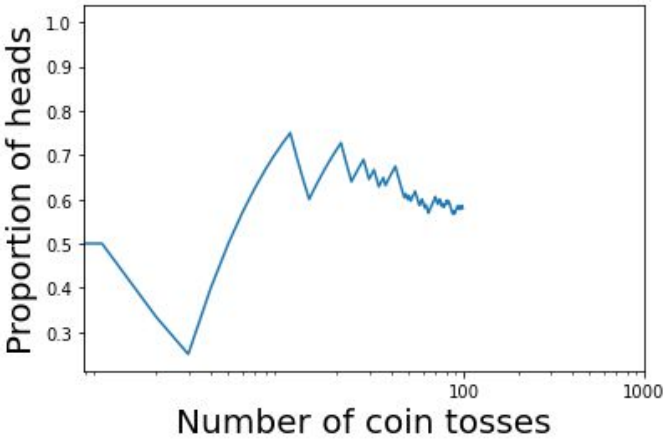

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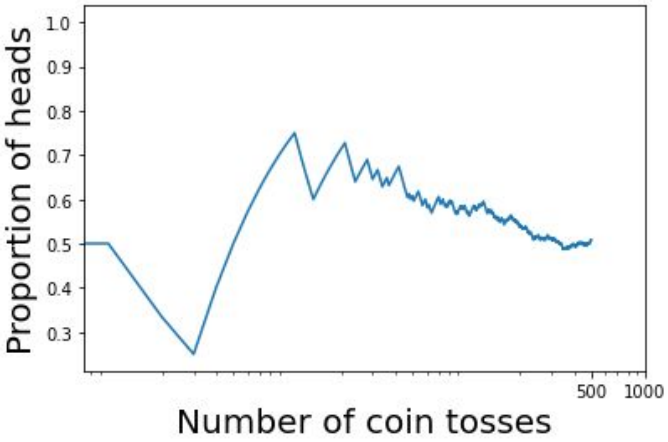

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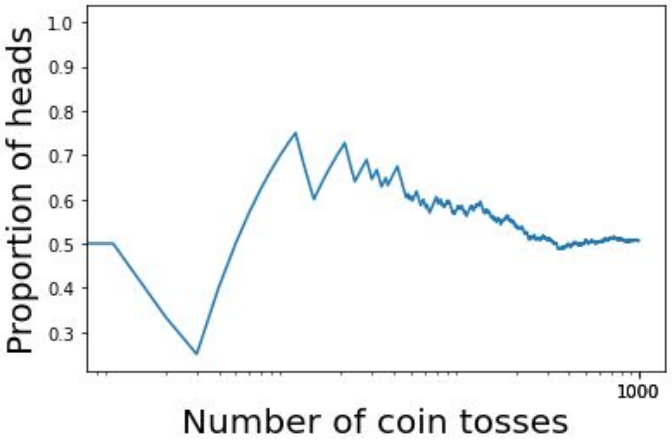
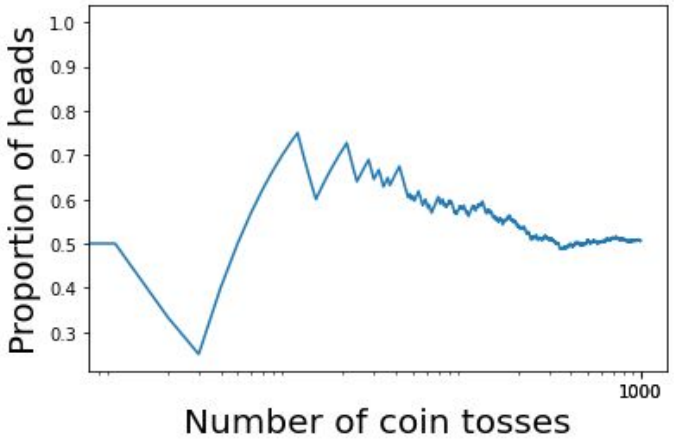


Probability

Example: Probability of heads after N coin tosses… After a sufficiently **large** number of times, the probability **converges** to the real value (here 0.5).



Probability models

A random phenomenon can be mathematically represented in a **probability model**, which has the following ingredients:

- Sample space: The list of all possible outcomes in a random phenomenon.

  e.g. : {Blue jeans, green jeans, grey jeans, black suit, blue track}

- Event: An outcome or a set of outcomes of a random phenomenon.

  e.g. {Blue jeans}

- Each event gets assigned a probability.

Probability distributions: Sample space

The sample space *S* is the set of **all possible outcomes** of a random phenomenon.

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{1, 2, 3, 4, 5, 6}

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Question: what is the sample space of tossing 2 coins?

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Question: what is the sample space of tossing 2 coins?

{HH, HT, TH, TT}

Probability vs Statistics

Question: How different are Probability and Statistics?

Probability vs Statistics

- What are the chances of a fair coin coming up heads 10 times in a row?
- If I roll two six sided dice, how likely is it that I'll roll two sixes?
- How likely is it that five cards drawn from a perfectly shuffled deck will all be hearts?
- What are the chances that I'll win the lottery?
- If my friend flips a coin 10 times and gets 10 heads, are they playing a trick on me?
- If five cards off the top of the deck are all hearts, how likely is it that the deck was shuffled?
- If the lottery commissioner's spouse wins the lottery, how likely is it that the lottery was rigged?

Probability vs Statistics

- Probabilistic questions assume a **known** model of the world (e.g. P(heads) = 0.5)
- We use this model to perform calculations.
- Here, the model is known, but the data are not.

- We do not know the truth about the world.
- We only have the data.
- We want to use the data to learn (infer) the truth about the world.

Probability vs Statistics

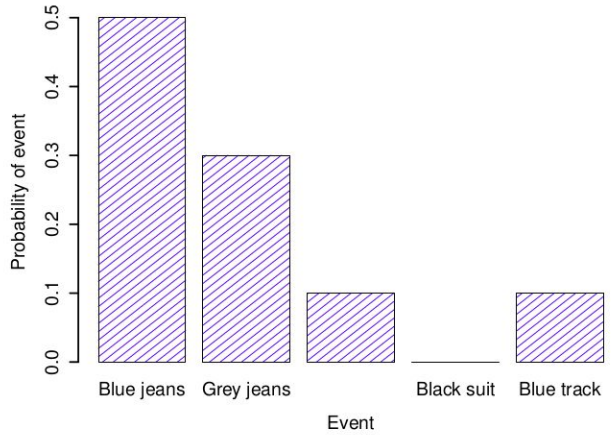
1. The world generates our data.
2. Statistical models and data generate probabilities.
3. With probabilities, we can make predictions about the world.

Probability vs Statistics

Statistical inference is to figure out **which** probability models are right. Although not the same, both are deeply **connected** to one another.

Probability distributions

- A **probability distribution** P(X) is a mathematical function that provides the probabilities of an event X from all different possible **outcomes (the sample space).**



Which pants?	Label	Probability
Blue jeans	X_1	$P(X_1) = .5$
Grey jeans	X_2	$P(X_2) = .3$
Black jeans	X_3	$P(X_3) = .1$
Black suit	X_4	$P(X_4) = 0$
Blue tracksuit	X_5	$P(X_5) = .1$

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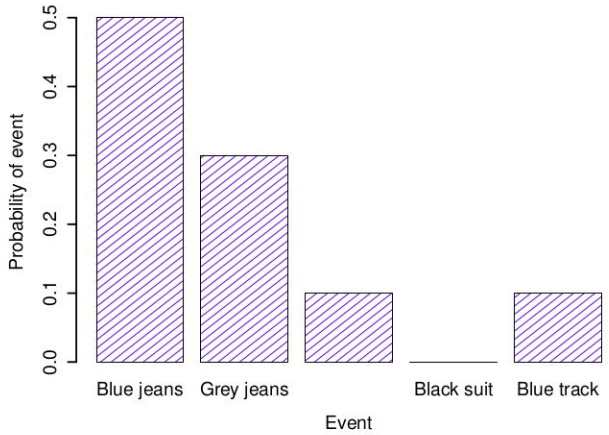

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Form of a density function → Their probabilities must sum 1.

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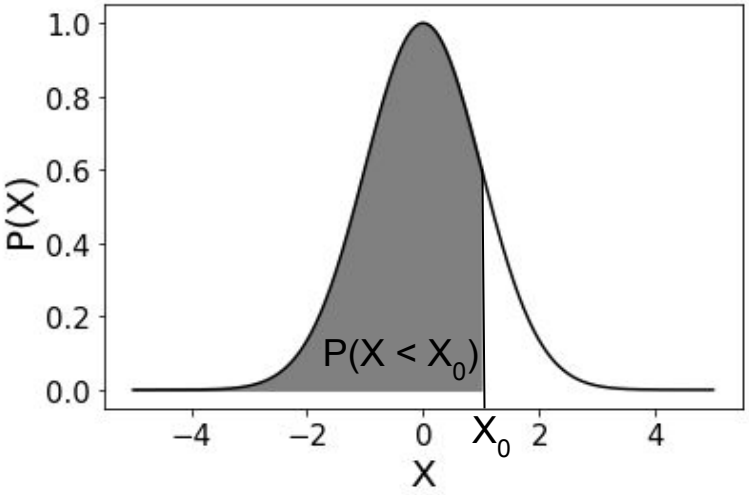
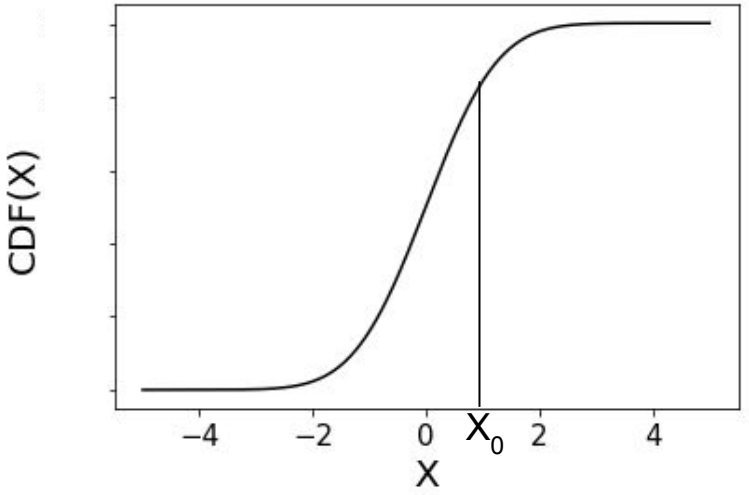
- As any function, it will in general depend on some underlying parameters θ_i
- Keep this in mind for the future: Probability distributions can be **discrete**, if the outcomes take only a set of finite values, or **continuous**, if it takes an infinite set of outcomes.

Probability distributions: some derived functions

➢ **Cumulative distribution function CDF(x)**:

It tells you the **probability** of obtaining an outcome smaller than or equal to X_0, that is, $P(X < X_0)$.

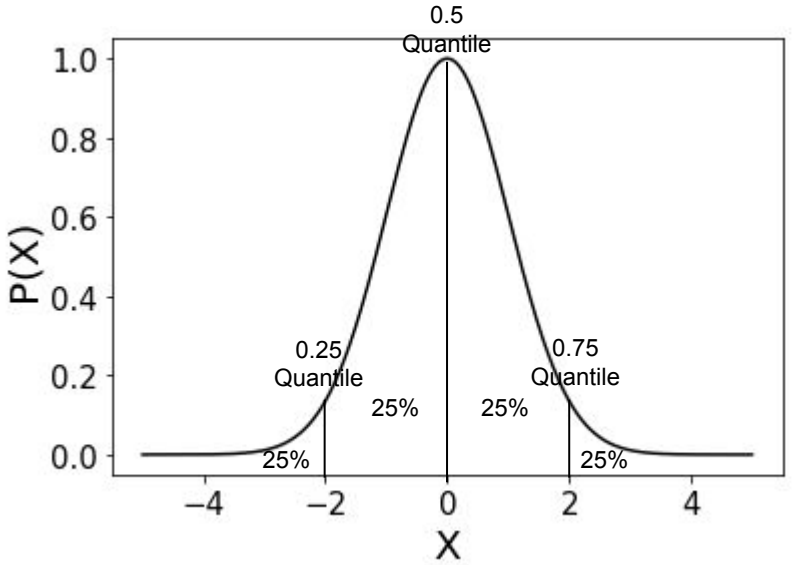




Probability distributions: some derived functions

➢ **Quantile function Q(p)**:

It's the inverse of the CDF, and tells you the particular X_0 for which the probability is less than or equal to a given value *p*. It is often given in terms of percentages (e.g. 25%, 75%, etc)



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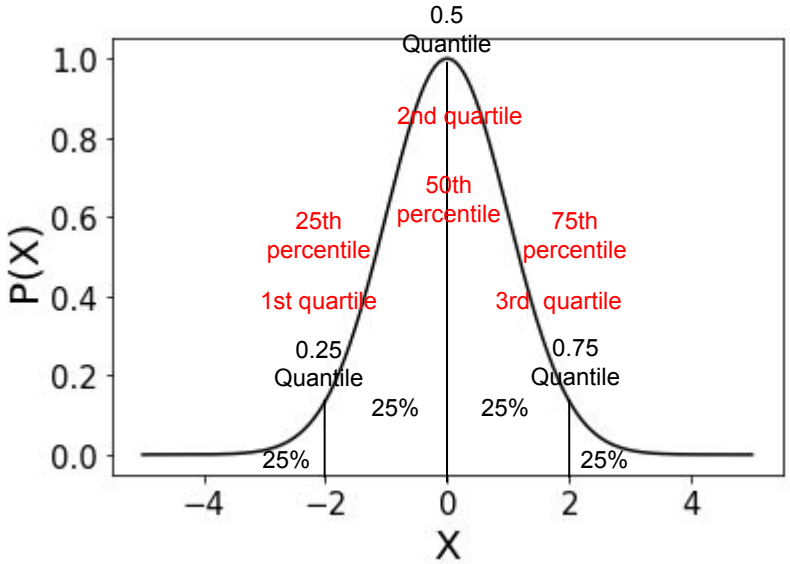


Binomial distribution

- It's a **discrete** distribution, that models the probability that **positive** events occur in a given sample of **repeated** independent experiments.

  Example:

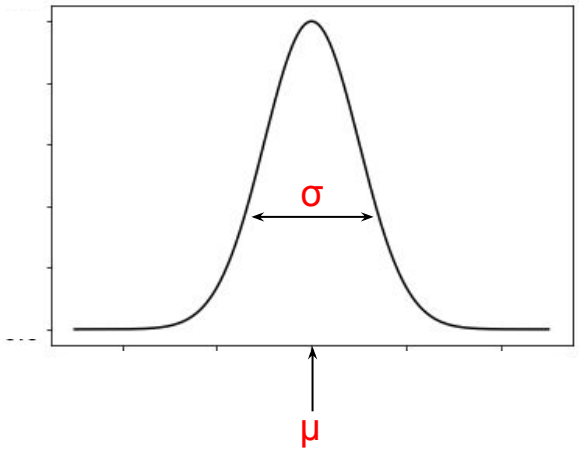
  - If I toss a coin 5 times, what's the probability of getting 3 heads?
  - If I roll a dice 10 times, what's the probability of getting 4 fives?

- Therefore, it will depend on the number of independent occurrences *N*, and the probability of the positive occurrence *p.*

$$\theta_i = \{p, N\}$$

Gaussian (or Normal) distribution

- It's a **continuous** distribution and probably the most important one in statistics. (Why? We'll see this in the future…)
- A gaussian distribution is described by two parameters: the mean of the distribution μ, and the standard deviation of the distribution σ. $\theta_i = \{\mu , \sigma\}$



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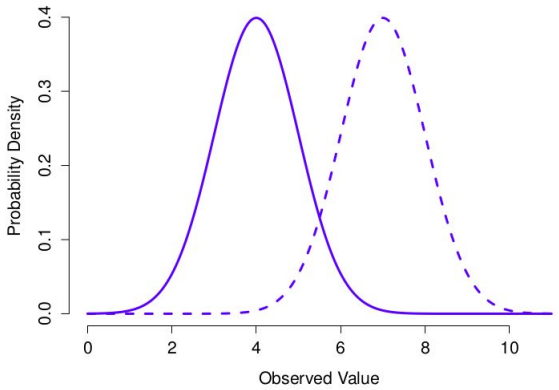

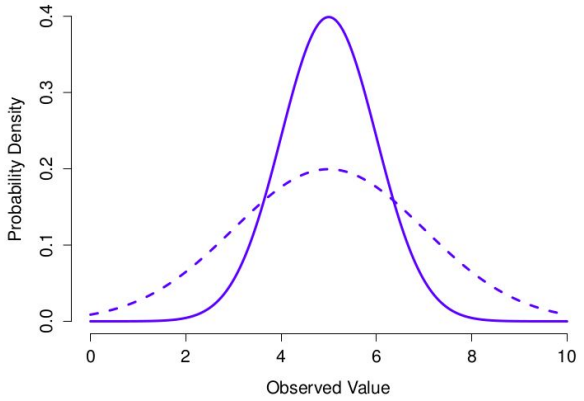

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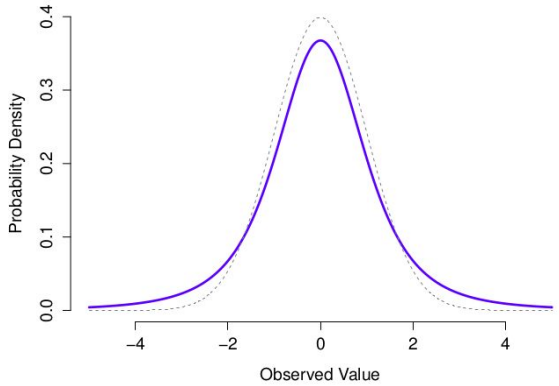
- The special case μ=0 and σ=1 has its own name that you'll often see: **The Standard Normal distribution.**

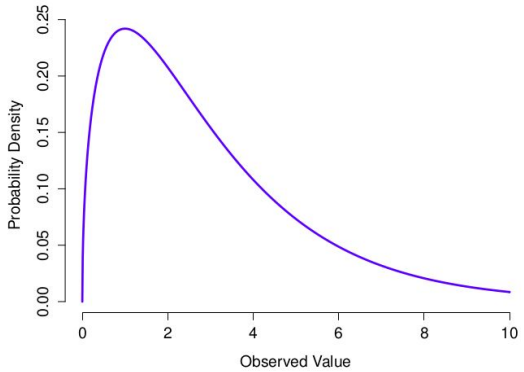
Other types: Student's *t*-distribution

- The Student's t-distribution is like a normal distribution but with **heavier tails**. θ_i = {df} (df stands for degrees of freedom).
- It is a very important distribution in statistics, particularly for things like assessing differences between the **means of two samples** and constructing **confidence intervals** (we'll see this in the future).



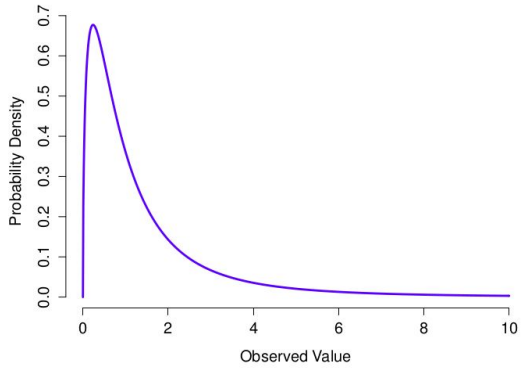
Other types: χ^2 distribution

- It is the distribution of the **sum of squares** (keep this in mind for the next slide) of independent variables **normally** distributed $\theta_i = \{df\}$.
- It is a widely used distribution in statistics, e.g. testing differences in **proportions** between groups, or **goodness fit** of the data.



Other types: F-distribution

- It is related to the χ^2 distribution; specifically as the **ratio** between two χ^2 statistics. $\theta_i = \{df_1, df_2\}$
- It usually arises as the ratio between variances (aha, here you have the “sum of squares” that I mentioned before). This ratio is common in testing **mean differences** across groups (ANOVA test).



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- Statistics develops on these probability models to quantify how **likely** our world is given a **certain testable condition** (This is basically the famous **p-value**!!!)
- There exists **many** probability distributions, each suitable for **specific situations/problems**, so you'll need to choose wisely.