Week 3:
Probability and distributions

Phase 1

Key Ideas

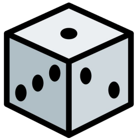
- Statistics is particularly useful when making inferences about data.

- We can never be 100% certain about our inferences. It’s all about quantifying the certainty of an observation using **probability**.

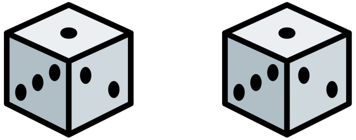
- In general, observations follow a **probability distribution**, i.e. they exhibit different probabilities.

Examples

Image you have one dice:



- ➢ Chance of getting 1 when rolling it? 1/6
- ➢ Chance of getting a 1 or 2 in the next roll? 2/6
- ➢ Chance of getting either 1, 2, 3, 4, 5, or 6 on the next roll? 1 (100%)
- ➢ Chance of not rolling a 2? 5/6



Now you have two dice:

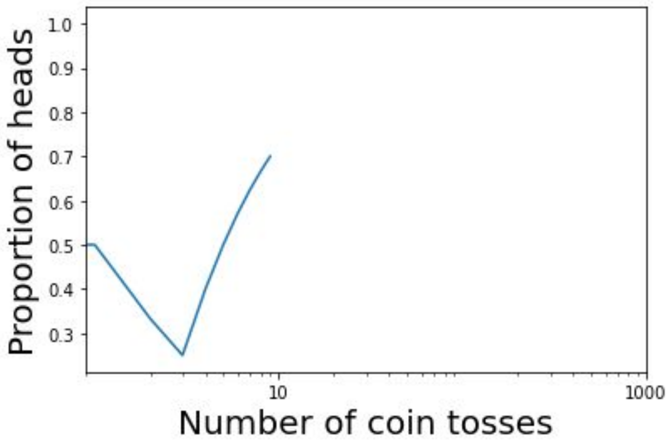
- ➢ What is the chance of getting two 1s? 1/6 x 1/6 = 1/36

Probability: a formal definition

The probability of an outcome is the **proportion** of times the outcome would occur if we observed the **random** process an **infinite** number of times.

Probability

<u>Example</u>**:** Probability of heads after N coin tosses…



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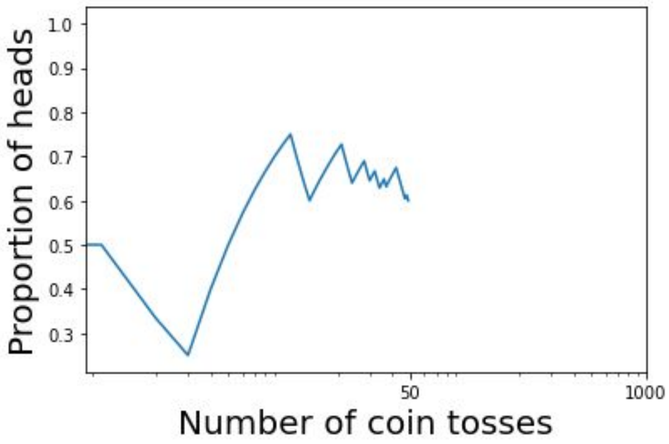

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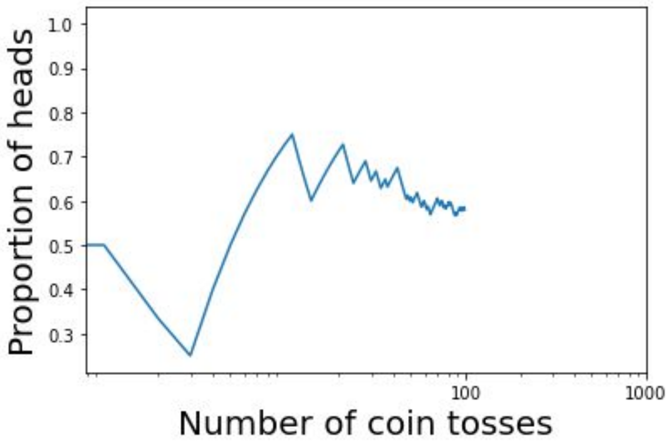

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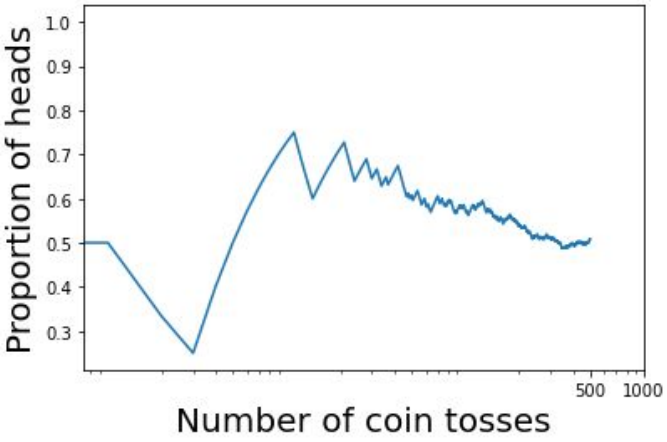

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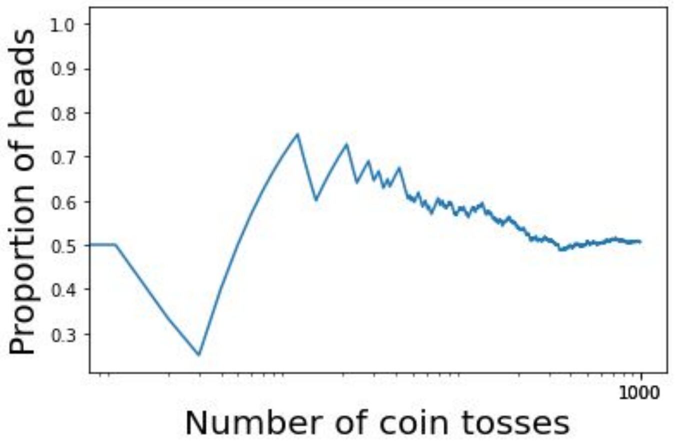
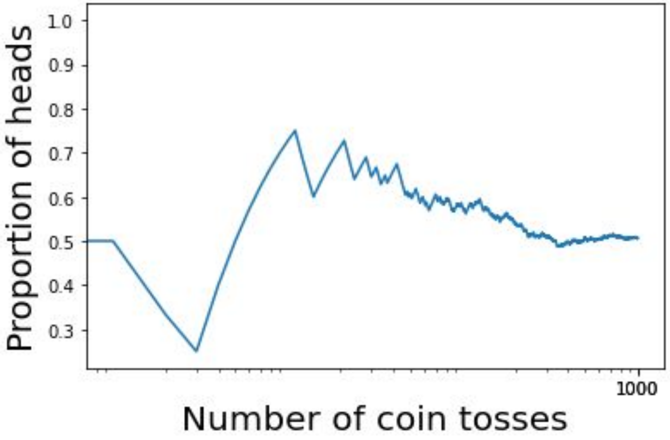


Probability

<u>Example</u>: Probability of heads after N coin tosses… After a sufficiently **large** number of times, the probability **converges** to the real value (here 0.5).



Probability models

A random phenomenon can be mathematically represented in a **probability model**, which has the following ingredients:

➢ Sample space: The list of all possible outcomes in a random phenomenon.

e.g. : {Blue jeans, green jeans, grey jeans, black suit, blue track}

➢ Event: An outcome or a set of outcomes of a random phenomenon.

e.g. {Blue jeans}

➢ Each event gets assigned a probability.

Probability distributions: Sample space

The sample space S is the set of **all possible outcomes** of a random phenomenon.

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$\{1, 2, 3, 4, 5, 6\}$

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Question: what is the sample space of tossing 2 coins?

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Question: what is the sample space of tossing 2 coins?

{HH, HT, TH, TT}

Probability vs Statistics

Question: How different are Probability and Statistics?

Probability vs Statistics

- What are the chances of a fair coin coming up heads 10 times in a row?
- If I roll two six sided dice, how likely is it that I'll roll two sixes?
- How likely is it that five cards drawn from a perfectly shuffled deck will all be hearts?
- What are the chances that I'll win the lottery?

- If my friend flips a coin 10 times and gets 10 heads, are they playing a trick on me?
- If five cards off the top of the deck are all hearts, how likely is it that the deck was shuffled?
- If the lottery commissioner's spouse wins the lottery, how likely is it that the lottery was rigged?

Probability vs Statistics

- Probabilistic questions assume a **known** model of the world (e.g. P(heads) = 0.5)
- We use this model to perform calculations.
- Here, the model is known, but the data are not.

- We do not know the truth about the world.
- We only have the data.
- We want to use the data to learn (infer) the truth about the world.

Probability vs Statistics

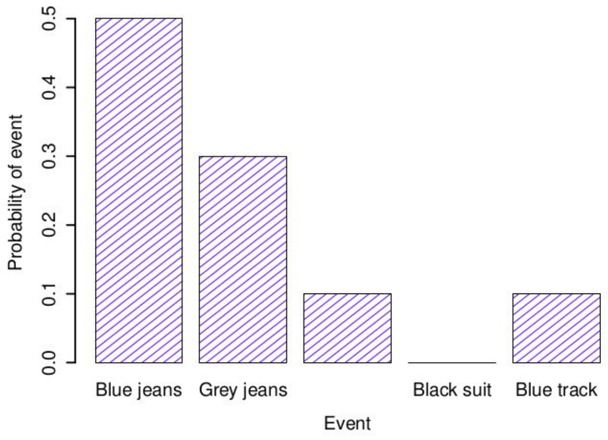
1. The world generates our <u>data</u>.
2. Statistical <u>models</u> and data generate probabilities.
3. With probabilities, we can make <u>predictions</u> about the world.

Probability vs Statistics

Statistical inference is to figure out **which** probability models are right.
Although not the same, both are deeply **connected** to one another.

Probability distributions

➢ A **probability distribution** P(X) is a mathematical function that provides the probabilities of an event X from all different possible **outcomes (the sample space).**



Which pants?	Label	Probability
Blue jeans	X_1	$P(X_1) = .5$
Grey jeans	X_2	$P(X_2) = .3$
Black jeans	X_3	$P(X_3) = .1$
Black suit	X_4	$P(X_4) = 0$
Blue tracksuit	X_5	$P(X_5) = .1$

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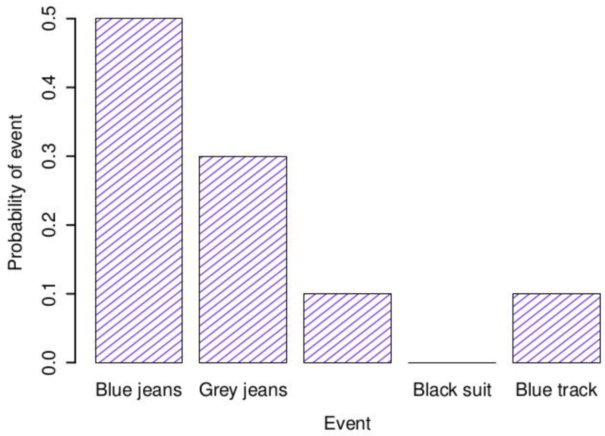

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Form of a density function ⟶ Their probabilities must sum 1.

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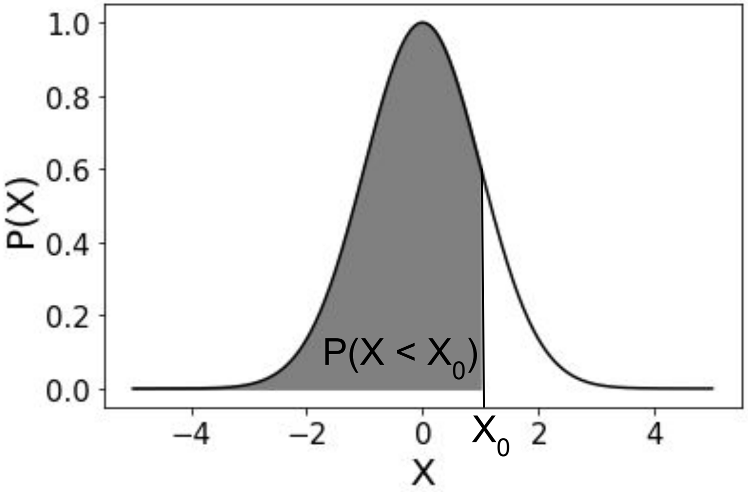
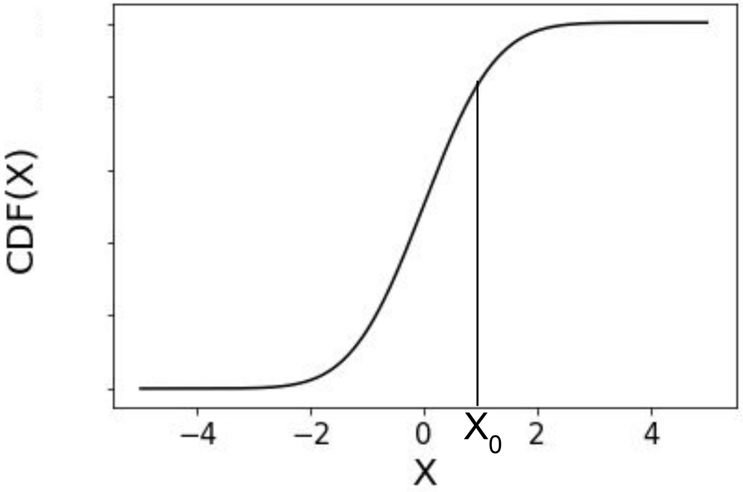
- As any function, it will in general depend on some underlying parameters θ_i
- <u>Keep this in mind for the future</u>: Probability distributions can be **discrete**, if the outcomes take only a set of finite values, or **continuous**, if it takes an infinite set of outcomes.

Probability distributions: some derived functions

➢ **Cumulative distribution function CDF(x)**:

It tells you the **probability** of obtaining an outcome smaller than or equal to X_0, that is, $P(X < X_0)$.

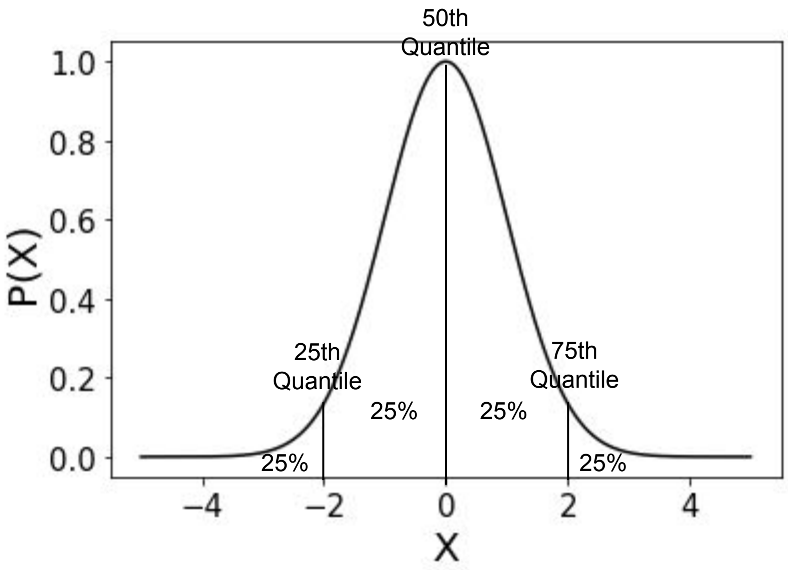




Probability distributions: some derived functions

➢ **Quantile function Q(p)**:

It's the inverse of the CDF, and tells you the particular X_0 for which the probability is less than or equal to a given value *p*. It is often given in terms of percentages (e.g. 25%, 75%, etc)



Binomial distribution

- It's a **discrete** distribution, that models the probability that **positive** events occur in a given sample of **repeated** independent experiments.

  Example:

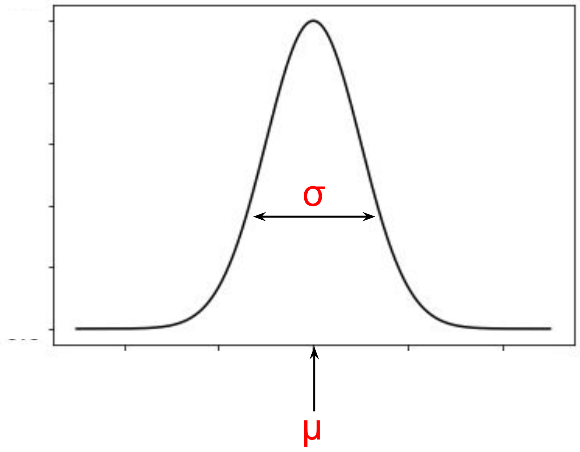
  - If I toss a coin 5 times, what's the probability of getting 3 heads?
  - If I roll a dice 10 times, what's the probability of getting 4 fives?

- Therefore, it will depend on the number of independent occurrences *N*, and the probability of the positive occurrence *p.*

$$\theta_i = \{p, N\}$$

Gaussian distribution

- It's a **continuous** distribution and probably the most important one in statistics. (Why? We'll see this in the future…)
- A gaussian distribution is described by two parameters: the mean of the distribution μ, and the standard deviation of the distribution σ. $\theta_i = \{\mu , \sigma\}$



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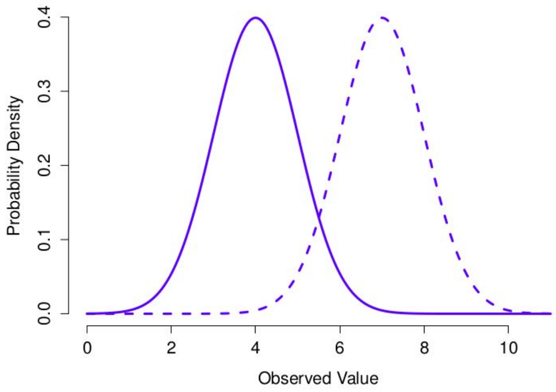

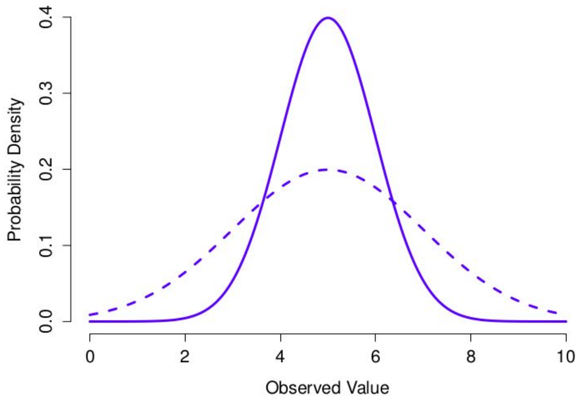

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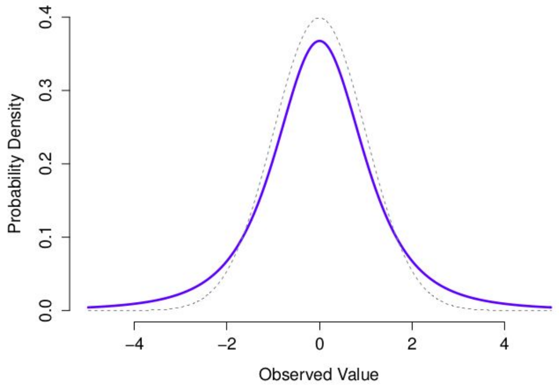
- The special case μ=0 and σ=1 has its own name that you'll often see: **The normal distribution.**

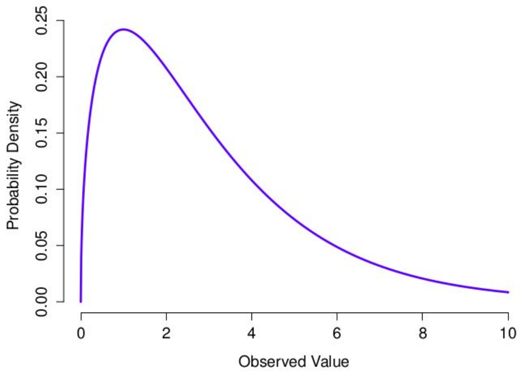
Other types: Student's *t*-distribution

- The Student's t-distribution is like a normal distribution but with **heavier tails**. θ_i = {df} (df stands for degrees of freedom).
- It is a very important distribution in statistics, particularly for things like assessing differences between the **means of two samples** and constructing **confidence intervals** (we'll see this in the future).



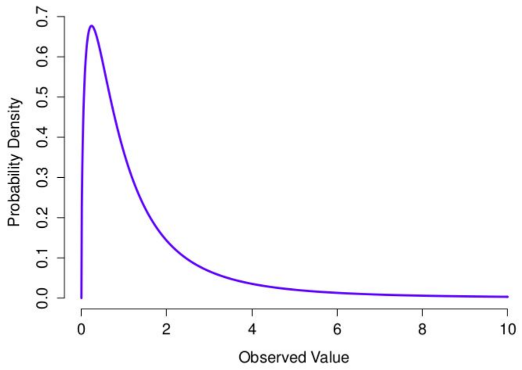
Other types: χ^2 distribution

- It is the distribution of the **sum of squares** (keep this in mind for the next slide) of independent variables **normally** distributed $\theta_i = \{df\}$.
- It is a widely used distribution in statistics, e.g. testing differences in **proportions** between groups, or **goodness fit** of the data.



Other types: F-distribution

- It is related to the χ^2 distribution; specifically as the **ratio** between two χ^2 statistics. $\theta_i = \{df_1, df_2\}$
- It usually arises as the ratio between variances (aha, here you have the “sum of squares” that I mentioned before). This ratio is common in testing **mean differences** across groups (ANOVA test).



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- We can use probability models to quantify the **certainty** of our observations.
- Statistics develops on these probability models to quantify how **likely** our world is given a **certain testable condition** (This is basically the famous **p-value**!!!)
- There exists **many** probability distributions, each suitable for **specific situations/problems**, so you'll need to choose wisely.