Point, Vectors, and Lines

CS 355: Interactive Graphics and Image Processing

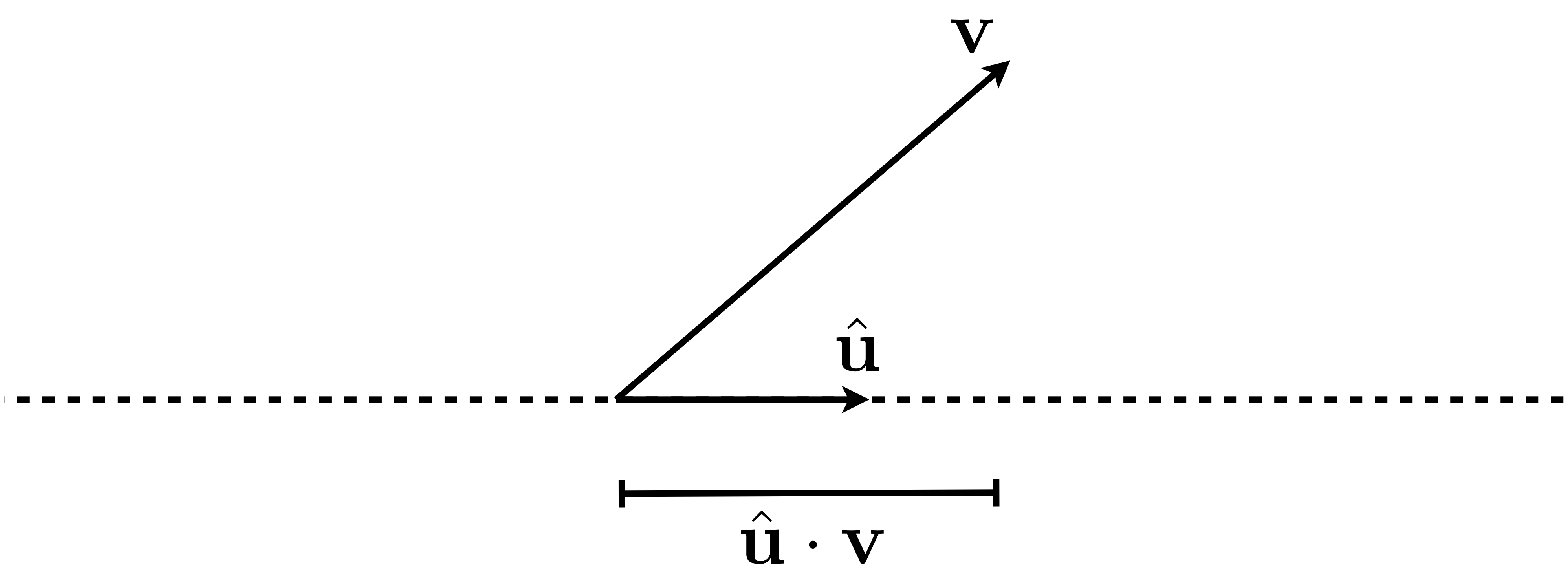
Continued

Vector Dot Products

$$\mathbf{a} \cdot \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^{n} a_i b_i$$

One of the most commonly used vector operations in graphics

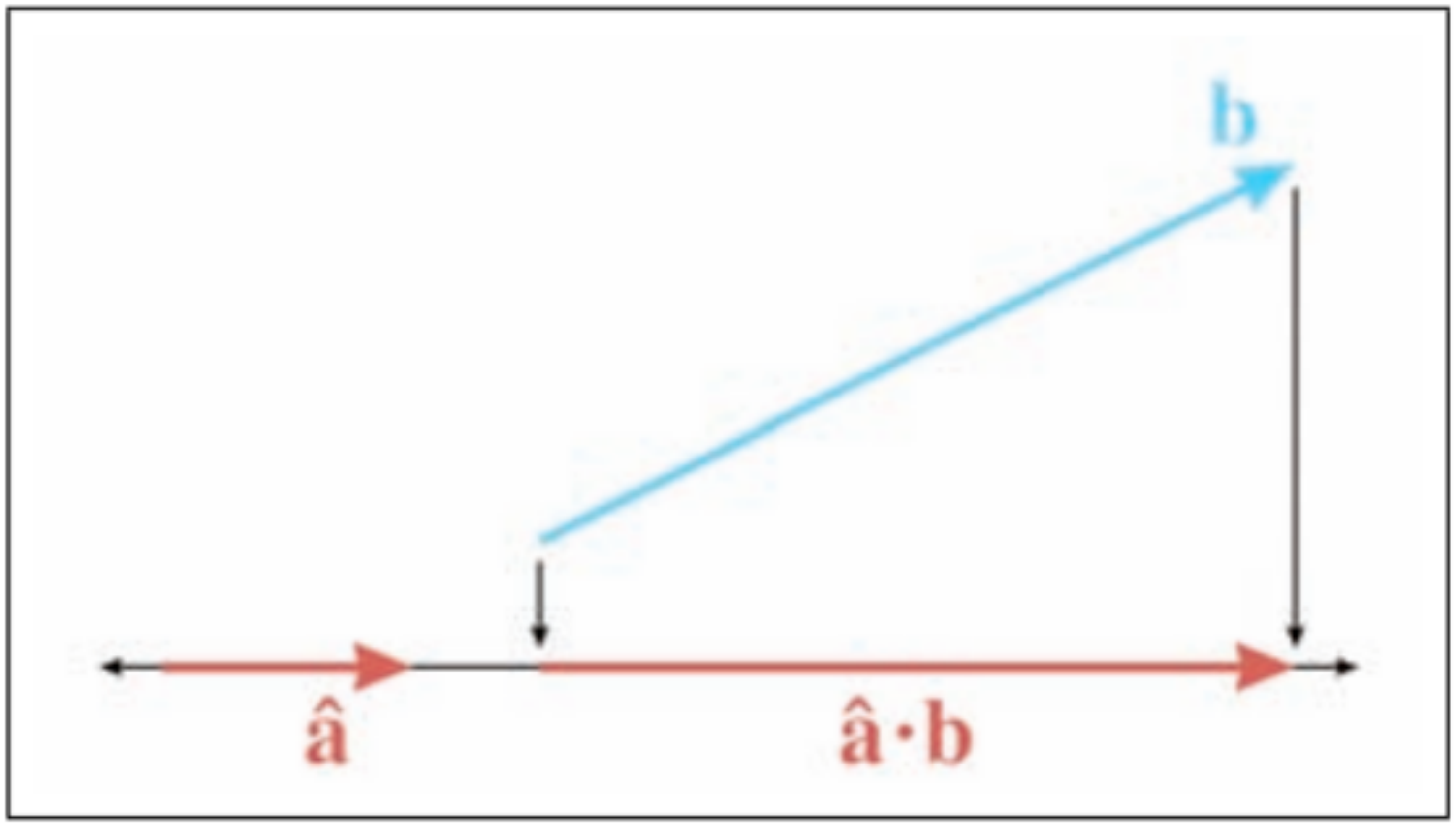
Geometric Interpretation



The dot product of a vector and a unit vector
is the length of the *projection* onto that unit vector

"How much of this vector lies in that direction?"

Geometric Interpretation



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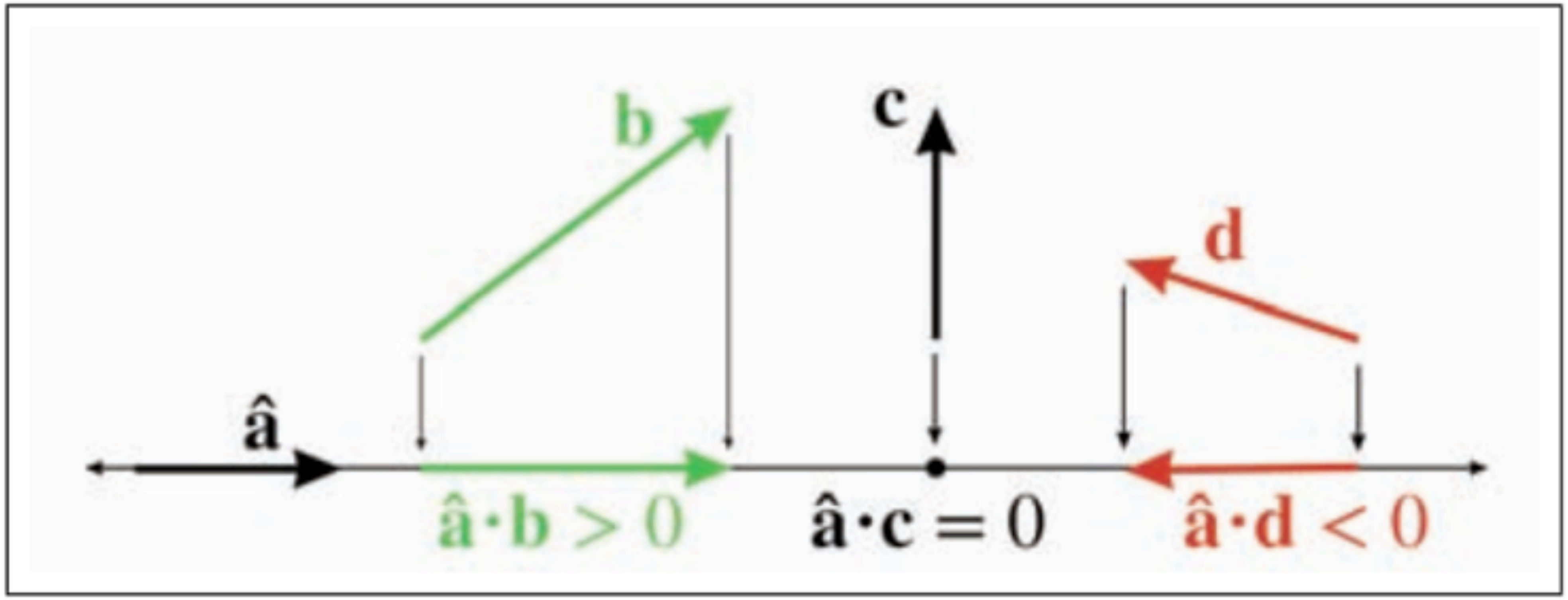
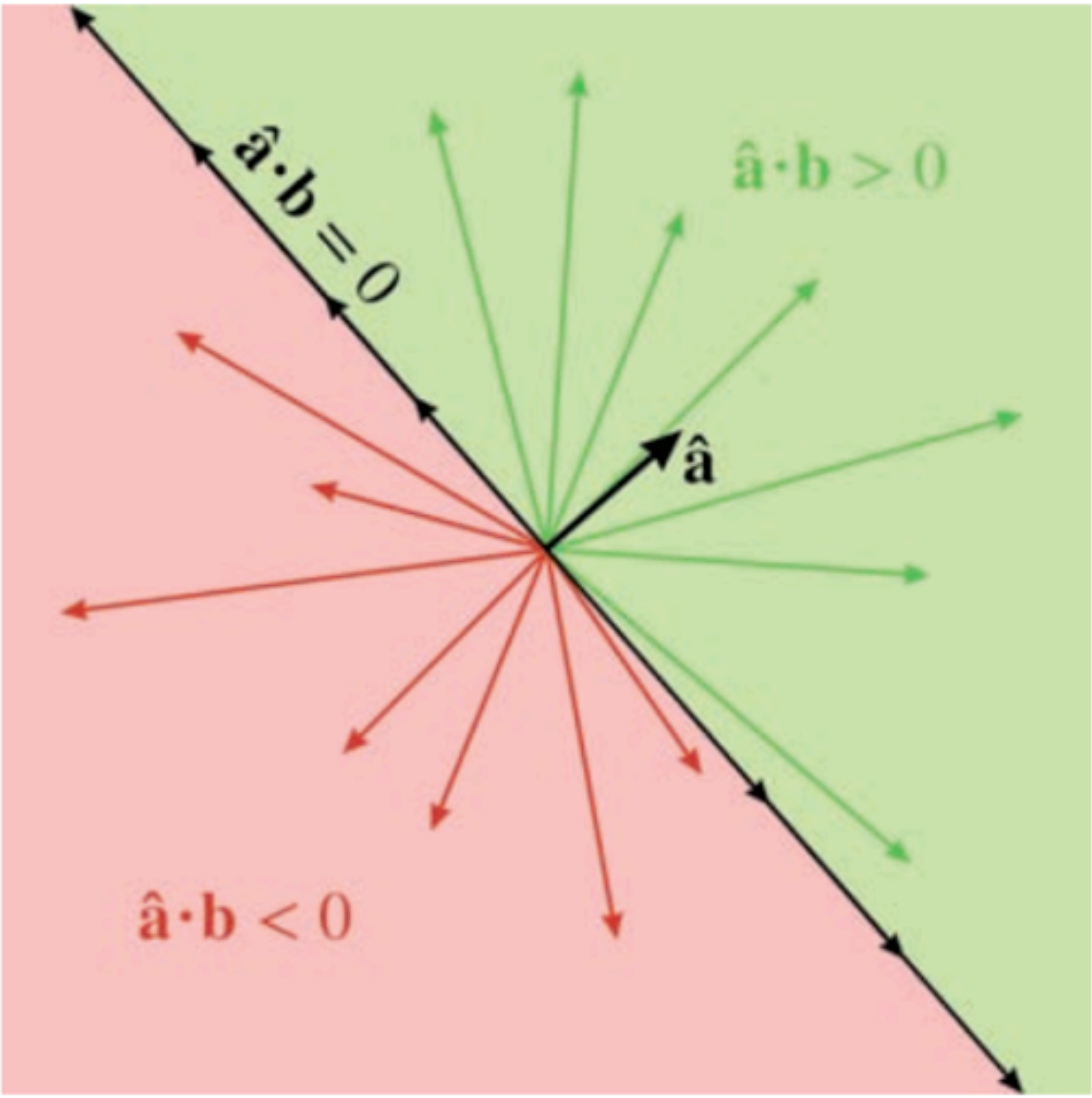
Sign of the Dot Product



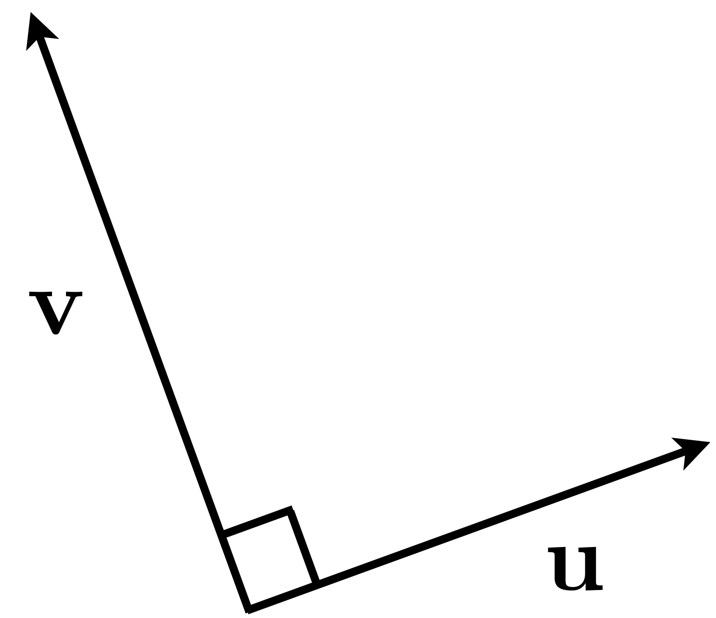
The sign of the dot product between two vectors tells whether the projection is in the same direction

Sign of the Dot Product



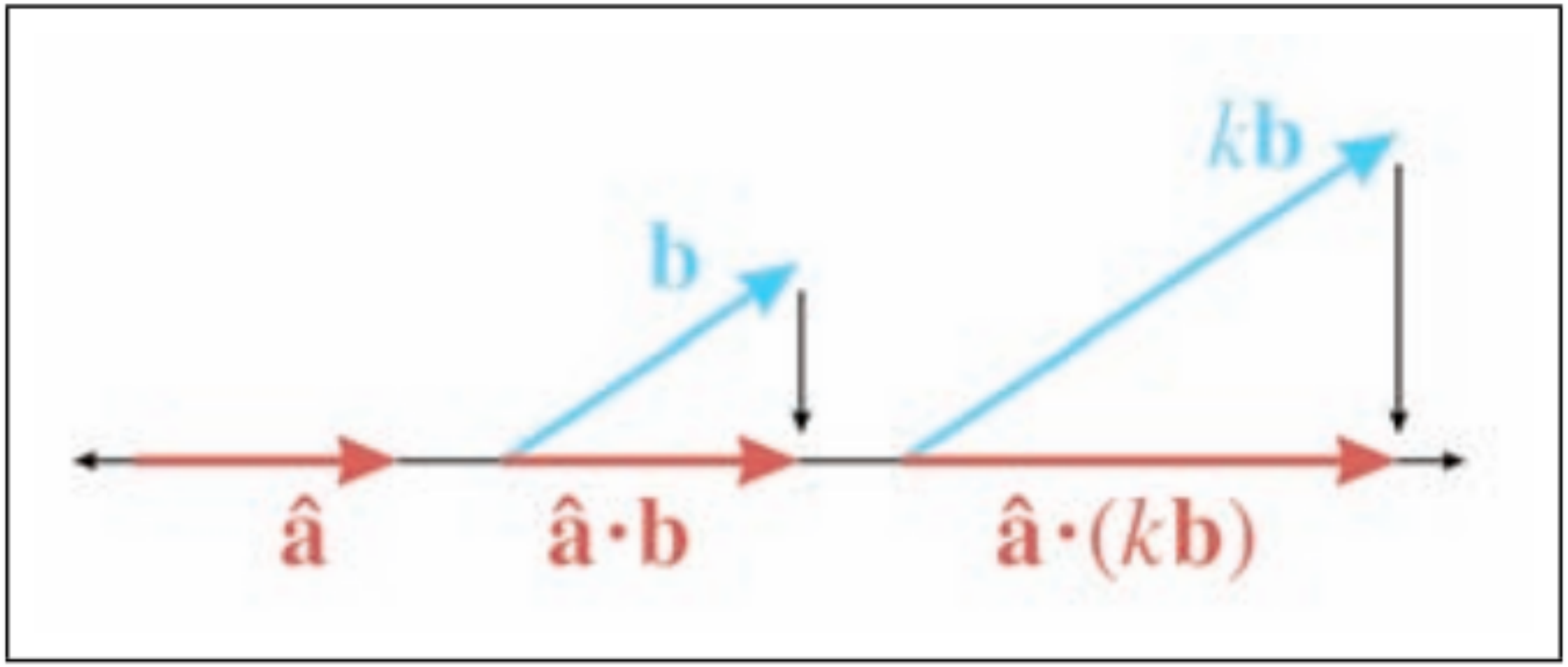
Orthogonality

- Vectors whose dot product is zero are said to be "orthogonal"
- "Right angle" to each other (regardless of length)



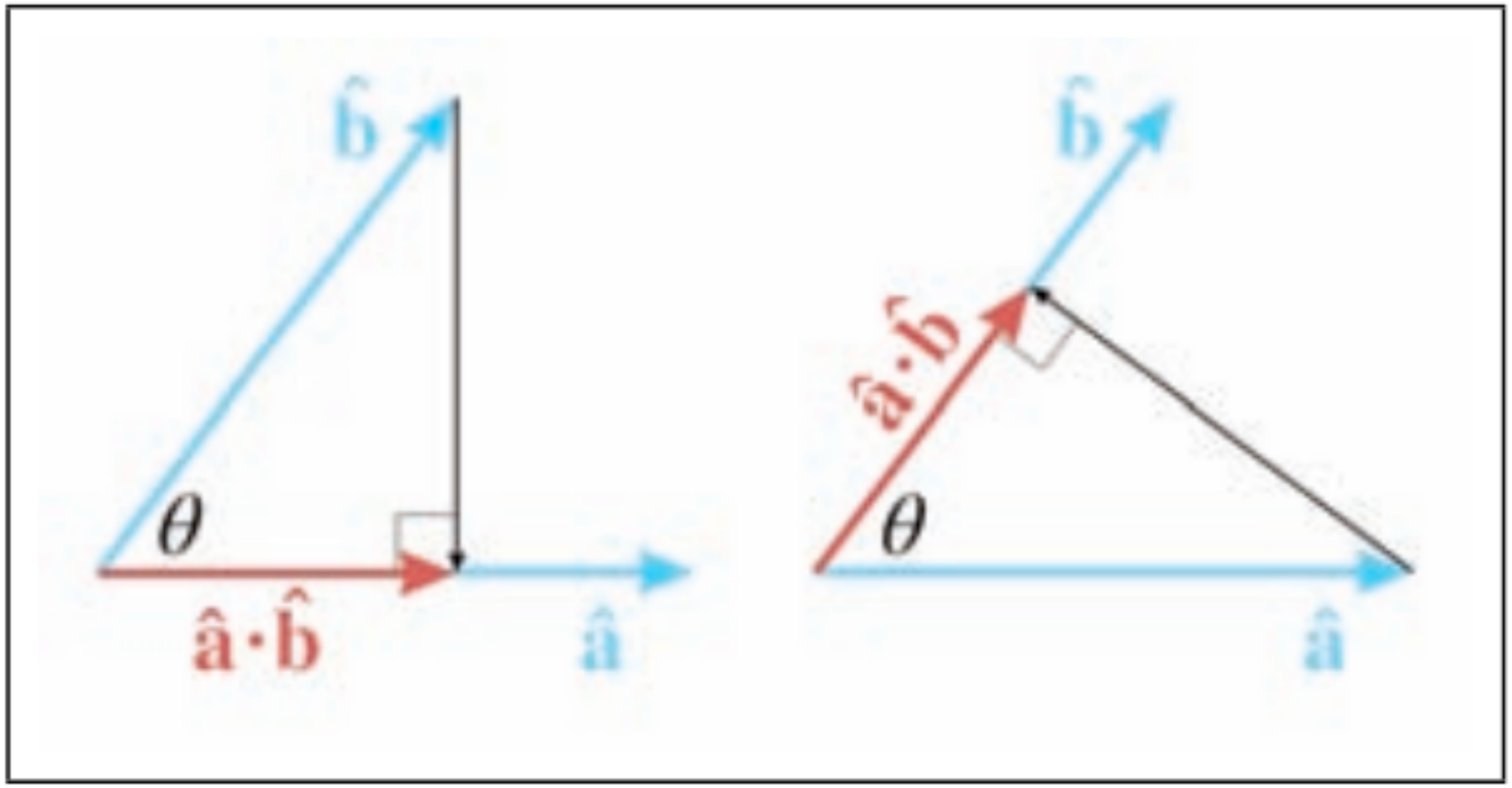
$$\mathbf{u} \cdot \mathbf{v} = 0$$

Scalar Multiplication



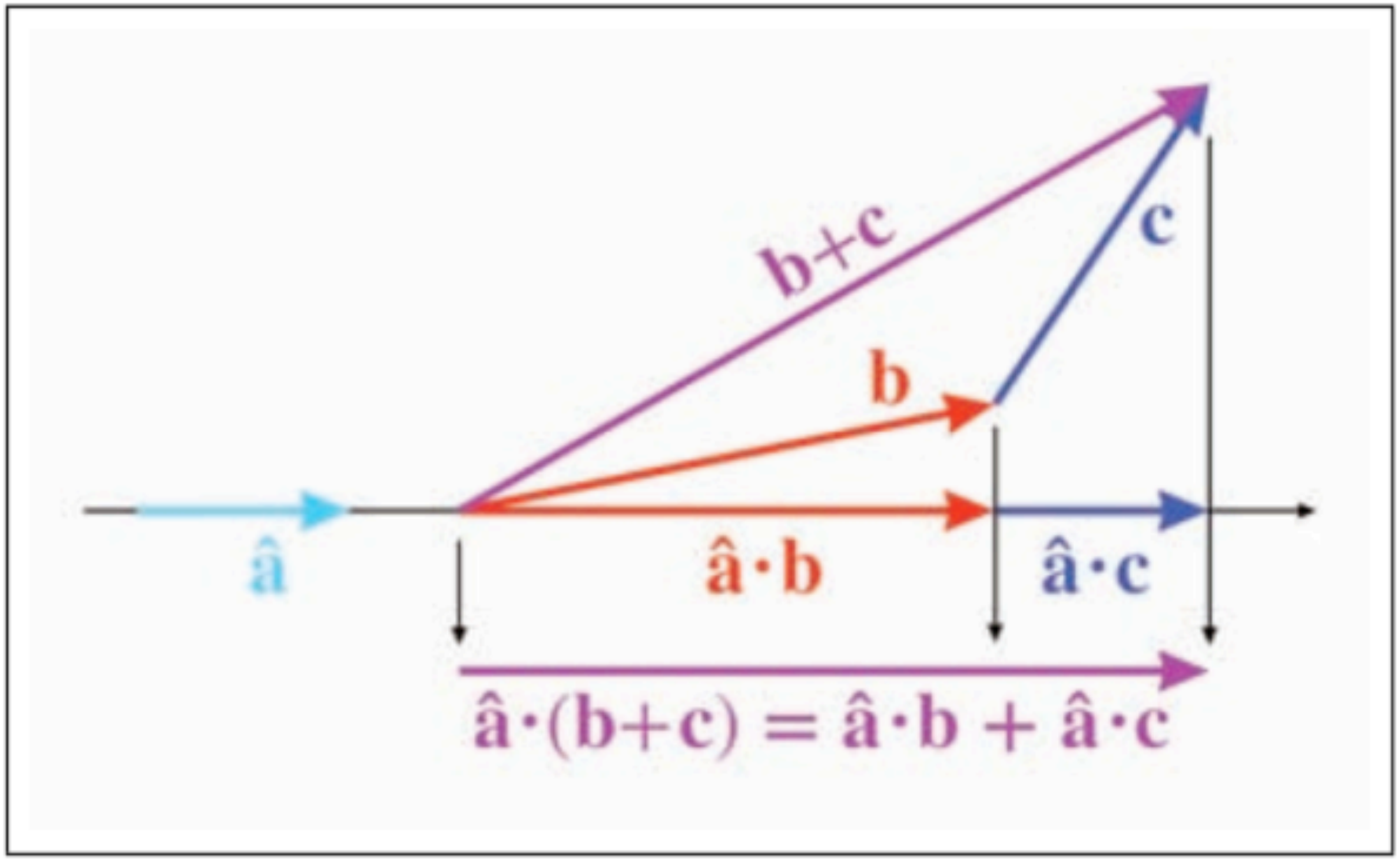
$$\mathbf{a}\cdot(k\mathbf{b}) = k(\mathbf{a}\cdot\mathbf{b}) = (k\mathbf{a})\cdot\mathbf{b}$$

Commutative



$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

Distributes Over Addition

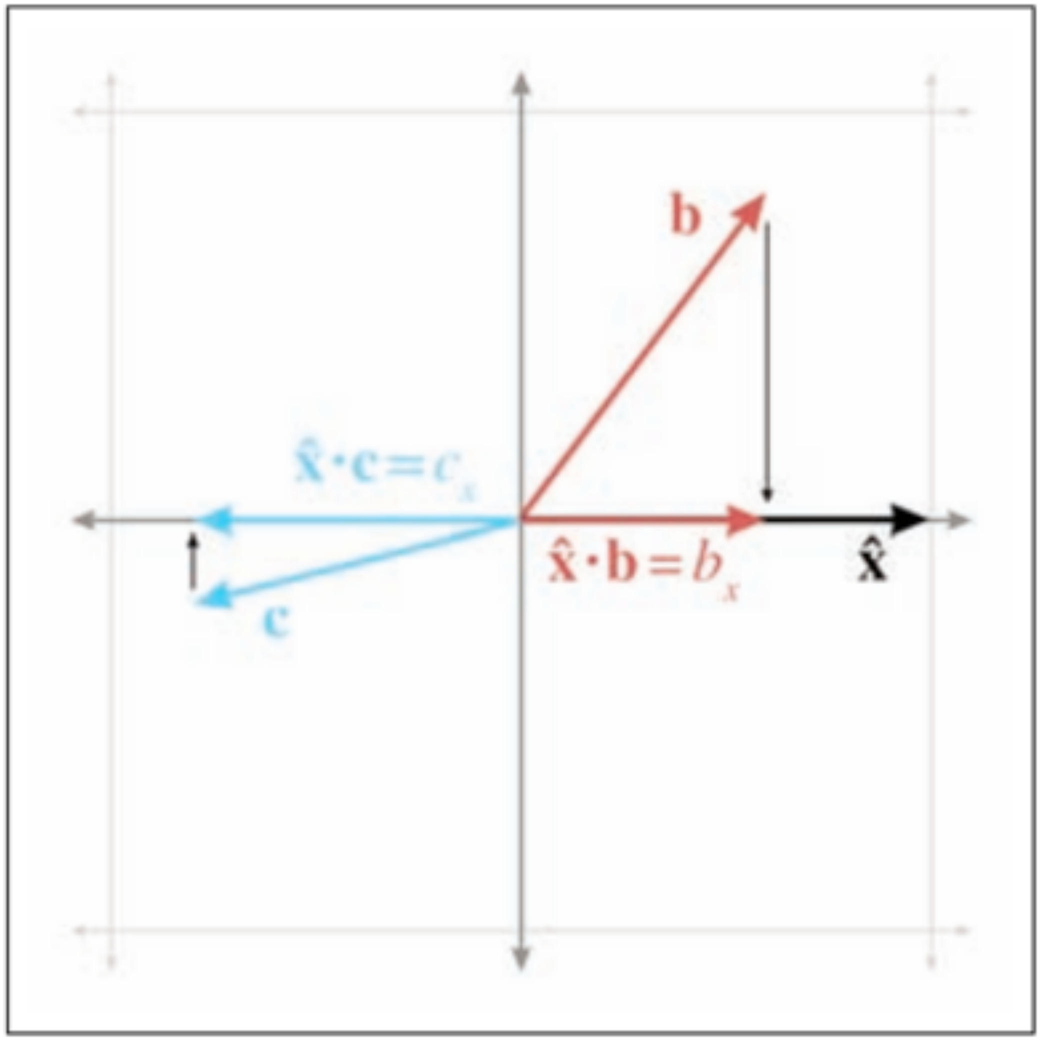


$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

Coordinates

- To get coordinates, just project onto the axis directions

- Really useful property that we'll come back to



Angles

$$\mathbf{u} = \|\mathbf{u}\| \ \hat{\mathbf{u}}$$

$$\mathbf{v} = \|\mathbf{v}\| \ \hat{\mathbf{v}}$$

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u}\| \, \|\mathbf{v}\| \, (\hat{\mathbf{u}} \cdot \hat{\mathbf{v}}) \\ &= \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta_{uv} \end{aligned}$$

The dot product of two vectors is the product of their lengths times the cosine of the angle between them

Lengths

$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$$

The dot product of something with itself
is its own length squared

Tip: lots of “distance” tests only need squared distance

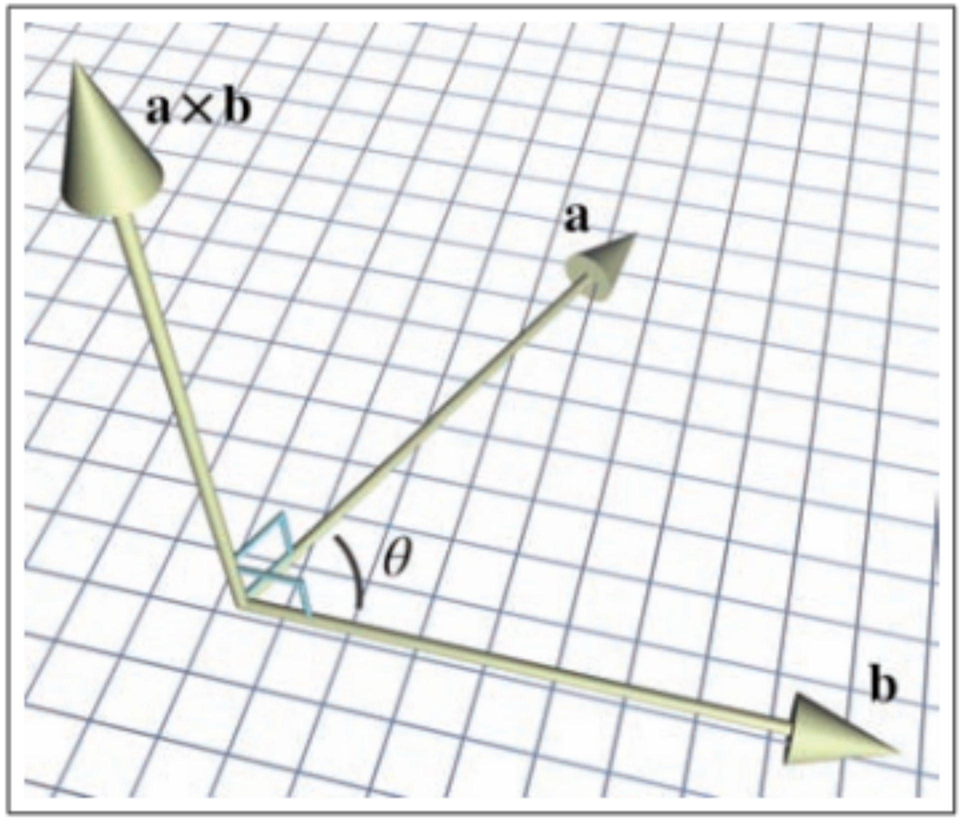
Cross Product

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

Result is a vector

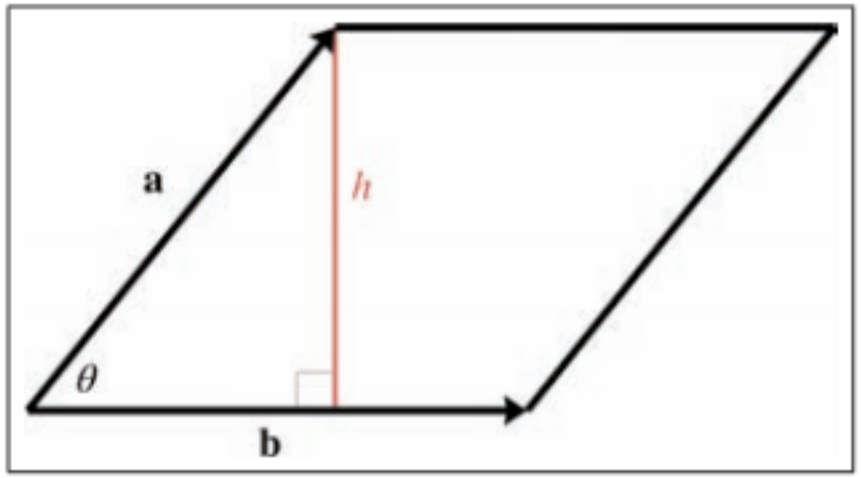
Only done in 3D

Geometric Interpretation



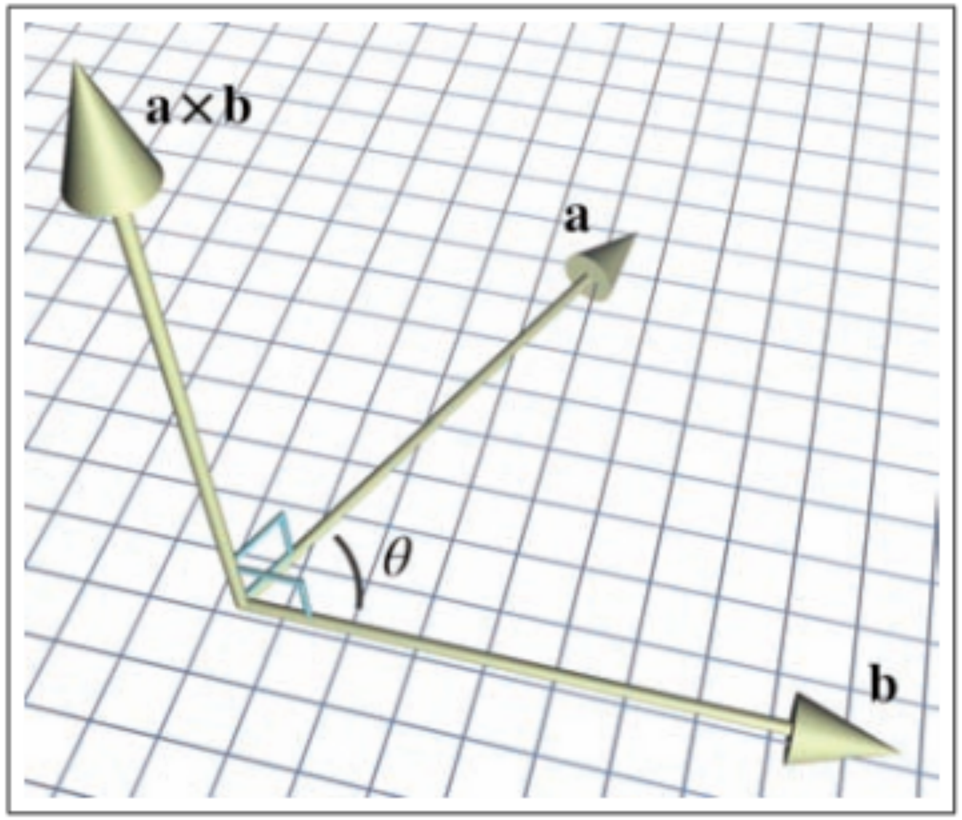
The cross product of two vectors is another vector orthogonal to the two

Geometric Interpretation



The length of the cross product of two vectors
is the area of the parallelogram spanned by the two

Geometric Interpretation



$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin \theta_{ab}$$

Linear Algebra Identities

Identity	Comments
$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	Commutative property of vector addition
$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$	Definition of vector subtraction
$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$	Associative property of vector addition
$s(t\mathbf{a}) = (st)\mathbf{a}$	Associative property of scalar multiplication
$k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$	Scalar multiplication distributes over vector addition
$\|k\mathbf{a}\| = \|k\|\|\mathbf{a}\|$	Multiplying a vector by a scalar scales the magnitude by a factor equal to the absolute value of the scalar
$\|\mathbf{a}\| \geq 0$	The magnitude of a vector is nonnegative
$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 = \|\mathbf{a} + \mathbf{b}\|^2$	The Pythagorean theorem applied to vector addition.
$\|\mathbf{a}\| + \|\mathbf{b}\| \geq \|\mathbf{a} + \mathbf{b}\|$	Triangle rule of vector addition. (No side can be longer than the sum of the other two sides.)
$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$	Commutative property of dot product
$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$	Vector magnitude defined using dot product
$k(\mathbf{a} \cdot \mathbf{b}) = (k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b})$	Associative property of scalar multiplication with dot product
$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$	Dot product distributes over vector addition and subtraction
$\mathbf{a} \times \mathbf{a} = \mathbf{0}$	The cross product of any vector with itself is the zero vector. (Because any vector is parallel with itself.)
$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$	Cross product is anticommutative.
$\mathbf{a} \times \mathbf{b} = (-\mathbf{a}) \times (-\mathbf{b})$	Negating both operands to the cross product results in the same vector.
$k(\mathbf{a} \times \mathbf{b}) = (k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b})$	Associative property of scalar multiplication with cross product.
$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$	Cross product distributes over vector addition and subtraction.

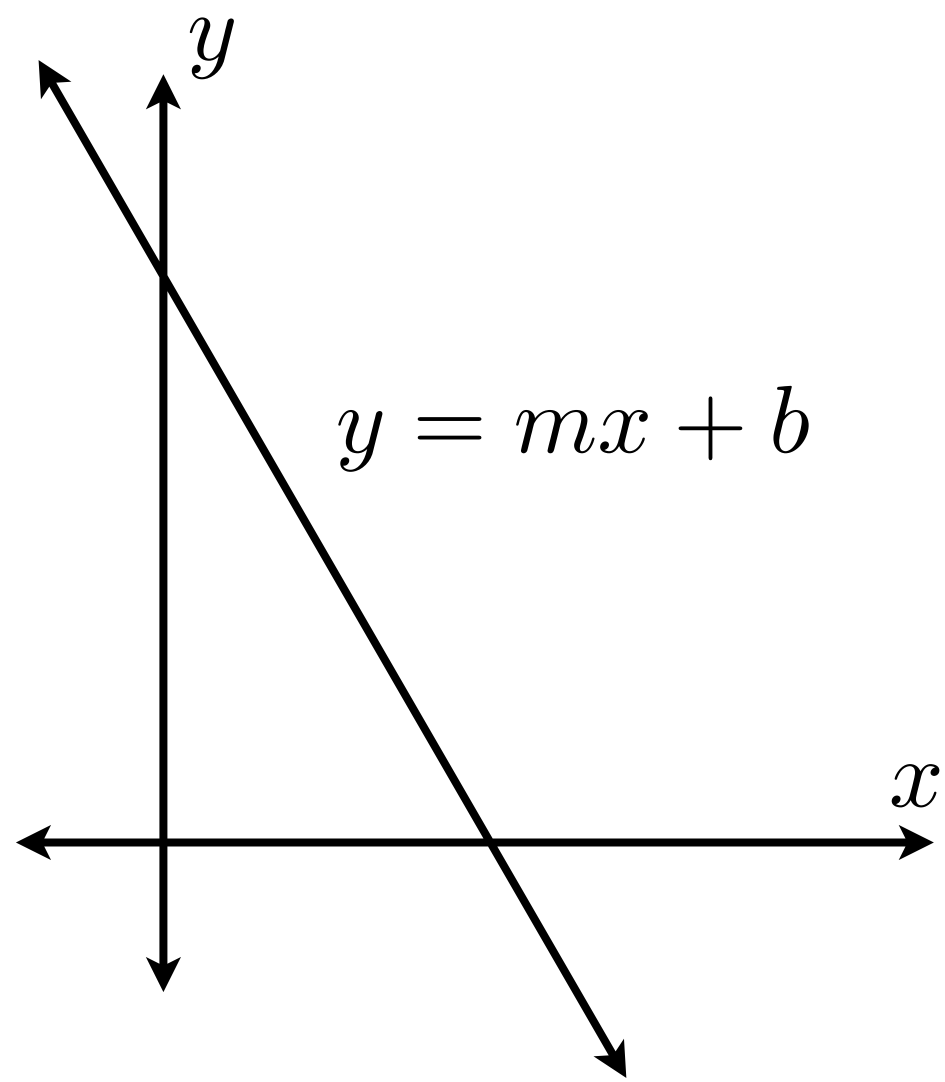
Lines

Lines extend infinitely in both directions

Line segments are finite

Rays extend infinitely in <u>one</u> direction

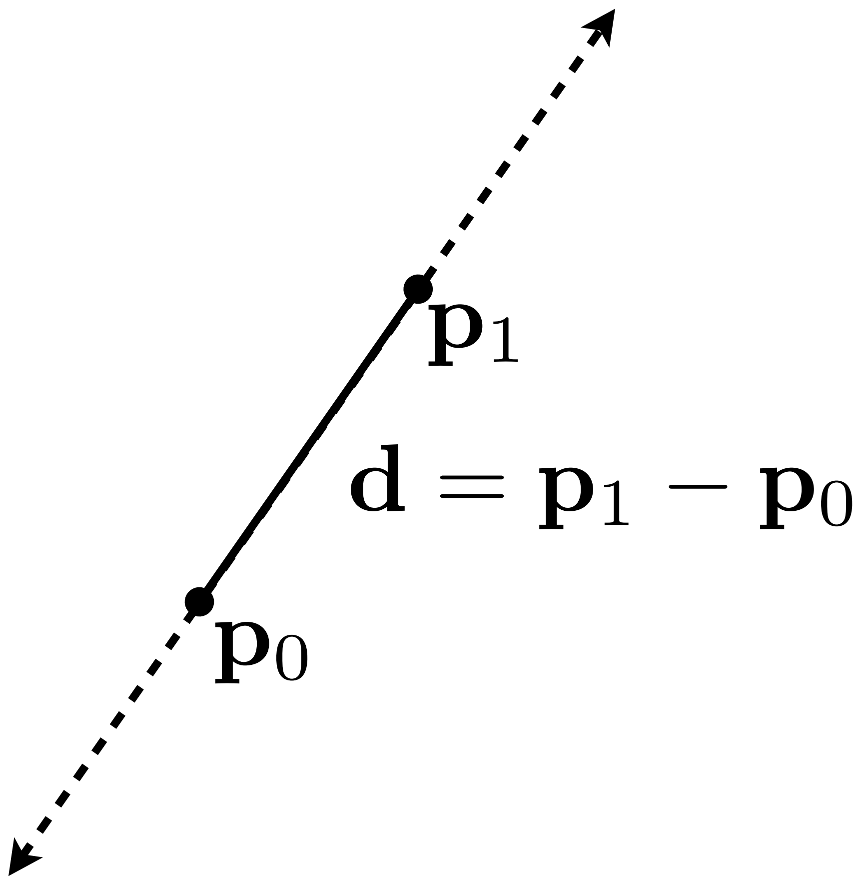
Slope / Intercept



Doesn't work for vertical lines

Hard to extend to 3D

Parametric Representation



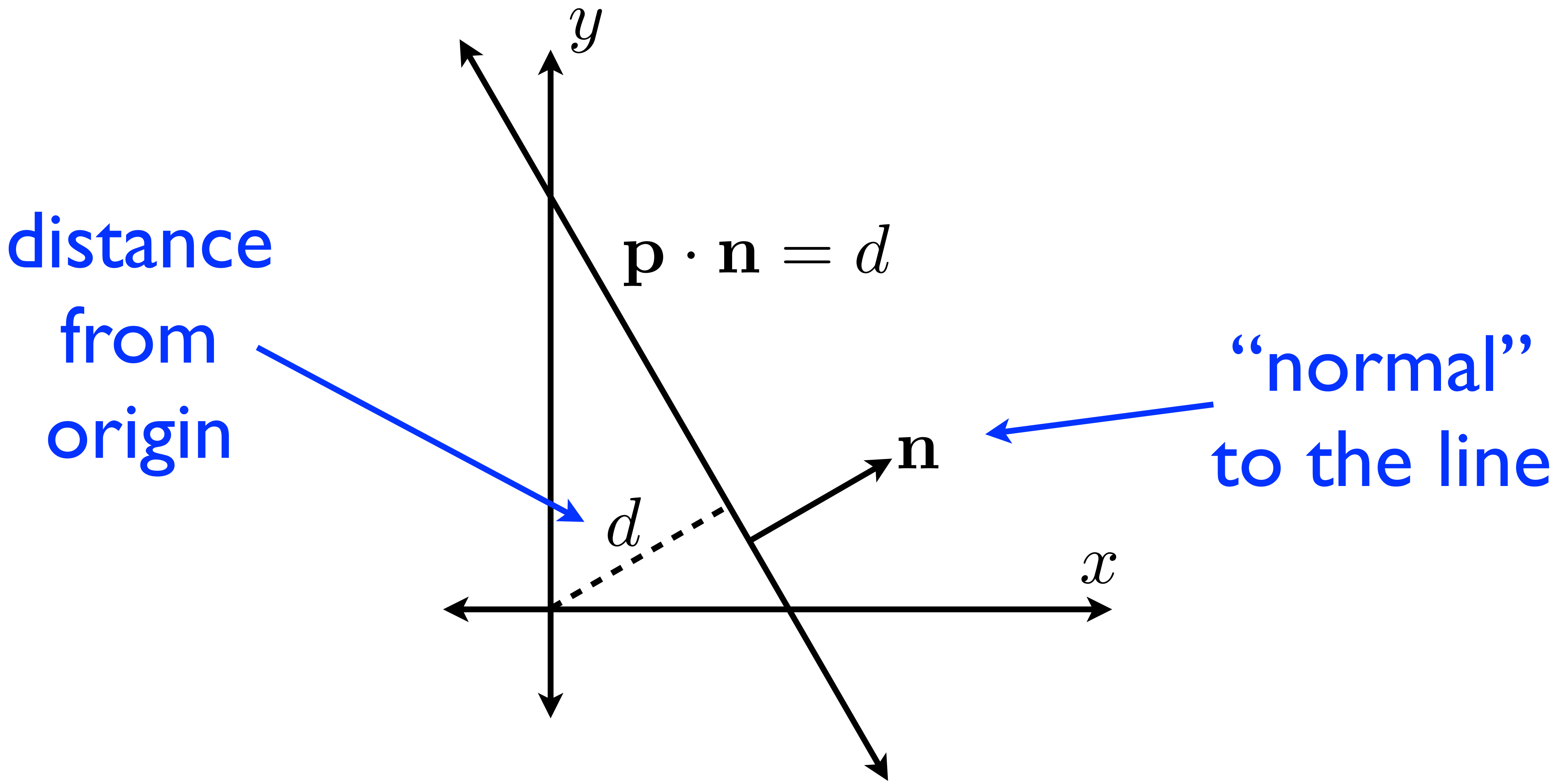
$$\mathbf{p}_0 + t\ \mathbf{d}$$

Line: $-\infty < t < \infty$

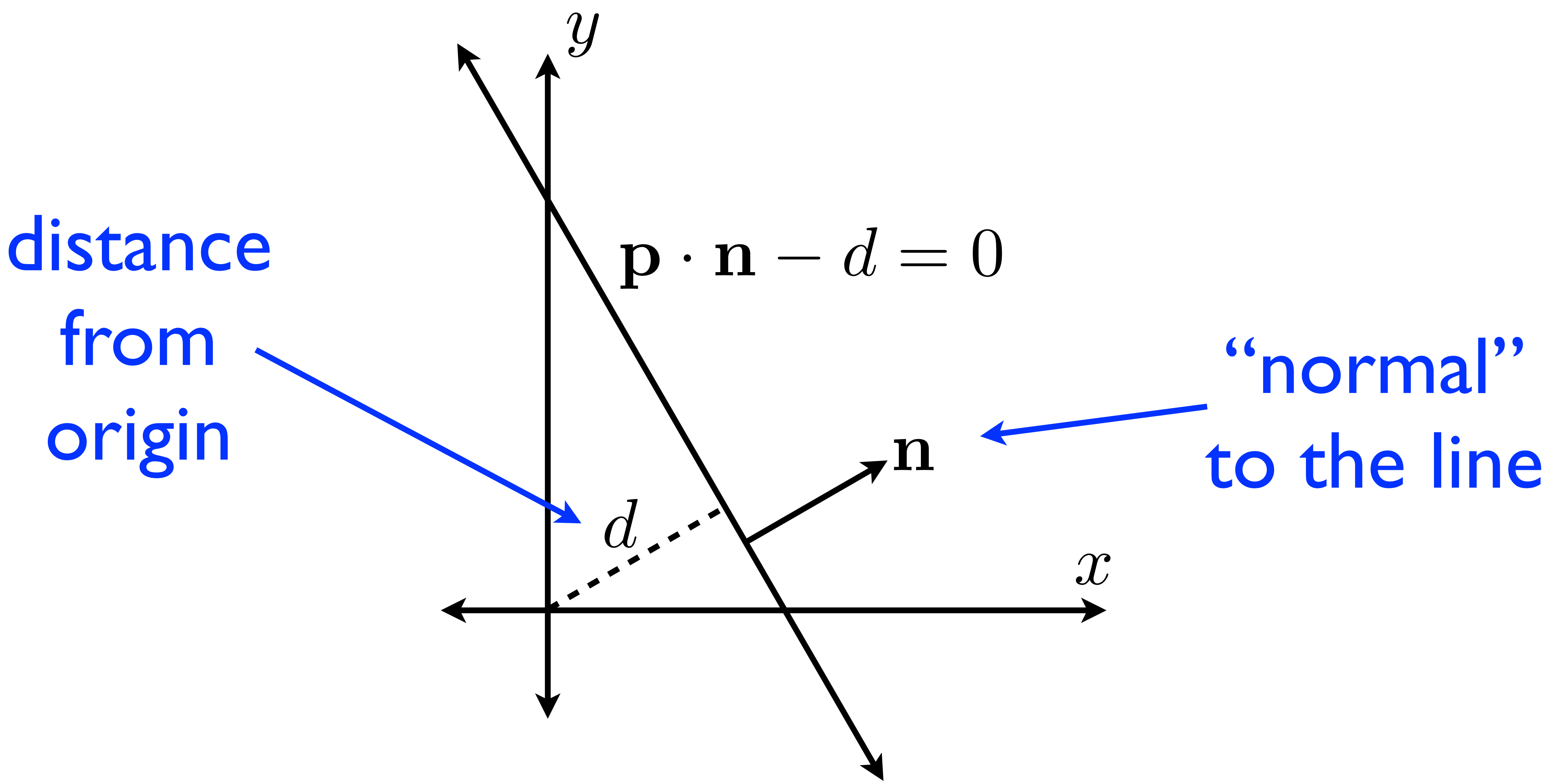
Line segment: $0 \leq t \leq 1$

Ray: $0 \leq t < \infty$

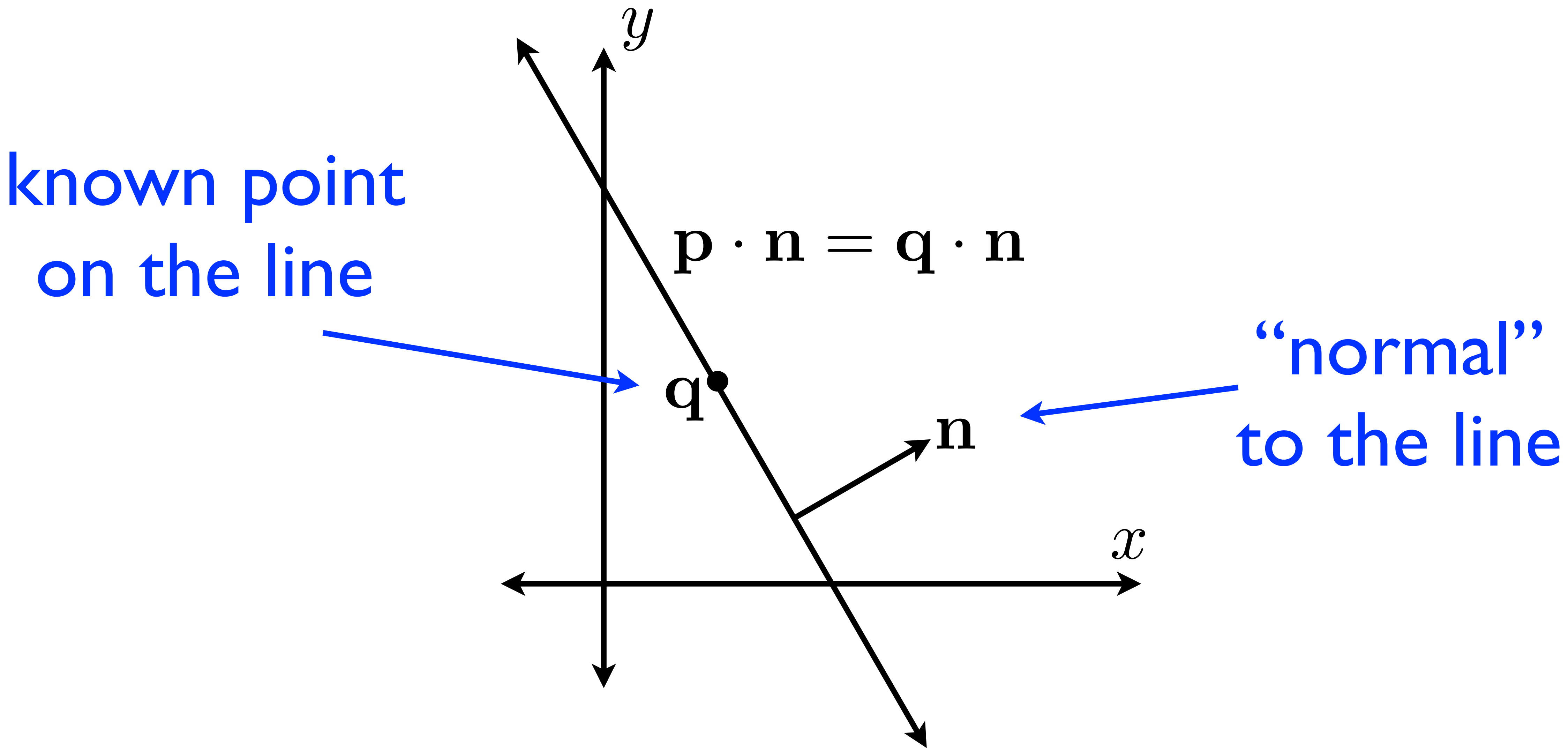
Normal + Distance



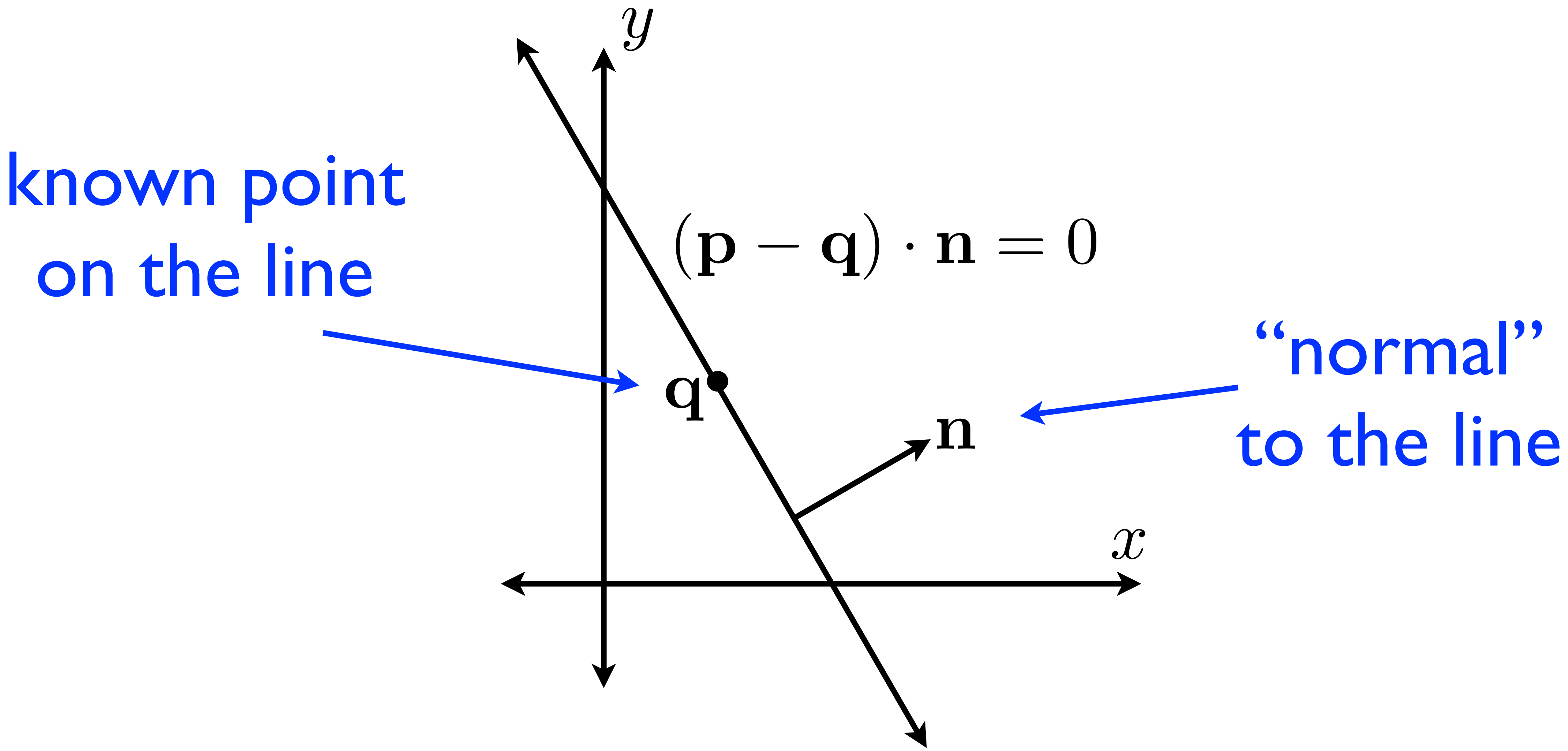
Implicit Representation



Normal + Point



Implicit Representation



Representing Shapes

- Parametric (sweeps as a function of some parameters) $\mathbf{p}(t)$
- Implicit (meets some test) $f(\mathbf{p}) = 0$
- Others...

Can usually convert between representations

Next time...

- Multiple coordinate systems
- May start on matrices