Hw#6 4.1,2 Ryan Bartley

① $z(t) = x(t) + jy(t)$ zero mean white Gaussian

$R_z(\tau) = N_0\,\delta(\tau)$ $f_m(t)$ $(m=1,2,\dots,M)$ are M orthogonal equivalent lowpass waveforms defined on $0 \le t \le T$

$N_{mr} = \mathrm{Re}\left[\int_0^T z(t) f_m^*(t)\,dt\right]$ $m = 1,2,\dots,M$

a) $\mathrm{var}[N_{mr}] = ?$

$E[N_{mr}] = E\left[\mathrm{Re}\left[\int_0^T z(t) f_m^*(t)\,dt\right]\right] = 0$

$\mathrm{var}\left[\int_0^T z(t) f_m^*(t)\,dt\right] = E[|\quad|^2] = E\left[\int_0^T z(t) f_m^*(t)\,dt \int_0^T z^*(t') f_m^*(t')\,dt'\right]$

$$\iint_0^T \underbrace{E[z(t) z^*(t')]}_{R_z(\tau)} f_m^*(t) f_m(t')\,dt\,dt'$$

$$\int_0^T\int_0^T N_0\,\delta(t-t')\, f_m^*(t)\, f_m(t')\,dt\,dt'$$

$$N_0 \underbrace{\int_0^T f_m^*(t') f_m^*(t')\,dt'}_{\mathcal{E}}$$

$N_0\,\mathcal{E}$ ← variance of complex R.V

$\mathrm{var}[N_m] = \dfrac{N_0\,\mathcal{E}}{2}$ ← Real part is $\frac{1}{2}$ of complex, if proper C.R.V.

$\mathrm{Var}[Z] = E[|Z|^2] = E[X^2] + E[Y^2]$

Ryan Bartley

b) show that $E[N_{mr} N_{kr}] = 0$ for $k \neq m$

$$E\left[\int_0^T z(t) f_m^*(t)\,dt \int_0^T z^*(t') f_k(t')\,dt'\right] =$$

$$\iint_0^T \underbrace{E[z(t) z^*(t')]}_{R_z(\tau)} f_m^*(t) f_k(t')\,dt\,dt'$$

$$\iint_0^T N_0 \delta(t-t') f_m^*(t) f_k(t')\,dt\,dt'$$

$$N_0 \int_0^T \underbrace{f_m^*(t') f_k(t')}\,dt'$$

orthogonal, so 0 when $m \neq k$
+ $\mathcal{E}$ when $m = k$

* Again real part is $\frac{1}{2}$ of complex
so

$$E[N_{mr} N_{kr}] = \begin{cases} \frac{N_0 \mathcal{E}}{2} & m = k \\ 0 & \text{else} \end{cases}$$

4.2

recall that a signal $s_m(t)$ can be constructed

$$s_m(t) = \sum_{k=1}^{K} s_{mk}\,\phi_k(t)$$

and that

$$\int_0^T \phi_n(t)\phi_m(t) = \begin{cases} 1 & m=n \\ 0 & \text{else} \end{cases}$$

$$C(\bar{r}, \bar{s}_m) = 2\int_0^T r(t)\,s_m(t)\,dt - \int_0^T s_m^2(t)\,dt$$

working on the left hand term of $C(\bar{r}, \bar{s}_m)$

$$2\int_0^T r(t)\,s_m(t)\,dt = 2\int_0^T \sum_{n=1}^{N} r_n \phi_n(t) \sum_{k=1}^{K} s_{mk}\phi_k(t)\,dt$$

1 $k=n$

$$= 2\sum_{n=1}^{N}\sum_{k=1}^{K} r_n s_{mk} \int_0^T \phi_n(t)\cancel{\phi_k(t)}\,dt$$

$$= 2\sum_{n=1}^{N} r_n s_{mn}$$

Now the right hand term

$$\int_0^T s_m^2(t)\,dt = \int_0^T \sum_{n=1}^{N} s_{mn}\phi_n(t) \sum_{k=1}^{K} s_{mk}\phi_k(t)\,dt$$

1 $k=n$

$$= \sum_{n=1}^{N}\sum_{k=1}^{K} s_{mn} s_{mk} \int_0^T \phi_n(t)\cancel{\phi_k(t)}\,dt$$

$$= \sum_{n=1}^{N} s_{mn}^2$$

$$\therefore\ C(\bar{r}, \bar{s}_m) = 2\sum_{n=1}^{N} r_n s_{mn} - \sum_{n=1}^{N} s_{mn}^2 \qquad m = 1, 2, \ldots, M$$

$S_0(t) = 0 \quad 0 \le t \le T$

$S_1(t) = A \quad 0 \le t \le T$

THE DEMOD CROSS CORRELATES THE RECEIVED SIGNAL $r(t)$ W/ $s(t)$ AND SAMPLES THE OUTPUT OF THE CORRELATOR AT $t+T$

$\int_0^T r(t)s(t-\tau)\,d(t)$

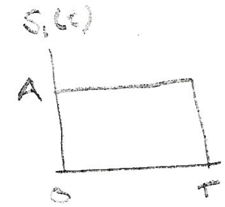
$\int_0^T (A+z)A\,dt$

$\int_0^T A^2 + zA\,dt$

$A^2t + zAt \Big|_0^T$

a) $r = S_m + z \qquad \sigma^2 = \frac{N_0}{2}$

$\underset{m}{\text{ARGMAX}}\; P(r|S_m)$



$P(r|S_0) > P(r|S_1)$

$$\frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{2\sigma^2}(r-0)^2} > \frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{2\sigma^2}(r-A\sqrt{T})^2}$$

$$e^{-\frac{1}{N_0}r^2} > e^{-\frac{1}{N_0}(r-A\sqrt{T})^2}$$

CONSTELLATION

$$-\frac{1}{N_0}r^2 > -\frac{1}{N_0}(r-A\sqrt{T})^2$$

$r^2 < r^2 - 2rA\sqrt{T} + A^2T$

$0 < -2rA\sqrt{T} + A^2T$

$2rA\sqrt{T} < A^2T$

$2r < A\sqrt{T}$

$r < \frac{A\sqrt{T}}{2}$

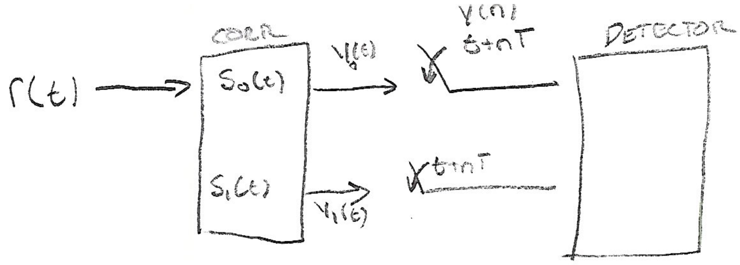
$\mathcal{E}_{S_1} = \int_{-\infty}^{\infty} s_1(t)s_1^*(t)\,dt$

$= \int_0^T A^2\,dt$

$= A^2T$

$$D_r = \begin{cases} S_0, & r < \frac{A\sqrt{T}}{2} \\ S_1, & r > \frac{A\sqrt{T}}{2} \end{cases}$$

HERE WE COMPARE THE <u>RECEIVED</u> SIGNAL TO A THRESHOLD



SECOND APPROACH TO ⓐ

$$Y_1(t) = \int_0^T r(\tau) s_1(t-\tau)\,d\tau$$

$$= \int_0^T (s_1(\tau) + n(\tau))\, s_1(t-\tau)\,d\tau$$

$$= \int_0^T s_1(\tau) s_1(t-\tau)\,d\tau + \int_0^T n(\tau) s_1(t-\tau)\,d\tau$$

$$= \int_0^T A^2\,d\tau + \underbrace{A\int_0^T n(\tau)\,d\tau}_{N}$$

$$= A^2 T + N(t)$$

$E[N] = 0$ $\quad E[N^2] = A^2 \int_0^T \int_0^T E[n(\tau)n(\tau')]\,d\tau\,d\tau'$

$$= A^2 \int_0^T \int_0^T \frac{N_0}{2}\delta(\tau-\tau')\,d\tau\,d\tau'$$

$$= A^2 \int_0^T \frac{N_0}{2}\,d\tau'$$

$$= \frac{A^2 N_0 T}{2}$$

$$P(e) = P_0 P(e|0) + P_1 P(e|1)$$

$$= P_0 P(Y_0 > \eta) + P_1 P(Y_1 < \eta)$$

$$= P_0 (1 - F_{Y_0}(\eta)) + P_1 F_1(\eta)$$

$$\frac{\partial}{\partial \eta} = -P_0 f_0(\eta) + P_1 f_1(\eta) = 0$$

$$\cancel{P_1} f_1(\eta) = \cancel{P_0} f_0(\eta) \qquad P_1 = P_0$$

$$\cancel{\frac{1}{\sqrt{2\pi\sigma^2}}}\, e^{-\frac{1}{2\sigma^2}(\eta - A^2T)^2} = \cancel{\frac{1}{\sqrt{2\pi\sigma^2}}}\, e^{-\frac{1}{2\sigma^2}(\eta)^2}$$

$$\therefore$$

$$(\eta - A^2T)^2 = \eta^2$$

$$\cancel{\eta^2} - 2\eta A^2 T + A^4 T^2 = \cancel{\eta^2}$$

$$2\eta A^2 T = A^4 T^2$$

$$\eta = \frac{A^2 T}{2}$$

$$D_Y = \begin{cases} S_0, & Y < \frac{A^2T}{2} \\ S_1, & Y > \frac{A^2T}{2} \end{cases}$$

HERE WE COMPARE
CORRELATOR OUTPUT
TO A THRESHOLD

(3) $P_e = P(Y_0 > \frac{A^2T}{2})$

$$\sigma^2 = \frac{A^2 N_0 T}{2}$$

$$= Q\left(\frac{\frac{A^2T}{2}}{\sigma}\right) = Q\left(\sqrt{\frac{E_{AVG}}{N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

$$\frac{A^2T}{2\sigma} = \sqrt{\frac{A^4T^2}{4\sigma^2}} = \sqrt{\frac{A^4T^2}{4}\frac{2}{A^2N_0T}} = \sqrt{\frac{A^2T}{2N_0}}$$

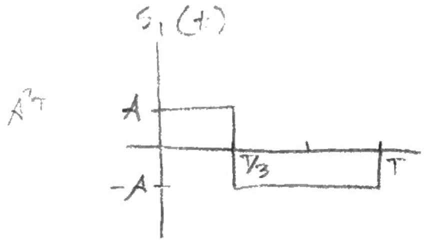
$E_{AVG} = \frac{1}{2}A^2T$

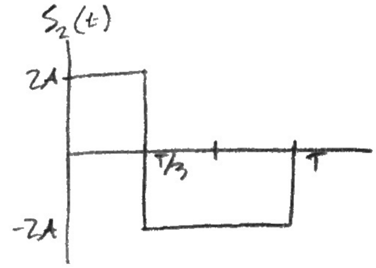
P_e FOR ANTIPODAL SIGNALING $Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

$\boxed{\text{THUS ANTIPODAL HAS 3 dB GAIN.}}$

[illegible]	EE 776 Dr. Rice	Chad Josephson

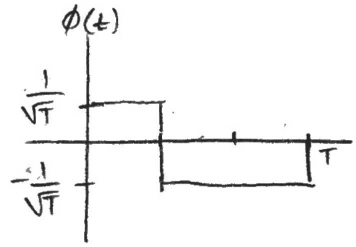
[illegible]) A communication system transmits one of three messages m_1, m_2, and m_3 using signals $s_1(t)$, $s_2(t)$, and $s_3(t)$. $s_3(t)$ is zero. The channel is additive white Gaussian noise with noise power spectral density equal to $N_0/2$.

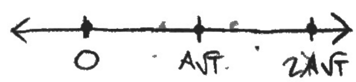




1. Determine an orthonormal basis for this signal set, and depict the signal constellation.

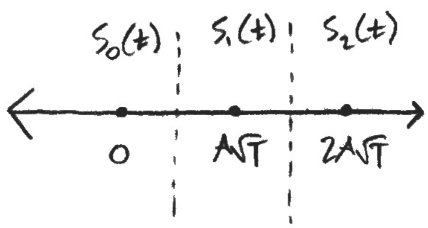
Orthonormal Basis:





2. If the three messages are equiprobable, what are the optimal decision rules for this system? Show the optimal decision regions on the signal constellation you plotted in part 1.

$$d(R) = \begin{cases} s_0(t) & \text{if } R < \frac{A\sqrt{T}}{2} \\ s_1(t) & \text{if } \frac{A\sqrt{T}}{2} \le R \le \frac{3A\sqrt{T}}{2} \\ s_2(t) & \text{if } R > \frac{3A\sqrt{T}}{2} \end{cases}$$



3. If the signals are equiprobable, express the error probability of the optimal detector in terms of the average SNR per bit.

$$E_{avg} = \frac{0 + A^2T + 4A^2T}{3} = \frac{5}{3}A^2T$$

$$E_{avg} = E_{bavg} \log_2 M$$

$$E_{bavg} = \frac{E_{avg}}{\log_2 3} = \frac{5/3\, A^2T}{\log_2 3}$$

We want to use the $Q(\cdot)$ function to find the probability of error. In order to do this we want to find the euclideen distance d between each point in terms of useful variables.

We know

$$d = A\sqrt{T}$$
$$d^2 = A^2T$$

so we have

$$d^2 = E_{bavg} \frac{\log_2 3}{5/3}$$

And we know...

$$\sigma_s^2 = \frac{N_0}{2}$$

$$\sigma_s = \sqrt{\frac{N_0}{2}}$$

$$P(E \mid s_0(t)) = P(E \mid s_2(t)) = Q\left(\frac{d/2}{\sigma_s}\right) = Q\left(\frac{d}{2}\sqrt{\frac{2}{N_0}}\right)$$

$$= Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$$

$$= Q\left(\sqrt{\frac{3}{5}(\log_2 3)\frac{E_{bavg}}{2N_0}}\right)$$

$$P(E \mid s_1(t)) = 2Q\left(\sqrt{\frac{3}{5}(\log_2 3)\frac{E_{bavg}}{2N_0}}\right)$$

Using total probability we get

$$P(E) = P(E|s_0(t))\,P(s_0(t)) + P(E|s_1(t))\,P(s_1(t)) + P(E|s_2(t))\,P(s_2(t))$$

$$= \frac{4}{3}\,Q\left(\sqrt{\frac{3}{5}(\log_2 3)\,\frac{E_{bavg}}{2N_0}}\right)$$

4. Assuming this system transmits 3,000 symbols per second, what is the resulting transmission rate (in bits per second)?

$$\frac{3{,}000 \text{ symbols}}{\text{sec}} \cdot \frac{\log_2 3 \text{ bits}}{1 \text{ symbol}} = \boxed{4755 \text{ bits/sec}}$$

4.6 BPSK

AWGN channel $\frac{1}{2} N_0 = 10^{-10}$ W/Hz

$E_b = \frac{1}{2} A^2 T$ (assumes equally likely symbols)

$P_b = 10^{-6}$

① $R_b = 10 \cdot 10^3$ bits/s

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = 10^{-6} \implies Q^{-1}(10^{-6}) = 4.7534$$ (matlab's qfuncinv function)

$$\sqrt{\frac{(2)(\frac{1}{2}A^2T)}{N_0}} = 4.7534$$

$$\sqrt{\frac{A^2T}{N_0}} = 4.7534$$

$$A = \sqrt{\frac{4.7534^2 \cdot N_0}{T}}$$

$T = 10^{-4}$

$N_0 = 2 \cdot 10^{-10}$

$$\boxed{A = 6.722 \cdot 10^{-3}}$$

② $T = 10^{-5}$ $\boxed{A = 21.26 \cdot 10^{-3}}$

③ $T = 10^{-6}$ $\boxed{A = 67.22 \cdot 10^{-3}}$

4.9) A ternary Communication system transmits one of three equiprobable signals $s(t)$, 0, or $-s(t)$ every T seconds. The recieved signal is $r_1(t) = s(t) + z(t)$, $r_\ell(t) = z(t)$, or $r_\ell(t) = -s(t) + z(t)$, where $z(t)$ is white Gaussian noise with $E[z(t)] = 0$ and $R_z(\tau) = E[z(t)z^*(\tau)] = 2N_0\delta(t-\tau)$. The optimum reciever computes the Correlation metric

$$U = Re\left[\int_0^T r_\ell(t)\, s^*(t)\, dt\right]$$

and Compares U with a threshold A and a threshold A-.
If $U > A$, the decision is made that $s(t)$ was sent.
If $U < -A$, the decision is made in favor of $-s(t)$.
If $-A < U < A$, the decision is made in favor of 0.

1.) Determine the three conditional probabilitres of error: Pe given $s(t)$ was sent, Pe given that $-s(t)$ was sent, and Pe given that 0 was sent.

$$P(error|s(t)) = P(U<A|s(t)) = P\left(Re\left[\int_0^T r_\ell(t) s^*(t)\, dt\right] < A \,\middle|\, s(t)\right)$$

$$= P\left(Re\left[\int_0^T (s(t)+z(t))(s^*(t))\, dt\right] < A\right)$$

$$= P\left(\underbrace{Re\left[\int_0^T s(t)s^*(t)\, dt\right]}_{E_s} + \underbrace{Re\left[\int_0^T z(t)s^*(t)\, dt\right]}_{N} < A\right)$$

- $E_s = Re\left[\int_0^T s(t)s^*(t)\, dt\right]$
- $N = Re\left[\int_0^T z(t)s^*(t)\, dt\right]$

$$E[N] = Re\left[\int_0^T E[z(t)]s^*(t)\,dt\right] = 0$$

$$E[NN^*] = E\left[Re\left[\int_0^T\int_0^T z(t)\,z^*(t')\,s(t')s^*(t)\,dt\,dt'\right]\right]$$

$$= Re\left[\int_0^T\int_0^T 2N_0\,\delta(t-t')\,s(t')s^*(t)\,dt\,dt'\right]$$

Sifting Property ; $t = t'$

$$= Re\left[\int_0^T 2N_0\, s(t')s^*(t')\,dt'\right] = 2N_0 E_s$$

So N is a Gaussian R.V. with zero mean and variance $2N_0E_s$.

AND $P(\text{error}|s(t)) = P(N < A - E_s) = \boxed{Q\left(\frac{E_s - A}{\sqrt{2N_0E_s}}\right)}$

$$P(\text{error}|-s(t)) = P(U > -A|-s(t)) = P\Big(\underbrace{Re\left[\int_0^T -s(t)s^*(t)\,dt\right]}_{-E_s} + \underbrace{Re\left[\int_0^T z(t)s^*(t)\,dt\right]}_{N} > A\Big)$$

$$= P(N > E_s - A) = \boxed{Q\left(\frac{E_s - A}{\sqrt{2N_0E_s}}\right)}$$

$$P(\text{error}/0) = P((U > A \text{ or } U < -A)/0) = P(U > A/0) + P(U < -A/0)$$

$$= P\left(Re\left[\int_0^T z(t)s^*(t)\,dt\right] > A\right) + P\left(Re\left[\int_0^T z(t)s^*(t)\,dt\right] < -A\right)$$

$$= P(N > A) + P(N < -A) = \boxed{2Q\left(\frac{A}{\sqrt{2N_0E_s}}\right)}$$

2.) Determine the average probability of error P_e as a function of the threshold A, Assuming that the three symbols are equally probable a priori.

Using Equation 4.1-13, $\boxed{P_e = \frac{2}{3}\left(Q\left(\frac{E_s - A}{\sqrt{2N_0E_s}}\right) + Q\left(\frac{A}{\sqrt{2N_0E_s}}\right)\right)}$

3) The value of A that minimizes P_E is found by

$$\frac{\partial P_E}{\partial A} = 0$$

where the derivative of $Q(x) = \int_{f(A)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$ with respect to A can be calculated using Leibnitz rule:

$$\frac{\partial}{\partial A}\left(\int_{f(A)}^{\infty} g(t)\,dt\right) = -\frac{\partial f}{\partial A}\, g(f(A))$$

$$0 = \frac{\partial P_E}{\partial A} = \frac{2}{3}\left[\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{E_s-A}{\sqrt{2N_0E_s}}\right)^2}\left[-\left(\frac{E_s-A}{\sqrt{2N_0E_s}}\right)(-1)\right] + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{A}{\sqrt{2N_0E_s}}\right)^2}\left(\frac{-A}{\sqrt{2N_0E_s}}\right)\right]$$

$$\Rightarrow \frac{E_s-A}{\sqrt{2N_0E_s}} e^{-\frac{1}{2}\left(\frac{E_s-A}{\sqrt{2N_0E_s}}\right)^2} - \frac{A}{\sqrt{2N_0E_s}} e^{-\frac{1}{2}\left(\frac{A}{\sqrt{2N_0E_s}}\right)^2} = 0$$

This holds true if $\boxed{A = E_s/2}$

When $A = E_s/2$

$$P_e = \frac{2}{3}\left(Q\left(\frac{E_s/2}{\sqrt{2N_0E_s}}\right) + Q\left(\frac{E_s/2}{\sqrt{2N_0E_s}}\right)\right)$$

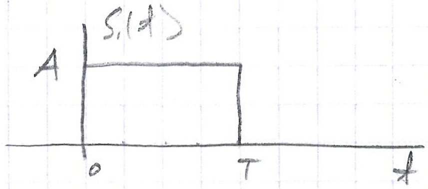
$$= \boxed{\frac{4}{3} Q\left(\frac{1}{2}\sqrt{\frac{E_s}{2N_0}}\right)}$$

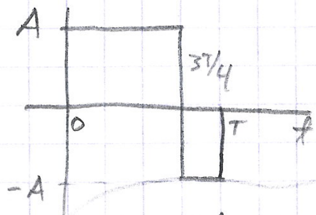
(4.10) The two equivalent lowpass signals shown in Figure P4.10 are used to transmit a binary information sequence. The transmitted signals, which are equally probable, are corrupted by additive zero-mean Gaussian noise having an equivalent lowpass representation $z(t)$ with an autocorrelation function

$$R_z(\tau) = E[z^*(t) z(t+\tau)] = 2N_0\, \delta(\tau)$$

1. what is the transmitted signal energy?

2. what is the probability of a binary digit error if coherent detection is employed at the receiver?





$$\mathcal{E}_1 = \frac{1}{2}\int_0^T s_1^2(t)\,dt$$

$$= \frac{1}{2}\int_0^T (A)^2\,dt$$

$$= \frac{1}{2}A^2 t\Big|_0^T$$

(1.) $$= \frac{1}{2}A^2 T$$

$$\boxed{\mathcal{E}_1 = \frac{A^2}{2}T}$$

why the $\frac{1}{2}$?

Pg 26
"The energy in the lowpass equivalent signal is twice the energy in the bandpass signal"

$$\mathcal{E}_2 = \frac{1}{2}\int_0^T s_2^2(t)\,dt$$

$$= \frac{1}{2}\int_0^T (A)^2\,dt$$

$$= \frac{1}{2}A^2 t\Big|_0^T$$

$$\boxed{\mathcal{E}_2 = \frac{A^2}{2}T}$$

What is P_b for this Equiprobable binary signaling scheme?

Using equation (4.2-37) and (4.2-38)

$$P_b = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right) \qquad d_{12}^2 = \int_0^T (s_1(t) - s_2(t))^2\,dt$$

$$= 2\mathcal{E}(1-\rho)$$

$$\rho_{12} = \frac{\langle s_1(t), s_2(t)\rangle}{\sqrt{\mathcal{E}_1\mathcal{E}_2}}$$

$$= \frac{1}{2\mathcal{E}}\int_0^T s_1(t)\, s_2^*(t)\,dt$$

$$= \frac{1}{2\mathcal{E}}\left(\int_0^{\frac{3T}{4}} A^2\,dt + \int_{\frac{3T}{4}}^{\frac{4T}{4}} -A^2\,dt\right)$$

$$= \frac{A^2}{2\mathcal{E}}\left(\frac{3T}{4} - \left(\frac{4T}{4} - \frac{3T}{4}\right)\right)$$

$$= \frac{A^2T}{2\mathcal{E}}\left(\frac{3}{4} - \frac{1}{4}\right) = \frac{A^2T}{4\mathcal{E}} = \frac{A^2T}{4\left(\frac{A^2}{2}T\right)} = \frac{1}{2}$$

(2.)

$$\boxed{P_b = Q\left(\sqrt{\frac{\mathcal{E}}{N_0}(1-\rho)}\right) = Q\left(\sqrt{\frac{\mathcal{E}}{2N_0}}\right)}$$

MARK RUZINDANA
EE 776
HW 7

4.4, 4.5, 4.6, 4.7, 4.9, 4.10(1-4), 4.13

(4.13) $r_i(t) = s_i(t) + z(t), \quad 0 \le t \le T, \quad i = 1, 2$

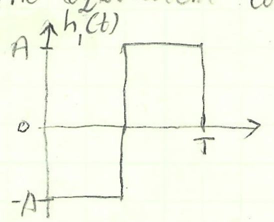
$z(t)$ is a zero-mean Gaussian noise process with $R_z(\tau) = E\{z^*(t)z(t+\tau)\} = 2N_0\delta(\tau)$.

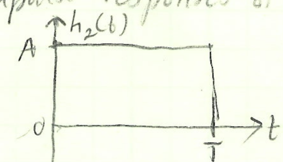
(1) $\mathcal{E}_1 = \frac{1}{2}\int_0^T |s_1(t)|^2 dt = \frac{A^2}{2}\int_0^T dt = \boxed{\frac{A^2T}{2}}$

$\mathcal{E}_2 = \frac{1}{2}\int_0^T |s_2(t)|^2 dt = \frac{A^2}{2}\int_0^T dt = \boxed{\frac{A^2T}{2}}$

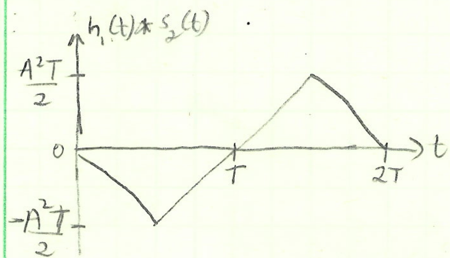
Since $\mathcal{E}_1 = \mathcal{E}_2$, then, $\rho_{12} = \frac{1}{2\mathcal{E}}\int_0^T s_1(t)s_2^*(t)\,dt = \boxed{0}$

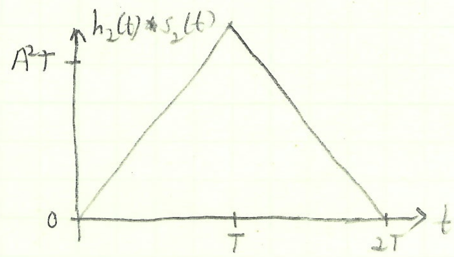
(2) The equivalent low pass impulse responses of the matched filters:





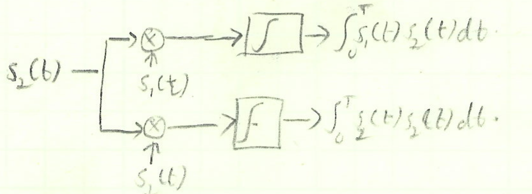
(3)

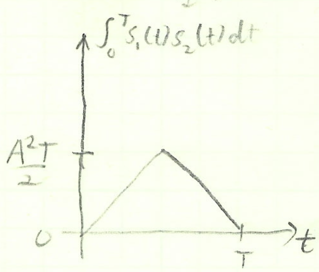


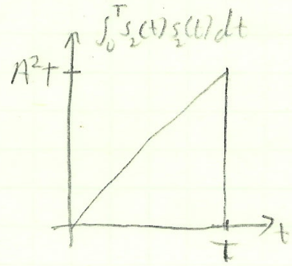


These are the noise-free responses of the two matched filters when $s_2(t)$ is transmitted.

(4)







(5) The sketches in part 3 and part 4 are different because of the difference in interval and the result of the convolution between $h_1(t)$ and $s_2(t)$ is different from $\int_0^T s_1(t)s_2(t)dt$ since $h_1(t)$ is $s_1(t)$ flipped about the horizontal axis.

(6) The signals are orthogonal so the probability of error is $\boxed{P_b = Q\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right)}$.