1/24: Andreas/Nikhil: Intro to Cohomology

Consider a chain complex

$$\cdots \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to \cdots$$

$C_n^* = \mathrm{Hom}(C_n, G)$ $\quad$ G-group (usually $\mathbb{Z}$)

$\delta = \partial^* : C_{n-1} \to C_n$

Def: $H^n(C, G) = \ker\delta / \mathrm{im}\,\delta$ $\quad$ (δ should have a subscript)

Def: Cohomology ring $\quad H^*(X, R) = \bigoplus_{n \in \mathbb{Z}} H^n(X, R)$

Elements can be written as finite sum $\sum_i \alpha_i \quad \alpha_i \in H^i$

$$\left(\sum_i \alpha_i\right)\left(\sum_j \beta_j\right) = \sum_{i,j} \underbrace{\alpha_i \beta_j}_{}$$

$\hookrightarrow$ cup product

$\sigma : \Delta^{k+l} \to X$ $\quad$ $\varphi \in C^{\ell}(X, R)$
$\psi \in C^{k}(X, R)$

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_k]})\,\psi(\sigma|_{[v_k,\dots,v_{k}]})$$

__Universal Coefficient Theorem__: Let C be a complex of free abelian groups. Then there is a split exact sequence

$$0 \to \mathrm{Ext}^1(H_{n-1}(C), G) \to H^n(C,G) \to \mathrm{Hom}(H_{n-1}(C),G) \to 0$$

Ext^1 is additive functorial in both variables

JK we'll do this with examples

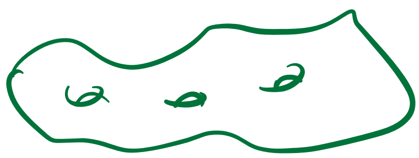
<u>Ex</u>] $\bullet = \{pt\}$ $\sigma : \Delta^0 \to X \; : \; \{0\} \to \{x\}$

$$H_i(X) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & \text{else} \end{cases}$$

$$H^i(X) = \mathrm{Hom}(H_i(X), \mathbb{Z})$$
$$= \begin{cases} \mathbb{Z} & i=0 \\ 0 & \text{else} \end{cases}$$

C_n: maps from simplices to $\mathbb{Z}$

Andrey:

$X =$ 

$H_i(X)$ always triangulate into simplices

Δ 2-dim simp

$|$ 1-dim simp

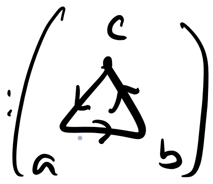
$\bullet$ 0-dim simp

$C_2 =$ Vect space / $\mathbb{C}$ w/ basis Δ

$C_1 =$ Vect. space / $\mathbb{C}$ w/ basis $/$

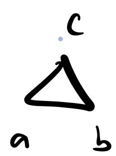
C_0 = vect space / $\mathbb{C}$ w/ basis •

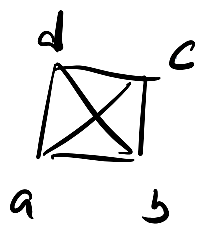
$\in C_1 \qquad \in C_1 \qquad \in C_1$

∂  $= [b\,c] + [c\,a] + [a\,b]$

$[c\,a] = -[c\,b]$?? see note

(as oriented simplices)

$[a,b,c] =$ 

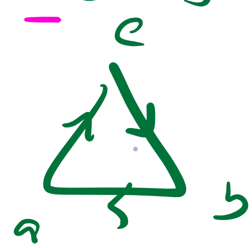
$[a,b,c,d] =$ 

remove i_k^{th} entry

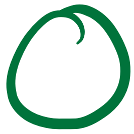
$$\partial[i_1, i_2, \dots, i_n] = \sum_{k=1}^{n} (-1)^k [i_1, i_2 \dots \cancel{i_k} \dots i_n]$$

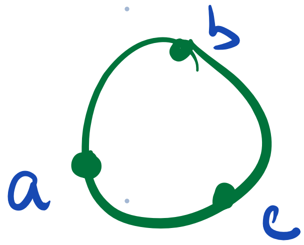
Ex

$$\partial[a\,b\,c] = \underline{-}[b\,c] \underline{+} [a\,c] \underline{-} [a\,b]$$



Signs are responsible for orientation

Ex) $S^1 =$ 



$$C_3 = 0 \xrightarrow{\partial} C_2 = 0 \xrightarrow{\partial} C_1 = \mathbb{C}^3 = \mathbb{C}^3([a,b],[b,c],[c,a]) \xrightarrow{\partial} C_0 = \mathbb{C}^3 = \mathbb{C}^3(a,b,c) \xrightarrow{\partial} C_{-1} = 0$$

The point is that the triangulation choice doesn't matter, Homology is topologically invariant

"You guys know linear algebra?"

$$\partial[ab] = -b + a$$

$$\partial[bc] = -c + b$$

$$\partial[ca] = -a + c$$

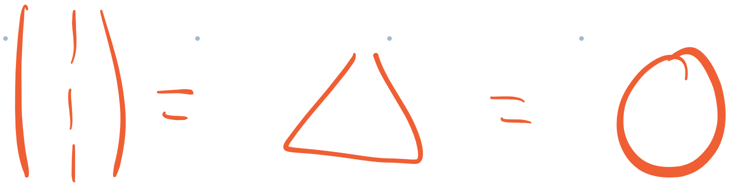
$$0 \xrightarrow{\partial_0} \mathbb{C}^3 \xrightarrow{\partial_1} \mathbb{C}^3 \longrightarrow 0$$

$$\begin{array}{c|ccc} & [ab] & [bc] & [ca] \\ \hline a & & & \\ b & & \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} & \\ c & & & \end{array}$$

as a matrix

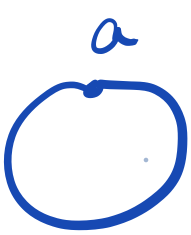
$H = \ker \partial_1$

$$H_1 = \frac{\ker \partial_1}{\text{im}\, \partial_0} = \mathbb{C}/0 = \mathbb{C}$$



no boundary so sent to 0

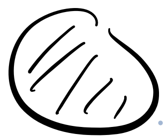
$$H_2 = \frac{\ker \partial_2}{\text{im}\, \partial_1} = \mathbb{C}^3/\mathbb{C}^2 = \mathbb{C}$$

$S^1 =$ 

∂ is trivial $= a - a = 0$

$$C_1 = \mathbb{C} \xrightarrow{\partial} C_0 = \mathbb{C}$$

Ex $\quad S^2 \simeq \mathbb{CP}^1$

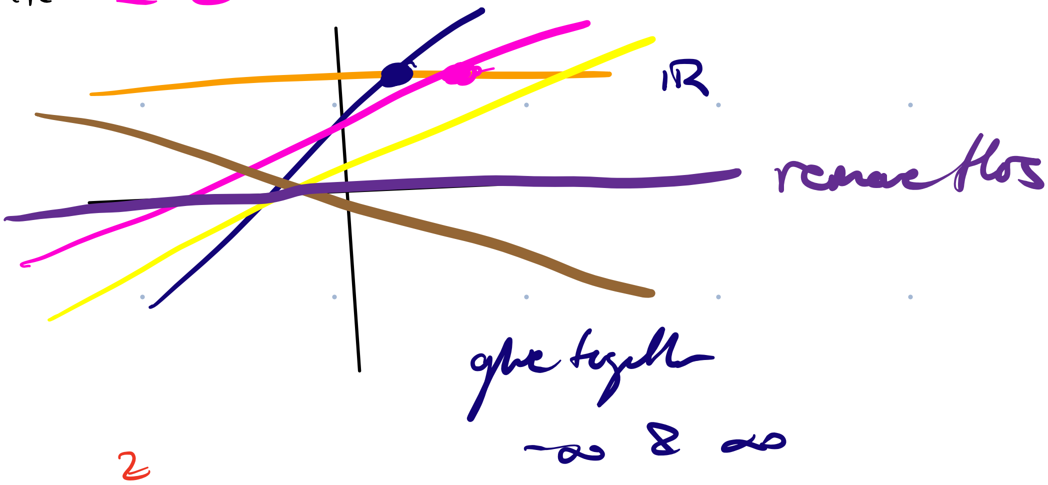
$C_2 = \mathbb{C}$ 

$C_0 = \bullet$

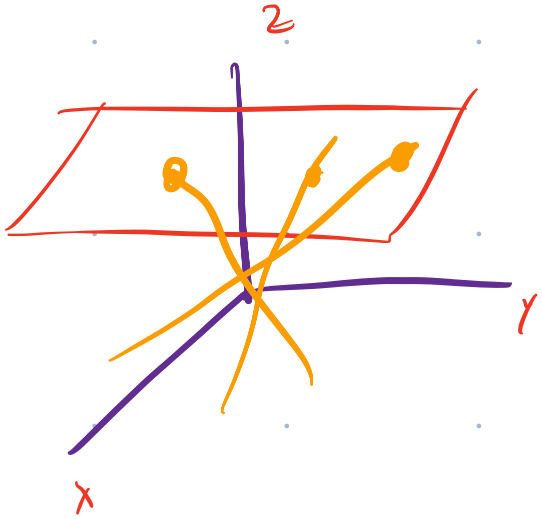
$$0 \to \underset{H_0}{\mathbb{C}} \to 0 \to \underset{H_2}{\mathbb{C}} \to 0$$

This is how $\mathbb{CP}^n$ is compute

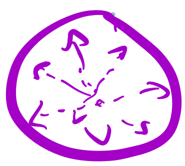
$\mathbb{RP}^1$ = lines in $\mathbb{R}^2$ $\cong S^1$



$\mathbb{RP}^3$



$\mathbb{RP}^2$ =



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equiv classes

$$\mathbb{CP}^n = \{[z_0, z_1, \dots, z_n] : (z_0, \dots, z_n) \neq 0\}$$

$\{z_0 = 0\}$

$$= \{[z_1, \dots, z_n] : (z_1, \dots, z_n) \neq 0\} \cong \mathbb{CP}^{n-1}$$

$\{z_0 \neq 0\}$

$$\hookrightarrow \{[1, z_1, \dots, z_n] : (z_1, \dots, z_n) \in \mathbb{C}^n\} \cong \mathbb{C}^n = e^{2n}$$

$\underbrace{e^{2n}}_{\text{2n-dim real disc}}$

$$\mathbb{CP}^n = \mathbb{CP}^{n-1} \cup e^{2n} = e^0 \cup e^2 \cup \dots \cup e^{2n}$$

$\Rightarrow$ Odd homologies die

$$\cdots \to \cancelto{\mathbb{Z}}{C_{2n}} \to \cancelto{0}{C_{2n-1}} \to \cdots \to \cancelto{\mathbb{Z}}{C_2} \to \cancelto{0}{C_1} \to \cancelto{\mathbb{Z}}{C_0} \to \cdots$$

$$H_{2n}(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z}$$

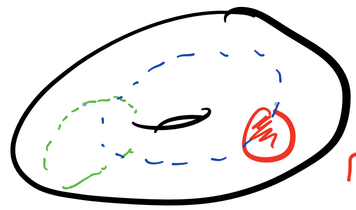
$$H_{2n-1}(\mathbb{CP}^n, \mathbb{Z}) \cong 0$$

$$H^*(\mathbb{CP}^n, \mathbb{Z}) = \bigoplus_{m \in \mathbb{Z}} H^m(\mathbb{CP}^n, \mathbb{Z})$$

$$= \bigoplus_{i=0}^{n} \mathbb{Z} \quad \text{as a group}$$

$$= \bigoplus_{i=0} \mathrm{Hom}(\mathbb{Z}\Delta^{?i}; \mathbb{Z})$$

A:



∂ of something not intersecting

not a ∂

Class of whole space generates top cohomology

intersection of hyperplanes $\mathbb{C}P^{n-1}$, cap product

$\hookrightarrow \mathbb{C}P^{n-2}$

$\vdots$

$\hookrightarrow \mathbb{C}P^{n-k}$ for intersect of k-hyperplanes

the generator $x^2 \in H^{n-2}$

$$H^*(\mathbb{C}P^n, \mathbb{Z}) \simeq \mathbb{Z}[\alpha]/(\alpha^{n+1})$$

hyperplane

$\mathbb{C}P^n \subset H^n$ intersect w/ something

generator of group is class of hyperplane identity of this ring is class of $\mathbb{CP}^n$

α^2 : intersection fellow class α & intersect w/ another

$\mathbb{CP}^{n-2}$ $\quad z_0=0,\ z_1=0$

intersection of 2 hyperplanes $\longleftrightarrow$ multiplication

perturb α a bit & compute

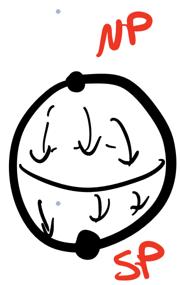
Grassmanians ??

A: $(\mathbb{C}^\times)^n \circlearrowright \mathbb{C}^n$

$a_1, a_2, \dots, a_n$ $\quad (z_1, \dots, z_n) \mapsto (a_1 z_1, \dots, a_n z_n)$

Induced action on $\mathbb{P}(\mathbb{C}^n)$ or on $Gr(k, \mathbb{C}^n)$

$\mathbb{P}(\mathbb{C}^2) \simeq \mathbb{CP}^1 \simeq S^2$



$\mathbb{C}^2 = (z_1, z_2) \mapsto (u_1 z_1, u_2 z_2)$

$(1,0) \mapsto (u_1, 0) \sim (1,0)$ by rescaling

$(0,1) \mapsto (0, u_2) \sim (0,1)$

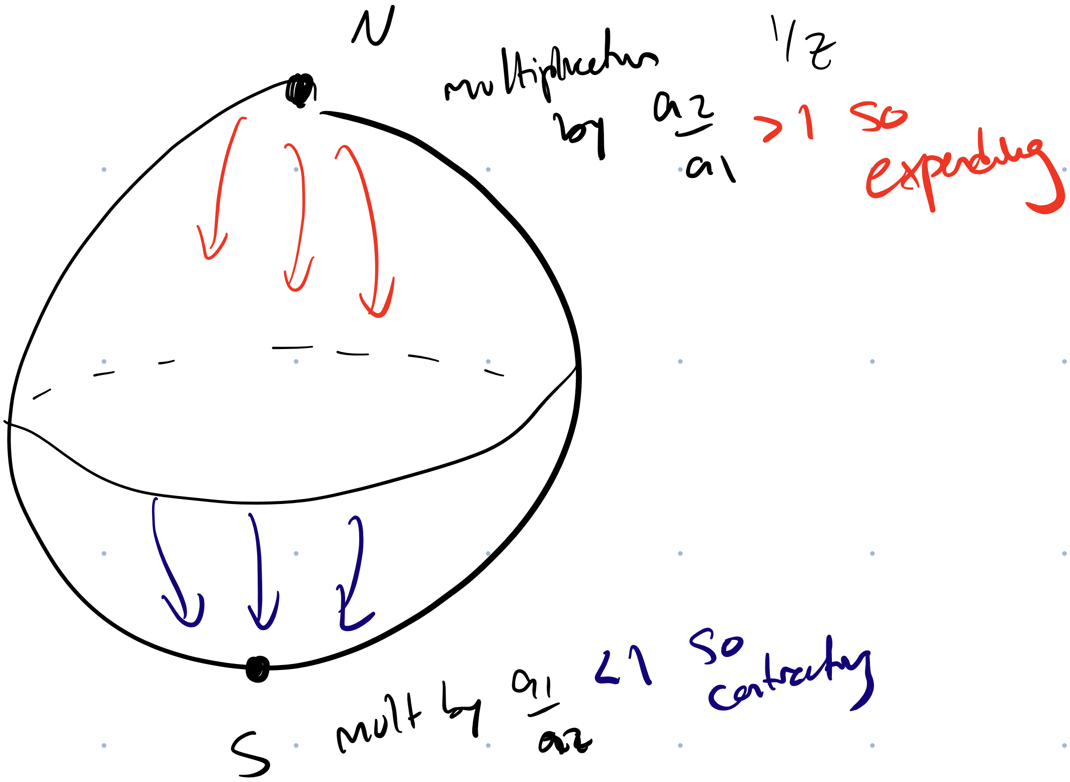
$T_{(1,0)}\mathbb{CP}^1 \to T_{(1,0)}\mathbb{CP}^1$ tangent space mapped to itself at fixed pts

$\Rightarrow$ representation of a group

What is the character?

$N = (1,0)$ local coord: $z_2/z_1 \sim [z_2 : z_1]$, $z_2/z_1 = z$

$S = (0,1)$ local coord: $z_1/z_2 \sim [z_1 : z_2]$, $z_1/z_2 = 1/z$



Choose chamber (cocharacter): $a_1 < a_2$

What is everything at $\lim_{\to\infty}$ goes to S? This is N = 1st cell in decomp

1 dim 2 = attractor of S (everything but NP)

1 dim 0 = attractor of N (just NP)

Determine fixed pts

Then fix flow

Ex) $\mathbb{CP}^2$, $\mathbb{C}^3 \circlearrowleft (\mathbb{C}^\times)^3$: $(z_1, z_2, z_3) \mapsto (z_1 a_1, z_2 a_2, z_3 a_3)$

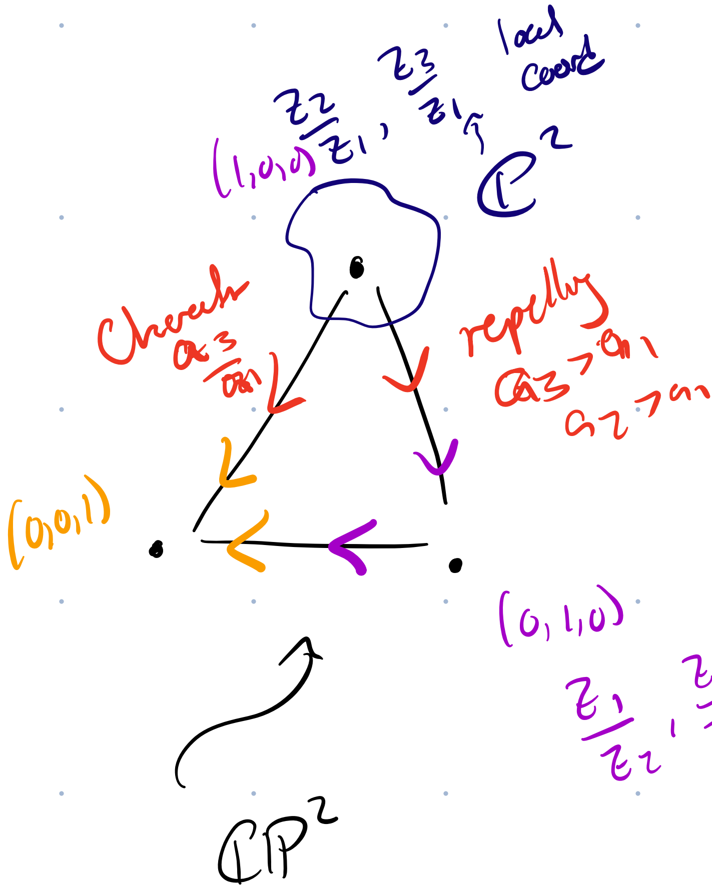
a_1, a_2, a_3

Induces action on $\mathbb{CP}^2 = \mathbb{P}(\mathbb{C}^3)$

fixed pts = coord. lines $(1,0,0) \sim (a_1,0,0)$ same up to rescaling

$(0,1,0)$

$(0,0,1)$



Choose $a_1 < a_2 < a_3$

Toric representation

Toric picture

Attracting set of $(1,0,0) = (1,0,0)$

$(0,1,0) = $ line $\mathbb{CP}^1 \setminus \{pt\}$
$\qquad\qquad\qquad\qquad\qquad \parallel$
$\qquad\qquad\qquad\qquad\qquad \mathbb{C}$

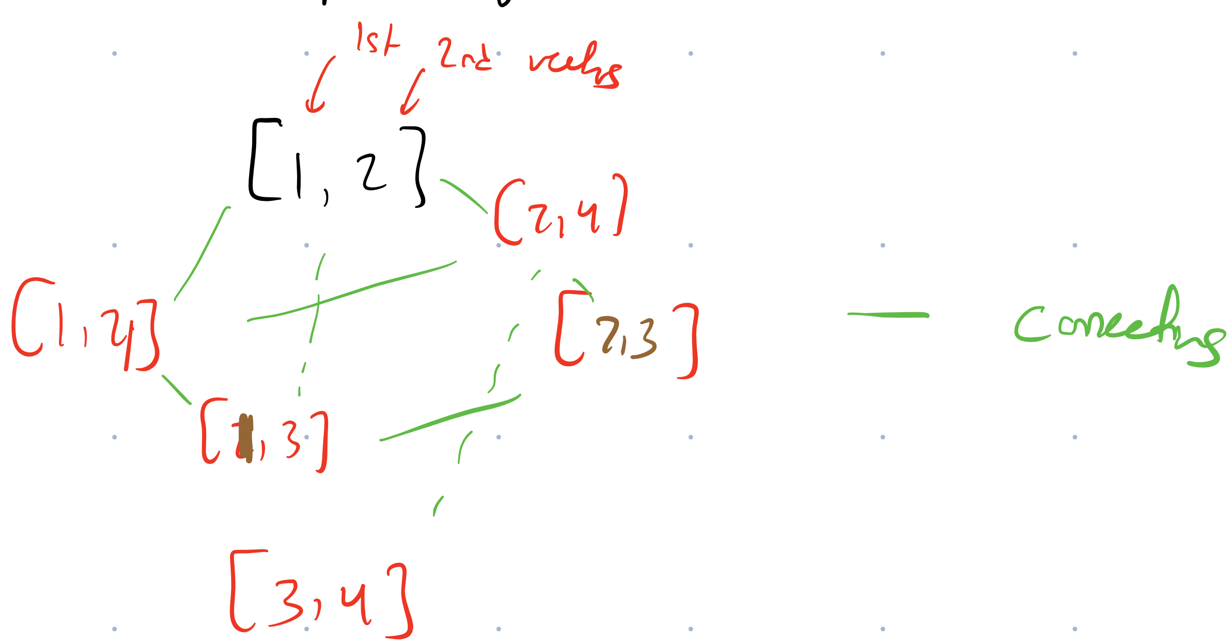
$(0,0,1) = \mathbb{C}^2$

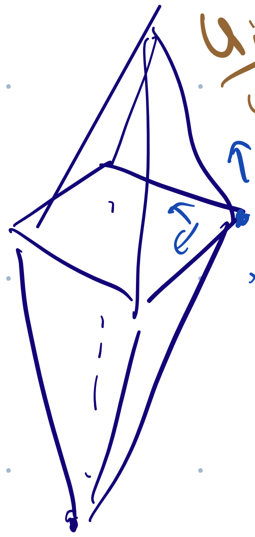
again the cell decomp

$Gr(2,4) \circlearrowleft (\mathbb{C}^\times)^4$ (equiv parameters — coordinates on torus action)

$\mathbb{C}^4 \circlearrowleft \quad (z_1, z_2, z_3, z_4) \to (z_1 a_1, z_2 a_2, z_3 a_3, z_4 a_4)$

Every coord plane gives a fixed pt $\binom{4}{2} = 6$





u_i / u_j

↑ dim of G_r = # lines at of a fixed pts

↓ 4-dim

∃ line between fixed pts if [x, y] & [u, v] share an element e.g.

$[1,4] \longleftrightarrow [1,3]$

both share 1

but no line

[1,2] & [3,4]

no shared numbers

(ratio of uncommon elements)

$\frac{q_2}{q_4}$ [1,2]

$\frac{q_4}{q_2}$

[1,4]