Feynman diagrams in equiv. enumerative geometry

Enumerative geometry is the study of counting geometric objects.
For me, this is counting certain curves into a special kind of variety
In particular K-theory, which is an invariant much like homology, allows us to use algebraic topology to study these counts in interesting ways.

<u>Plan:</u>
- What kinds of varieties are we going to study?
- What is the K-theory of these objects & why should you care? (Bethe equations, interesting algebraic actions)
- What are we counting?
- How to diagrammatically model these objects (i.e. Feynman diagrams)

Part 1: Nakajima Varieties

In the 90's Hiraku Nakajima discovered an amazing connection between QFT & certain geometric objects arising from graph theory.

Def: A quiver is a directed graph

Ex:

$$
\begin{tikzcd}
v_1 \arrow[r] & v_2 \arrow[r] & v_3 \arrow[r, shift left] \arrow[r, shift right] & v_4 \arrow[loop right] \\
\boxed{w_1} \arrow[u] & \boxed{w_2} \arrow[u] & \boxed{w_3} \arrow[u] & \boxed{w_4} \arrow[u]
\end{tikzcd}
$$

Taking this quiver we may assign

$$\{\text{vertices } v_i\} \longleftrightarrow \left\{ \begin{matrix}\text{Vector spaces } V_i \\ \dim V_i = v_i\end{matrix} \right\}$$

$$\{w_i\} \longleftrightarrow \{W_i,\ \dim W_i = w_i\}$$

$$\left\{ \begin{matrix}\text{edges} \\ \overset{v_i}{\bullet} \longrightarrow \overset{v_j}{\bullet}\end{matrix} \right\} \longleftrightarrow \{ \mathrm{Hom}(V_i, V_j) \}$$

$$\mathrm{Rep}_Q(v) = \bigoplus_{i \to j} \mathrm{Hom}(V_i, V_j) \oplus \bigoplus_i \mathrm{Hom}(W_i, V_i)$$

Quiver Rep Theory studies group actions of these spaces

We want a geometric object. It turns out the way to do this is

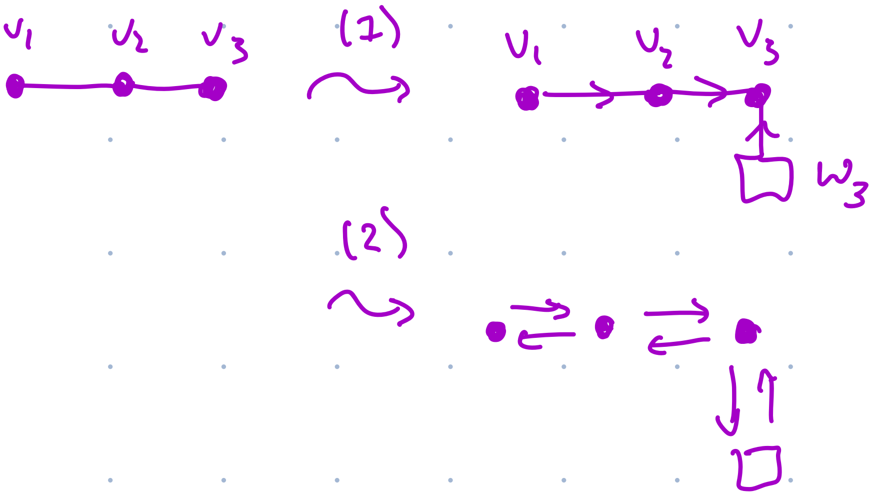
1) Add "framing vertices" and arrows
   ↳ vect. space W_i w/ $\dim W_i = w_i$

2) Double all arrows

See Ginzburg's notes for precise reason

Answer: Nonempty quotients & cotangent bundles

Ex:



Now the rep space

$$T^*\mathrm{Rep}_Q(v,w) = \bigoplus_{i\to j}\mathrm{Hom}(V_i,V_j)\oplus\bigoplus_i \mathrm{Hom}(W_i,V_i)$$

$\oplus$ duals

Actually cotangent bundle

Group action $\quad G_v = \prod_i GL(V_i)$

$$G_w = \prod GL(w_i)$$

$\exists$ Moment Map

$$\mu : T^* Rep_Q(v,w) \longrightarrow \mathfrak{g}^* = (\mathrm{Lie}\ G_v)^*$$

Traverse the arrows to get the formula

The last important piece is something called a Stability parameter

Again it's a super important part but why would take too long. It's a Map

$$\theta : G_v \longrightarrow \mathbb{C}^*$$
$$g_i \longmapsto \prod \det(g_i)$$

Changing $\theta \longleftrightarrow$ Different Nakajima Variety

So what is a Nakajima Variety?

"twisted GIT quotient"

$$\boxed{X = \mu^{-1}(0) /\!/_{\theta} G_v}$$

If you've never seen these before here are some examples:

Ex: $T^*Gr(k,n)$, $T^*\text{Partial Flag}$, $Hilb^n(\mathbb{C}^2)$

$w = (0,\dots,0,n)$

$\theta = \pm(1,\dots,1)$

There is a torus action $T = \underbrace{A}_{\text{Max torus of } Aut(X)} \times \underbrace{\mathbb{C}^\times_\hbar}_{\text{Scaling symplectic form}} \curvearrowright X$

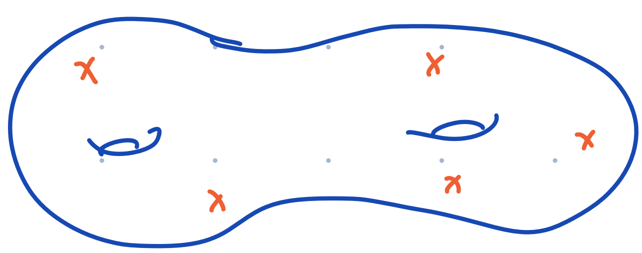
$(\uparrow, \downarrow) = (\uparrow, \underbrace{\hbar\downarrow}_{\text{Scale opposite arrows}})$

What are we Counting?

A: Quasimaps

$\underbrace{\text{Nakajima Varieties}}_{\text{Stable locus Smooth, symplectic}} \underset{\text{dense open subset}}{\subseteq} \mathfrak{X} = \left[Rep\, G_v / G_v \right]$

A quasimap

$C =$  $\xrightarrow{f} \mathfrak{X}$

$\mathfrak{X} \supseteq X$

"Def": Maps $f: C \to \mathcal{X}$ are called quasimaps if $f(p) \in X$ for all but finitely many pts $p \in C$. Pts $p \in C$ where $f(p) \notin X$ are called singularities of f

$$QM^d(X) = \left\{ C = \mathbb{P}^1 \overset{f}{\dashrightarrow} X \;\middle|\; \underset{d \in H_2(X,\mathbb{Z})}{\deg(f) = d} \right\}$$

$\exists$ well-defined $\hat{\mathcal{O}}^d_{vir} \in K_G(QM^d(X))$

We are really in each of formulas counting the "number of degree d quasimaps"

$\chi(\hat{\mathcal{O}}^d_{vir} \otimes \cdots)$ $\left[= \text{analog of } \int_{QM^d} (\text{volume form}) \text{ in cohomology} \right]$

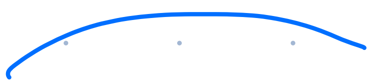
natural classes on $QM^d(X)$

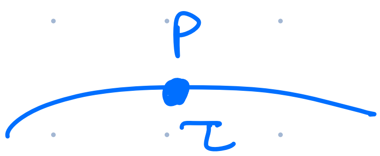
There's an evaluation map

$$\begin{array}{ccc} QM^d(X) & \xrightarrow{ev_p} & \mathcal{X} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f & \longmapsto & f(p) \end{array}$$

If $\tau \in K_T(\mathcal{X})$ then $ev_p^*(\tau) \in K_T(QM^d(X))$

There are useful shorthand notation for describing quasimaps "Feynman Diagrams"

 – Component of source curve $C = \mathbb{P}^1$



– $ev_p^*(\tau)$ descendant insertion

$G = GL(n)$

$$K([X] = T^*V/G) \simeq K([pt/G]) \simeq K_G(pt) = Rep(G) = Char(G)$$

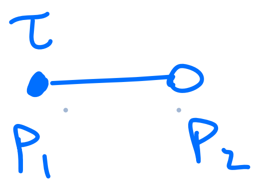
Another structure

$$
\begin{array}{c}
QM^d(X) \\
\cup \\
QM^d_{ns\,p}(X) = \{ f \in QM^d : f(p) \in X \} \\
\Big\downarrow {\scriptstyle ev_p} \\
X
\end{array}
$$

$= \; ev_{p_*} : K_G(QM_{ns\,p}) \longrightarrow K_G(X)$

$+$ Sum over all degrees

<u>Ex:</u>



$$V^{(\tau)}(z) = \sum_d ev_{p_2,*}\left(QM^d_{ns\,p},\; ev^*_{p_1}(\tau) \otimes \hat{\mathcal{O}}^d_{vir} \otimes \cdots \right) z^d$$

$$\in K_{T\times \mathbb{C}_q^\times}(X)_{loc}[[z]]$$

Vertex function w/ descendent τ

<u>Important pts:</u>

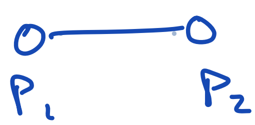
1) $G = T \times \mathbb{C}_q^\times$

rotation of $\mathbb{P}^1$ symmetry

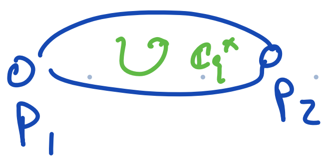
2) $ev_{p*} : K(QM_{nsp}) \longrightarrow K(X)$ not defined

it's nonproper. To fix this we need to use equivariant localization, this is where $\mathbb{C}_q^*$ comes in.

This is why we land in localized equivariant K-theory.

Claim: 

would only get constant quasimaps



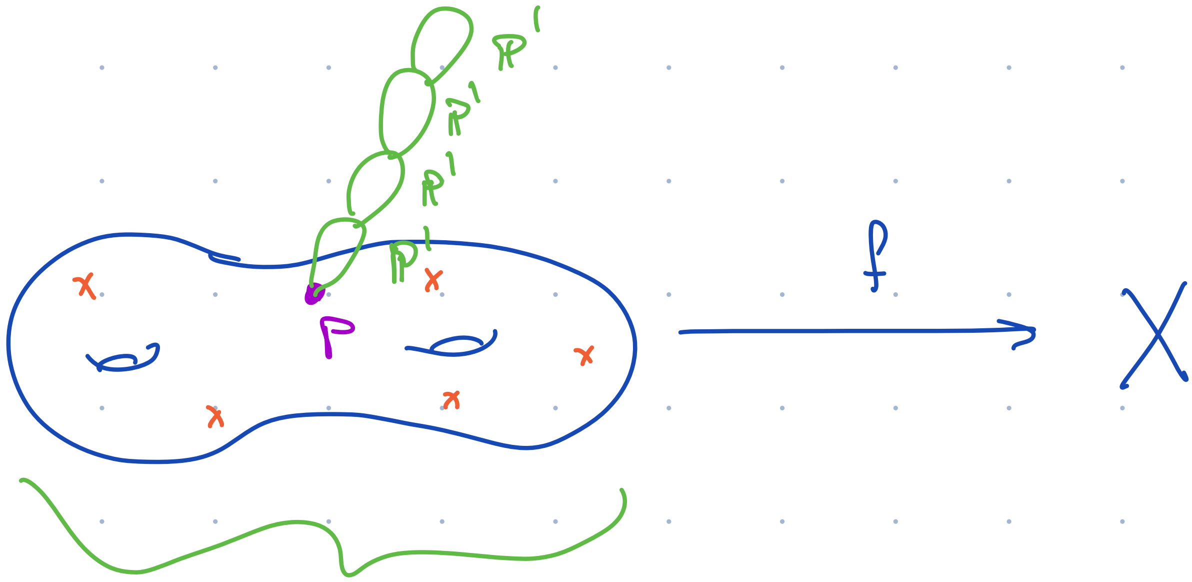
No singularities on $\mathbb{P}^1 \setminus \{p_1, p_2\}$

So we need localization, but the upshot is that this means $V^{(\tau)}$ is easy to compute w/ localization

In fact, these are solutions of certain q-difference equations (q-differential in H_T) & generalize ${}_{n+1}\phi_n$ hypergeometric functions.

How do we avoid localization?

Relative Quasimaps

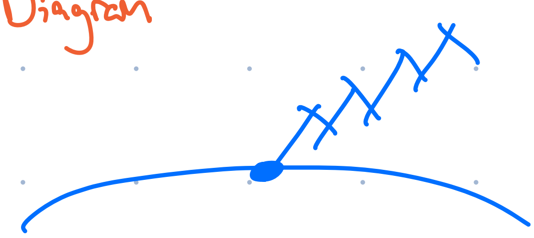


$\tilde{C} \xrightarrow{\pi} C$

collapsing the bubbles

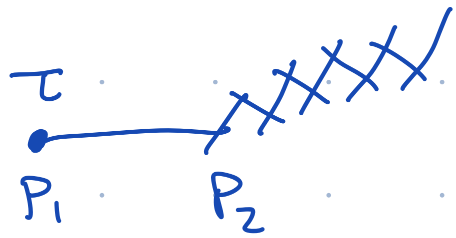
$$\begin{array}{c} QM^{d}_{rel\, p} = \{ \tilde{C} \dashrightarrow X \} \\ \text{proper} \Big\downarrow ev_p \\ X \end{array}$$

Feynman Diagram



$:= ev_{p_*} +$ sum of all degrees

__Ex:__



$$\hat{V}^{(\tau)}(z) := \sum_d ev_{p_2,*}\left(QM^d(X)_{\mathrm{rel}\, p_2},\ ev_{p_1}^*(\tau) \otimes \hat{\mathcal{O}}_{\mathrm{vir}}^{d} \otimes \cdots \right) z^d$$

$$\in K_T(X)[[z]]$$

nonlocalized K-theory

This is very difficult to compute since we don't have localization, and is the main focus of my research.

Many deep properties:

__Conjecture:__ This is a rational function

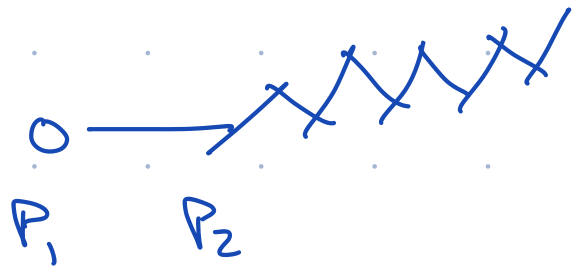
Thm [Pandharipande - Pixton, '13]: This is rational for the cohomological version in the case of certain stable pairs

Thm [Smirnov, '16]: This is true for $X = \mathcal{M}(n,r)$

Thm [A - Smirnov, '24] This is true for $X = \mathrm{Hilb}^n(\mathbb{C}^2)$ and compute an explicit combinatorial formula for this $\hat{V}^{(\tau)}$

Thm [A - Dinkins, To appear soon] This is true for $X = \mathrm{Hilb}^n([\mathbb{C}^2/\mathbb{Z}_n\mathbb{Z}])$ and compute an explicit combinatorial formula for $\hat{V}^{(\tau)}$

Ex:



$$\sum_d ev_{p_1,*}\otimes ev_{p_2,*}\left(QM^d_{\substack{ns\ p_1\\ rel\ p_2}}(X),\ \hat{\mathcal{O}}^d_{vir}\otimes\cdots\right)z^d \in K_T^{\otimes 2}(X)_{loc}[[z]]$$

$$||$$

$$\Psi(a,z) \qquad \text{"capping operator"}$$

$\Psi(a,z)$ is the fund. solution matrix of q-diff equations in both a & z

a: qKZ

z: dynamical equations

"a is a spectral parameter arising from RT"

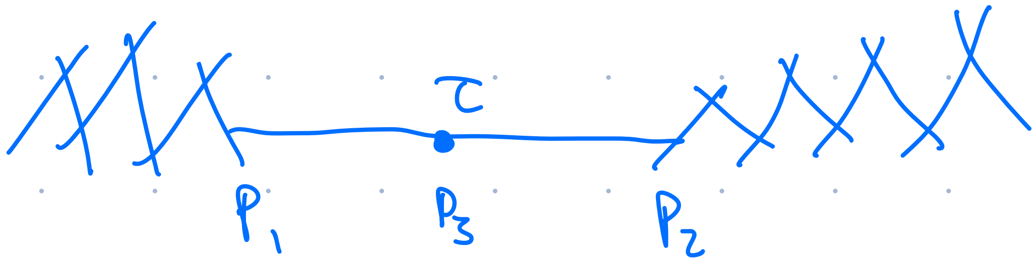
There is a nice symmetry between a & z called 3D - Mirror Symmetry

Why are diagrams useful? Bare vertex is easy to compute, capped isn't b/c no localization but

$\Longrightarrow$

Very Very important thm in K-theoretic Enum. Geo proven by Okounkov. It connects quasimap geometry, geometric rep theory & mathematical physics into one equation.

Ex:



$$G = \sum_d ev_{p_1,*} \otimes ev_{p_2} \left(QM^d_{rel\, p_1, p_2}(x),\ ev^*_{p_3}(\tau) \otimes \hat{Q}^d_{vir} \otimes \cdots \right) z^d$$

"Glue Matrix"

Important Connection to RT: $G = \tau * -$

quantum multiplication by τ, these operators commute & form Bethe algebra

In particular they're H_τ = Hamiltonian whose eigenvalues are controlled by τ for an XXZ-spin chain

Integrable Spin chains & connections to geometry

$$\underbrace{U_q(\hat{g})}_{\text{quantum loop group}}, \quad \underbrace{V(a)}_{\text{fund rep of } U_q(\hat{g})}$$

$$\text{Fock} = V(a_1) \otimes \cdots \otimes V(a_n)$$

$\exists$ system of commuting Hamiltonians $H_1, H_2, \ldots \curvearrowright \text{Fock}$

$$\text{Fock} = \bigoplus_k \text{Fock}_k$$

$\uparrow$ weight k subspace

$$\text{Fock}_k \overset{\text{stable envelope}}{\simeq} K_T(X_k)$$

X_k is a Nakajima variety

Under stable envelopes operators of quantum mult

$\longleftrightarrow$ Hamiltonians