Ricci Flat Manifolds

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1 Ricci Flat Manifolds

Ricci flat manifolds are a vast area of research spanning much of modern day geometry and analysis. The study of these manifolds have resulted in some of the most impressive advances in General Relativity, and Differential Geometry. Even today certain Ricci flat manifolds are though to be the reason why String Theory works so well. I wanted to outline these two crucial areas; General Relativity, and Differential Geometry related to Theoretical Physics. To start with the former we specialize the types of manifolds we are interested in.

Definition (Pseudo-Riemannian Manifold)**.** A pseudo-Riemannian manifold is a pair (M, g) where M is a smooth manifold, and g is an everywhere non-degenerate, smooth, symmetric metric tensor.

Note that we just leave off the condition of the metric being positive definite, this relaxation allows for some very interesting structure. Now we define Ricci Curvature from a bilinear form perspective.

Definition (Ricci Curvature/Ricci Flat Manifolds)**.** The Ricci Curvature for a pseudo-Riemannian manifold is

$$\mathrm{Ric}_p = Tr(Z \to R(X, Z)Y)$$

If the Ricci curvature of a pseudo-Riemannian manifold (M, g) is zero we call (M, g) a **Ricci Flat Manifold**

Our above definition of Ricci curvature via trace is equivalent to the one given in class via an orthonormal basis. We now investigate General Relativity, and the Ricci flat manifolds that arise in this field.

2 General Relativity

Definition. A pseudo-Riemannian manifold (M, g) is called **Lorentzian** is the metric has signature (1,n-1)

Ricci flat Lorentzian manifolds are of fundamental importance in General Relativity since these are the vacuum solutions of Einstein's Field Equation. To see this we let (M, g) be a 4-dimensional Ricci Flat Lorenzian Manifold. Then we have Einstein's Field Equation.

Theorem. Einstein's Field Equations

$$\mathrm{Ric} - \frac{1}{2}Kg = T$$

Let's break down this statement. First it would be dishonest of me to say that this is the full Field Equation of Einstein. As one would see in any article on this equation the full Field Equation is

Theorem.

$$\mathrm{Ric} - \frac{1}{2}Kg + \Lambda g = \frac{8\pi G}{c^4}T$$

Each term in the above equation has importance.

- The $\mathrm{Ric} - \frac{1}{2}Kg$ is always included in statements about this equation. Often this is packaged as "Einstein's Tensor", usually denoted G. Ric is the Ricci tensor, and K is the scalar curvature.
- Λg is a constant Λ times the metric g, this constant is called the cosmological constant. Originally this was not included in Einstein's paper, later it was added in, but again taken out to suggest a non-expanding universe. As is now well known the universe is expanding so this term is typically now included. For us though it will always be zero since Ricci Flat manifolds are those with $\Lambda g = 0$, since they are special kinds of Einstein Manifolds.
- The term $\frac{8\pi G}{c^4}$ is a constant of gravitation, it's equal to $2.077 \times 10^{-43}N^{-1}$. G is Newton's gravitation constant, $G = 6.674 \times 10^{-11}m^2kg^{-1}s^{-2}$. c is the speed of light in a vacuum, $c \approx 300,000km/s$. This is a constant, and so we can renormalize this to just be 1, it will not affect what will happen.

The last term deserves a lengthy explanation. T is called the stress/energy tensor. Physically speaking it encodes the energy momentum of matter. Mathematically there is no definite formulation of T, rather we have a few requirements for this tensor:

- It's a symmetric 2-tensor
- The energy density measured by u is $T(u, u)$ for $u \in T_pM$
- $div(T) = 0$, this physically represents conservation of energy-momentum.

With this in mind, we now say more about this $\mathrm{Ric} - \frac{1}{2}Kg$. As mentioned above Einstein used physical models to get this, one of them being that if something were to equal T, and $div(T) = 0$, then the right hand side should have 0 divergence. Now we show that in fact this is the case for the right hand side.

Proposition. $div(Ric) = \frac{1}{2}\nabla S$

Proof. The idea: Use the second Bianchi identity and express each term in coordinates, getting to a form by switching indices yields the result. See [3] page 88 for details. $\square$

With all this in mind we now show that Ricci-Flat manifolds are solutions of Einstein's equation in a vacuum.

Proof. First, since we are working with a vacuum we have both that $\Lambda = 0$ and $T = 0$, the former is due to the idea that there is no "expanding universe", the second is because there is no energy density in a vacuum. Einstein's Field Equation becomes

$$\mathrm{Ric} - \frac{1}{2}Kg = 0$$

By a solution to Einsteins' equation we mean a metric g and a stress-energy tensor T, since $T = 0$ we need a metric g that makes the above equation true. By expressing the above in local coordinates:$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}Kg_{\mu\nu} = 0$, and multiplying by the inverse metric $g^{\mu\nu}$, we'll arrive at the desired result.

$$\begin{aligned} g^{\mu\nu}\left(\mathrm{Ric}_{\mu\nu} - \frac{1}{2}Kg_{\mu\nu}\right) &= 0 \\ g^{\mu\nu}\mathrm{Ric}_{\mu\nu} - \frac{1}{2}Kg^{\mu\nu}g_{\mu\nu} &= 0 \\ Ric^{\mu}_{\mu} - \frac{1}{2}K \cdot 4 &= 0 \\ K - 2K &= 0 \\ K &= 0 \end{aligned}$$

Plugging this back into the original Einstein's Equation yields $Ric = 0$, meaning that Ricci-Flat manifolds are exactly the solutions of the Field Equation in a vacuum. $\square$

Although we've now seen that Ricci-flat metrics are the solutions of the Field Equation in vacuum, we haven't seen an explicit Ricci-flat metric which solves this equation. One of the most famous solutions of the $\Lambda = 0$ Einstein equation $\mathrm{Ric} = 0$ is due to Schwarzschild in 1916. The solution models the gravitational field outside an isolated static spherically symmetric star. The set up is the following: Let $M = \mathbb{R} \times I \times S^2$ with any warped product metric

$$g = F^2(\rho)\,dt^2 - d\rho^2 - G^2(\rho)\,d\sigma^2$$

Where I is an open interval to be specified later, t and ρ are coordinates on $\mathbb{R}$ and I, $d\sigma^2$ is the constant curvature 1 metric of the sphere, and t denotes time. The warped product metric is so called because we "warp" the sphere as t increases. The solution of this problem is due to Schwarzschild:

$$g = \left(1 - \frac{2m}{\tilde{\rho}}\right)dt^2 - \left(1 - \frac{2m}{\tilde{\rho}}\right)^{-1} d\tilde{\rho}^2 - \tilde{\rho}^2\,d\sigma^2$$

This resolution has a few new terms: $\tilde{\rho}$ is the radical coordinates of a sphere centered around a massive body, $2m$ is the Schwarzschild radius of the body. A full derivation of this is given in [1], but we give an outline of the idea: One can show that the Ricci tensor is diagonal relative to the product metric on M, so one only needs to compute sectional curvatures and sum them up to get the diagonal terms. Once these are computed we know that we get the eigenvalues of the Ricci curvature. From this solving Einstein's Field Equation $\mathrm{Ric} = 0$ for the warped product metric amounts to solving equations which yield the coefficents of the warped product metric to be the Schwarzschild metric.

3 Calabi's Conjecture and Yau's Resolution

Now we move onto the second part, on Differential Geometry. We start with a definition.

Definition (Flat Manifold)**.** A Riemannian Manifold (not necessarily pseudo) is called **flat** is the Riemann Curvature Tensor is 0

Clearly from the definition of Ricci-Flat manifolds, all flat manifolds are Ricci-Flat, but does the converse hold? The answer is yes, but finding such metrics are in general very hard. The Schwarzschild metric is one such metric, Kerr gave another such metric in 1963. To help with this search we restrict our question: Is every Ricci-flat Riemannian metric on a closed manifold flat? Again the answer is complicated, so complicated that the resolution resulted in a Fields Medal.

This begins with Calabi's conjecture first posed in 1954. It states the following:

> If the first Chern class vanishes there is a unique Kähler metric in the same class with vanishing Ricci curvature.

There are a lot of words here. First, the first chern class is a special sequence of elements of the second Cohomology group: $H^2(X, \mathbb{Z})$, where X is the Riemannian manifold.
A Kähler metric is a metric arising in the following way:

Definition. A Hermitian metric on a Riemannian manifold M is g such that

$$g(Jx, Jy) = g(x, y)$$

Where J is a smooth map such that $J^2 = -Id$

Definition. The Kähler form of a Hermitian metric is the 2-form associated to the metric, it's equal to -1/2 the imaginary part.

Definition. A Hermitian metric is a Kähler metric is the complex operator J is parallel to the Levi-Civita connection: $\nabla J = 0$

Corollary. A Hermitian metric is a Kähler metric if and only if the Kähler form is closed.

This conjecture was resolved in 1977/1978 by Shing-Tung Yau, who received the Fields Medal, in part due to this work. The proof involved a heavy use of PDE theory, specifically a non-linear PDE, then showed the uniqueness of the solution to give uniqueness of the Kähler metric. Then he actually constructed a solution to this equation to get an example. The result of his proof is that there exist Ricci-flat metrics in the special case of Kähler metrics on closed complex manifolds.

The resulting Ricci-flat manifolds with the Kähler metric and vanishing first Chern class are called **Calabi-Yau Manifolds**. These objects are quite hard to demonstrate examples of, partically because they require a lot of Algebraic Geometry knowledge for interesting ones. Nonetheless they are extremely important, particularly showing up as the proposed "hidden" 6 dimensions in String Theory.

Altogether the study of Ricci-flat manifolds is integral to modern day Physics and Differential Geometry, and their importance is hopefully apparent to anyone who reads this and wishes to work in these fields.

4 Bibliography

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2) Carmo, Manfredo P. Riemannian Geometry. Boston: Birkhäuser, 1993. Print.

3) O'Neill, Barrett. Semi-riemannian Geometry with Applications to Relativity. Burlington: Elsevier, 1983.