Wigner Quasi-probability Distribution of the Cosecant-squared Potential Well

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The Wigner distribution of a particle subject to the cosecant squared potential, $V(x) = Csc^2(x)$, is calculated. Cosecant squared potential is among the non-trivial potential that cannot be solved exactly. However, this potential is known to be the supersymmetric (SUSY) partner potential of the infinite square well. Thus, the position and momentum wavefunctions of Cosecant squared potential for some values of n are evaluated with the use of Mathematica 5.2. With these, the Wigner distribution can be determined. In this work, this distribution is plotted in phase-space wherein negative probabilities are observed. It is shown that several properties of Wigner function are observed to be similar to probability density; i.e. it is called Wigner Quasi-probability distribution.

Cosecant squared is a non trivial potential; it is among the potentials that are not exactly solvable under the conventional calculation using Schroedinger equation, i.e. its wave functions cannot be determined. In this work, however, we show that the probabilty distribution of this potential can be solved through supersymmetry (SUSY) quantum mechanics. It is known in SUSY quantum mechanics that if the eigenenergies and the wavefunctions for a given potential is known, the eigenenergies and wavefunctions of its supersymmetric partner potential can also be determined [1]. The cosecant squared potential qualifies as a supersymmetry partner potential of infinite square well. This means that both potentials share the same eigenenergies except for their ground state and that their wavefunctions are related. Therefore the wavefunctions of the cosecant squared potential can be derived from the infinite square well. Using the wavefunctions of cosecant squared potential, we can derive the corresponding Wigner function [2]. Basing on the Wigner function, the phase space formulation of this distribution can be determined.

It is for the first time that the quasi probability distribution for this potential is graphically represented through the phase-space formulation. It must however compensate for the Heisenberg uncertainty principle; as a consequence the function is not positive definite [3]. Furthermore, the probability densities will be compared graphically to Wigner function.

1. SUSY Quantum Mechanics

In supersymmetric quantum mechanics[1], for a given potential $V_-(x)$, there exist partner potentials $V_+(x)$ with precisely the same eigenvalue spectra except for the ground state. Supersymmetry formulates a partner potential $V_+(x)$ for the given potential $V_-(x)$. In this work we show that the partner potential of the infinite square well is the cosecant squared.

Given the potential

$$\begin{aligned} V(x) &= 0, \quad 0 \le x \le L \\ &= \infty, \quad otherwise \end{aligned} \tag{1}$$

we have the well-known results for the quantized energies

$$E_n = \frac{(n+1)^2 h^2}{8mL^2} \tag{2}$$

and the corresponding wave functions

$$\psi_n = \left(\frac{2}{L}\right)^{1/2} sin\frac{(n+1)\pi x}{L} \quad (0 \le x \le L) \tag{3}$$

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where n=0,1,2,3,....

A superpotential $W(x)$ is used in SUSY mechanics instead of ground state eigenfunctions ψ_0. The superpotential $W(x)$ and ground state eigenfunction ψ_0 are related by

$$W(x) = -\frac{h}{\sqrt{2m}}(\frac{\psi_0'}{\psi_0}) \tag{4}$$

Also, the supersymmetric partner potentials $V_{\pm}(x)$ are given by

$$V_{\pm}(x) = W^2(x) \pm \frac{h}{\sqrt{2m}}W'(x) \tag{5}$$

where $W'(x) = \frac{dW}{dx}$

Formally, using Eq.5, we obtained the supersymmmetric partner potential $V_+(x)$ of infinite square well $V_-(x)$

$$V_+(x) = \frac{h^2[2csc^2(\frac{\pi x}{L}) - 1]}{8L^2m} \tag{6}$$

This is a non-trivial potential which is not exactly solvable in wave function formulation.

Also, the eigenstates of a given potential are related to eigenstates of its partner potential by

$$E_n^{(+)} = E_{n+1}^{(-)} \tag{7}$$

$$\psi_n^{(+)} = [E_{n+1}^{(-)}]^{-1/2} A \psi_{n+1}^{(-)} \tag{8}$$

where $\psi_{n+1}^{(-)} = \psi_{n+1}$ and $E_{n+1}^{(-)}$ is the adjusted energy state as a consequence of shifted potential of infinite square well

$$V_-(x) = V(x) - \frac{h^2}{8mL^2} \tag{9}$$

so that $E_0^{(-)} = 0$. Figure 1 shows the comparison of the eigenstates and eigenvalues for the infinite square well and the $csc^2(x)$ potential.

The A in Eq.8 is a defined operator that converts a wavefunction of a given potential to its partner potential's wavefunction with the same energy

$$A = \frac{\hbar}{\sqrt{2m}}(\frac{d}{dx} - \frac{\psi_0'}{\psi_0}) \tag{10}$$

Using Eq.8, the wavefunction of supersymmetric partner potential $V_+(x)$ is given by

$$\psi_n^{(+)} \propto [(1+n)cos\frac{(2+n)\pi x}{L} - \frac{sin\frac{(1+n)\pi x}{L}}{sin\frac{\pi x}{L}}] \tag{11}$$

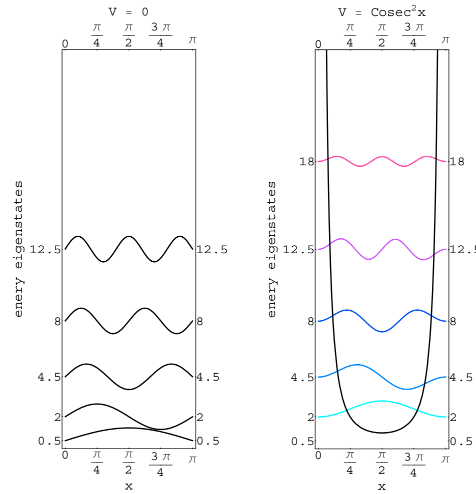

Figure 1. The enegy eigestates of the infinite square well of width π and its SUSY partner potential $Csc^2(x)$.

2. Wigner Function

The Wigner function arises in the context of a relatively less-known formulation of quantum mechanics, the deformation quantization [4]. This quantization developed over the years. Groenewold showed that

$$\vartheta(f)\vartheta(g) = \vartheta(f \star g) \tag{12}$$

where ϑ is known as the Weyl map which takes functions on phase space to operators. Eq.12 means that multiplying two Weyl operators is the same as multiplying the corresponding phase space functions, and then applying ϑ. The $\star$ product takes the form

$$f(x,p) \star g(x,p) = f(x,p)e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x\overrightarrow{\partial}_p - \overrightarrow{\partial}_p\overleftarrow{\partial}_x)}g(x,p) \tag{13}$$

where the arrow indicates which function is to be differentiated. Then we introduce an operator W acting on operators to give phase space functions, $W(\vartheta(f)) = f$ for any $f = f(x,p)$. Having operator G, then we have

$$W(G) = \hbar \int dy e^{-ipy} \left\langle x + \frac{\hbar y}{2} \left| G \right| x - \frac{\hbar y}{2} \right\rangle \tag{14}$$

where $W(G)$ is called the Weyl transform of the operator G. Since the quantum state must be described, we need density matrix $\hat{\rho}$ to specify a mixed state. Mixed state is related to pure state ψ as

$$\hat{\rho} = |\psi\rangle\langle\psi| \tag{15}$$

Finding the Weyl transform of density matrix $\hat{\rho}$ can be declared as

$$W(\hat{\rho}) = \hbar \int dy e^{-ipy} \psi^*(x - \frac{\hbar y}{2}) \psi(x + \frac{\hbar y}{2}) \tag{16}$$

After normalization, it is known as the Wigner function $\rho \equiv \frac{W(\hat{\rho})}{2\pi\hbar}$ or

$$P_W(x,p) \equiv \frac{1}{\pi\hbar} \int_{-\infty}^{+\infty} \psi^*(x+y)\psi(x-y) e^{2ipy/\hbar} dy$$

In terms of the momentum-space function φ, the Wigner function can be written as

$$P_W(x,p) \equiv \frac{1}{\pi\hbar} \int_{-\infty}^{+\infty} \varphi^*(p+q)\varphi(p-q) e^{-2iqx/\hbar} dq$$

The position and momentum probability densities are obtained from P_W by

$$\int_{-\infty}^{+\infty} P_W(x,p) dp = |\psi(x)|^2 \tag{17}$$

$$\int_{-\infty}^{+\infty} P_W(x,p) dx = |\varphi(p)|^2 \tag{18}$$

3. Visualization

From Eq.11, we evaluate the Wigner function for some values of n. We graph the Wigner distribution of time independent one dimensional supersymmetric partner potential of infinite square well, which is the cosecant squared potential, using Mathematica 5.2. The following figures show the plots of Wigner distribution for some energy eigenstates. In Figures 2-4, where only positive probability is shown, "flat" areas are observed; this signifies the locality of small waves due to the non-positive definite Wigner function. Note that the Wigner function is in phase space, according to Heisenberg's uncertainty principle [5], the x (position) and p (momentum) cannot be measured simultaneously. To compensate for this, the Wigner function is not positive definite. Figures 5-6 shows the comparison between momentum space probability densities and the cross sectional views of Wigner function along momentum axes at $x = 0.5$. It is found that this similarity of the Wigner function and probability density is not restricted to momentum axes but also exhibited in the position axes. The oscillations in Wigner function, which even goes into negative values, are explicitly visualized in Figure 6. These measures corresponds to the joint probability distribution of the Wigner function. Negative probabilities take place to reconstruct the correlation between the marginal distribution of position and momentum. For higher energies, $n = 10$ in Figure 7, central spines along the $p = 0$ are shown. This arises from the limits of Wigner function.

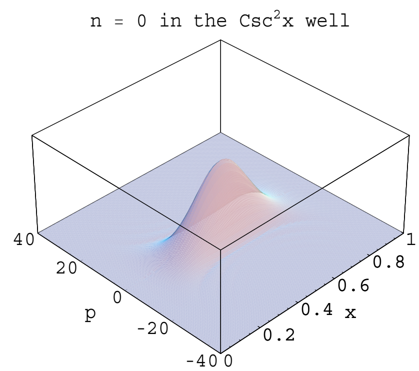

Figure 2. Wigner distribution for n=0 energy eigenstate in the $Csc^2(x)$.

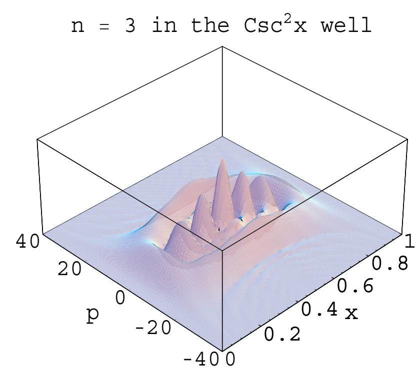

Figure 3. Wigner distribution for n=3 energy eigenstate in the $Csc^2(x)$.

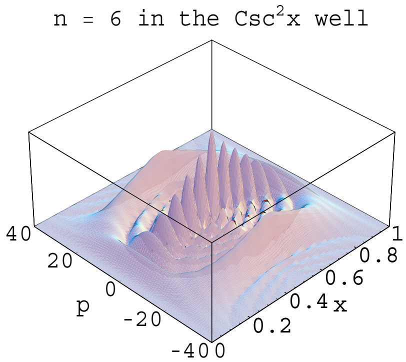

Figure 4. Wigner distribution for n=6 energy eigestate in the $Csc^2(x)$.

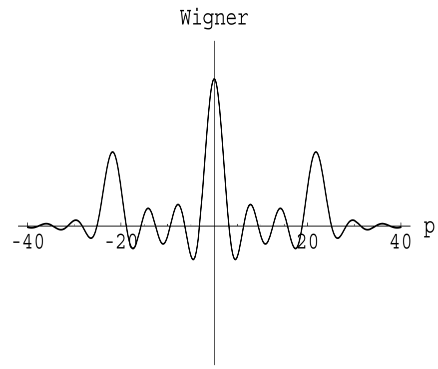

Figure 6. Cross sectional view of Wigner function for n=6 energy eigestate of the $Csc^2(x)$ well at $x = 0.5$.

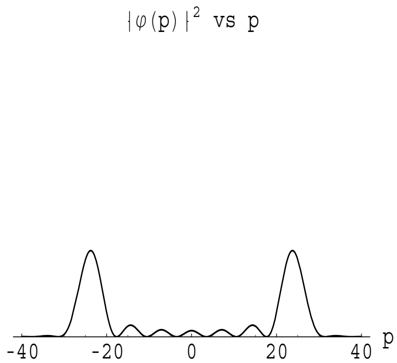

Figure 5. Momentum space probability density for n=6 energy eigenstate of the $Csc^2(x)$ well.

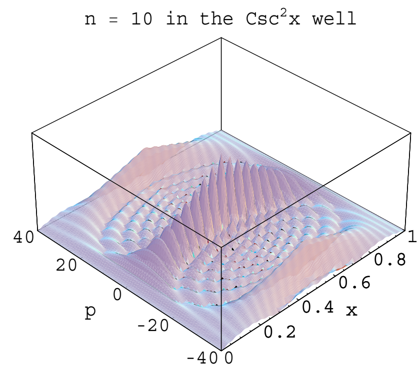

Figure 7. Wigner distribution for n=10 energy eigestate in the $Csc^2(x)$.

4. Conclusion

In this work, we obtained the qualitative description for the Wigner function. It was shown that the Wigner function shares same properties with the probability density only that Wigner experiences negative values. It can therefore be said, like any other probability density, Wigner function can be used to predict the behavior of the wavefunction under a given potential. Moreover, the advantage in using the Wigner function lies in its capability to show the probability distribution along the position and the momentum at the same time, which should have been impossible under Heisenberg's uncertainty principle. This work can be extended in getting the time of arrival of the wavepackets under $Csc^2(x)$.

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