Effect of a Linear Potential on the Temporal Diffraction of Particle in a Box

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Abstract

Diffraction in time of a particle initially confined in a box is studied under linear potential. Moshinsky's shutter problem is generalized to include new initial conditions with linear potential, which show double temporal diffraction for each opposite-moving plane wave with occurrence of reflected wave at later times. Density profiles at transient times and later times are discussed. Twofold Moshinsky's diffraction in time for high energy state of the particle is also analyzed. Classical limit as box width L tends to zero is illustrated.

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1. Introduction

Diffraction in time of matter waves has been studied analytically by Moshinsky [1]. This temporal diffraction for the transient current implies similarity for the intensity of light in the Fresnel diffraction by a straight edge. Moshinky's shutter problem forms the hallmark of the diffraction effect. However, the initial condition of the problem is unrestained with the width of the initial wave function, i.e. particle's momentum is not constant. In accordance with the time-energy uncertainty relation, we want to consider the time evolution of monoenergetic particles which demands an initial finite width in space.

Moshinsky's problem considers the free evolution of wavefunction after the shutter is opened. However, due to experimental imperfections, the influence of unavoidable potential is significant. Sensitivity of ultra-cold atoms in gravitational field is one [2]. Time evolution of probability density profile of a particle has been analyzed in the presence of a linear potential showing a profile bifurcation after the semiclassical time t_n [3]. In this work, we deal with the probability densities at transient and later times of Moshinsky's shutter problem with new initial conditions and under a linear potential, i.e a monoenergetic beam of particles initially trapped in an infinite square well, and free at time $t = 0$ with linear potential turned on at that instant. This can reproduce the system of confined monoenergetic particles with a constant electric field switched on simulanteously with the opening of the inifinite square well walls. As the width of the box L approaches zero, we study the classical limit of probability densities.

2. The Shutter Problem with a Linear Potential

Consider the one-dimensional problem of free particles trapped in an infinite square well with perfect reflecting walls at $x = -L$ and $x = 0$. At time $t = 0$, the walls are opened and a linear potential is turned on instantaneously. What will be the transient density observed at $x > 0$?

For non-relativistic particles trapped in an inifinite square well, the stationary wave functions are given by

$$\Psi_n(x,0) = \begin{cases} \sqrt{\frac{2}{L}} Sin\,(k_n x) & for \quad -L \leq x \leq 0 \\ 0 & Otherwise \end{cases} \tag{1}$$

where $k_n = \frac{n\pi}{L}$, $n = 1, 2, 3....$

The evolved wave function has an integral form

$$\Psi(x,t) = \int_{-\infty}^{\infty} K(x,t|x',t')\Psi(x',0)dx' \tag{2}$$

where $K(x,t|x',t')$ is the propagator for a Hamiltonian with a linear potential given by

$$\hat{H} = \frac{p^2}{2m} + fx, \tag{3}$$

where f is constant. The propagator of this potential is known [4],

$$K(x,t|x',t') = \sqrt{\frac{m}{2\pi i\hbar(t-t')}}e^{i\frac{m(x-x')^2}{2\hbar(t-t')}-i\frac{f(x+x')(t-t')}{2\hbar}-i\frac{f^2(t-t')^3}{24\hbar}} \tag{4}$$

Plugging (4) and (1) into (2), we have

$$\Psi_n(x,t) = \sqrt{\frac{m}{4\pi i^3\hbar tL}}\left[\int_{-L}^{0} e^{\frac{im}{2\hbar t}(x-x')^2-\frac{ift}{2\hbar}(x+x')-\frac{if^2t^3}{24\hbar}+ikx'}\,dx' - \int_{-L}^{0} e^{\frac{im}{2\hbar t}(x-x')^2-\frac{ift}{2\hbar}(x+x')-\frac{if^2t^3}{24\hbar}-ikx'}\,dx'\right] \tag{5}$$

Since these integrals are straightforward, wavefunction can be worked out to be

$$\Psi_n(x,t) = \frac{1}{2}\sqrt{\frac{i}{L}}e^{-\frac{ift}{\hbar}x-\frac{ift^3}{24\hbar}-\frac{if^2t^3}{8m\hbar}-\frac{ip^2t}{2m\hbar}} \tag{6}$$

$$X\left\{[F_+(t,x+L)-F_+(t,x)]\,e^{\frac{ifpt^2}{2m\hbar}+\frac{ipx}{\hbar}}+[F_-(t,x)-F_-(t,x+L)]\,e^{-\frac{ifpt^2}{2m\hbar}-\frac{ipx}{\hbar}}\right\}$$

Here, $F_+(t,x)$ is the Fresnel integral for the right-moving plane wave given by

$$F_+(t,x) \equiv \int_0^\alpha e^{\frac{i\pi\alpha^2}{2}}\,d\alpha \tag{7}$$

where

$$\alpha = \sqrt{\frac{m}{\pi\hbar t}}\left(vt - x - \frac{ft^2}{2m}\right) \tag{8}$$

and $F_-(t,x)$ is the Fresnel integral for the left-moving plane wave given by

$$F_-(t,x) \equiv \int_0^\alpha e^{\frac{i\pi\sigma^2}{2}}\,d\alpha \tag{9}$$

where

$$\sigma = \sqrt{\frac{m}{\pi\hbar t}}\left(-vt - x - \frac{ft^2}{2m}\right) \tag{10}$$

The evolving wavefunction, according to (6), is a superposition of two opposite moving plane waves, each exhibiting a temporal diffraction. From this, we can easily uncover the probability density in the form

$$|\Psi_n(x,t)|^2 = \frac{1}{4L}\left|[F_+(t,x+L)-F_+(t,x)]\,e^{\frac{ifpt^2}{2m\hbar}+\frac{ipx}{\hbar}}+[F_-(t,x)-F_-(t,x+L)]\,e^{-\frac{ifpt^2}{2m\hbar}-\frac{ipx}{\hbar}}\right|^2 \tag{11}$$

Theoretically, the free time evolution of monoenergetic particles caused by suddenly removal of infinite square well walls has been discussed by Godoy [5]. But if we have linear potential at which is switched on instantaneously at the time of shutting the walls off, the quantum dynamic is governed by (6) with the corresponding probability density (11).

3. Diffraction in Time

In this section, we visualize the quantum dynamics of a particle initially trapped in an box. Considering a low-energy excited state, $n = 6$, Figure 1 shows diffraction in time of plane waves where T_0 is the classical time of flight of wave coming from $x = 0$ and T_{-L} is the classical time of flight of wave coming from $x = -L$. In Figure 2, times of flight for reflected wave occur because of the presence of linear potential fx. This quantum dynamic for reflected wave is not present for free time propagation.

Now, for the case of high-energy state, $n = 500$, double Moshinsky diffraction in time appears. Figure 3 shows the probability density of plane waves in time under the action of linear potential for this energy state. This time evolution of probability density profile is well defined during transient times

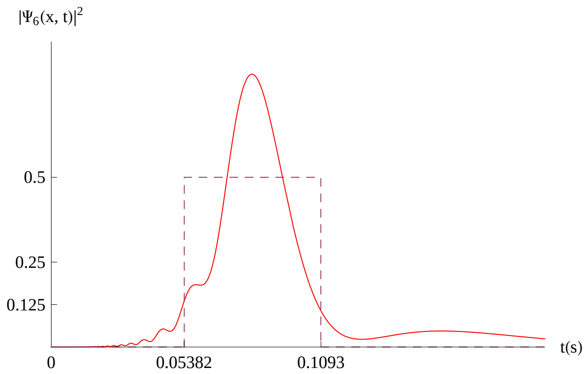

Figure 1: Diffraction in time for state $n = 6$ of incident beam at observation point $x = 1$, with box width $L = 1$ and $f = 10$. The dashed line corresponds to classical density. Classical times of flight $T_0 = 53.82$ ms, $T_{-L} = 109.3$ ms are shown.

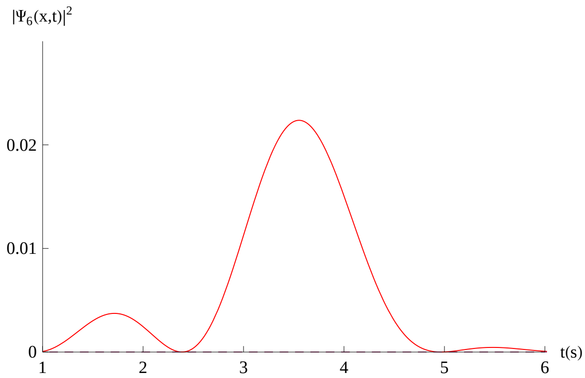

Figure 2: Reflected beam for state $n = 6$ due to potential at observation point $x = 1$, with box width $L = 1$ and $f = 10$. The classical times of flight are $T_{-L} = 3660.64$ ms and $T_0 = 3716.09$ ms.

$$\left(\frac{mv - m\sqrt{v^2 - \frac{2fx}{m}}}{f} \right) \leq t \leq \left(\frac{mv - m\sqrt{v^2 - \frac{2f(x+L)}{m}}}{f} \right) \tag{12}$$

with oscillation around the classical probability density value. After these non-trivial short times, it is steadily decreases to zero. In later time, probability density for reflected wave arises.

Both from Figure 1 and 3, classical probability density leaps rapidly from zero to 0.5 at $t = T_0$, while quantum mechanically a set of oscillations especially in high energy states are illustrated. At $t = T_{-L}$, classical probability density suddenly falls to zero, while quantum mechanical probability density gradually falls to zero as a result of destructive interference made by the arrival of plane wave coming from $x = -L$. This is a double diffaction effect in time generated by the superposition of two plane waves coming from $x = 0$ and $x = -L$.

When we make $L \to 0$, we see from (11) that the probability density profile is turned into a peak as shown in Figure 4. In this case, the energy of the particle approaches infinity. The probability density reduces to classical probability density, as if we shift a wave into its particle counterpart, by making the width of the box as small as possible. This shows the arrival of quantum phenomenon to classical regime.

4. Conclusion

Temporal diffraction of particle initially confined in a box has been investigated under linear potential. Furthermore, reflected wave exists at later times as a consequence of linear potential which switch on at

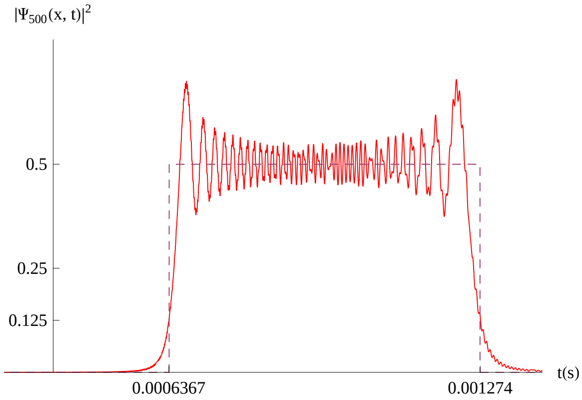

Figure 3: Diffraction in time for state $n = 500$ of an incident beam at observation point $x = 1$, with box width $L = 1$ and $f = 1000$. The dashed line corresponds to classical density. Classical times of flight $T_0 = 0.6367$ ms, $T_{-L} = 1.274$ ms are shown.

Figure 4: Diffraction in time for state $n = 6$ of an incident beam at observation point $x = 1$, with box width $L = \frac{1}{8}$ and $f = 10$. The classical limit of probability density $\left|\Psi(x,t)\right|^2$ is shown by the peak. Classical times of flight are $T_0 = 7.462$ ms, $T_{-L} = 6.633$ ms.

$t = 0$. This attribute is only connected with linear potential. Moreover, since the new initial conditions depend on the width of the box L, a peak is observed in quantum mechanical density as L approach to zero. The classical limit of this quantum system corresponds to $L \to 0$.

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