A Class of PL-Homeomorphism Groups with Irrational Slopes

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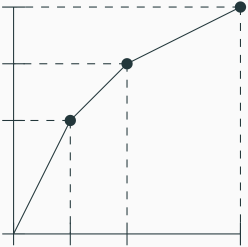
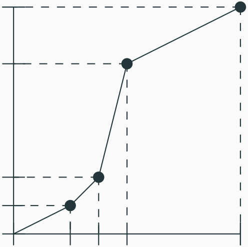
Definition ($\mathcal{F}_\alpha$)

Given $\alpha \in \mathbb{R}^+$, we define $\mathcal{F}_\alpha$ to be the set of homeomorphisms $f : [0,1] \to [0,1]$ satsifying:

1. f is piecewise linear
2. The gradient of f on each linear segment is a power of α
3. f has finitely many singularities , which all lie in the ring $\mathbb{Z}[\alpha, \alpha^{-1}]$





Some Examples

$\mathcal{F}_1$
The trivial group

$\mathcal{F}_2$
Thompson's Group F. This group can equivalently be defined as:

1. The group of associative laws on binary operations [GG06].
2. The automorphism group of elements in the free monoidal category generated by a single object A and an isomorphism $A \otimes A \to A$ [FL10].
3. The group of order preserving automorphisms of the free Jónsson-Tarski algebra on one generator.

$\mathcal{F}_n$ **for** $n \in \mathbb{Z}_{>2}$
These groups share many properties with $\mathcal{F}_2$ [Ste92].

Properties of $\mathcal{F}_2$

- Torsion-free
- Finitely presented
- Infinite cohomological dimension
- Simple commutator subgroup
- Dense in $\mathrm{Aut}([0,1])$
- FP_∞
- Not elementary amenable
- Does not contain the free group of rank 2
- Solvable word problem
- QFA [LAS14]

Many of these properties hold for $\mathcal{F}_n$ for $n \in \mathbb{Z}_{>1}$.

General $\mathcal{F}_\alpha$

Properties of $\mathcal{F}_\alpha$ for all $\alpha \in \mathbb{R}^+ \backslash \{1\}$

- Torsion-free ✓
- Finitely presented × [BS85]
- Infinite cohomological dimension **?**
- Simple commutator subgroup **?**
- Dense in $\mathrm{Aut}([0,1])$ ✓
- *FP*$_\infty$ ×
- Not elementary amenable **?**
- Does not contain the free group of rank 2 ✓ [BS85]
- Solvable word problem **?**
- QFA ×

✓ : True for all $\alpha \neq 1$ × : Not true for some $\alpha \neq 1$ **?** : unclear

Some known results

Let τ denote the unique positive solution to $x^2 + x = 1$: $\frac{-1+\sqrt{5}}{2} \approx 0.61803$

Properties of $\mathcal{F}_\tau$

- Torsion-free ✓
- Finitely presented ✓ [Cle00]
- Infinite cohomological dimension ✓ [BNR18]
- Simple commutator subgroup ✓ [BNR18]
- Dense in $\mathrm{Aut}([0,1])$ ✓
- FP_∞ [Cle00] ✓
- Not elementary amenable ✓ [BNR18]
- Does not contain the free group of rank 2 ✓
- Solvable word problem ✓ [BNR18]
- QFA **?**

Definition (τ_k)
For $k \in \mathbb{Z}_{>0}$ we let τ_k denote the positive solution to the quadratic equation: $x^2 + kx = 1$ which is given by:

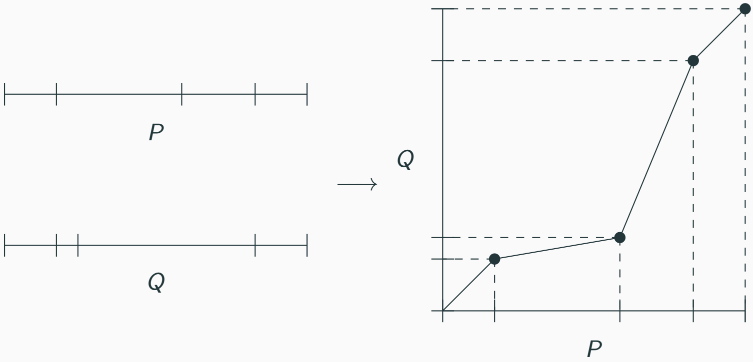
$$\tau_k := -\frac{k}{2} + \sqrt{\left(\frac{k}{2}\right)^2 + 1}$$

$\mathcal{F}_{\tau_k}$ is the group of piecewise-linear homeomorphisms of $[0, 1]$ with slopes powers of τ_k and singularities in $\mathbb{Z}[\tau_k, \tau_k^{-1}]$.

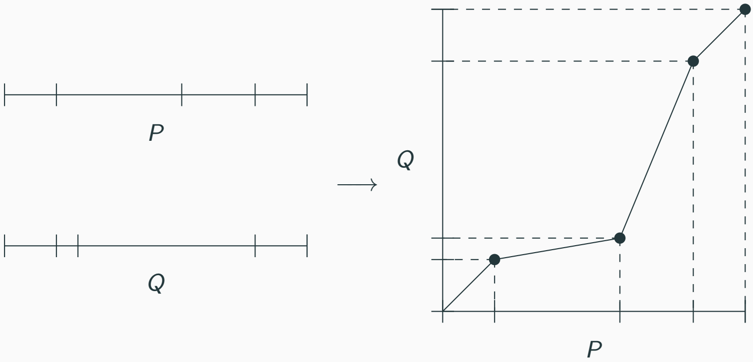
A combinatorial approach

$$\begin{tikzcd}
\text{Elements of } \mathcal{F}_{\tau_k} \arrow[r, squiggly] & \text{Pairs of Trees} \\
\text{Composition in } \mathcal{F}_{\tau_k} \arrow[r, squiggly] & \text{Tree Operations} \arrow[d, "\textit{Combinatorial arguments}"] \\
\text{Presentations and normal form for } \mathcal{F}_{\tau_k} & \text{Generating process for trees and equivalence of trees} \arrow[l, squiggly]
\end{tikzcd}$$

Breakpoint sets



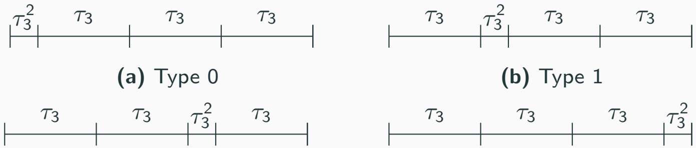
Pairs of subsets of $\mathbb{Z}[\tau_k] \not\twoheadrightarrow \mathcal{F}_{\tau_k}$



$$\begin{tikzcd}
\text{Pairs of } k\text{Subs} \arrow[r, hook] \arrow[rr, bend right, two heads] & \text{Pairs of subsets of } \mathbb{Z}[\tau_k] \arrow[r, two heads, "/" marking] & \mathcal{F}_{\tau_k}
\end{tikzcd}$$

k-partitions

Recall that $\tau_k^2 + k\tau_k = 1$.



(a) Type 0

(b) Type 1

(c) Type 2

(d) Type 3

Figure 3: The four 3-partitions of the unit interval

Definition (k-subdivision)

A k-subdivision is either:

1. $[0, 1]$
2. A collection of intervals obtained by replacing an interval in a k-subdivision with a k-partition of that interval (of any type)

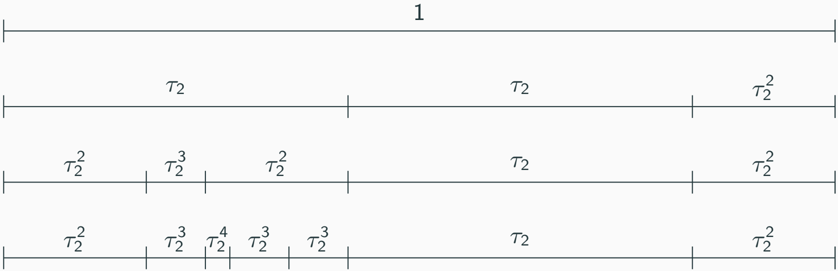


Figure 4: A 2-subdivision obtained by three 2-partitions

k-trees

(a)

(a)

Figure 6: A 2-subdivision obtained by three 2-partitions

(a)

(b)

(a)

(b)

Figure 6: A 2-subdivision obtained by three 2-partitions

k-trees

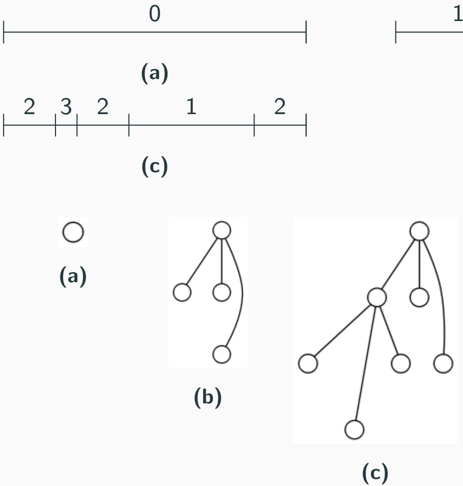
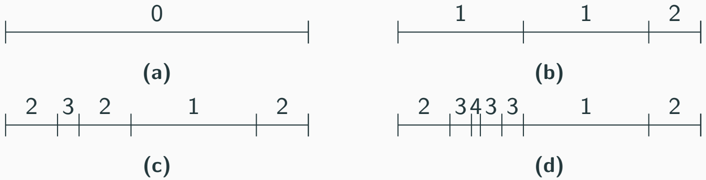


Figure 6: A 2-subdivision obtained by three 2-partitions

k-trees



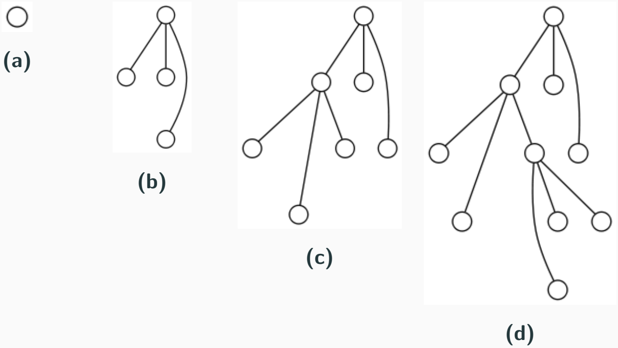
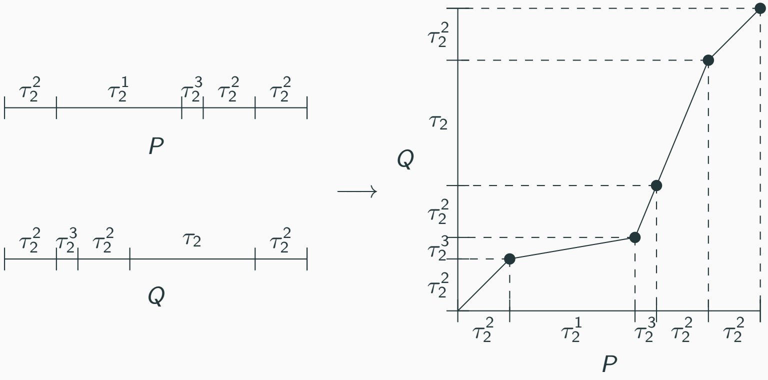


Figure 6: A 2-subdivision obtained by three 2-partitions

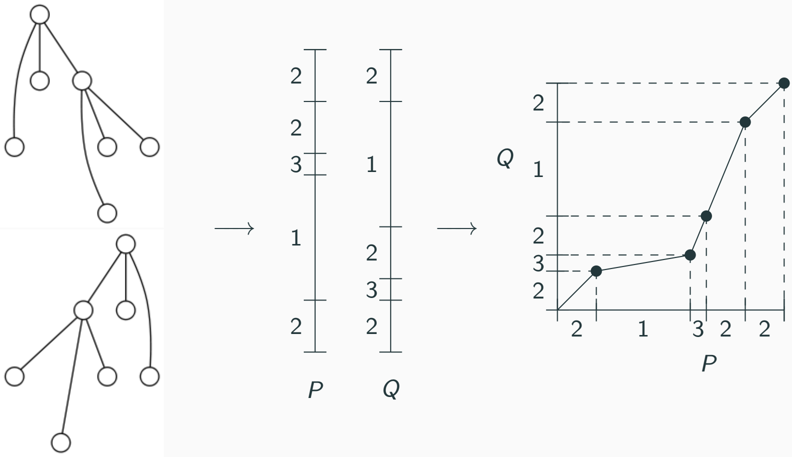
The connection to $\mathcal{F}_{\tau_k}$

Theorem

For any two k-subdivisions P, Q with the same number of breakpoints the interpolation of P and Q is an element of $\mathcal{F}_{\tau_k}$. Moreover, every element of $\mathcal{F}_{\tau_k}$ can be obtained in this way.



From k-trees to $\mathcal{F}_{\tau_k}$



Equivalent k-trees

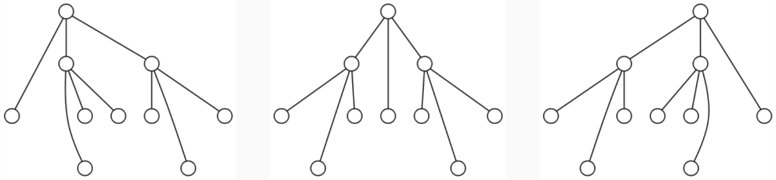
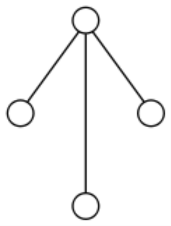


Figure 9: Three distinct 2-trees representing the same 2-subdivision

Definition (Node type)

For a non-leaf node, the number of children to the left of the long child.

(a) Type 0



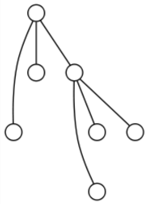
(b) Type 1

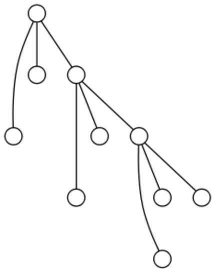
(c) Type 2

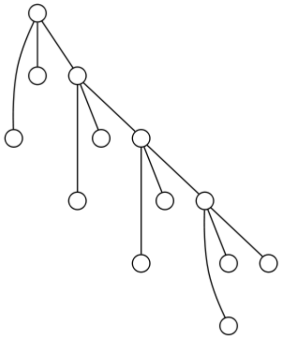
Figure 11: All node types for $k = 2$

Spines

Definition (Spine)
A k-tree where all but the right-most child of each node are leaves. All nodes are of type 0.



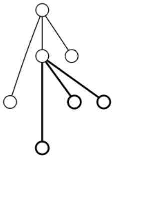




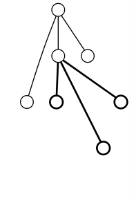
Elementary k-trees

Definition (Elementary tree)

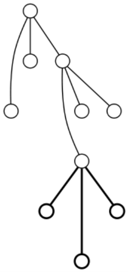
Obtained from a spine by type 0 or 1 expansion a single node which is not the right-most node. Written as $[\mathrm{type}]_{\#\mathrm{node}}$.



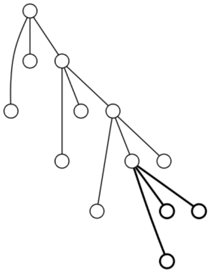
(a) $[0]_1$



(b) $[1]_1$



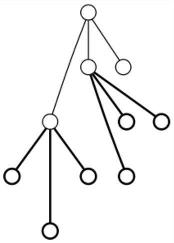
(c) $[1]_2$



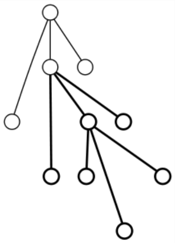
(d) $[0]_5$

Regular k-trees

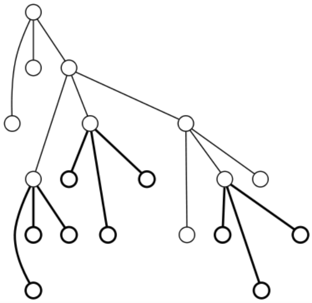
Definition (Regular k-tree)
Obtained from a spine by a sequence of type 0 or 1 expansions of nodes other than the right-most node.



(a) $[0]_1[1]_0$



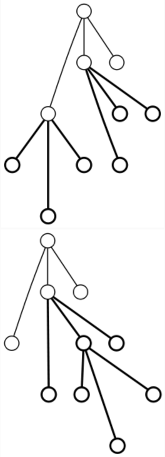
(b) $[0]_1[1]_2$



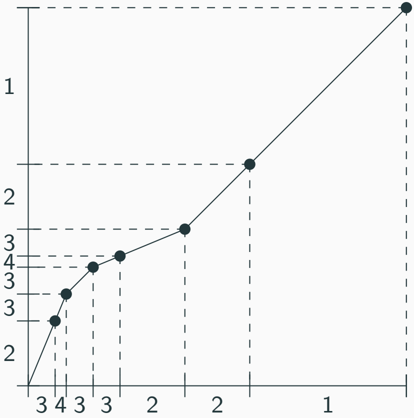
(c) $[1]_5[1]_3[0]_2$

Theorem

Every element of $\mathcal{F}_{\tau_k}$ is representable by a pair of regular k-trees.

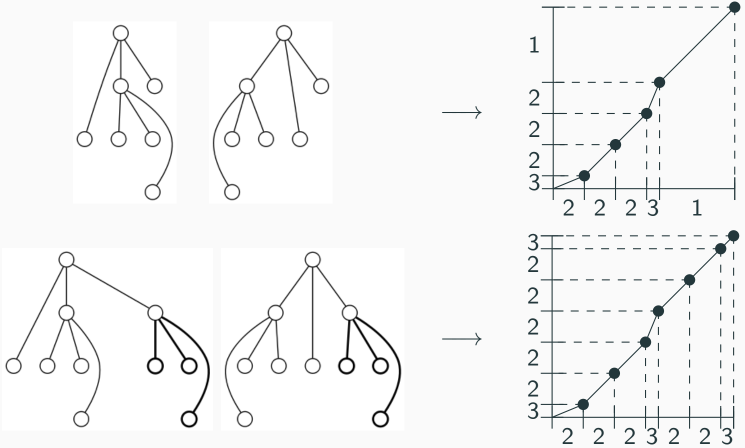


$\longrightarrow$

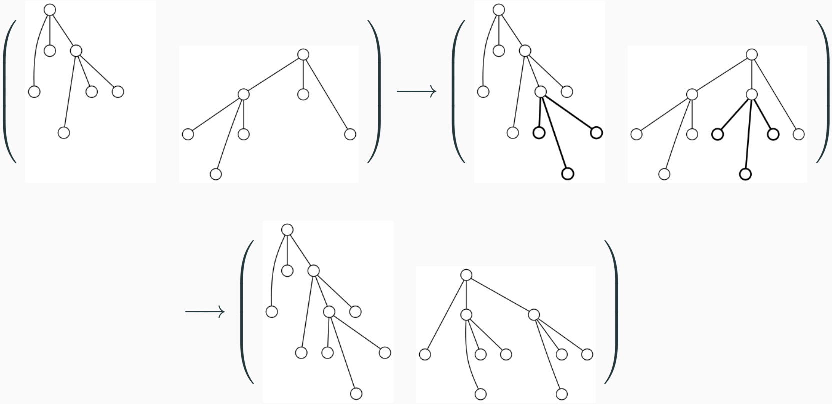


The Proof

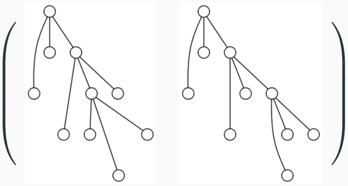
Common expansion leaves the element of $\mathcal{F}_{\tau_k}$ unchanged:



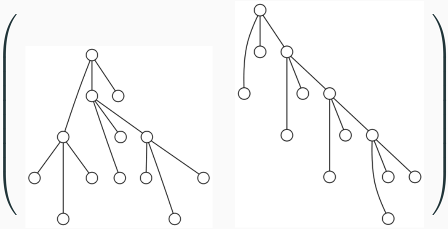
An example



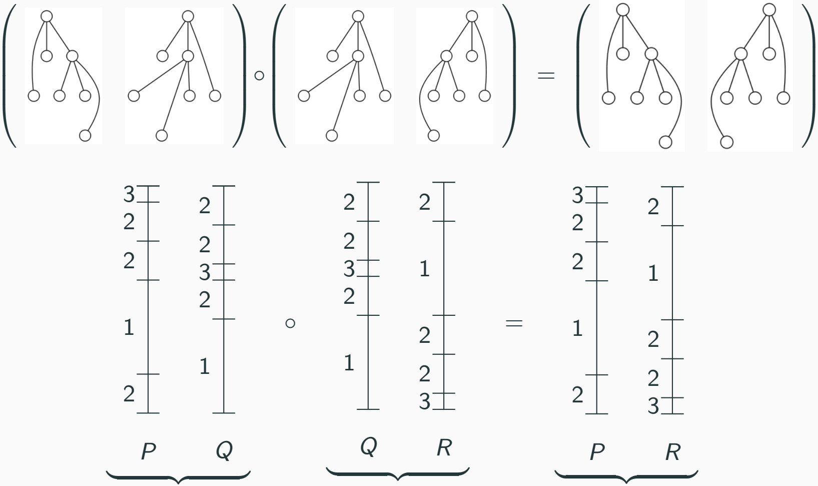
Definition (Spine pair)
Given a k-tree T, we construct its *spine pair* by pairing it with a spine of the appropriate size on the right. Denoted $\mathrm{sp}(T)$.



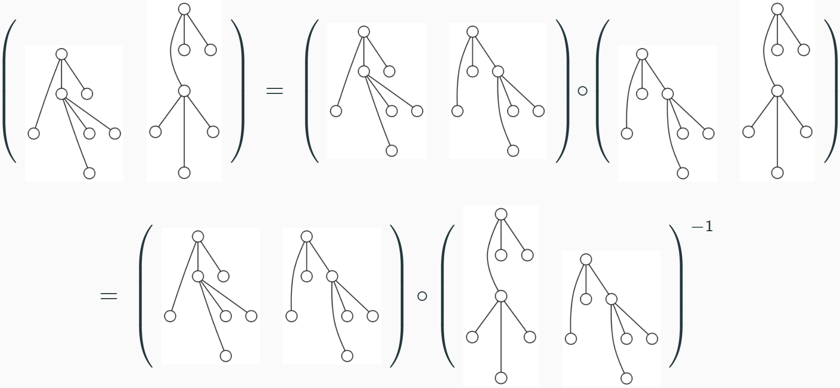
(a) $\mathrm{sp}\big([1]_3\big)$



(b) $\mathrm{sp}\big([0]_1[1]_3[1]_0\big)$


3 2 2 1 2
2 2 3 2 1
2 2 3 2 1
2 1 2 2 3
3 2 2 1 2
2 1 2 2 3
P Q
Q R
P R

Spine Pair Factorisation



$$\big([0]_1, [1]_0\big) = \mathrm{sp}\big([0]_1\big) \circ \mathrm{sp}\big([1]_0\big)^{-1}$$

Lemma

$(T_1, T_2) = \mathrm{sp}(T_1) \circ \mathrm{sp}(T_2)^{-1}$ *for k-trees* T_1, T_2 *with the same number of leaves.*

Products of Tree pairs

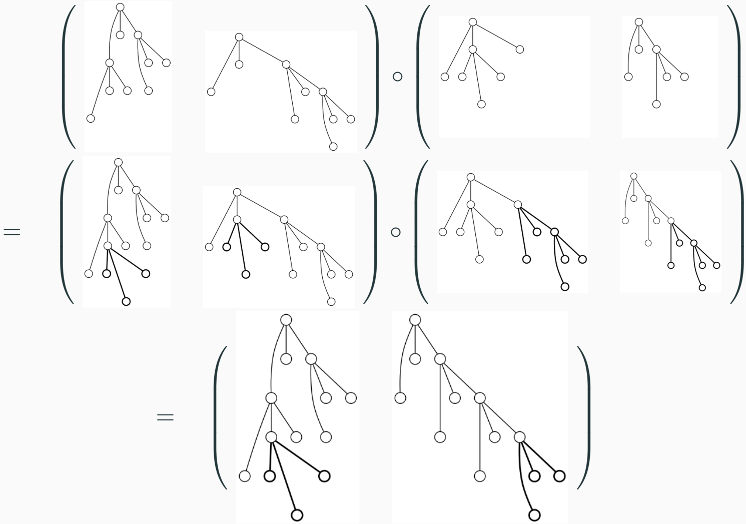


Figure 21: $\mathrm{sp}\big([0]_0\big) \circ \mathrm{sp}\big([1]_1\big) = \mathrm{sp}\big([0]_0[1]_1\big)$

A Generating Set

Theorem

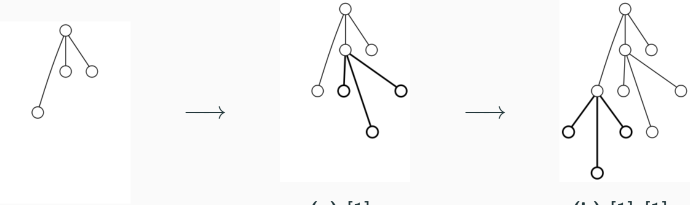
For w and w' words over $\{[n]_i \mid n \in \{0,1\}, i \in \mathbb{Z}_{\geq 0}\}$,
$\mathrm{sp}(w) \circ \mathrm{sp}(w') = \mathrm{sp}(ww')$.

Let a_i denote $\mathrm{sp}([0]_i)$ and let b_i denote $\mathrm{sp}([1]_i)$.

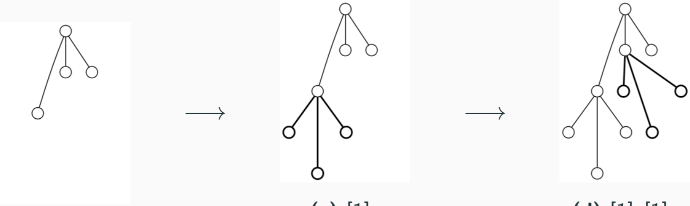
Corollary

$\mathcal{F}_{\tau_k}$ *is generated by* $\{a_i, b_i \mid i \in \mathbb{Z}_{\geq 0}\}$.

Relations of the first kind



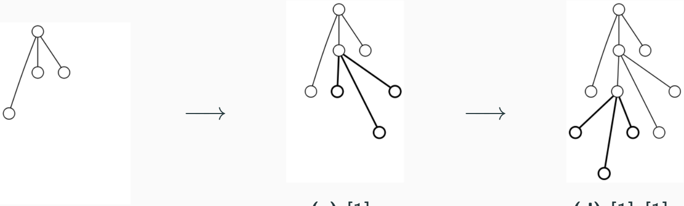
(a) $[1]_1$ **(b)** $[1]_1[1]_0$



(c) $[1]_0$ **(d)** $[1]_0[1]_3$

$$[n]_j[m]_i = [m]_i[n]_{j+k} \implies x_j y_i = y_i x_{j+k} \qquad i < j,\ x, y \in \{a, b\}$$

(a) $[0]_1$ **(b)** $[0]_1[0]_2$



(c) $[1]_1$ **(d)** $[1]_1[1]_1$

$$[0]_i[0]_{i+1} \sim [1]_i[1]_i \implies a_i a_{i+1} = b_i^2 \qquad i \in \mathbb{Z}_{\geq 0}$$

An infinite presentation for $\mathcal{F}_{\tau_k}$

Theorem

The relations

$$\begin{aligned} a_i a_{i+1} &= b_i^2 && i \in \mathbb{Z}_{\geq 0} \\ x_j y_i &= y_i x_{j+k} && i < j,\ x, y \in \{a, b\} \end{aligned}$$

over $\{a_i, b_i \mid i \in \mathbb{Z}_{\geq 0}\}$ *define a presentation for* $\mathcal{F}_{\tau_k}$*.*

Redundant generators

$$x_j y_i = y_i x_{j+k} \implies y_i^{-1} x_j y_i = x_{j+k}$$

A finite presentation for $\mathcal{F}_{\tau_k}$

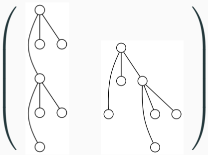
Theorem

The relations

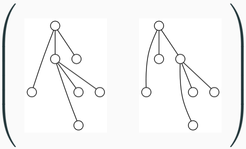
$$\begin{aligned}
&a_i a_{i+1} = b_i^2 && 0 \leq i < k \\
&a_k a_0^{-1} a_1 a_0 = b_k^2 && 0 \leq i < k \\
&x_i^{-1} y_j x_i = a_0^{-1} y_j a_0 && 0 \leq i < j \leq k,\ x, y \in \{a, b\} \\
&x_i^{-1} a_0^{-s} y_j a_0^s x_i = a_0^{-s-1} y_j a_0^{s+1} && 0 \leq i, j \leq k,\ x, y \in \{a, b\},\ s \in \{1, 2, 3\}
\end{aligned}$$

over $\{a_i, b_i \mid 0 \leq i \leq k\}$ *define a (finite) presentation for* $\mathcal{F}_{\tau_k}$.

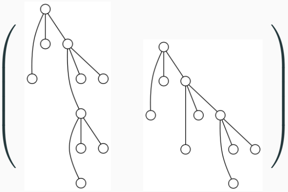
k-tree generators



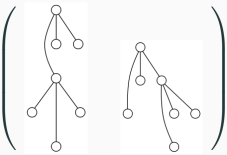
(a) a_0:Type 0, Leaf 0



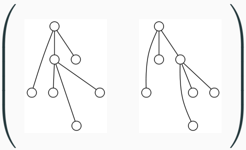
(b) a_1:Type 0, Leaf 1



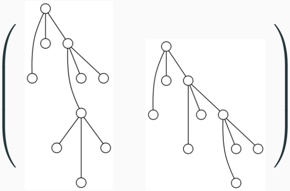
(c) a_2:Type 0, Leaf 2



(d) b_0:Type 1, Leaf 0



(e) b_1: Type 1, Leaf 1



(f) b_2: Type 1, Leaf 2

Theorem

The abelianisation of $\mathcal{F}_{\tau_k}$ *is isomorphic to* $\mathbb{Z}_2 \oplus \mathbb{Z}^{k+1}$

Corollary

$\mathcal{F}_{\tau_n} \ncong \mathcal{F}_{\tau_m}$ *for* $n \neq m$

Theorem

The abelianisation of $\mathcal{F}_n$ *for* $n \in \mathbb{Z}_{\geq 2}$ *is torsion free*

Corollary

$\mathcal{F}_{\tau_n} \ncong \mathcal{F}_m$ *for* $n, m \in \mathbb{Z}_{\geq 1}$

Theorem

The commutator subgroup of $\mathcal{F}_{\tau_k}$ *is simple.*

A normal form for $\mathcal{F}_{\tau_k}$

Definition (Normal form)

We will say that a word w over Φ and symbolic inverses $\overline{\Phi}$ is in *normal form* if it can written as $w \equiv uv^{-1}$ where:

1. u and v are both monotone words over Φ
2. If a type b generator occurs in u, the following generator in u must have level greater than that of the type b generator.
3. All the generators in v are of type a
4. If w contains a subword of the form $a_i u a_i^{-1}$ for some word u, then $\mathcal{L}(u) \leq i + k$.
5. If w contains a subword of the form $a_i b_i a_{i+k+1} u a_{i+1}^{-1} a_i^{-1}$ then $\mathcal{L}(u) \leq i + 2k$

Theorem

Every element of $\mathcal{F}_{\tau_k}$ has a unique representation as a normal form word over $\Phi \cup \overline{\Phi}$.

A normal form for $\mathcal{F}_{\tau_k}$

Theorem

If $n-1$ divides k then there exists a map from the generators of $\mathcal{F}_n$ to the generators of $\mathcal{F}_{\tau_k}$ that preserves normal forms and induces a group homomorphism.

Connections to Thompson's group

Corollary

If n divides k then $\mathcal{F}_{\tau_k}$ contains a subgroup isomorphic to $\mathcal{F}_{n+1}$.

Corollary

Each group $\mathcal{F}_{\tau_k}$ contains a subgroup isomorphic to Thompson's group $\mathcal{F}_2$.

Corollary

Each group $\mathcal{F}_{\tau_k}$ is not elementary amenable.

Corollary

If $\mathcal{F}_{\tau_k}$ is amenable for any $k \in \mathbb{Z}_{\geq 1}$, then so is $\mathcal{F}_2$.

Corollary

$\mathcal{F}_{\tau_k}$ has infinite cohomological dimension.

Unanswered Questions

1. Is $\mathcal{F}_{\tau_n}$ embeddable in $\mathcal{F}_{\tau_m}$ for $n \neq m$?
2. Is $\mathcal{F}_{\tau_k}$ embeddable in $\mathcal{F}_n$ for any $k, n \in \mathbb{Z}_{\geq 1}$?
3. What about $\mathcal{F}_\alpha$ where α is some other algebraic integer? For example, solutions to the equations:

$$x^n + kx = 1, \qquad nx^2 + kx = 1$$

Thank you

References

José Burillo, Brita Nucinkis, and Lawrence Reeves. "An Irrational-slope Thompson's Group". In: *arXiv preprint arXiv:1806.00108* (2018).

Matthew G Brin and Craig C Squier. "Groups of piecewise linear homeomorphisms of the real line". In: *Inventiones mathematicae* 79.3 (1985), pp. 485–498.

Sean Cleary. "Regular subdivision in $\mathbf{Z}\left(\frac{1+\sqrt{5}}{2}\right)$". In: *Illinois Journal of Mathematics* 44.3 (2000), pp. 453–464.

Marcelo Fiore and Tom Leinster. "An abstract characterization of Thompsons group F". In: *Semigroup Forum*. Vol. 80. 2. Springer. 2010, pp. 325–340.

Ross Geoghegan and Fernando Guzmán. "Associativity and Thompson's group". In: *Contemporary Mathematics* 394 (2006), pp. 113–136.

CLMENT LASSERRE. "R.J. THOMPSON'S GROUPS F AND T ARE BI-INTERPRETABLE WITH THE RING OF THE INTEGERS". In: *The Journal of Symbolic Logic* 79.3 (2014), pp. 693–711. ISSN: 00224812, 19435886. URL: http://www.jstor.org/stable/43303756.

Melanie Stein. "Groups of Piecewise Linear Homeomorphisms". In: *Transactions of the American Mathematical Society* 332 (1992).