Weights for Oplax Colimits

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Defn. (Cat-weighted colimit): For a Cat-presheaf $W: \mathcal{A}^{op} \to \text{Cat}$ and 2-functor $F: \mathcal{A} \to \mathcal{B}$, the W-weighted colimit of F is a representation:

$$\mathcal{B}(W * F, -) \cong [\mathcal{A}^{op}, Cat](W, \mathcal{B}(F, -))$$

Defn. (Oplax colimit): (For $W : \mathcal{A}^{op} \to Cat$, $F : \mathcal{A} \to \mathcal{B}$, the W-weighted oplax colimit of F is a representation:

$$\mathcal{B}(W \otimes F, -) \cong \left[\mathcal{A}^{op}, \mathsf{Cat}\right]_{oplax}(W, \mathcal{B}(F, -))$$

When $W = \Delta 1 : \mathcal{A}^{op} \to \mathcal{B}$ we say the the (oplax) colimit is conical.

Examples

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Coproducts: (\Delta 1: X \rightarrow Cat, X \text{ discrete})
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Copowers (aka tensors): for C locally-discrete, $\Delta_C \mathbb{1} \otimes \Delta a \cong C \odot a$ for a in some 2-category \mathcal{A} .

Cographs of functors: $(W = \Delta 1 : (\bullet \rightarrow \bullet) \rightarrow Cat)$

CoKleisli objects: $(W = \Delta 1 : \Sigma(\Delta_+^{op}) \to \mathcal{R})$

Grothedieck constructions: $(W = \Delta 1 : C \rightarrow Cat)$

Defn. (oplax-morphism classifier): of $W : \mathcal{A}^{op} \to Cat$ is a representation:

$$[\mathcal{A}^{op}, \mathsf{Cat}](W^{\sharp}, -) \cong [\mathcal{A}^{op}, \mathsf{Cat}]_{\mathsf{oplax}}(W, -)$$

Assuming W^{\sharp} exists for a given $W: \mathcal{A}^{op} \to Cat$:

$$\mathcal{B}(W \circledast F, -) \cong \left[\mathcal{A}^{op}, \mathsf{Cat} \right]_{\mathsf{oplax}} (W, \mathcal{B}(F, -))$$
$$\cong \left[\mathcal{A}^{op}, \mathsf{Cat} \right] \left(W^{\sharp}, \mathcal{B}(F, -) \right)$$
$$\cong \mathcal{B} \left(W^{\sharp} * F, - \right)$$

So oplax colimits are special Cat-weighted colimits.

Constructing **W**[‡]

(Street 1972) for $W: A \rightarrow Cat$ an oplax functor from a 1-category

(Bozapalidès 1980) for arbitrary presheaves $W: \mathcal{A} \to \mathbf{Cat}$ using (op)lax (co)ends

(Blackwell, Kelly and A. J. Power 1989) for the general case of oplax morphism classifiers for the algebras of certain 2-monads

(Lack 2002) as *codescent* objects, again for general 2-monad algebras

Idea:

$$\begin{split} \left[\mathcal{A}^{\text{op}},\mathsf{Cat}\right]_{\mathsf{oplax}}(W,X) &\cong \oint_{a \in \mathcal{A}^{\text{op}}} \left[W_a,X_a\right] \\ &\cong \oint_{a \in \mathcal{A}^{\text{op}}} \left[W_a,\int_{x \in \mathcal{A}} \left[\mathcal{A}(x,a),X_x\right]\right] \\ &\cong \int_{x \in \mathcal{A}} \oint_{a \in \mathcal{A}^{\text{op}}} \left[W_a \times \mathcal{A}(x,a),X_x\right] \\ &\cong \int_{x \in \mathcal{A}} \left[\oint^{a \in \mathcal{A}} W_a \times \mathcal{A}(x,a),X_x\right] \\ &\cong \left[\mathcal{A}^{\text{op}},\mathsf{Cat}\right] \left(\oint^{a \in \mathcal{A}} W_a \times \mathcal{A}(-,a),X\right) \end{split}$$

Explicitly: for $a \in \mathcal{A}$, W_a^{\sharp} is the category with:

o-cells pairs $(u: a \rightarrow b, x \in W_b)$

1-cells from $(u: a \rightarrow b, x)$ to $(v: a \rightarrow b', y)$ given by pairs:

$$a \xrightarrow{\psi \alpha} b \\ f \\ h' \\ x \xrightarrow{\beta} X_f y \in X_b$$

modulo the equivalence relation generated by:

$$a \xrightarrow[v]{b}_{W'} X \xrightarrow{\beta} X_{W'} Y \sim a \xrightarrow[v]{b}_{W} X \xrightarrow{\beta} X_{W'} Y \xrightarrow{X_{\theta} Y} X_{W} Y$$

Two questions:

- (a) What is the saturation of weights of the form \mathbf{W}^{\sharp} ?
- (b) What are the *coalgebras* for the comonad $W \mapsto W^{\sharp}$?

Saturation

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For a class \Phi of weights, the saturation \Phi^* contains all (small) weights W: \mathcal{A}^{op} \to \mathsf{Cat} such that \Phi-complete/continuous \Longrightarrow W-complete/continuous. If \Phi = \Phi^*, the class is said to be saturated.
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Examples:

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{non-empty finite coprods} \subseteq {binary coprods}*
{representables} = \emptyset^*
{small weights} = \begin{cases} coproducts \\ coequalisers \\ tensors by 2 \end{cases}
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Coalgebras

$$[\mathcal{A}^{op}, Cat] \xrightarrow{\downarrow} [\mathcal{A}^{op}, Cat]_{oplax}$$

$$[\mathcal{A}^{op}, Cat] \Big(W^{\sharp}, X \Big) \cong [\mathcal{A}^{op}, Cat]_{oplax} (W, X)$$

This is an *oplax-idempotent comonad* (Blackwell, Kelly and A. J. Power 1989; Lack and Shulman 2012).

$$[\mathcal{A}^{op},\mathsf{Cat}] \xrightarrow{\bot} [\mathcal{A}^{op},\mathsf{Cat}]_{\mathsf{pseudo}}$$

$$\overbrace{\mathsf{forget}}^{\natural}$$

$$[\mathcal{A}^{op}, Cat](W^{\natural}, X) \cong [\mathcal{A}^{op}, Cat]_{pseudo}(W, X)$$

ξ-coalgebras are precisely the PIE weights, i.e.:

- (a) the saturation of {products, inserters, equifiers}
- (b) weights W such that $el(W_0)$ has terminal objects in each connected component.

Coalgebra characterisation: (Lack and Shulman 2012)

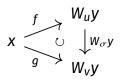
 $(a) \Leftrightarrow (b)$: (J. Power and Robinson 1991)

Defn. (2-category of elements, el W): for W: $\mathcal{A}^{op} \to Cat$, the 2-category el W has:

o-cells: pairs $(a \in \mathcal{A}, x \in Wa)$

1-cells: $(a,x) \rightarrow (b,y)$ are pairs $(u: a \rightarrow b, f: x \rightarrow W_u y)$

2-cells: $(u,f) \Rightarrow (v,g) \colon (a,x) \to (b,y)$ are 2-cells $\sigma \colon u \Rightarrow v$ in $\mathcal A$ such that $W_{\sigma}y f = g$:



A 2-functor |W|: el $W \to \mathcal{A}$ is then given by projection onto the first component, e.g. |W|(a,x) = a.

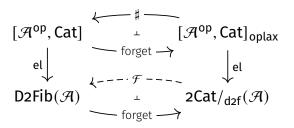
Discrete 2-fibrations

A discrete 2-fibration is a split 2-fibration which is a discrete opfibration on hom-categories.

Claim: every discrete 2-fibration $F: \mathcal{A} \to \mathcal{B}$ in Cat is isomorphic to $|W|: el W \to \mathcal{B}$ for some $W: \mathcal{B}^{op} \to Cat$.

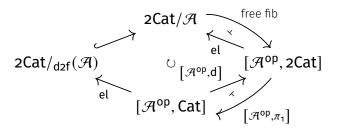
Moreover, $el: [\mathcal{B}^{op}, Cat] \to D2Fib(\mathcal{B})$ underlies an equivalence of 2-categories, where $D2Fib(\mathcal{B}) \subseteq 2Cat/\mathcal{B}$ is the locally-full subcategory of discrete 2-fibrations and split-cartesian functors (Lambert 2020).

The equivalence $el: [\mathcal{A}^{op}, Cat] \to D2Fib(\mathcal{A})$ extends to an equivalence from $[\mathcal{A}^{op}, Cat]_{oplax}$ to the **full** subcategory of discrete 2-fibrations in $\mathbf{2Cat}/\mathcal{A}$, denoted $\mathbf{2Cat}/_{d2f}(\mathcal{A})$.

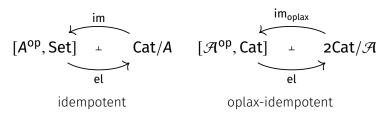


Conclusion: The map $el: [\mathcal{A}^{op}, Cat] \to 2Cat/_{d2f}(\mathcal{A})$ is (up to equivalence) the coKleisli map for \sharp .

A "larger" adjunction generates the same comonad:



Compare with the 1-categorical situation:



 $\operatorname{im}_{\operatorname{oplax}} F$ is the oplax image presheaf of $F : \mathcal{B} \to \mathcal{A}$, defined as the **oplax** colimit of $\mathcal{B} \xrightarrow{F} \mathcal{A} \xrightarrow{\sharp} [\mathcal{A}^{\operatorname{op}}, \operatorname{Cat}]$

$$C(\mathsf{im}_{\mathsf{oplax}}F * G, c) \cong [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}](\mathsf{im}_{\mathsf{oplax}}F, C(G-, c))$$

$$\cong \int_{x \in \mathcal{A}} \left[\oint^{b \in \mathcal{B}} \mathcal{A}(x, Fb), C(Gx, c) \right]$$

$$\cong \oint_{b \in \mathcal{B}^{\mathsf{op}}} \int_{x \in \mathcal{A}} \left[\mathcal{A}(x, Fb), C(Gx, c) \right]$$

$$\cong \oint_{b \in \mathcal{B}^{\mathsf{op}}} C(GFb, c)$$

$$\cong C(\Delta 1 \circledast GF, c)$$

In particular, $W \circledast G \cong W^{\sharp} * G \cong \operatorname{im}_{\operatorname{oplax}} |W| * G \cong \Delta \mathbb{1} \circledast G |W|$.



For a 2-functor $p: \mathcal{B} \to \mathcal{A}$, the free *split* 2-fibration is given by a lax comma 2-category (λ is lax):

$$\mathcal{A} \Downarrow p \xrightarrow{\xrightarrow{\lambda}} \mathcal{B}$$

The free discrete 2-fibration $p^* : \widehat{p} \to \mathcal{A}$ is constructed by quotienting out the π -vertical 2-cells of $\mathcal{A} \parallel p$.

Explicitly, for $p:\mathcal{B}\to\mathcal{A}$ the 2-category \widehat{p} has:

o-cells given by pairs $(x \in \mathcal{B}, u: a \rightarrow px)$

1-cells equivalence classes of lax squares:

$$\begin{array}{cccc}
a \xrightarrow{u} px & a \xrightarrow{u} px \\
s \downarrow \stackrel{\sigma}{\Longleftrightarrow} \downarrow pf & \sim & s \downarrow \stackrel{\sigma}{\Longleftrightarrow} pf \stackrel{p\alpha}{\Longleftrightarrow} pg \\
b \xrightarrow{v} py & b \xrightarrow{v} py
\end{array}$$

2-cells $(s, f, \sigma) \Rightarrow (t, g, \tau)$ are 2-cells κ : $s \Rightarrow t$ such that:

A \star -coalgebra structure $G: p \to p^{\star}$ on a discrete 2-fibration involves a section of the counit:

$$x \overset{G}{\mapsto} \left(px \xrightarrow{g_x} pG_x \right) \overset{\epsilon}{\mapsto} g_x^*(G_x)$$

which picks for each $x \in \mathcal{B}$ some chosen cartesian arrow

$$X = u^*(G_X) \xrightarrow{\gamma_X = \bar{G_X}(G_X)} G_X$$

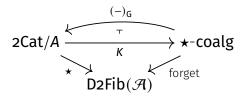
Because **G** preserves chosen cartesian morphisms:

so $G_{u^*x} = G_X$ and $\gamma_{u^*x} = \gamma_X \bar{u}_X$.

Conclusion: a \star -coalgebra structure on $p: \mathcal{B} \to \mathcal{A}$ involves choosing a terminal object in each connected component of the wide sub-1-category $p_{\text{cart}} \subseteq \mathcal{B}$ of chosen cartesian 1-cells.

When p = |W| for some $W : \mathcal{A}^{op} \to \mathsf{Cat}$, $p_{\mathsf{cart}} \cong \mathsf{el} \ W_{\mathsf{o}}$, so \sharp -coalgebras are PIE weights.

We call objects of the form G_x generic, and the restriction of p to the full subcategory of \mathcal{B} containing these objects is the generic core, $p_{\Gamma}: \mathcal{B}_{\Gamma} \to \mathcal{A}$.



The right adjoint to K must send $G: p \to p^*$ to the $(\star\text{-split})$ equaliser $E \subseteq \stackrel{e}{-} \to p \subset \stackrel{G}{\eta_p} \not\supset p^*$

So $p_{\Gamma} \stackrel{\text{incl}}{\longleftrightarrow} p \stackrel{G}{\leadsto} p^*$ is an equaliser, and $(-)_{\Gamma}$ is right-adjoint to $K: 2\text{Cat}/\mathcal{A} \to \star\text{-coalg}$.

The counit of $K: \mathbf{2Cat}/\mathcal{A} \rightleftharpoons \star\text{-coalg}: (-)_{\Gamma}$ has component at $G: p \to p^{\star}$ given by:

$$p_{\Gamma}^{\star} \xrightarrow{\mathsf{incl}^{\star}} p_{\Gamma} \xrightarrow{\epsilon_p} p$$

Proposition: $p_{\Gamma}^{\star} \xrightarrow{\text{incl}^{\star}} p_{\Gamma} \xrightarrow{\epsilon_p} p$ is an isomorphism.

Proof. The coalgebra structure map $G: p \to p^*$ forms an adjunction $\epsilon_p \dashv G$ with identity counit (from the general theory of coalgebras for oplax-idempotent monads). So G is fully-faithful and thus restricts to an isomorphism to its image, which is p_{Γ}^* . The restriction of ϵ_p to p_{Γ}^* is a left-inverse, and thus an inverse to this map.

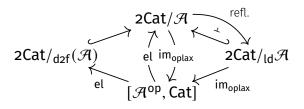
Corollary: $(-)_{\Gamma}: \star\text{-coalg} \to 2\text{Cat}/\mathcal{A}$ is a reflective subcategory.

In fact, this adjunction underlies a *comprehensive* factorisation system (Berger and Kaufmann 2017) on **2Cat** whose *covering* morphisms (i.e. right class) are "local discrete opfibrations" and whose *connected* morphisms are b.o.o locally initial 2-functors.

This orthogonal factorisation system lifts the comprehensive factorisation system of (Street and Walters 1973) on **Cat** to **2Cat** locally.

$$\mathcal{B} \xrightarrow{r_p} (\mathcal{B}^{\star})_{\Gamma} \xrightarrow{(p^{\star})_{\Gamma}} \mathcal{A}$$

Returning to presheaves:



A presheaf in $[\mathcal{A}^{op}, Cat]$ admits a \sharp -coalgebra structure precisely if:

- (a) it is the oplax image presheaf of a 2-functor $F:\mathcal{B}\to\mathcal{A}$
- (b) it is the oplax image presheaf of a local discrete opfibration $p:\mathcal{B}\to\mathcal{A}$
- I.e. #-coalgebras are the oplax colimits of representables.



Set-presheaves which are coproducts of representables are those whose category of elements have "enough component-terminal objects".

Cat-presheaves which are PIE weights are those whose category of elements has a enough component-terminal objects in its chosen-cartesian sub-1-category.

For $p:\mathcal{B}\to\mathcal{A}$ a discrete 2-fibration, an object $x\in\mathcal{B}$ is oplax generic if it is "cartesian-component-terminal" and:

for any $f: y \to x$ and chosen-cartesian $g: y \to z$, $(y \Downarrow \mathcal{B})(g,f)$ has a single connected component.

Prop: Cat-presheaves are #-coalgebras if their category of elements has enough oplax-generic objects.

Saturation

Let δ , θ and Θ denote the classes of $\Delta 1^{\sharp}$'s, W^{\sharp} 's and \sharp -coalgebras respectively. Note: $\delta \subset \theta \subset \Theta$.

Thm: (Kelly and Schmitt 2005) for a class of small weights Φ , the weights in the saturation Φ^* with domain $\mathcal A$ are those in the closure of the representables in $[\mathcal A^{op}, \mathsf{Cat}]$ under Φ -colimits (henceforth denoted $\Phi \mathcal A$).

Cor: $\Theta \subseteq \delta^*$, and so $\Theta^* \subseteq \delta^*$, and so $\Theta^* = \delta^* = \theta^*$.

Prop: Θ is saturated.

Proof. It suffices to show that $\Theta_{\mathcal{A}} = \Theta_{\mathcal{A}}^*$; i.e. that $\Theta_{\mathcal{A}} \subseteq [\mathcal{A}^{op}, \mathsf{Cat}]$ contains the representables and is closed in $[\mathcal{A}^{op}, \mathsf{Cat}]$ under oplax colimits.

Now \sharp is an oplax-idempotent comonad, so $U \colon \sharp\text{-coalg}_{\text{oplax}} \to [\mathcal{A}^{\text{op}}, \text{Cat}]$ is fully-faithful. The repletion of U's image is $\Theta_{\mathcal{A}}$. Because U creates oplax colimits (Thm. 4.8, Lack 2005) $\Theta_{\mathcal{A}}$ is indeed closed under oplax colimits in $[\mathcal{A}^{\text{op}}, \text{Cat}]$.

See also a more general proof for saturation of *rigged* weights in (Lack and Shulman 2012).

Corollary: $\delta^* = \theta^* = \Theta$.

Corollary: $\Theta_{\mathcal{A}} \simeq \sharp\text{-coalg}_{\text{oplax}}$ is the free cocompletion of \mathcal{A} under oplax colimits.

Other questions

oplax versions of (semi)-flexible weights? small generating class of weights, as for PIE weights?

Thanks

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