2-Categorical Fam Constructions

A thesis presented for the degree of Doctor of Philosophy

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This work has not previously been submitted for a degree or diploma in any university. To the best of my knowledge and belief, the thesis contains no material previously published or written by another person except where due reference is made in the thesis itself.

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Abstract

This thesis provides constructions for 2-categories of families indexed by 1-categories, which are shown to share properties with the Fam construction for ordinary categories. In particular, we describe the free completion of a 2-category under oplax colimits of oplax functors from 1-categories as such a 2-category of families. Submonads of this free cocompletion with objects given by pseudofunctors or by strict functors are described and shown to be instances of higher-dimensional analogues of Weber's *Familial 2-monads*. A related (2,1)-categorical fam construction is shown to be both a free cocompletion and a higher-dimensional familial monad. Alongside these monads we consider related notions of familial functors and show that left extensions along such functors have a weak lifting property for lax transformations, as well as 2-natural transformations. This extension property endows the free oplax cocompletion with a lax-Gray functorial structure.

In the course of this investigation we provide a characterisation of coalgebras for oplax-transformation classifiers on presheaves as the saturation of the class of weights for conical oplax colimits. From this characterisation we construct the free completion under this class of colimits, of which our 2-categorical fam constructions are all subfunctors. We also consider representable notions of 2-fibrations, and conclude that horizontally split 2-fibrations are the appropriate Cat-representable notion, whereas general 2-fibrations are Gray-representable.

Acknowledgements

It has been a privilege to spend the last four years thinking about interesting problems and working within a community of wonderful people. This experience was made possible by support from a number of sources, some of which I'd like to briefly acknowledge.

To start, this work has been funded by the Macquarie Research Excellence Scholarship, which I was very grateful to receive. Macquarie University has continued to provide a supportive and stimulating environment for research, including during a difficult period of lock-downs.

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Thanks above all go to my supervisor, Richard Garner, who has been so generous with his time, advice and other support offered over the course of nearly 200 weekly meetings. His rapid insights have been instrumental in identifying promising directions, perspectives and connections to other research.

The thesis itself has been typeset with LaTeX2e and contains diagrams produced in *TikZ* through the help of applications *Quiver* and *TikZiT* which are made openly available respectively by Nathanael Arkor and Aleks Kissinger (among others) to the benefit of many.

Finally, thank you to friends and family who have supported me beyond the scope of my research, and who remind me that there is indeed a "beyond". In particular: to Edmund for our discursive conversations, to Mandy for her role in setting me on this path, to Rachel and Ryan for the much-needed fun, and to my parents for their unconditional support in everything I do.

Contents

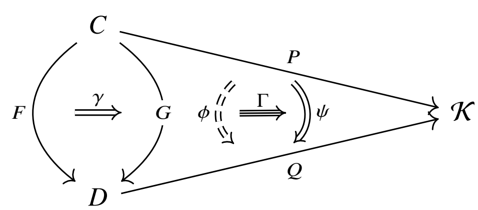
Chapter 1

Introduction

The Fam construction produces from a category C another category $\mathsf{Fam}(C)$ whose objects are given by a set I and family of objects $A\colon I \to \mathsf{ob}(C)$. A morphism from $A\colon I \to \mathsf{ob}(C)$ to $B\colon J \to \mathsf{ob}(C)$ is given by a function $f\colon I \to J$ and a family of maps $g_i\colon A_i \to B_{fi}$. This construction was described under the name Fam at least as early as 1985 by Bénabou, [Bén85], though its descriptions there and in [MP89] seem to treat it as category theory folklore. It is, after all, a fairly simple construction that arises inevitably as a description for the free completion of a category under coproducts.

This description of the free completion under coproducts naturally leads one to consider similar constructions for 2-categories. An obvious approach is to construct from a 2-category $\mathcal{K}$ another 2-category with objects given by a set I and an I-indexed family of objects in $\mathcal{K}$, $A\colon I \to \mathsf{ob}(\mathcal{K})$. The 1-cells can be defined as in the 1-categorical case, and 2-cells given by families of 2-cells in $\mathcal{K}$ of the form $\theta_i\colon g_i \Rightarrow h_i\colon A_i \to B_{fi}$. This is indeed a very good 2-category of families. It looks like ordinary Fam and shares many of its properties, including the property of being the free coproduct completion. We shall denote it by $\mathsf{2Fam}$.

However, one might take the view that since the ordinary (i.e. Set-categorical) Fam construction has families indexed by sets, a Cat-categorical fam construction should have families indexed by categories. So, perhaps we should define our 2-category of families to have objects given by a *category* C and a *functor* $P\colon C \to \mathcal{K}$. A morphism from $P\colon C \to \mathcal{K}$ to $Q\colon D \to \mathcal{K}$ should then be a functor $F\colon C \to D$ along with a natural transformation $\phi\colon P \Rightarrow QF$, and the 2-cells from (F, ϕ) to (G, ψ) can be given by a natural transformation $\gamma\colon F \Rightarrow G$ and a modification $\Gamma\colon Q\gamma\ \phi \Rrightarrow \psi$ like so:



However, one then needs to decide what *sort* of functors and transformations are appropriate. For $\mathcal{K}$ a 2-category, we could choose P and ϕ to independently be either strict, pseudo, oplax or lax. This seems to give sixteen possible constructions, though on closer inspection choosing P and Q to be (op)lax makes the definition of 2-cells by modifications $Q\gamma\ \phi \Rrightarrow \psi$ dubious. Perhaps some of these combinations are therefore not viable, and others — like taking P to be pseudo and ϕ to be lax — seem unpromising for aesthetic reasons. But at the very least, choosing to take the P and ϕ components to both be strict or both be pseudo seems to provide a natural construction, and either choice does indeed define a 2-category with compositions defined in the obvious way. These constructions can then be extended to define endo-functors on $\mathsf{2Cat}$, and in fact pseudomonads. Variants of both constructions have been described elsewhere — the strict version applied to 1-categories is called the "foncteur diagramme" $\mathcal{D}$ in

[Gui73], later named Diag in [PT22], and a dual of the pseudo version is referred to by Buckley in [Buc14] simply as "$\mathsf{Fam}(\mathcal{K})$". We shall use our own notation, denoting the strict version by $\mathfrak{S}$ and the pseudo version by $\mathfrak{P}$.

Given the structural similarities between Fam, $\mathfrak{S}$, and $\mathfrak{P}$, we might wonder whether $\mathfrak{S}$ and $\mathfrak{P}$ share any of Fam's interesting properties. Some good comparisons can be made, for example $\mathsf{Fam}(!)\colon \mathsf{Fam}(C) \to \mathsf{Fam}(\mathbb{1}) \cong \mathsf{Set}$ is a fibration, and both $\mathfrak{S}(!)\colon \mathfrak{S}(\mathcal{K}) \to \mathfrak{S}(\mathbb{1}) \cong \mathsf{Cat}$ and $\mathfrak{P}(!)\colon \mathfrak{P}(\mathcal{K}) \to \mathfrak{P}\mathbb{1} \cong \mathsf{Cat}$ are 2-fibrations, though $\mathfrak{S}(!)$ is split and $\mathfrak{P}(!)$ is not. However, there are also differences — neither $\mathfrak{P}$ nor $\mathfrak{S}$ are free completions for any class of Cat-colimits. We know that the coproduct completion for 2-categories is $\mathsf{2Fam}$, but it seems plausible that some other 2-category of families might be a free cocompletions for a different class of Cat-weights. In fact, of the sixteen constructions described above precisely two are free cocompletions. By taking the functors to be oplax and the transformations to be lax, one obtains the free completion under the class of conical oplax colimits of oplax functors from 1-categories, $\mathsf{F}_\Omega\mathcal{K}$. Predictably, then, taking the functors to be lax and the transformations to be oplax gives the free completion under lax colimits of lax functors from 1-categories, $\mathsf{F}_\Lambda\mathcal{K}$. Though as we've already observed, such a construction can't possibly have 2-cells described by genuine modifications. Instead, we will find they are given by the related notion of *modulations*, a notion that coincides with modifications when the functors involved are at least pseudonatural. We therefore have three (up to duality) promising 2-categorical fam constructions — $\mathfrak{S}$, $\mathfrak{P}$, and F_Ω — which seem to share between them 2-categorical versions of some of Fam's special properties.

For each of these 2-categorical fam constructions there are corresponding notions of *familial functors*. Such functors can be viewed as a generalisation of adjoint functors; indeed, those corresponding to the ordinary Fam-construction has been given the names "multi-adjoint" by Diers in [Die77] and "parametric adjoint" by Street in [Str00]. Some 2-categorical notions of familial functors already exist in the literature: there are the *familial 2-functors* introduced by Mark Weber in [Web07] as well as Charles Walker's *lax familial 2-functors* from [Wal20]. Weber's notion coincides (under some dualisation) with the familial functors for $\mathfrak{S}$, whereas Walker's notion is slightly more general than the familial functors for $\mathfrak{P}$. The familial functors for F_Ω, on the other hand, seem not to have been studied directly, though they are examples of *admissible 1-cells* for a KZ doctrine as described by Bunge and Funk in [BF99]. These F_Ω-functors don't have all of the nice properties possessed by $\mathfrak{P}$-functors and $\mathfrak{S}$-functors, but they do have a different property in common with parametric left adjoints. We shall see that pointwise left extensions along F_Ω-familial functors (and also the yet-to-be defined F_Θ-familial functors) have a canonical lifting property with respect to *lax* transformations which extends the usual universal property of left extensions. The analogous property for parametric left adjoints is that pointwise left extensions along parametric left adjoints lift *unnatural* transformations.

The investigation into these (and other) 2-categorical fam constructions and their accompanying familial functors provides an over-arching theme to the thesis, though along the way we explore a number of topics in 2-category theory that are only tangentially related. Indeed, we don't encounter our first 2-categorical fam construction, $\mathsf{F}_\Omega\mathcal{K}$, until half-way through Chapter 4, and Chapter 6 contains no discussion of either fam constructions or familial functors. While the results of our 2-categorical digressions are often later applied in some way to the study of fam constructions, we hope that they are also interesting in their own right.

The ordering of topics in this thesis is as follows:

2. Preliminaries

We establish our notational conventions, and use of string diagrams in particular. A brief exposition is given for some foundational concepts in 2-category theory.

3. Weights for Oplax Colimits

This chapter provides theory to support the later discussion of $\mathsf{F}_\Omega\mathcal{K}$, though no 2-categorical fam constructions are mentioned here. We begin by giving an explicit description of the *oplax-morphism classifier* on a presheaf category $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$. This is a comonad $\mathcal{Q}$ on $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ with the property that *oplax* transformations out of a presheaf $X\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ are "classified" by 2-natural transformations out of $\mathcal{Q}X$. Such comonads are special cases of weak morphism classifiers for T-algebras described by Blackwell, Kelly and Power in [BKP89]. We provide a brief summary of the general results for weak morphism classifiers described there and elsewhere, though this level of generality will be unnecessary for our purposes. Our primary interest is in the *coalgebras* for the oplax-morphism classifier on $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$, which are shown to correspond under the fibred-indexed equivalence to 2-functors $P\colon \mathcal{X} \to \mathcal{A}$ that are discrete opfibrations on hom-categories. This description is used to prove that such coalgebras are weights for conical oplax colimits. Observing that this class of coalgebras also contain weights for all general oplax colimits, we conclude that the saturation of oplax colimits coincides with the saturation of conical oplax colimits, though this is not formally stated in these terms until after the general theory of free cocompletions is summarised at the being of Chapter 4.

4. The Oplax Fam Construction

Here we introduce $\mathsf{F}_\Omega\mathcal{K}$ via a description of the free completion $\mathsf{F}_\Theta\mathcal{K}$ of a 2-category $\mathcal{K}$ under *all* oplax colimits (whereas $\mathsf{F}_\Omega\mathcal{K}$ is the free completion under oplax colimits of oplax functors from 1-categories).

We first show that the class of coalgebras for oplax-morphisms classifiers, Θ, is saturated, and thus equal to the saturation of the class of weights for oplax colimits. The free cocompletion $\mathsf{F}_\Theta\mathcal{K}$, of a 2-category $\mathcal{K}$ is constructed as a Kleisli category for a monad on $\mathsf{2Cat}/\mathcal{K}$ related to the fibred-category version of the oplax-morphism classifier. The 2-categorical fam construction $\mathsf{F}_\Omega\mathcal{K}$, (the "oplax fam" of the chapter title) is then shown to be equivalent to a full subcategory of $\mathsf{F}_\Theta\mathcal{K}$. The majority of the chapter from this point is spent demonstrating that $\mathsf{F}_\Omega\mathcal{K}$ is closed in $\mathsf{F}_\Theta\mathcal{K}$ under oplax colimits of oplax functors from 1-categories, from which it follows that $\mathsf{F}_\Omega\mathcal{K}$ is the free completion with respect to colimits in this class. The proof involves a lengthy combinatorial argument in Section 4.5. Examples and related constructions (e.g. the free-completion under *lax* colimits of *lax* functors, as well as the coKleisli completion) are described at the end of the chapter.

5. Oplax-Familial 2-Functors

We define 2-categorical versions of parametric left adjoints that are related to F_Θ and F_Ω in the same way parametric left adjoints are related to Fam, i.e. as F_Θ-admissible and F_Ω-admissible 1-cells, in the sense of [BF99]. We call these maps F_Θ-functors and F_Ω-functors respectively. Functors of these two types can be characterised by the existence of "generic factorisations" for morphisms in a manner similar to how generic factorisations are shown to characterise parametric right adjoints in [Web04]. We show that pointwise left extensions along F_Θ-functors have a lifting property with respect to lax transformations that extends the isomorphism on 2-natural transformations which characterises the left extension. The free completion maps $\mathsf{Z}_\mathcal{K}\colon \mathcal{K} \to \mathsf{F}_\Theta\mathcal{K}$ are examples of F_Θ-functors, and their ability to lift lax transformations is shown to endow F_Θ with the structure of a $\mathsf{Gray}_\mathscr{L}$-functor, where $\mathsf{Gray}_\mathscr{L}$ is the monoidal category structure on $\mathsf{2Cat}$ with the lax Gray tensor product.

6. 2-Fibrations

This chapter is a self-contained investigation into properties of 2-fibrations which are applied in the following chapter. We extend Buckley's definition of 2-fibrations in $\mathsf{2Cat}$ from [Buc14] to describe 2-fibrations in general 3-categories representably, and observe that the 2-fibrations in $\mathsf{2Cat}$ are not necessarily representably 2-fibrations. Instead, we see that it is the *horizontally split* 2-fibrations which are representably (horizontally split) 2-fibrations, whereas general 2-fibrations are Gray-representable fibrations, which we also define. In the second half of this chapter we give a detailed description of the fibred-indexed equivalence for *globally split* 2-fibrations and show that the corresponding indexed categories can be expressed by certain Gray-functors into Gray.

7. Familial Monads and Submonads

We consider properties analogous to Weber's *familial 2-monad* property [Web07] and determine whether these are satisfied by F_Ω. Upon observing that they are not, we describe the submonads $\mathfrak{P}$ and $\mathfrak{S}$ of F_Ω and show that these have better familial monad properties. In particular, they are parametric right adjoints in a suitable sense, and have their unit and multiplication maps given by cartesian natural transformations (in a weak sense for the case $\mathfrak{P}$). We then define notions of familial functors corresponding to $\mathfrak{S}$ and $\mathfrak{P}$, and observe that these notions are related to Weber's familial 2-functors and Walker's lax familial functors [Wal20]. Finally, we give a (2,1)-categorical fam construction, $\mathfrak{G}$, and observe that it both satisfies the familial-monad properties of $\mathfrak{P}$ and is a free cocompletion.

Chapter 2

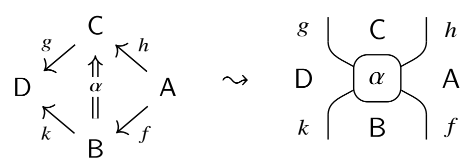
Preliminaries

2.1 Notation

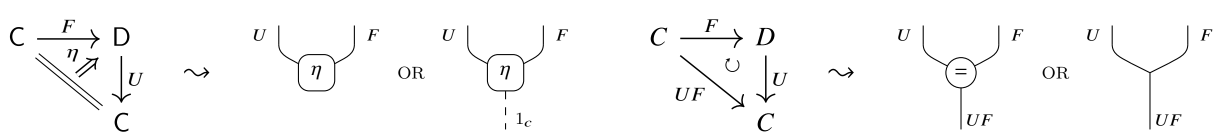
We use standard notation for 2-categorical objects and operations when possible, and when this is not possible — e.g. due to collision with other notation — we indicate our alternative clearly. In some cases there are a number of different notations or conventions used in the literature, so we've had to make choices. Among such choices, our use of string diagrams and notation for composition and application require some explanation, which we provide below.

2.1.1 String Diagrams

String diagrams will be used to represent configurations of 0, 1, and 2-cells in the various 2-categories encountered. Strings will be written right-to-left and flow upwards, as demonstrated in the pasting-diagram/string-diagram translation below:



This has the advantage that ordering of 1-cells in diagrams matches the ordering in which the composition is usually written. The Labelling each region of the string diagram with the corresponding 0-cell is done here only to clarify the correspondence with the pasting diagram and will usually be suppressed. Occasionally the labelling of the edges of string diagram with the corresponding 1-cell will be omitted as well if it is otherwise clear from the context. Identity 1-cells will either be invisible or indicated by a dashed line, and identity 2-cells will be indicated by an unlabelled juncture of 1-cells, or a =:



2.1.2 Composition and Application

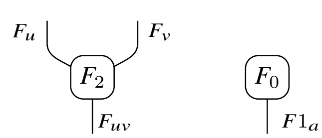
Composition of 1-cells or horizontal composition of 2-cells will usually be indicated by concatenation, e.g. $f\,g$ represents the composite of $f\colon y \to z$ with $g\colon x \to y$. The symbol "$\circ$" is typically reserved for vertical composition of 2-cells, though in situations where 3-cells are involved it usually indicates the notion of composition on the "higher" level, and vertical composition of 2-cells — viewed as 1-cells in a hom-category — may be indicated with concatenation. Application is either indicated by concatenation or, when this may lead to ambiguity with composition, subscripts. The use of subscripts also serves to disambiguate iterated application. For example, if $F\colon \mathcal{A} \to \mathsf{Cat}$ is a 2-functor and $u\colon a \to b$ a 1-cell in $\mathcal{A}$, either Fa or F_a will indicate the image of a in Cat, but for $x \in Fa$, $F_u x$ will be used to indicate the image of x under the functor F_u which is the image of u under F. Similarly, for a 2-cell $\alpha\colon u \Rightarrow v$ in $\mathcal{A}$, F_α will indicate the natural transformation $F_u \Rightarrow F_v$ in Cat whose component at $x \in F_a$ will be written $F_\alpha x\colon F_u x \to F_v x$.

2.2 2-Category Theory

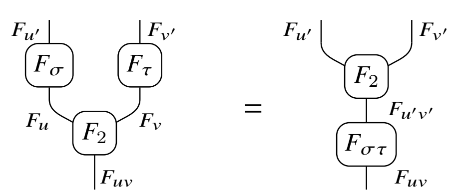
This section provides brief description of some of the elements of 2-category theory which we will typically assume without explanation in later chapters. We also further establish our chosen notation and terminology (e.g. colax vs. oplax) among various options present in the literature, adopting the most prevalent usage where possible. Our definitions will reflect a preference for working with oplax functors and colimits over lax functors and colimits, though this choice is arbitrary from a mathematical standpoint as all statements in one context can be dualised to the other. Our main reason for this preference is that the canonical projections from *lax* colimit of *lax* functors are of form $\int^{\mathsf{lax}} F \to C^{\mathsf{op}}$, and so we find that keeping track of the direction of morphisms in the lax case to be slightly more complicated. However, examples of lax functors, lax colimits and lax cocompletions are more familiar (e.g. monads, Kleisli objects and Kleisli completions vs. comonads, co-Kleisli objects, co-Kleisli completions) so occasionally in later chapters we will explicitly dualise our results from the oplax setting to place them in a more recognisable context.

2.2.1 Oplax, Lax and Pseudo Functors

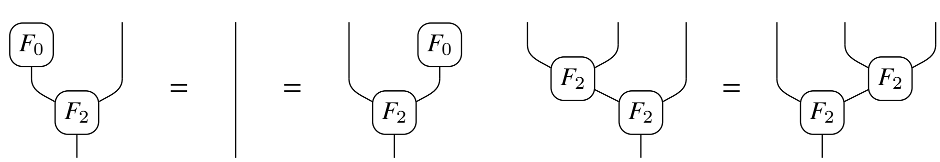
An *oplax functor* $F\colon \mathcal{K} \to \mathcal{L}$ between strict 2-categories maps objects of $\mathcal{K}$ to those of $\mathcal{L}$ and maps hom-categories between objects of $\mathcal{K}$ to the appropriate hom-category between objects in $\mathcal{K}$ via a functor (i.e. preserving vertical composition of 2-cells). It fails to strictly preserve horizontal composition in that for a pair, u, v of composable 1-cells in $\mathcal{K}$ the equality $F_{uv} = F_u F_v$ is replaced by a chosen 2-cell $F_2^{u,v}\colon F_{uv} \Rightarrow F_u F_v$. Similarly, rather than equality $F1_a = 1_{Fa}$ we have a chosen 2-cell $F_0^a\colon F1_a \Rightarrow 1_{Fa}$. We usually suppress the superscript for these "oplax functoriality" 2-cells, denoting them simply by $F_2\colon F_{uv} \Rightarrow F_u F_v$, $F_0\colon F1_a \Rightarrow 1_{Fa}$:



The oplax-functoriality F_2 2-cells are natural with respect to 2-cells in $\mathcal{K}$ in the following sense:



They also satisfy coherence conditions (for any compatible labelling of the 1-cells):

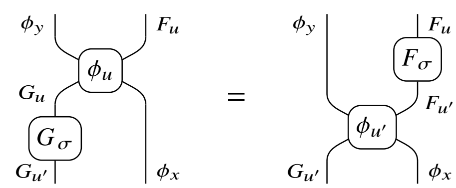


From repeated uses of F_2 and F_0 2-cells, we could define general n-ary 2-cells of the form $F_n\colon F_{u_1\ldots u_n} \Rightarrow F_{u_1}\ldots F_{u_n}$. The coherence relations above ensure that any of the possible definitions of F_n in terms of F_0 and F_2 are equal.

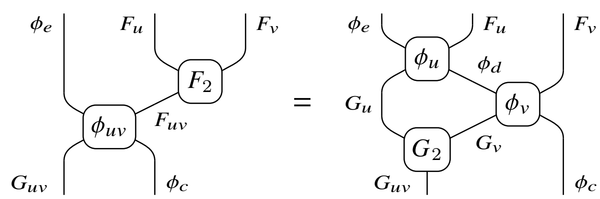
A *lax functor* $F\colon \mathcal{K} \to \mathcal{L}$ is an oplax functor $F\colon \mathcal{K}^{\mathsf{co}} \to \mathcal{L}^{\mathsf{co}}$. That is, the given data and conditions for a lax functor are obtained by vertically reflecting the string diagrams in this section. A *pseudofunctor* is an oplax (or lax) functor where all 2-cells F_2 and F_0 are invertible and is moreover a *(strict) 2-functor* if F_2 and F_0 are identities.

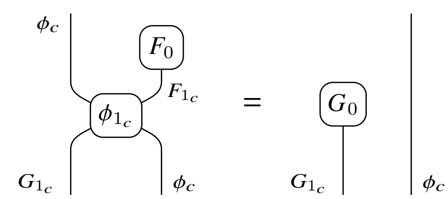
2.2.2 Lax, Oplax and Pseudo Transformations

Lax transformations provide a weak notion of natural transformation between (op)lax functors. A lax transformation between oplax functors between 2-categories, $\phi\colon F \Rightarrow G\colon \mathcal{K} \to \mathcal{L}$ is given by a collection of 1-cells $\phi_x\colon F_x \to G_x$ for each $x \in \mathcal{K}$ and a collection of 2-cells $\phi_u\colon G_u\ \phi_x \Rightarrow \phi_y\ F_u$ for each $u\colon x \to y$ in $\mathcal{K}$. Note that these 2-cells fill the squares that would be required to commute in the case of strict naturality, so we refer to them as "lax naturality 2-cells". The 2-cells ϕ_u are required to be natural with respect to the 2-cells of $\mathcal{K}$ in that the below diagrams must be equal for any $\sigma\colon u' \Rightarrow u\colon x \to y$ in $\mathcal{K}$:



The 2-cells ϕ_u must also respect the oplax-functoriality 2-cells of F and G:

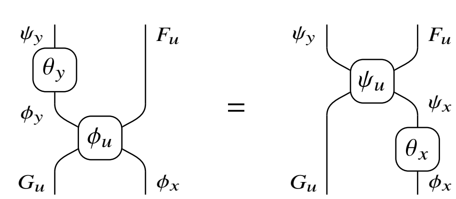




By changing the direction of the two cells ϕ_u filling the naturality squares and appropriately modifying the naturality and coherence conditions, one obtains the definition of an *oplax transformation.* That is, an oplax transformation between oplax functors $\phi\colon F \Rightarrow G\colon \mathcal{K} \to \mathcal{L}$ is the same as a lax transformation $\phi\colon G^{\mathsf{op}} \Rightarrow F^{\mathsf{op}}\colon \mathcal{K}^{\mathsf{op}} \to \mathcal{L}^{\mathsf{op}}$ (reflect all the string diagrams horizontally and swap F and G). We can also define oplax transformations between lax functors $F \Rightarrow G\colon \mathcal{K} \to \mathcal{L}$ as a lax transformation between the corresponding oplax functors $F' \Rightarrow G'\colon \mathcal{K}^{\mathsf{co}} \to \mathcal{L}^{\mathsf{co}}$ (vertically reflect all string diagrams). A *pseudo transformation* is an oplax (or lax) transformation where each of the oplax-naturality 2-cells is invertible and a (strictly) *2-natural transformation* has oplax-naturality 2-cells given by identities.

2.2.3 Modifications

Modifications are morphisms between (op)lax transformations. Given oplax functors $F, G\colon \mathcal{K} \to \mathcal{L}$ and lax transformations $\phi, \psi : F \Rightarrow G$ a modification $\theta\colon \phi \to \psi$ is a collection of 2-cells $\theta_x\colon \phi_x \to \psi_x$ in $\mathcal{L}$ indexed by objects $x \in \mathcal{K}$. These 2-cells must satisfy the following naturality condition with respect to 1-cells $u\colon x \to y$ of $\mathcal{K}$:



Note that if ϕ and ψ are strictly 2-natural, so that ϕ_u and ψ_u are identities, then this naturality condition for modification θ is equivalent to requiring that $\theta_y\, F_u = G_u\, \theta_x$.

2.2.4 Oplax Functor Classifiers

From any 2-category $\mathcal{K}$ one can construct another 2-category $\mathcal{K}^\dagger$ such that strict 2-functors out of $\mathcal{K}^\dagger$ classify oplax functors out of $\mathcal{K}$. This 2-category $\mathcal{K}^\dagger$ is called the *oplax functor classifier* for $\mathcal{K}$.

By "classify" we mean that for every 2-category $\mathcal{L}$ there is an isomorphism between the 2-category $\mathsf{Oplax}(\mathcal{K}, \mathcal{L})$ of oplax functors, lax transformations and modifications and the 2-category $\mathsf{Gray}_{\mathscr{L}}(\mathcal{K}, \mathcal{L})$ of strict 2-functors, lax transformations and modifications (we will say more about $\mathsf{Gray}_{\mathscr{L}}$ in Section 2.2.7). Moreover, these isomorphisms are natural in $\mathcal{L}$ with respect to 2-functors, *lax*-transformations and modifications (i.e $\mathsf{Gray}_{\mathscr{L}}$ natural). It follows that these isomorphisms are induced by precomposition with a "unit" for the classifier: an oplax functor $\lambda_{\mathcal{K}} \colon \mathcal{K} \to \mathcal{K}^\dagger$.

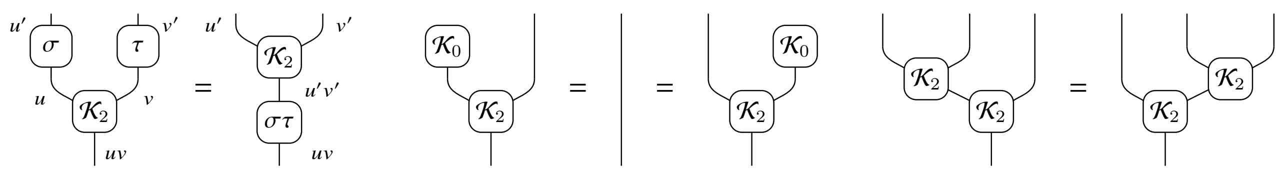
For $\mathcal{K}$ a 2-category, we can describe $\mathcal{K}^\dagger$ via a presentation. The underlying 1-category of $\mathcal{K}^\dagger$ is given by the free category on the underlying graph of $\mathcal{K}_0$. The 2-cells of $\mathcal{K}^\dagger$ are generated by generators of three types:

(a) $[\theta] \colon [f] \Rightarrow [g]$ for $\theta \colon f \Rightarrow g$ in $\mathcal{K}$

(b) $\mathcal{K}_2^{f,g} \colon [fg] \Rightarrow [f][g]$ for f and g composable 1-cells in $\mathcal{K}$

(c) $\mathcal{K}_0^x \colon [1_x] \Rightarrow 1_x$ for x an object of $\mathcal{K}$.

The type (a) generators satisfy their relations under vertical composition in $\mathcal{K}$. That is, for $\theta \colon f \Rightarrow g$ and $\zeta \colon g \Rightarrow h$ in $\mathcal{K}$, $[\zeta \circ \theta] = [\zeta] \circ [\theta]$ and $\left[1_f\right] = 1_{[f]}$. The remaining relations correspond to the axioms on oplax functors described in Section 2.2.1:



The square bracket notation is redundant in the string diagrams, and is therefore suppressed.

Comparing these relations to the coherence conditions on oplax functors described above shows how a strict 2-functor $F' : \mathcal{K}^\dagger \to \mathcal{L}$ might be constructed from an oplax functor $F \colon \mathcal{K} \to \mathcal{L}$. The 2-functor F' acts on objects as F does, and sends a morphism $[f_1], \ldots, [f_n]$ to $F_{f_1} \ldots F_{f_n}$. It maps the generating 2-cells of $\mathcal{K}^\dagger$ as follows:

$$[\theta] \mapsto F_\theta \qquad \mathcal{K}_2^{f,g} \mapsto F_2^{f,g} \qquad \mathcal{K}_0^x \mapsto F_0^x$$

Given a 2-functor $G \colon \mathcal{K}^\dagger \to \mathcal{L}$, one recovers an oplax functor $G' \colon \mathcal{K} \to \mathcal{L}$ by precomposing with the canonical oplax functor $H \colon \mathcal{K} \to \mathcal{K}^\dagger$ which acts as the identity on objects, sends a 1-cell $f : x \to y$ in $\mathcal{K}$ to the atomic 1-cell $[f] \colon x \to y$ and sends $\theta \colon f \Rightarrow g$ to the generator $[\theta] \colon [f] \Rightarrow [g]$ in $\mathcal{K}^\dagger$. The coherence data for H are given by the corresponding generators in $\mathcal{K}^\dagger$.

The *lax* functor classifier for $\mathcal{K}$, denoted $\mathcal{K}^\ddagger$, can be defined as $\left(\mathcal{K}^{\mathsf{co}\dagger}\right)^{\mathsf{co}}$, meaning essentially that the $\mathcal{K}_2$ and $\mathcal{K}_0$ generators are vertically reflected as string diagrams. Requiring $\mathcal{K}_2$ and $\mathcal{K}_0$ to be invertible produces the pseudofunctor classifier, denoted by $\mathcal{K}^\natural$. We will recall these notations when they are first used in the text. We note that the pseudofunctor classifier can also be defined as the bijective-on-objects 2-fully-faithful factorisation of the map from the free category on the underlying graph of $\mathcal{K}_0$ into $\mathcal{K}$. This means that 2-cells from $[f_1] \ldots [f_n]$ to $[g_1] \ldots [g_m]$ in $\mathcal{K}^\natural$ can be identified with 2-cells from $f_1 \ldots f_n$ to $g_1 \ldots g_m$ in $\mathcal{K}$.

We will be particularly interested in oplax functor classifiers for locally discrete 2-categories, i.e. those with only trivial 2-cells. Such 2-categories can be identified with their underlying 1-categories, so by a slight abuse of notation we will often refer to $C^\dagger$ for C a 1-category. The 2-cell generators of type (a) are all trivial for these 2-categories, so only the C_2 and C_0 generators are relevant and only the two right-most relations from the diagrams above need be considered. It follows from the nature of the C_2 and C_0 generators that 2-cells only exist between different strings of atomic 1-cells whose compositions in C are equal, and that each 1-cell is the codomain of a 2-cell from the atomic 1-cell given by its composite in C. That is, a 1-cell $[f_1]\dots[f_n]$ admits a 2-cell from $[f_1 \dots f_n]$. The relations on the generators ensure that these 2-cells are unique, so that atomic 1-cells are initial in their connected components of each hom-category. As a consequence of this, or the alternative description of $\mathcal{K}^\natural$ above, the pseudofunctor classifier $C^\natural$ for a 1-category has equivalence relations as hom-categories, whose equivalence classes are the sets of strings $[f_1]\dots[f_n]$ which are equal under composition in C.

2.2.5 Oplax Colimits and the Grothendieck Construction

An *oplax colimit* of an oplax functor $F\colon \mathcal{K} \to \mathcal{L}$ is a universal *lax* transformation from F to a constant functor ΔX for some $X \in \mathcal{L}$. We can call such a transformation a *lax cocone* from F to X. The reason the name *oplax* colimit is given to a universal *lax* cocone is that a lax cocone from F to X is equivalent to a *oplax* cone from the terminal category $\mathbb{1}$ to the contravariant functor $\mathcal{L}(F-, X)\colon \mathcal{K}^{\mathsf{op}} \to \mathsf{Cat}$, which exhibits the universal oplax cocone as an oplax weighted colimit. The oplax colimit can also be expressed as an ordinary weighted colimit with respect to weights which we will describe in detail in Chapter 3.

Of particular relevance will be the oplax colimits in Cat. When $F\colon C \to \mathsf{Cat}$ is an oplax functor from a 1-category, its oplax colimit is given by the *Grothendieck construction*[1] of F, $\int F$. This is equivalently the oplax colimit of the strict 2-functor $F'\colon C^\dagger \to \mathsf{Cat}$ which classifies F.

Construction 2.2.1 (The Grothendieck Construction, $\int F$)**.** For $F\colon C \to \mathsf{Cat}$ oplax from a 1-category, $\int F$ is defined as follows:

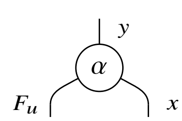
Objects are pairs of the form (c, x) with $c \in C$ and $x \in F_c$

Morphisms from $(c, x) \to (d, y)$ are pairs of the form $\left(c \xrightarrow{u} d, F_u x \xrightarrow{\alpha} y\right)$

Identities are given by $1_{(c,x)} = \left(1_c, F_0^c x\right)$

Composition is given by $(v, \beta)(u, \alpha) = \left(uv, \alpha\ F_u\beta\ F_2^{u,v} z\right)$

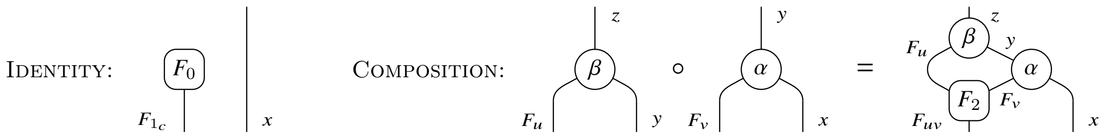
We can give a (hopefully) simpler description of the identities and composition by use of string diagrams. This representation arises from the observation that morphisms of *any* category C can be represented with string diagrams by identifying objects as 1-cells $\mathbb{1} \to C$ in Cat from the terminal category, and a morphism $f\colon x \to y$ is then a 2-cell $\langle f \rangle : \langle x \rangle \Rightarrow \langle y \rangle : \mathbb{1} \to C$. So, for a morphism $(u, \alpha)\colon (c, x) \to (d, y)$ in $\int F$, we can construct a string diagram representation of the component $\alpha\colon F_u x \to y$ which is a 1-cell in F_d:



We can recover the underlying morphism $u\colon c \to d$ from the diagram when written in this form, so such a diagram equally represents the 1-cell (u, α) in $\int F$. In terms of these diagrams, the composition and identities in $\int F$ are given

[1] The term Grothendieck construction is also used to refer to the lax colimit of a lax functor. Both constructions can be applied to pseudofunctors, with the lax version perhaps more commonly described. The oplax version produces an opfibration over the domain, whereas the lax version produces a fibration over the opposite of the domain

as follows:



◊

The universal lax cocone $\rho\colon F \rhd \int F$ has component $\rho_c\colon F_c \to \int F$ at $c \in C$ which sends $x \in F_c$ to $(c, x) \in \int F$ and sends $\alpha\colon x \to y$ to $(1_c, \alpha\ F_0 x)$. The lax-naturality 2-cell $\rho_u\colon \rho_c \Rightarrow \rho_d\ F_u$ is itself a natural transformation whose component $\rho_u x$ at $x \in F_c$ is given by $\big(u, 1_{F_u x}\big)\colon (c, x) \to (d, F_u x)$.

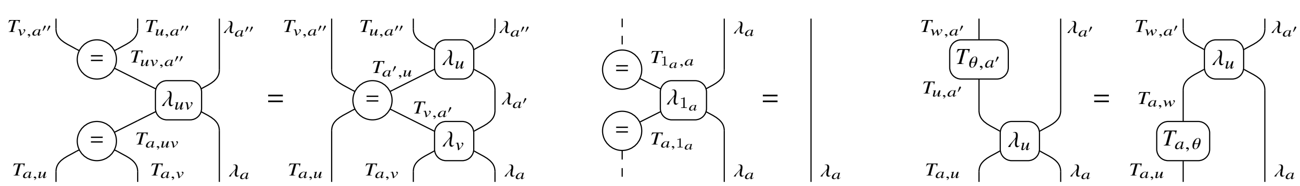
There is a canonical projection $\Pi_F\colon \int F \to C$ from the Grothendieck construction which corresponds to the unique lax cocone from F to $\Delta\mathbb{1}$. Explicitly, this maps n-cells of $\int F$ onto their first component, i.e. $(c, x) \mapsto c$, $(u, \alpha) \mapsto u$. If the functor F is *pseudo*-functorial, rather than merely oplax, this projection will be an opfibration. In fact, there is a 2-categorical equivalence between 2-category $\mathsf{Pseudo}(C, \mathsf{Cat})$ of pseudofunctors, pseudonatural transformations and modifications, and $\mathsf{OpFib}(C)$ of opfibrations, opcartesian functors (i.e. functors preserving opcartesian morphisms) and vertical 2-cells between them.

The *lax* colimit of a *lax* functor $F\colon C \to \mathsf{Cat}$ from a 1-category is the opposite of the oplax colimit of the composite $C \xrightarrow{F} \mathsf{Cat} \xrightarrow{\mathsf{op}} \mathsf{Cat}^{\mathsf{co}}$. That is, $\mathsf{op}\big(\int^{\mathsf{oplax}} F\big) \cong \int^{\mathsf{lax}}(\mathsf{op} \circ F)$. More generally[2], the (op)lax colimit of a strict 2-functor $F\colon \mathcal{A} \to \mathsf{Cat}$ from a 2-category is given by taking the 2-dimensional category of elements $\int F \in 2\mathsf{Cat}$, or its dual as just described for the lax case, and then quotienting out 2-cells to obtain $\pi_1(\int F) \in \mathsf{Cat}$ — the category whose morphisms are connected components of hom-categories of $\int F$. The 2-dimensional category of elements will be described in detail in Construction 3.3.6.

2.2.6 (Op)lax (Co)ends

Lax ends are a 2-dimensional analogue of ordinary ends in 1-categories (or, more generally, $\mathcal{V}$-categories). Many basic notions of category theory can be expressed in terms of (co)ends, notably: weighted (co)limits, pointwise Kan extensions and (extra) natural transformations. The same is true for (op)lax (co)ends in describing (op)lax notions in 2-dimensional category theory. The basic concepts and results of lax end calculus are given in [Hir22]. Below we reproduce from this paper the parts relevant to our use of lax ends in later chapters.

Definition 2.2.2 ((Op)lax (co)wedge)**.** Given a 2-functor $T\colon \mathcal{A}^{\mathsf{op}} \times \mathcal{A} \to \mathcal{B}$, a *lax wedge* from an object $b \in \mathcal{B}$ to T is a family $\lambda_a\colon b \to T(a, a)$ indexed by $a \in \mathcal{A}$ along with a family $\lambda_u\colon T_{a,u}\ \lambda_a \Rightarrow T_{u,a'}\ \lambda_{a'}$ indexed by $u\colon a \to a' \in \mathcal{A}$ satisfying 2-naturality and coherence conditions given by the following equalities of 2-cells for arbitrary $v\colon a \to a'$, $u\colon a' \to a''$ and $\theta\colon u \Rightarrow w$:



A *lax cowedge* T to b is a lax wedge from b to T^{op}. The coherence and naturality conditions are those above reflected across the vertical axis (with the direction of v and u correspondingly reversed). A *oplax wedge* from b to T is a wedge from b to T^{co}, whose axioms are those above reflected across the horizontal axis (with the direction of θ correspondingly reversed). Equivalently, it is a lax wedge from b to T', where $T'\colon \mathcal{A} \times \mathcal{A}^{\mathsf{op}} \to \mathcal{B}$ is defined as

[2](Op)lax colimits of strict 2-functors subsume oplax colimits of (op)lax functors from 1-categories, since the latter are equally given by oplax colimits of the corresponding strict 2-functors from the (op)lax functor classifier.

$T'_{a,b} \coloneqq T_{b,a}$. Note that swapping the order of the arguments of T and reversing the 1-cells of A has the same effect on the above string diagrams as a vertical reflection. ◊

For a given 2-functor $T\colon \mathcal{A}^{\mathsf{op}} \times \mathcal{A} \to \mathcal{B}$ and $b \in \mathcal{B}$ there exists a category of wedges from b to T. A 1-cell between wedges λ and ρ is a family of 2-cells $\alpha_a\colon \lambda_a \to \rho_a$ satisfying certain obvious coherence conditions and with the obvious notions of identity and composition. We call this category $\mathsf{Wedge}_{\mathsf{lax}}(b, T)$.

Remark 2.2.3. Given a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$, we can define $\overline{F}\colon \mathcal{A}^{\mathsf{op}} \times \mathcal{A} \to \mathcal{B}$ as $\overline{F}_{a,a'} \coloneqq F_{a'}$. In this case, a lax wedge from $b \in \mathcal{B}$ to $\overline{F}$ is the same as a lax cone from b to F, i.e. a lax transformation $\Delta b \to F$. However, a lax *cowedge* from F to b is equivalent to an *oplax* transformation $F \to \Delta b$. This is an instance of the general fact that a lax transformation $\alpha\colon F \to G$ between 2-functors is equivalent to an *oplax* transformation from G^{op} to F^{op}. ◊

Definition 2.2.4 ((Op)lax (co)end)**.** Given a 2-functor $T\colon \mathcal{A}^{\mathsf{op}} \times \mathcal{A} \to \mathcal{B}$, a *lax end* $\lambda_a : \sqint_{a\in\mathcal{A}} T_{a,a} \to T_{a,a}$ is a universal lax wedge to T, in the sense that there is an isomorphism $\mathsf{Wedge}_{\mathsf{lax}}(b, T) \cong \mathcal{B}(b, \sqint_a T_{a,a})$ natural in b. The lax *coend* of T — denoted $\sqint^a T_{a,a}$ — is the lax end of T^{op}; the *oplax* end — denoted $\oint_a T_{a,a}$ — is the lax end of T^{co}. These dualised variants also admit definitions in terms of universal instances of the corresponding (op)lax (co)wedges. ◊

Remark 2.2.5. There is an important feature of lax ends not present with ordinary ends. For a 2-functor $T\colon \mathcal{A}^{\mathsf{op}} \times \mathcal{A} \to \mathcal{B}$, it is usually unnecessary to distinguish between the ordinary ends $\int_{a\in\mathcal{A}} T_{a,a}$ and $\int_{a\in\mathcal{A}^{\mathsf{op}}} T'_{a,a}$ as they are isomorphic. However, this isn't true for (op)lax coends. Instead, we have:

$$\sqint_{a\in\mathcal{A}} T_{a,a} \cong \oint_{a\in\mathcal{A}^{\mathsf{op}}} T'_{a,a}$$

That is, the lax end over $\mathcal{A}$ is isomorphic to the oplax end over $\mathcal{A}^{\mathsf{op}}$. We will generally not distinguish between T and T', since the appropriate choice for which argument to T is "first" can be inferred from the index of the end. ◊

We now state some of the predictable results from lax end calculus which we will use in later chapters. Proofs for these results are given in [Hir22].

Lemma 2.2.6 (Lax ends in Cat are categories of lax wedges)**.** *For $T\colon \mathcal{A}^{\mathsf{op}} \times \mathcal{A} \to \mathsf{Cat}$, $\mathsf{Wedge}_{\mathsf{lax}}(\mathbb{1}, T)$ is the lax end $\sqint_{a\in\mathcal{A}} T_{a,a}$ with projection maps $\mathsf{E}_a\colon \mathsf{Wedge}_{\mathsf{lax}}(\mathbb{1}, T) \to T_{a,a}$ given by sending a wedge λ from $\mathbb{1}$ to T to the element of $T_{a,a}$ chosen by $\lambda_a\colon \mathbb{1} \to T_{a,a}$.*

Lemma 2.2.7 (Lax hom-categories are lax ends)**.** *For 2-functors $F, G\colon \mathcal{A} \to \mathcal{B}$, the category $[F, G]_{\mathsf{lax}}$ of lax natural transformations and modifications is a lax end $\sqint_{a\in\mathcal{A}} \mathcal{B}(Fa, Ga)$ whose projection maps $\mathsf{E}_a\colon [F, G]_{\mathsf{lax}} \to \mathcal{B}(Fa, Ga)$ are given by sending a lax transformation or modification to its component at $a \in \mathcal{A}$. Similarly, $[F, G]_{\mathsf{oplax}} \cong \oint_{a\in\mathcal{A}} \mathcal{B}(Fa, Ga)$.*

Lemma 2.2.8 (Representables preserve lax ends)**.** *For $T\colon \mathcal{A}^{\mathsf{op}} \times \mathcal{A} \to \mathcal{B}$, $b \in \mathcal{B}$:*

$$\sqint_{a\in\mathcal{A}} \mathcal{B}(b, T_{a,a}) \cong \mathcal{B}\left(b, \sqint_{a\in\mathcal{A}} T_{a,a}\right) \qquad \sqint_{a\in\mathcal{A}} \mathcal{B}(T_{a,a}, b) \cong \mathcal{B}\left(\oint^{a\in\mathcal{A}} T_{a,a}, b\right)$$

when the relevant (co)ends exist.

Lemma 2.2.9 (Fubini rule)**.** *For $T\colon \mathcal{A}^{\mathsf{op}}\times\mathcal{A}\times\mathcal{B}^{\mathsf{op}}\times\mathcal{B}\to\mathcal{C}$, we have the following isomorphisms when the relevant (co)ends exist:*

$$\oint_{a\in\mathcal{A}}\int_{b\in\mathcal{B}} T_{a,a,b,b} \cong \int_{b\in\mathcal{B}}\oint_{a\in\mathcal{A}} T_{a,a,b,b} \qquad \oint_{a\in\mathcal{A}}\oint_{b\in\mathcal{B}} T_{a,a,b,b} \cong \oint_{b\in\mathcal{B}}\oint_{a\in\mathcal{A}} T_{a,a,b,b}$$

By duality, it follows that oplax ends commute with oplax, lax and ordinary ends as well, and that oplax, lax and ordinary coends commute with each other.

2.2.7 The Gray Tensor Product(s)

The map $-\times\mathcal{A}\colon \mathsf{2Cat}\to\mathsf{2Cat}$ which sends a 2-category $\mathcal{B}$ to its product with a given 2-category $\mathcal{A}$ can be defined as the left adjoint to the hom-functor $[\mathcal{A},-]\colon\mathsf{2Cat}\to\mathsf{2Cat}$. This hom-functor sends the 2-category $\mathcal{B}$ to the 2-category $[\mathcal{A},\mathcal{B}]$ of 2-functors, strict 2-natural transformations and modifications. If we instead consider the hom-functor $[\mathcal{A},B]_{\mathsf{pseudo}}$ of 2-functors, *pseudonatural* transformations and modifications, then the left adjoint defines the *Gray tensor product* with $\mathcal{A}$, denoted $-\otimes\mathcal{A}$. There are also lax and oplax variants of the Gray tensor products corresponding to the hom-functors $[\mathcal{A},B]_{\mathsf{lax}}$ and $[\mathcal{A},B]_{\mathsf{oplax}}$ whose 1-cells are lax and oplax transformations respectively. We will use $\boxminus$ to denote the lax variant and $\ominus$ to denote the oplax variant. The monoidal category given by $\mathsf{2Cat}$ with the monoidal product $\otimes$ (resp. $\boxminus$, $\ominus$) will be denoted Gray (resp. $\mathsf{Gray}_{\mathscr{L}}$, $\mathsf{Gray}_{\mathscr{O}}$).

We will give a presentation for the lax Gray tensor product $\mathcal{A}\boxminus\mathcal{B}$ for 2-categories $\mathcal{A}$ and $\mathcal{B}$, as this variant is most relevant to our purposes. Afterwards, we note briefly how this presentation can be altered to obtain presentations for $\mathcal{A}\ominus\mathcal{B}$ and $\mathcal{A}\otimes\mathcal{B}$.

Definition 2.2.10 (The lax Gray tensor product, $\mathcal{A}\boxminus\mathcal{B}$)**.** For 2-categories $\mathcal{A}$ and $\mathcal{B}$, the 2-category $\mathcal{A}\boxminus\mathcal{B}$ admits the following presentation:

0-cells are pairs $(a\in\mathcal{A}, b\in\mathcal{B})$.

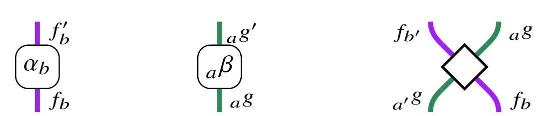
1-cells are generated by 1-cells of two sorts:

(a) $(a,b)\xrightarrow{[f]_b}(a',b)$ for $f\colon a\to a'\in\mathcal{A}$, $b\in\mathcal{B}$

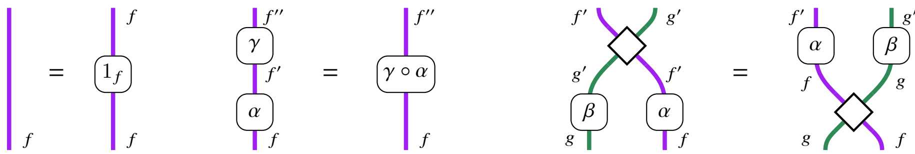
(b) $(a,b)\xrightarrow{{}_a[g]}(a,b')$ for $a\in\mathcal{A}$, $g\colon b\to b'\in\mathcal{B}$

Though we will usually omit the subscript notation when the value of the subscript is clear from context. The relations on these generating 1-cells are given by those in $\mathcal{A}$ and $\mathcal{B}$. That is, for $f: x\to x'$, $f'\colon x'\to x''$ in either $\mathcal{A}$ of $\mathcal{B}$, we have $[f'][f]=[f'f]$ (for any choice of subscripts), and $1_{(a,b)}=[1_a]_b={}_a[1_b]$. The square brackets are omitted from the labelling of 1-cells in our string diagrams below, since each string always represents a single generator. The type of the generator is indicated by the colour of the string: type (a) is purple and type (b) is green.

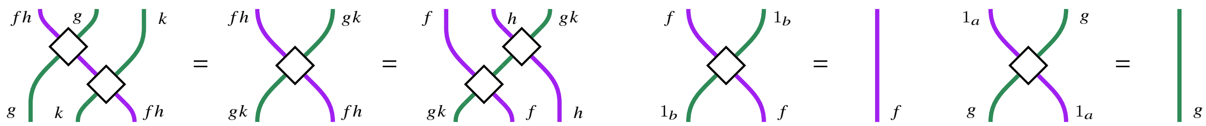
2-cells are generated by 2-cells of three sorts, for arbitrary $\alpha\colon f\Rightarrow f'\colon a\to a'$ in $\mathcal{A}$ and $\beta\colon g\Rightarrow g'\colon b\to b'$ in $\mathcal{B}$:



The generators of the third sort are determined by their inputs (or outputs), thus remain unlabelled. We also typically suppress the subscripts for generators of the first two types, either inferring them from context or leaving them polymorphic. The generators of the first two types merely transport the 2-dimensional structure of $\mathcal{A}$ and $\mathcal{B}$ to $\mathcal{A}\boxminus\mathcal{B}$. They satisfy all relations that hold in $\mathcal{A}$ and $\mathcal{B}$ respectively, e.g. the vertical composite of generators $\alpha\colon f \Rightarrow f'$ with $\gamma\colon f' \to f''$ is equal to the generator 2-cell $\gamma \circ \alpha$. 2-cell generators of the first two types can also "pass through" 2-cell generators of the third type:



There also exist coherence relations on generators of the third type which guarantee that the various ways in which the 1-cell generators can be shuffled by use of ◇'s produce the same 2-cell:



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Given a 2-functor $H : \mathcal{A} \to [\mathcal{B}, X]_{\mathsf{lax}}$, one obtains a functor $h\colon \mathcal{A} \boxminus \mathcal{B} \to X$ which acts on the generating data as follows:

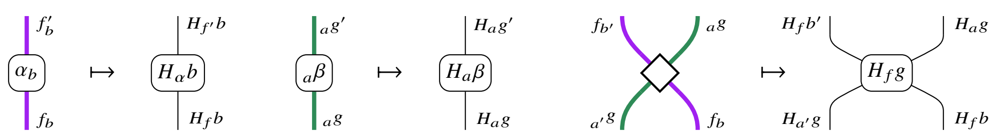
0-cells $(a, b) \mapsto H_a b$

1-cells according to the following mapping of generators:

$(a, b) \xrightarrow{[f]_b} (a', b)$ is sent to $H_f b$ (the component of H_f at b).

$(a, b) \xrightarrow{{}_a[g]} (a, b')$ is sent to $H_a g$.

2-cells according to the following mapping of generators:



The 2-cell generators of the third type thus correspond to the lax naturality 2-cells for the lax transformations in the image of H (cf. the string diagrams in Section 2.2.2). Oplax transformations have naturality 2-cells pointing in the opposite direction, so we obtain a presentation for $\mathcal{A} \ominus \mathcal{B}$ by flipping the ◇'s vertically (or horizontally). This vertical–horizontal symmetry in fact demonstrates that $\mathcal{A} \ominus \mathcal{B} \cong \mathcal{B} \boxminus \mathcal{A}$. The naturality 2-cells for a pseudo-natural transformation are invertible, and so by requiring the ◇'s to have inverses we produce a presentation for $\mathcal{A} \otimes \mathcal{B}$. If we instead replace the ◇ generator with an equality between its domain and codomain, we obtain a presentation for the ordinary product of 2-categories. Removing this generator entirely gives a presentation for the so-called "funny" tensor product.

2.2.8 The Fam Construction

We conclude this chapter with an explicit description of the Fam construction on a 1-category.

Construction 2.2.11 (The Fam construction, $\mathsf{Fam}(C)$)**.** For a category C, $\mathsf{Fam}(C)$ is the category with:

0-cells given by a set I and a function $A\colon I \to \mathsf{ob}(C)$

1-cells from (I, A) to (J, B) given by a function $f\colon I \to J$ and a family of maps $\phi_i\colon A_i \to B_{Ji}$ indexed by $i \in I$.

The composition $(I,A) \xrightarrow{(f,\gamma)} (J,B) \xrightarrow{(g,\psi)} (K,C)$ is given by $(gf, \psi J\ \phi)$, where $(\psi J\ \phi)_i = \psi_{Ji}\ \phi_i$. ◊

This mapping $C \mapsto \mathsf{Fam}(C)$ extends to a 2-functor $\mathsf{Fam}\colon \mathsf{Cat} \to \mathsf{Cat}$ which sends functors and natural transformations to their "post-composition" maps. That is, for a functor $F\colon C \to D$, the functor $\mathsf{Fam}(F)\colon \mathsf{Fam}(C) \to \mathsf{Fam}(D)$ maps objects as $(I,A) \mapsto (I,FA)$ and 1-cells as $(f,\phi) \mapsto (f,F\phi)$. For $\alpha\colon F \Rightarrow G$ a natural transformation, $\mathsf{Fam}(\alpha)\colon \mathsf{Fam}(F) \Rightarrow \mathsf{Fam}(G)$ has component at (I,A) given by $(1_I, \phi A)$, where $(\phi A)_i = \phi_{Ai}$. This 2-functor is endowed with the structure of a pseudomonad on Cat by the following natural transformations:

Unit $\eta\colon 1_{\mathsf{Cat}} \Rightarrow \mathsf{Fam}$ the 2-natural transformation whose component at C sends $c \in C$ to $(*, * \xrightarrow{\langle c\rangle} \mathsf{ob}(C))$ and sends $u\colon c \to d$ to $(1_*, \langle u\rangle)$.

Multiplication $\mu\colon \mathsf{Fam}^2 \Rightarrow \mathsf{Fam}$ the 2-natural transformation which sends $(I, (J_i, A_i)_{i\in I})$ to $\left(\sum_{i\in I} J_i, (A_i j)_{(i,j)}\right)$. The mapping on morphisms sends $\left(f, (g_i, \phi_i)_{i\in I}\right)\colon (I, (J_i, A_i)_{i\in I}) \to (K, (L_k, B_k)_{k\in K})$ to the morphism whose first part is the function $\sum_i J_i \to \sum_k L_k$ which sends (i,j) to $(f_i, g_i j)$ and whose second part has component at (i,j) given by $\phi_i j\colon A_i j \to B_{f_i}(g_i j)$.

The associativity and unit laws are the only non-strict features of this pseudomonad. The isomorphisms replacing strict equalities for these laws are essentially given by those for the associativity of coproducts. For example the image of (I,A) under $\mu \circ \mathsf{Fam}\,\eta$ has indexing set $\sum_{i\in I} *$, which is not literally *equal* to I, merely isomorphic. Similarly, we have mere isomorphisms $\sum_{x\in *} I \cong I$ and $\sum_{(i,j)\in\sum_{i\in I} Ji} K_i j \cong \sum_{i\in I}\left(\sum_{j\in J_i} K_i j\right)$ which underlie the isomorphisms for the unit and associativity laws for Fam.

We will describe some of the interesting properties of the Fam construction as we explore their 2-categorical counterparts for the 2-categorical fam constructions described in later chapters.

Chapter 3

Weights for Oplax Colimits

In relation to the investigation into 2-categorical fam constructions, the purpose of this chapter is to introduce a class of weights, Θ, with the property that the free completion of a 2-category $\mathcal{K}$ under Θ-colimits, $\mathsf{F}_\Theta\mathcal{K}$, contains the 2-categorical fam construction $\mathsf{F}_\Omega\mathcal{K}$. Properties of weights in the class Θ established in this chapter are used in the next chapter to demonstrate that $\mathsf{F}_\Theta\mathcal{K}$ — and by extension $\mathsf{F}_\Omega\mathcal{K}$ — admit fam-like descriptions. The path to describing the class Θ — referred to in the chapter title as "weights for oplax colimits" — goes via a description of the *oplax morphism classifier*, $\mathcal{Q}$, which is a comonad on presheaf categories $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ whose coalgebras form the class Θ. An understanding of these comonads (one for each 2-category $\mathcal{A}$) is instrumental in establishing some important properties of the class Θ, but will also have other applications in later chapters. For example, the explicit description of $\mathcal{Q}$ which we obtain in Section 3.2 will be used in Chapter 5 when we describe F_Θ-functors as certain coalgebras for related comonads, and the characterisation of categories of elements for $\mathcal{Q}$-coalgebras by the existence of certain "generic" objects will be used in Chapters 5 and 7. These applications notwithstanding, this chapter is also intended to stand alone as an investigation into oplax-morphism classifiers and their coalgebras, providing answers to questions which we believe require no external motivation. Our investigation begins with a brief review of general *weak morphism classifiers*, of which the oplax morphism classifier on presheaves is a special case.

3.1 Weak Morphism Classifiers

For any 2-category $\mathcal{A}$ there is a forgetful functor from the 2-category $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ of strict 2-functors, strict transformations and modifications, to the 2-category $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}$ of strict functors, *oplax* transformations and modifications. This 2-functor has a left adjoint and thus induces a comonad $\mathcal{Q}$ on $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ with the property that the category of strict transformations $\mathcal{Q}X \Rightarrow Y$ is naturally isomorphic to the category of oplax transformations $X \Rightarrow Y$.

This adjunction was first described in [Str72] for the case where $\mathcal{A}$ is a 1-category. Later, a more general theory of weak morphism classifiers for algebras of 2-monads was described in [BKP89]. There it is shown that for any 2-monad T with rank[1] on a complete and cocomplete category $\mathcal{K}$, the forgetful functors from the 2-category of strict T-algebras and strict T-algebra morphisms, $T\text{-}\mathsf{alg}_{\mathsf{s}}$, to the 2-category of strict T-algebras and *oplax* morphisms, $T\text{-}\mathsf{alg}_{\mathsf{o}}$, (or lax morphisms $T\text{-}\mathsf{alg}_{\mathsf{l}}$, or pseudo morphisms $T\text{-}\mathsf{alg}_{\mathsf{p}}$) has a left adjoint. This theory yields the oplax morphism classifier on $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ because extension along $\mathsf{ob}(\mathcal{A}) \hookrightarrow \mathcal{A}$ induces a 2-monad with rank, T, on $[\mathsf{ob}(\mathcal{A}), \mathsf{Cat}]$ which satisfies $T\text{-}\mathsf{alg}_{\mathsf{s}} \cong [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ and $T\text{-}\mathsf{alg}_{\mathsf{o}} \cong [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}$. It also yields the oplax functor classifier, $\mathcal{K}^\dagger$, described in Section 2.2.4 via a 2-monad on the 2-category of graphs in Cat.

In [Lac02b] the conditions on T and $\mathcal{K}$ are weakened to $T\text{-}\mathsf{alg}_{\mathsf{s}}$ having oplax codescent objects, and in [LS12, Lemma 2.5] it is shown (based on an earlier argument given in [BKP89]) that the oplax morphism classifier comonad

[1] A monad *has rank* or is *with rank* if its underlying endo-functor preserves α-filtered colimits for some regular cardinal α.

will be oplax-idempotent whenever $\mathcal{K}$ admits lax limits of morphisms, which holds for the relevant example of $\mathcal{K} = [\mathsf{ob}(\mathcal{A}), \mathsf{Cat}]$. Also discussed in [LS12] (and elsewhere) are the *coalgebras* for weak morphism classifiers. The T-algebras which are additionally pseudo-coalgebras, normalised-pseudo-coalgebras and strict pseudo-coalgebras of the pseudo-morphism classifier are respectively called semi-flexible, flexible, and pie algebras for T, though alternative characterisations for these classes of T-algebras are given in [BKP89], [Bir+89] and [BG13]. The name "pie algebra" refers to the fact that in the case of the pseudo-morphism classifier on presheaves, these algebras are the PIE weights, i.e. weights in the saturation of products, inserters and equifiers. On the other hand, the *strict* coalgebras for w-morphism classifiers on $[\mathcal{A}, \mathsf{Cat}]$, where $w \in \{\mathsf{p} = \text{pseudo}, \mathsf{l} = \text{lax}, \mathsf{o} = \text{oplax}\}$, are shown in [LS12] to be characterised as the classes of weights for limits created by the forgetful functors from $T\text{-}\mathsf{alg}_\mathsf{p}$, $T\text{-}\mathsf{alg}_\mathsf{o}$ and $T\text{-}\mathsf{alg}_\mathsf{l}$ to $\mathcal{K}$ respectively, for arbitrary 2-monads T on $\mathcal{K}$. These coalgebras are the restriction to the 2-categorical setting of the more general $\mathcal{F}$-categorical notion of w-rigged weights. Such classes of weights are moreover shown to be saturated [LS12, Thm. 6.13].

We do not attempt to work from the level of abstraction of weak morphism classifiers for T-algebras when considering the oplax-morphism classifiers $\mathcal{Q}$ on $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$, but it will be useful to keep these more general results in mind to compare with and confirm observations we make for our particular case. The next section gives an explicit description of $\mathcal{Q}$ based on an expression for $\mathcal{Q}$ in terms of oplax coends. This is then translated via the fibred-indexed category equivalence to a description for a comonad on a 2-category of (certain special) 2-fibrations over the indexing 2-category $\mathcal{A}$. The fibred category setting will prove to be the most convenient for the purpose of describing the coalgebras of the oplax morphism classifier. From the fibred category perspective we will observe that the category of coalgebras is equivalent to the full subcategory of $2\mathsf{Cat}/\mathcal{A}$ whose objects are 2-functors which locally are discrete opfibrations. The corresponding coalgebras for the oplax morphism classifier on $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ are obtained by taking what we call the *oplax image presheaf* of such 2-functors. In particular, all coalgebras for the oplax morphism classifier are oplax colimits of representables. We will show moreover that any oplax colimit of representables indexed by an arbitrary 2-category naturally inherits a coalgebra structure, which is moreover unique up-to-isomorphism because the oplax morphism classifier is an oplax-idempotent 2-monad. Consequently, these coalgebras are contained in the saturation of the class of conical oplax colimits, which we show is equivalent to the saturation of the class of general oplax colimits.

3.2 The Oplax Morphism Classifier

For a 2-category $\mathcal{A}$, we wish to provide an elementary description of the comonad $\mathcal{Q}$ on $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ induced by the inclusion $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}] \hookrightarrow [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}$. This is more or less equivalent to obtaining a description for the left adjoint to this inclusion, which is what we shall do first before summarising the properties of the corresponding comonad in Section 3.2.3. We use $\mathcal{Q}$ to denote both the comonad and the left adjoint, since the comonad is really just the restriction of the left adjoint $\mathcal{Q} \colon [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ to $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}] \subseteq [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}$. We call this comonad *the* oplax morphism classifier with the understanding that this term might elsewhere be used with the more general meaning discussed in the previous section.

In [Hir22] Hirata constructs the *lax*-morphism classifier (and co-classifier) using lax ends and coends (Definition 4.1 for the construction, and Theorem 4.2 for the universal property). We find the use of lax ends and coends to be a simplifying abstraction of the construction in terms of lax codescent objects described in [Lac02b], though lax ends and coends are themselves constructed as lax codescent objects in [Hir22]. Our reproduction of Hirata's computation below to derive instead the *oplax*-morphism classifier using oplax ends and coends ($\oint_{\mathcal{A}}$ and $\oint^{\mathcal{A}}$ respectively) should resemble standard (co)end calculus, though we explicitly indicate the relevant lemmas from Section 2.2.6 as we go. In fact, replacing all the oplax (co)ends with ordinary (co)ends gives the usual proof for the "co-Yoneda lemma".

Note that each isomorphism is natural in $Y \in [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$:

$$
\begin{aligned}
[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}(X, Y) &\cong \oint_{a \in \mathcal{A}^{\mathsf{op}}} [X_a, Y_a] && \text{(Lemma 2.2.7)} \\
&\cong \oint_{a \in \mathcal{A}^{\mathsf{op}}} \left[X_a, \int_{b \in \mathcal{A}} [\mathcal{A}(b, a), Y_b] \right] && \text{(Yoneda Lemma)} \\
&\cong \oint_{a \in \mathcal{A}^{\mathsf{op}}} \int_{b \in \mathcal{A}} [X_a, [\mathcal{A}(b, a), Y_b]] && \text{(continuity of representables)} \\
&\cong \int_{b \in \mathcal{A}} \oint_{a \in \mathcal{A}^{\mathsf{op}}} [\mathcal{A}(b, a) \times X_a, Y_b] && \text{(Lemma 2.2.9)} \\
&\cong \int_{b \in \mathcal{A}} \left[\oint^{a \in \mathcal{A}} \mathcal{A}(b, a) \times X_a, Y_b \right] && \text{(Lemma 2.2.8)} \\
&\cong [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}] \left(\oint^{a \in \mathcal{A}} \mathcal{A}(-, a) \times X_a, Y \right)
\end{aligned}
$$

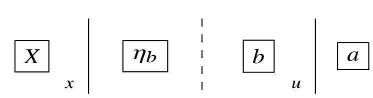
The 2-functor $\mathcal{Q} \colon [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ therefore sends $X \colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ to the oplax coend $\oint^{x \in \mathcal{A}} \mathcal{A}(-, x) \times X_x$, which exists by the fact $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ is cocomplete, and (op)lax coends are colimits. We can use the fact that colimits in $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ are moreover computed pointwise to give a presentation for each category $\mathcal{Q}X_a \cong \oint^{b \in \mathcal{A}} \mathcal{A}(a, b) \times X_b$ and describe the action of $\mathcal{Q}X$ on 1-cells and 2-cells.

3.2.1 A Presentation for $\mathcal{Q}X_a$

Recall from Definition 2.2.4 that an oplax coend of a 2-functor $T \colon C^{\mathsf{op}} \times C \to \mathcal{D}$ is a universal *oplax cowedge*. An oplax cowedge from T to $d \in \mathcal{D}$ comprises a family of coprojections $m_c \colon T(c, c) \to d$ for objects $c \in C$, and a family of natural transformations $M_u \colon m_c\ T(u, c) \Rightarrow m_{c'}\ T(c', u) \colon T(c', c) \to d$ indexed by 1-cells $u \colon c \to c'$ which satisfy axioms dual to those described for lax wedges in Definition 2.2.2. When $\mathcal{D} = \mathsf{Cat}$, a universal such oplax cowedge always exists and a presentation for its apex can be given in terms of the action of T, as we describe below for the case $T = \mathcal{A}(a, -) \times X$.

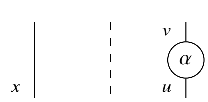
There are no equations the coprojections m_b for an oplax cowedge are required to strictly satisfy, so our universal cowedge $\mathcal{Q}X_a = \oint^{b \in \mathcal{A}} \mathcal{A}(a, b) \times X_b$ will have objects given by the coproduct $\coprod_{b \in \mathcal{A}} \mathsf{ob}(\mathcal{A}(a, b)) \times \mathsf{ob}X_b$. That is, the objects of $\mathcal{Q}X_a$ are pairs of the form $(u \colon a \to b,\ x \in X_b)$. Before we describe the 1-cells in $\mathcal{Q}X_a$ it will be convenient to establish a string diagram notation. To this end, notice that an object $x \in X_b$ can be identified with a natural transformation $x \colon \eta_b \to X$, where $\eta_b = \mathcal{A}(-, b)$ is the representable presheaf at $b \in \mathcal{A}$. So we can equally think of the objects of $\mathcal{Q}X_a$ as pairs $(u \colon a \to b,\ x \colon \eta_b \to X)$. We can represent each component of such objects by a string — one in $\mathcal{A}$ and the other in $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ — and we concatenate the two strings around a dashed line to obtain a graphical representation for the objects of $\mathcal{Q}X_a$ like so:

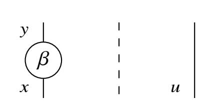
$$\left(X \xleftarrow{x} \eta_b\ ,\ b \xleftarrow{u} a \right) \quad \rightsquigarrow$$



The labelling of 0-cells (e.g. with $\boxed{X}$) will henceforth be suppressed, so the dotted line is necessary to distinguish the 1-cells in $\mathcal{A}$ from those in $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$.

Our presentation for $\mathcal{Q}$ has three kinds of generating 1-cells. The first two are just the morphisms from each factor of the $X_b \times \mathcal{A}(a, b)$ product categories. That is, for $\alpha \colon u \Rightarrow v \colon a \to b$ and $\beta \colon x \to y$ in X_b we have generating 1-cells (α, x) and (u, β) with the following string-diagram representations:

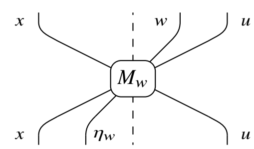




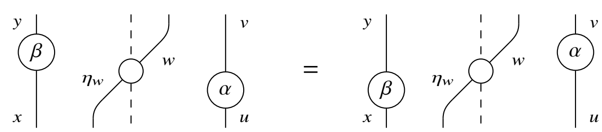
which satisfy all the relations coming from those product categories. For example, a X_b morphism commutes with a $\mathcal{A}(a,b)$ morphism, and vertical and horizontal composition of the generators is computed as in $\mathcal{A}(a,b)$ and X_b:

(3.1)

The third kind of generating morphism comes from the "M_w" data for the oplax cowedge. For such an M_w to exist for each $w\colon b \to c$ in $\mathcal{A}$, we need to freely add a morphism for each $u\colon a \to b$ and $x \in X_c$ from $(u, X_w x)$ to (wu, x). The object $X_w x \in X_b$ expressed as a presheaf $X_w x\colon \eta_b \Rightarrow X$ is given by $x\ \eta_w$, where $\eta_w = \mathcal{A}(-,w)\colon \eta_b \to \eta_c$, so a string diagram for such a morphism should have the following form:

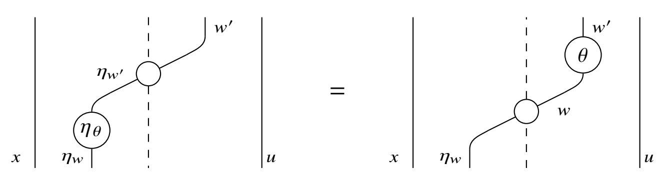


However, because M_w morphisms are required to be 2-natural in $u \in \mathcal{A}(a,b)$ and $x \in X_b$ we should "detach" the strings x and u from the node representing M_w. This makes it clear that 1-cells $\alpha : u \to v$ and $\beta\colon x \to y$ in $\mathcal{A}(a,b)$ and X_b can "slide past" the M_w generators (suppress the labelling of M_w, since it can be inferred from the attached strings):



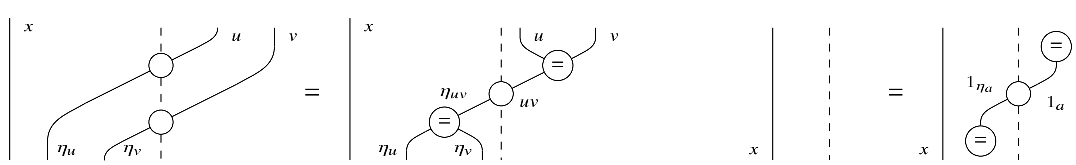
(3.2)

On the other hand, the M_w's are natural in w, (cf. the diagram on the right in Definition 2.2.2) so we also have a relation that allows us to slide a 2-cell $\theta\colon w \to w'$ across the M_w generators:



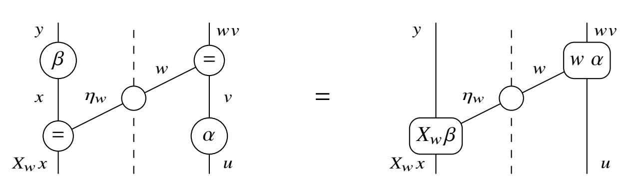
(3.3)

And finally we have relations corresponding to the coherence conditions (cf. the diagrams on the left in Definition 2.2.2) which more or less say that the M_w's behave functorially:

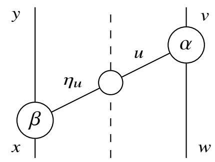


(3.4)

Given a string diagram representation for a 1-cell in $\mathcal{Q}X_a$ we can first consolidate all the generators of the first two types (the "α's and β's") using the relations in (3.1) and (3.2) to slide them respectively above and below all the M_w's and then compose them into a single generator:

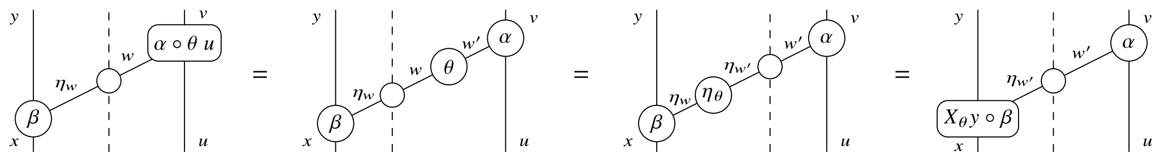


We can then consolidate all the M_w's into a single M_w using the coherence relations in (3.4). So every 1-cell in $\mathcal{Q}X_a$ from (w,x) to (v,y) has a string diagram of form shown in (3.5). Also shown is the underlying data, (α,u,β) of such a representation in "globular-diagram" notation:



$$\leftrightarrow \quad \left(\begin{tikzcd} & b \arrow[dd, "u"] \\ a \arrow[ur, "w"] \arrow[dr, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & \\ & b' \end{tikzcd} \;,\quad x \xrightarrow{\beta} X_u y \in X_b \right) \tag{3.5}$$

We will call such representations *consolidated*. Consolidated representations of 1-cells in $\mathcal{Q}X_a$ are not unique; the remaining relations on these consolidated forms are those shown in (3.3):



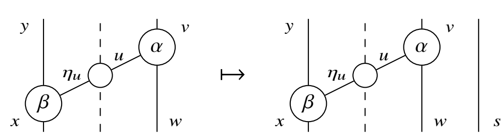
(3.6)

In terms of pasting diagrams, this relation is expressed as:

$$\left(\begin{tikzcd} & b \arrow[dd, dashed, "w"] \arrow[dd, bend left=60, "w'"] \arrow[dd, phantom, shift left=3ex, "\overset{\theta}{\Leftarrow}"] \\ a \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & \\ & b' \end{tikzcd} \;,\quad x \xrightarrow{\beta} X_{w'} y \right) \sim \left(\begin{tikzcd} & b \arrow[dd, "w"] \\ a \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & \\ & b' \end{tikzcd} \;,\quad x \xrightarrow{\beta} X_{w'} y \xrightarrow{X_\theta y} X_w y \right)$$

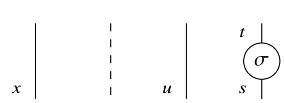
Our consolidated representation therefore doesn't constitute a "normal form" for 1-cells. In general, such a normal form for 1-cells doesn't exist, though normal forms may exist for special values of $\mathcal{A}$. For example, if $\mathcal{A}$ is locally discrete (i.e. a 1-category) then each 1-cell admits a unique expression of the form above since there are no non-trivial 2-cells "θ" to induce non-trivial relations on consolidated representations. More generally, if a 2-category $\mathcal{A}$ has initial objects in each connected component of each hom-category, then making a choice of initial object in each connected component and requiring the 1-cell component "w" in a consolidated representation to be one of these chosen 1-cells produces a normal form. We will consider 2-categories with this "component-initial 1-cells" property in more detail later.

This concludes the description of the category $\mathcal{Q}X_a$ for $a \in \mathcal{A}$. The map on objects $a \mapsto \mathcal{Q}X_a$ extends to a 2-functor $\mathcal{Q}X \colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ with the action of $\mathcal{A}^{\mathsf{op}}$ given by precomposition on the $\mathcal{A}(-,b)$ component. That is, for $s\colon a' \to a$ the functor $\mathcal{Q}X_s \colon \mathcal{Q}X_a \to \mathcal{Q}X_{a'}$ is given on objects by $(w,x) \mapsto (ws,x)$ and on morphisms by:



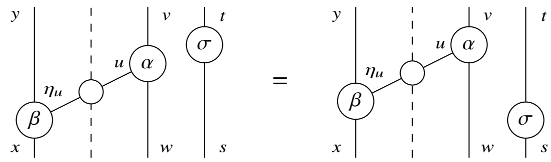
(3.7)

For a 2-cell $\sigma\colon s \Rightarrow t\colon a \to a'$, the natural transformation $\mathcal{Q}X_\sigma \colon \mathcal{Q}X_s \Rightarrow \mathcal{Q}X_t \colon \mathcal{Q}X_a \to \mathcal{Q}X_{a'}$ has component at (u,x) in $\mathcal{Q}X_a$ given by the following 1-cell in $\mathcal{Q}X_{a'}$:



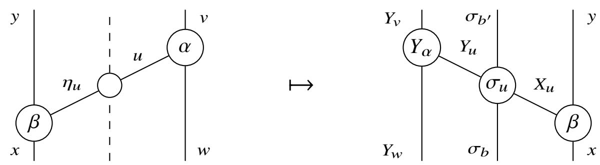
(3.8)

The naturality of these components is easily verified by a string-diagram argument:



3.2.2 $\mathcal{Q}X$ classifies oplax transformations

The presheaf $\mathcal{Q}X\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ classifies oplax transformations out of X by construction, but it will be convenient to see directly how this works. If $\sigma\colon X \Rightarrow Y$ is an oplax transformation, then the corresponding strict transformation $\overline{\sigma}\colon \mathcal{Q}X \Rightarrow Y$ has component at $a \in \mathcal{A}$ given by the functor $\overline{\sigma}_a\colon \mathcal{Q}X_a \to Y_a$ which sends object $\left(a \xrightarrow{u} b, x \in X_b\right)$ to $Y_u\,\sigma_b\,x$, and acts on morphisms as:



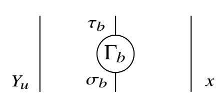
(3.9)

The string diagram on the right-hand side represents a 1-cell in the category Y_a by identifying an object $x \in X_b$ with the functor $\langle x \rangle : \mathbb{1} \to X_b$. This 1-cell is written more conventionally as:

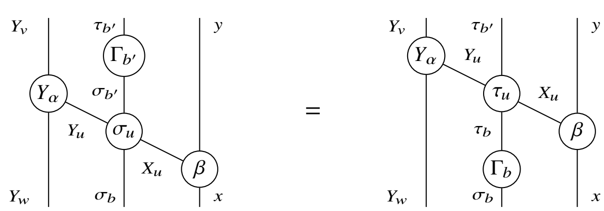
$$Y_w\sigma_b x \xrightarrow{Y_w\,\sigma_b\beta} Y_w\sigma_b X_u y \xrightarrow{Y_w\,\sigma_u y} Y_w Y_u \sigma_{b'}\,y \xrightarrow{Y_\alpha\,\sigma_{b'}\,y} Y_w\sigma_{b'}\,y$$

From the string diagram representation of this 1-cell, it is clear that the relations on the 1-cells of $\mathcal{Q}X$ are translated by $\overline{\sigma}$ to the familiar coherence and naturality conditions on the oplax transformation σ (compare the diagrams in Section 2.2.2 with those in (3.3) and (3.4)). The 2-naturality of the components $\overline{\sigma}_a\colon \mathcal{Q}X_a \to Y_a$ follows essentially from the 2-naturality of $Y_{a,b}\colon \mathcal{A}(b,a) \to \mathsf{Cat}(Y_a, Y_b)$ given how the action of $\mathcal{Q}X$ is defined on 1-cells and 2-cells in (3.7) and (3.8).

From a modification between oplax transformations $\Gamma\colon \sigma \Rrightarrow \tau\colon X \Rightarrow Y$ we obtain a modification $\overline{\Gamma}\colon \overline{\sigma} \Rrightarrow \overline{\tau}\colon \mathcal{Q}X \Rightarrow Y$ whose 2-cell component at $a \in \mathcal{A}$ is the natural transformation $\overline{\Gamma}_a\colon \overline{\sigma}_a \Rightarrow \overline{\tau}_a$ with component at object $(u\colon a \to b, x \in X_b)$ given by the morphism $Y_u\Gamma_b x\colon Y_u\sigma_b x \to Y_u\tau_b x$:



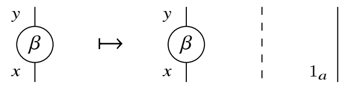
Considering the naturality square for $\overline{\Gamma}_a\colon \overline{\sigma}_a \Rightarrow \overline{\tau}_a$ for the morphism in $\mathcal{Q}X_a$ shown in (3.9) above, we see that the naturality of $\overline{\Gamma}_a$ follows directly from the fact that $\Gamma\colon \sigma \to \tau$ is a modification:



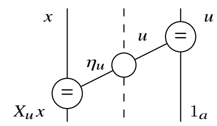
This mapping of oplax transformations and modifications described above provides a functor:

$$\overline{(-)} : [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}(X, Y) \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}](\mathcal{Q}X, Y)$$

which is one half of the isomorphism which exhibits $\mathscr{Q}$ as a left adjoint. The map going in the other direction is (necessarily) given by precomposition with an oplax transformation $\mathsf{q}(X)\colon X \Rightarrow \mathscr{Q}X$ which will form the unit of the adjunction. The component $\mathsf{q}(X)_a\colon X_a \longrightarrow \mathscr{Q}X_a$ acts on objects by sending $x \in X_a$ to $(1_a, x) \in \mathscr{Q}X_a$, and on morphisms as:

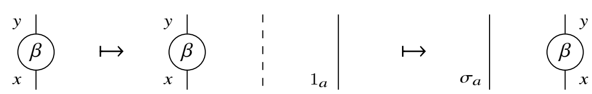


For $u\colon a \longrightarrow b$ in $\mathcal{A}$, the oplax-naturality 2-cell $\mathsf{q}(X)_u\colon \mathsf{q}(X)_a\ X_u \Rightarrow \mathscr{Q}X_u\ \mathsf{q}(X)_b$ is the natural transformation with component at $x \in X_b$ given by:

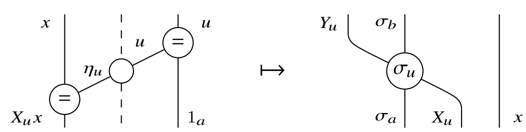


The naturality of $\mathsf{q}(X)_u$ with respect to morphisms in X_b corresponds to being able to "slide" a morphism in the X_b string past the M_u 2-cell. The oplax naturality and coherence of $\mathsf{q}(X)$ correspond to the naturality and coherence relations on the M_u 2-cells.

It can be verified by string-diagram arguments that precomposition by $\mathsf{q}(X)$ is, in fact, inverse to the 2-functor $\overline{(-)}$. Observe, for example, that for $\sigma\colon X \Rightarrow Y$ oplax, precomposing $\overline{\sigma}$ by $\mathsf{q}(X)$ gives an oplax transformation whose component at $a \in \mathcal{A}$ has the following action on objects and morphisms:

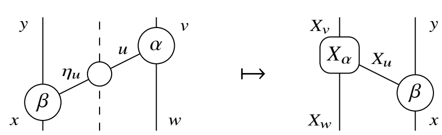


and whose oplax-naturality 2-cell at $u\colon a \longrightarrow b$ is a natural transformation with the required component at $x \in X_b$:



Conversely, any 2-natural transformation $\tau\colon \mathscr{Q}X \Rightarrow Y$ is determined by the images of objects $(1_a, x)$ and 1-cells $(1_a, X_u x) \longrightarrow (u, x)$ under the components τ_a, and these respectively determine the components and oplax-naturality 2-cells of $\tau\,\mathsf{q}(X)$ whose image under $\overline{(-)}$ is again τ. In particular, $\overline{\mathsf{q}(X)} = 1_{\mathscr{Q}X}$. Precomposition by $\mathsf{q}(X)$ is clearly 2-natural in Y, so we conclude that $\overline{(-)}$ is as well.

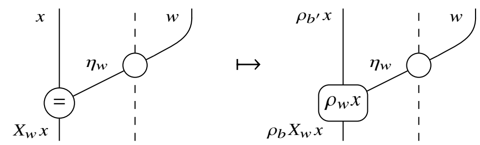
We now have a direct argument for what we had already observed to be true by coend calculus: the mapping $X \mapsto \mathscr{Q}X$ extends to a left adjoint to the forgetful functor $\mathsf{U}\colon [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}] \longrightarrow [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}$. The unit for this adjunction has components $\mathsf{q}(X)\colon X \longrightarrow \mathscr{Q}X$. The counit $\mathsf{s}(X)\colon \mathscr{Q}X \longrightarrow X$ is given by the strictification of the identity (oplax) transformation on X. Thus, it has component functor $\mathsf{s}(X)_a\colon \mathscr{Q}X_a \longrightarrow X_a$ given by the map $(w, x) \mapsto X_w x$ on objects and the following action on morphisms:



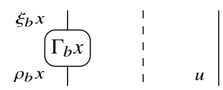
(3.10)

The map on objects $X \mapsto \mathscr{Q}X$ canonically extends to a 2-functor $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}} \longrightarrow [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ which maps an oplax transformation $\rho\colon X \Rightarrow Y$ to the strict 2-natural transformation $\mathscr{Q}\rho\colon \mathscr{Q}X \longrightarrow \mathscr{Q}Y$ whose component functor $\mathscr{Q}\rho_a\colon \mathscr{Q}X_a \longrightarrow \mathscr{Q}Y_c$ maps object (u, x) to $(u, \rho_b x)$ and acts on 1-cell generators of the first two types as $(\alpha, x) \mapsto$

$(\alpha, \rho_b x)$ and $(u, \beta) \mapsto (u, \rho_b \beta)$. The action on generators of the third type (the M_w's) is given by:



For $\Gamma\colon \rho \Rrightarrow \xi\colon X \Rightarrow Y$ a modification between oplax transformations, $\mathcal{Q}\Gamma_a\colon \mathcal{Q}\rho_a \Rightarrow \mathcal{Q}\xi_a$ is the natural transformation whose component at object (u, x) in $\mathcal{Q}X_a$ is:



3.2.3 The Comonad $\mathcal{Q}$

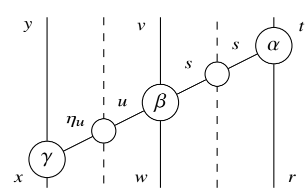
The comonad induced by the adjunction $\mathcal{Q}\colon [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}} \rightleftharpoons [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}] :\mathsf{U}$ is also denoted $\mathcal{Q}$, and its action as an endo-functor is the restriction of the action of the left adjoint $\mathcal{Q}$ to $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}] \subseteq [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}$. The counit for this comonad is the counit of the adjunction described in (3.10). Before we describe the comultiplication, let's consider for $X\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ the presheaf $\mathcal{Q}^2X$. We can identify the objects of $\mathcal{Q}^2X_a$ with triples $(u\colon a \to b, v : b \to c, x \in X_c)$ for any b and c in $\mathcal{A}$. Morphisms from $\left(a \xrightarrow{r} b \xrightarrow{w} c, x\right)$ to $\left(a \xrightarrow{t} b' \xrightarrow{v} c', y\right)$ can be expressed as triples:

$$b \xrightarrow{s} b' \in \mathcal{A}, \qquad (x, w) \xrightarrow{(u,\gamma,\beta)} (vs, y) \in \mathcal{Q}X_b, \qquad sr \xRightarrow{\alpha} t \in \mathcal{A}$$

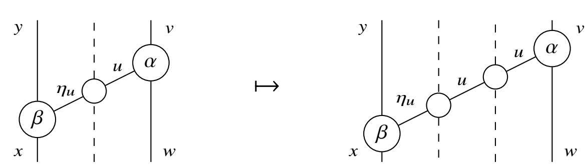
The second component is itself a triple, representing a morphism in $\mathcal{Q}X_b$. We can expand these data to obtain a quintuple:

$$b \xrightarrow{s} b', \qquad c \xrightarrow{u} c', \qquad x \xrightarrow{\gamma} X_u y, \qquad uw \xRightarrow{\beta} vs, \qquad \alpha\colon sr \Rightarrow t$$

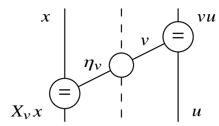
we can represent these data by string diagrams with two 'dividers':



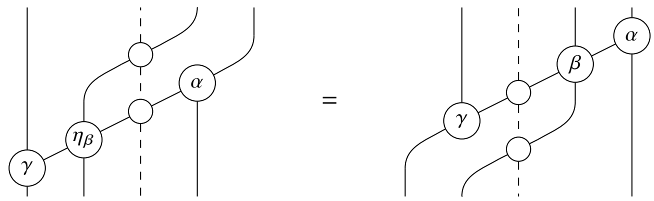
The comultiplication is given by $\delta_X \coloneqq \mathcal{Q}\mathsf{q}(X)\colon \mathcal{Q}X \to \mathcal{Q}^2X$. Its component at $a \in \mathcal{A}$ is the functor which sends an object $\left(a \xrightarrow{u} b, x\right)$ in $\mathcal{Q}X_a$ to $\left(a \xrightarrow{u} b = b, x\right)$ in $\mathcal{Q}^2X_a$ and which acts on morphisms in terms of the above string notation as:



By Lemma 2.5 of [LS12] we know that the comonad $\mathcal{Q}$ is *oplax-idempotent*. One way to demonstrate this is to exhibit a modification $\theta\colon \mathcal{Q}\mathsf{s} \to \mathsf{s}\mathcal{Q}$ such that $\theta\,\delta$ is the identity on $\mathcal{Q}\mathsf{s}\,\delta = \mathsf{s}\mathcal{Q}\,\delta$ and $\mathsf{s}\,\theta$ is the identity on $\mathsf{s}\,\mathcal{Q}\mathsf{s} = \mathsf{s}\,\mathsf{s}\mathcal{Q}$. Such a modification θ will be composed of a family of modifications $\theta_X\colon \mathcal{Q}\mathsf{s}X \to \mathsf{s}\mathcal{Q}X$ each composed of a family of natural transformations $\theta_{X,a}\colon \mathcal{Q}\mathsf{s}X_a \to \mathsf{s}\mathcal{Q}X_a$ which themselves have component morphisms $\theta_{X,a}(u, v, x)$ for each object in $\mathcal{Q}^2 X_a$. We thus define θ by declaring $\theta_X a_{(u,v,x)} \colon (u, X_v x) \to (vu, x)$ to be:



This choice is natural in (u, v, x) by the equality of the 1-cells represented by the diagrams of the form:



It is not difficult to verify that this θ does indeed satisfy the additional coherence conditions required for an oplax-idempotent comonad, though we won't appeal to this explicit construction of θ at any later point. Merely knowing that $\mathcal{Q}$ is oplax-idempotent will be sufficient for our purposes, and this property follows from more general arguments alluded to in Section 3.1.

Our explicit description of $\mathcal{Q}$ as both a left adjoint and a comonad will corroborate the construction of free split locally discrete 2-fibrations introduced in Section 3.3.2, though the string diagram notation for $\mathcal{Q}$ won't be used again until Chapter 5, as most further analysis of the oplax morphism classifier and its coalgebras will be done from the fibred category perspective.

3.3 The Fibred Category Perspective

To achieve a "fibred category perspective" on the oplax morphism classifier $\mathcal{Q}$ we transport it across a fibred-indexed category equivalence.

For a 1-category, C, the Grothendieck construction establishes an equivalence between Cat-presheaves on C and fibrations over C. When the presheaf $F \colon C^{\mathrm{op}} \to \mathsf{Cat}$ is a strict functor the corresponding fibration is split, and when the presheaf is a presheaf of sets, the corresponding fibration is discrete. The situation for 2-categories is similar, though there are very many notions of 2-dimensional presheaves, each with its own fibred/indexed category correspondence for a related notion of 2-dimensional fibred category. A few of these correspondences are described in detail in [Buc14], and we will appeal to results from that paper in our discussion. For example:

Theorem 3.3.1 (2.2.11 from [Buc14]). *For $\mathcal{A}$ a 2-category, the Grothendieck construction extends to a 3-functor equivalence between $[\mathcal{A}^{\mathrm{op}}, \mathsf{2Cat}]$ and the 3-category of split 2-fibrations over $\mathcal{A}$ with split-cartesian functors.*

Actually, Theorem 2.2.11. is about the 3-category $[\mathcal{A}^{\mathrm{cop}}, \mathsf{2Cat}]$, but we will redefine 2-fibrations to make the above statement true:

Definition 3.3.2 (2-fibration). A 2-functor $F \colon \mathcal{C} \to \mathcal{D}$ is a *2-fibration* if:

(a) 1-cells $u \colon d \to Fc$ have cartesian lifts to 1-cells $\bar{u} \colon u^*c \to c$

(b) 2-cells $\alpha \colon Ff \Rightarrow g$ have *opcartesian* lifts to 2-cells $\underline{\alpha} \colon f \Rightarrow \alpha_* g$

(c) The horizontal composition of opcartesian 2-cells is opcartesian

A 2-fibration with chosen (op)cartesian lifts of 1-cells and 2-cells (i.e. a cleavage) is *split* if these chosen lifts are preserved by vertical and horizontal composition in the obvious sense. ◊

This differs from Definition 2.1.6 in [Buc14] where the hom-functors are required to be fibrations, rather than opfibrations. Buckley suggests the term *co-2-fibrations* for what we describe in Definition 3.3.2, but we avoid this for the sake of simplicity and to prevent confusion with the notion of a 2-fibration in $\mathsf{2Cat}^{\mathsf{op}}$.

Remark 3.3.3. The F-cartesian 1-cells for a 2-functor $F\colon \mathcal{C} \to \mathcal{D}$ aren't merely cartesian 1-cells for the underlying 1-functor, F_0. Instead, F-cartesian is meant in a Cat-*enriched* sense: $f\colon x \to y$ is F-cartesian if the following square is a pullback in Cat for every $c \in \mathcal{A}$:

$$\begin{tikzcd}
\mathcal{C}(c,x) \arrow[r, "\mathcal{C}(c,f)"] \arrow[d, "F_{c,x}"'] & \mathcal{C}(c,y) \arrow[d, "F_{c,y}"] \\
\mathcal{D}(Fc,Fx) \arrow[r, "\mathcal{D}(Fc,Ff)"'] & \mathcal{D}(Fc,Fy)
\end{tikzcd}$$

This implies a lifting property for f with respect to 2-cells as well as 1-cells. For details see [Buc14, Definition 2.1.1]. ◊

Definition 3.3.4 (Split cartesian functor, $\mathsf{Fib_s}(\mathcal{A})$)**.** Given two cloven 2-fibrations over the same base, $F\colon C \to \mathcal{A}$, $G\colon D \to \mathcal{A}$, a *split cartesian functor* from F to G is a functor $H\colon C \to D$ that preserves chosen cartesian lifts of 1-cells and opcartesian lifts of 2-cells, as well as satisfying $GH = F$. The 3-category $\mathsf{Fib_s}(\mathcal{A})$ of split 2-fibrations has split cartesian functors as 1-cells, vertical transformations as 2-cells (i.e. those $\sigma\colon H \Rightarrow H'$ with $G\sigma = 1_F$) and vertical modifications as 3-cells. ◊

Theorem 3.3.1 asserts that the 3-category of split 2-fibrations over $\mathcal{A}$ is equivalent to $[\mathcal{A}^{\mathsf{op}}, \mathsf{2Cat}]$, but for the purpose of understanding the oplax-morphism classifier we are more interested in Cat-valued presheaves, i.e. $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$. We can view $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ as the full sub-3-category of $[\mathcal{A}^{\mathsf{op}}, \mathsf{2Cat}]$ whose objects are presheaves with image in locally discrete 2-categories. The Grothendieck construction for the presheaves in this sub-category will consequently have hom-functors with discrete fibres; or, equivalently, the hom-functors will be *discrete* opfibrations. We will call split 2-fibrations whose hom-functors have discrete fibres *locally discrete split 2-fibrations*, and let $\mathsf{Fib_{lds}}(\mathcal{A})$ denote the full sub-3-category of $\mathsf{Fib_s}(\mathcal{A})$ with objects locally discrete split 2-fibrations. Both $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ and $\mathsf{Fib_{lds}}(\mathcal{A})$ have only trivial 3-cells as 3-categories, so we will treat them as 2-categories. We obtain the fibred-indexed correspondence for Cat-presheaves by restricting the correspondence described in Theorem 3.3.1:

Corollary 3.3.5. *For $\mathcal{A}$ a 2-category, the Grothendieck construction extends to a 2-functor equivalence between $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ and $\mathsf{Fib_{lds}}(\mathcal{A})$.*

The fibred-indexed equivalence of Corollary 3.3.5 is described in detail in [Lam20], where it appears as Theorem 3.15. The terminology used is slightly different: Lambert refers to locally discrete 2-fibrations as *discrete 2-fibrations*, and the restricted Grothendieck construction as the *2-category of elements*.

Construction 3.3.6 (The 2-category of elements, $\int F$)**.** For any presheaf $F\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ we can construct a locally discrete split 2-fibration $\pi_F\colon \int F \to \mathcal{A}$ with domain the 2-category defined as follows:

0-cells: pairs $(a \in \mathcal{A}, x \in Fa)$

1-cells: $(a,x) \to (b,y)$ are pairs $(u\colon a \to b, f\colon x \to F_u y)$

2-cells: $(u,f) \Rightarrow (v,g)\colon (a,x) \to (b,y)$ are 2-cells $\sigma\colon u \Rightarrow v$ in $\mathcal{A}$ such that $F_\sigma y\ f = g$:

The 2-functor $|F|\colon \int F \to \mathcal{A}$ is then given by projection onto the first component, e.g. $|F|(a,x) = a$. ◊

This construction extends the 1-categorical Grothendieck construction (cf. Section 2.2.5) in that if $\mathcal{A}$ were a 1-category (i.e. locally discrete 2-category) then $\int F$ would also be locally discrete, and isomorphic to the 1-categorical Grothendieck construction on F. In the other direction, Construction 3.3.6 can be seen as a special case of the Grothendieck construction for a 2-functor $F\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{2Cat}$, whose definition we obtain from the one above by replacing the equality in the description of 2-cells with a given 2-cell $\gamma\colon F_\sigma y\ f \Rightarrow g$ in F_b. It follows that Construction 3.3.6 is equally the 2-categorical Grothendieck construction of $\mathcal{A}^{\mathsf{op}} \xrightarrow{F} \mathsf{Cat} \hookrightarrow \mathsf{2Cat}$.

The 2-functor $|F|$ of Construction 3.3.6 is clearly locally discrete by the definition of the 2-cells, and it is a split 2-fibration with the following chosen liftings of the 1-cells and 2-cells:

$$\begin{tikzcd}
(a, F_u x) \arrow[r, "{(u, 1_{F_u x})}"'] & (b, x) & (a, x) \arrow[r, bend left, "{(v, f)}"] \arrow[r, bend right, "{(u, F_\sigma y\ f)}"'] \arrow[r, phantom, "\Downarrow\sigma"] & (b, y) \\
a \arrow[r, "u"'] & b & a \arrow[r, bend left, "v"] \arrow[r, bend right, "u"'] \arrow[r, phantom, "\Downarrow\sigma"] & b
\end{tikzcd}$$

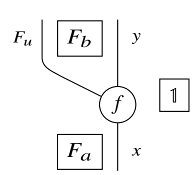
Conversely, we can construct a presheaf $\mathsf{P}_F\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ corresponding to any locally discrete split 2-fibration $F\colon C \to \mathcal{A}$ by sending $a \in \mathcal{A}$ to the fibre C_a and with action on the 1-cells and 2-cells of $\mathcal{A}$ determined by the chosen lifting data as described in [Lam20, Construction 3.2]. This map on objects $F \mapsto \mathsf{P}_F$ extends to a pseudo-inverse to the 2-category of elements construction.

The other relevant 2-dimensional category of presheaves is $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}$. The corresponding category of fibred categories has the same objects as $\mathsf{Fib}_{\mathsf{lds}}(\mathcal{A})$, but the 1-cells and 2-cells are those of $\mathsf{2Cat}/\mathcal{A}$, rather than $\mathsf{Fib}_{\mathsf{s}}(\mathcal{A})$. We will prove this directly, as it is not one of the cases described in [Buc14] or [Lam20].

3.3.1 The fibred-indexed equivalence for oplax morphisms

We would like to extend Construction 3.3.6 to a 2-functor with domain $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}$ and codomain the *full* sub-2-category of $\mathsf{2Cat}/\mathcal{A}$ with objects locally discrete split 2-fibrations, which we will denote by $\mathsf{2Cat}/_{\mathsf{lds}}\mathcal{A}$ (not to be confused with $\mathsf{Fib}_{\mathsf{lds}}(\mathcal{A})$ which has only the *split cartesian* maps as 1-cells).

Construction 3.3.7 ($\textstyle\int\colon [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}} \to \mathsf{2Cat}/_{\mathsf{lds}}\mathcal{A}$)**.** The action of $\int$ on objects is given by Construction 3.3.6. In describing the action of $\int$ on hom-categories, it will be convenient to represent 1-cells in categories of the form $\int F$ with string diagrams. Our convention will be to represent the 1-cell $(u, f)\colon (a, x) \to (b, y)$ by the usual string diagram representation for the 1-cell $f \in F_a$, where an object $x \in F_a$ is cast as a functor $x\colon \mathbb{1} \to F_a$ (cf. Section 2.2.5). The underlying 1-cell u can be read left-most branch at the top:

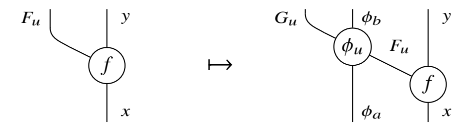


(3.11)

The 0-cells ("faces") of the string diagram (3.11) have been labelled, but will henceforth be anonymous as their labelling can be determined from the 1-cells.

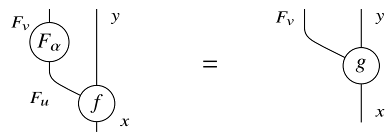
Returning to our definition of $\int$ — for an oplax transformation $\phi\colon F \Rightarrow G\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$, we define the functor

$\int\phi\colon \int F \to \int G$ to send $(a,x) \in \int F$ to $(a, \phi_a x) \in \int G$, and act on 1-cells as follows:

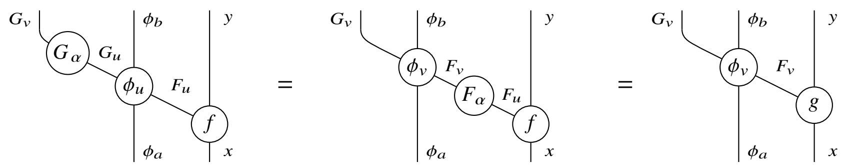


(3.12)

The functoriality of $\int\phi$ corresponds to the coherence axioms on the ϕ_u data. There is no choice available in defining the action of $\int\phi$ on 2-cells, because lifts of 2-cells in $\mathcal{A}$ to $\int G$ with fixed domain in $\int G$ are unique and $\int\phi$ is required to commute with $|F|$ and $|G|$. So, if $\alpha\colon u \Rightarrow v$ is the underlying data of a 2-cell (u,f) to (v,g), then α must also define the image of this 2-cell under $\int\phi$ from $\int\phi(u,f)$ to $\int\phi(v,g)$. We need only check that this is a well-defined 2-cell in $\int G$. Recall that $\alpha\colon u \Rightarrow v$ defines a 2-cell in $\int F$ from (u,f) to (v,g) if and only if it satisfies the following condition:

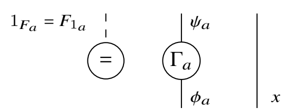


Under this assumption we see that α also defines a 2-cell between the images of (u,f) and (v,g) under $\int$:



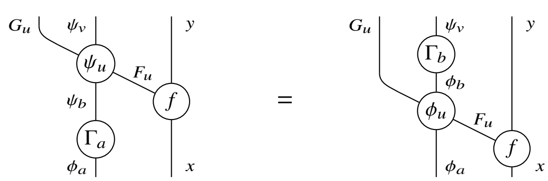
The 1-functoriality of this action of $\int$ on oplax transformations is immediately from the definition of composition for oplax transformations.

Now, given a modification $\Gamma\colon \phi \to \psi\colon F \Rightarrow G$ between oplax transformations we obtain the 2-natural transformation $\int\Gamma\colon \int\phi \Rightarrow \int\psi\colon \int F \to \int G$ by declaring its component at object $(a,x) \in \int F$ to be:



(3.13)

The inclusion of the identity functor F_{1_a} on the left indicates that this 1-cell lies over 1_a (i.e. is vertical) as required. The 1-naturality of these component morphisms with respect to a 1-cell (u,f) in $\int F$ amounts to the following equality of 1-cells:



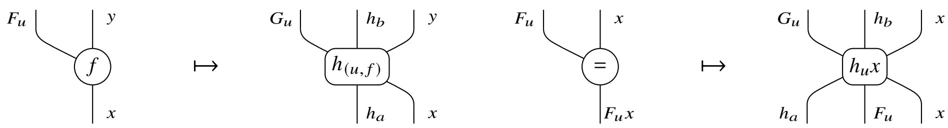
which corresponds to the naturality of the modification Γ. The 2-naturality of $\int\Gamma$ in $\int F$ is automatic because $\int G$ is locally a discrete opfibration. ◊

Proposition 3.3.8. *The 2-functor $\int\colon [A^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}} \to \mathsf{2Cat}/_{\mathsf{lds}}\mathcal{A}$ is an equivalence.*

Proof. The 2-functor $\int$ is essentially surjective on objects because its action on objects is the same as that of the 2-category of elements construction $\int\colon [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}] \to \mathsf{Fib}_{\mathsf{lds}}(\mathcal{A})$, which is shown to be essentially surjective in [Lam20, Corollary 3.14]. It remains to show that it is 2-fully-faithful, i.e. an isomorphism on hom-categories.

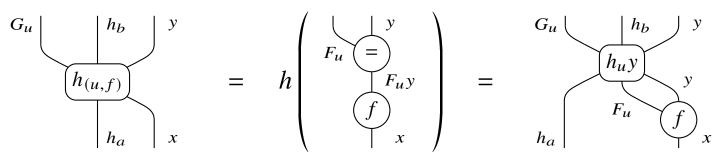
We first define an inverse map on the 1-cells. We will assume we have a 2-functor $h\colon \int F \to \int G$ such that $|G|\ h = |F|$

and from this 2-functor construct an oplax transformation $h' : F \Rightarrow G : \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$. By restricting $h : \oint F \to \oint G$ to each of the fibres of $\oint F$ and $\oint G$ we obtain for each $a \in \mathcal{A}$ a functor $h_a : F_a \to G_a$, such that $h(a, x) = (a, h_a x)$. Being a map in the slice over $\mathcal{A}$, h must additionally preserve the first component of 1-cells, so that for $(u, f) : (a, x) \to (b, y)$, $h(u, f) = \big(u, h_{(u,f)}\big)$ for some $h_{(u,f)} : h_a x \to G_u h_b y$. In particular, we have for any $u : a \to b$ and $x \in F_b$ the 1-cell $h_{u,1_{F_u x}} : h_a\, F_u x \to G_u\, h_b x$, to which we give the special name $h_u x$:



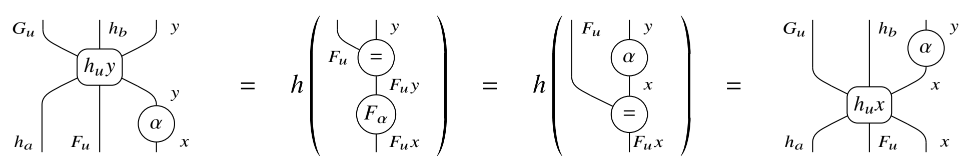
(3.14)

In fact, the general $h_{(u,f)}$ 1-cells are determined by the functors h_a and the 1-cells $h_u x$, since any morphism $(u, f) : (a, x) \to (b, y)$ in $\oint F$ factorises as a vertical 1-cell followed by a chosen cartesian 1-cell: $(u, f) = \big(u, 1_{F_u y}\big)\,(1_a, f)$, so we have:



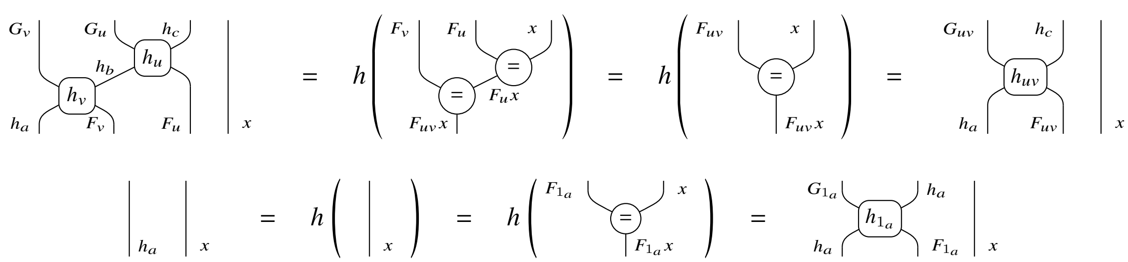
(3.15)

If f above has the form $f = F_u\alpha$ for some $\alpha : x \to y$ in F_b, then $(u, F_u\alpha)$ can also be factorised as $(1_b, \alpha)\,\big(u, 1_{F_u x}\big)$, from which we observe that the $h_u x$ 1-cells form a natural transformation $h_u : h_a\, F_u \Rightarrow G_u\, h_b$:

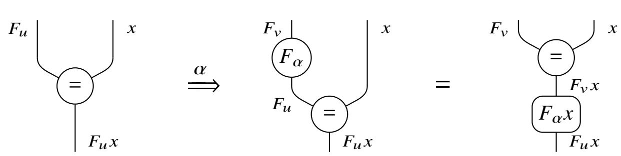


These functors h_a and natural transformations h_u are precisely the data we need to define the components and oplax-naturality data for an oplax transformation $h' : F \Rightarrow G$. It remains to check that the h_u 2-cells satisfy the oplax-naturality axioms.

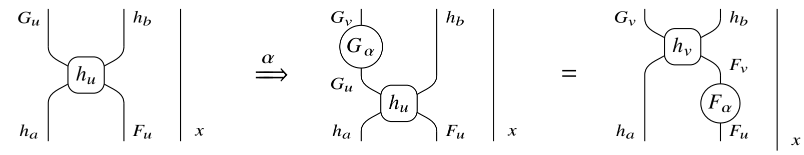
The coherence axioms follow from the functoriality of G, F and h:



For the naturality of the h_u data with respect to 2-cells, observe that any $\alpha : u \Rightarrow v : a \to b$ there is a 2-cell $\alpha : \big(u, 1_{F_u x}\big) \Rightarrow (v, F_\alpha x)$:



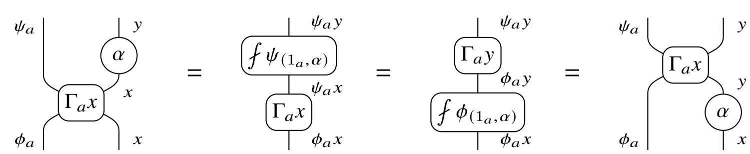
Being a map in $2\mathsf{Cat}/\mathcal{A}$, h must send this 2-cell to a 2-cell over α in $\textstyle\int G$, which produces the required equality:



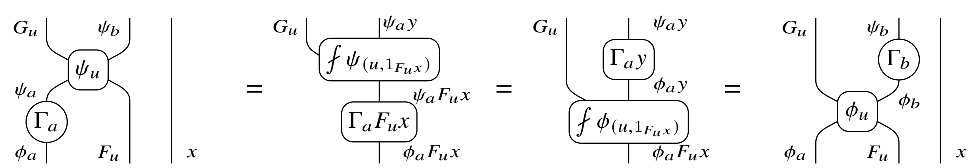
We can thus produce an oplax natural transformation $h'\colon F \Rightarrow G\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ from any 2-functor $h\colon \textstyle\int F \to \textstyle\int G$. It is straightforward to check that this map does indeed define an inverse to the underlying map on objects of $\textstyle\int_{F,G}\colon [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}(F, G) \to 2\mathsf{Cat}/\mathcal{A}(\textstyle\int F, \textstyle\int G)$ by comparing (3.15) with (3.12).

It remains to show that for any vertical 2-natural transformation $\Gamma\colon \textstyle\int\phi \Rightarrow \textstyle\int\psi\colon \textstyle\int F \to \textstyle\int G$ in $2\mathsf{Cat}/_{\mathsf{lds}}\mathcal{A}$ there exists a unique modification $\Gamma'\colon \phi \Rrightarrow \psi$ with $\textstyle\int\Gamma' = \Gamma$.

The data for such a $\Gamma\colon \textstyle\int\phi \Rightarrow \psi$ amount to a choice for each $(a, x) \in \textstyle\int F$ of a vertical morphism $(1_a, \Gamma_a x)\colon (a, \phi_a x) \to (a, \psi_a x)$ which is determined by the component 1-cell $\Gamma_a x\colon \phi_a x \to \psi_a x$. These are equally the data required for a modification $\Gamma'\colon \phi \Rrightarrow \psi$. The naturality of Γ with respect to vertical morphisms, i.e. those of the form $(1_a, \alpha)$ corresponds to the naturality of each Γ_a as a transformation $\phi_a \Rightarrow \psi_a\colon F_a \to G_a$:



On the other hand, the naturality of $\Gamma\colon \textstyle\int\phi \Rightarrow \textstyle\int\psi$ with respect to the chosen cartesian 1-cells of $\textstyle\int F$ corresponds to modification axioms that the collection of 2-cells Γ_a in Cat must satisfy:



So the $\Gamma_a x$ data do indeed define a modification $\Gamma'\colon \phi \to \psi$. Moreover, this construction provides an inverse to the action of $\textstyle\int$ on 2-cells shown in (3.13). □

3.3.2 Free locally discrete split 2-fibrations

The purpose of establishing the fibred-indexed equivalence for presheaves $\mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ is to gain a new perspective on the comonad $\mathcal{Q}$ by transporting it along the equivalence $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}] \simeq \mathsf{Fib}_{\mathsf{lds}}(\mathcal{A})$ to a comonad $\mathfrak{Q}$ on $\mathsf{Fib}_{\mathsf{lds}}(\mathcal{A})$. In this section we observe that this comonad $\mathfrak{Q}$ is induced by the free locally discrete 2-fibration adjunction between $2\mathsf{Cat}/\mathcal{A}$ and $\mathsf{Fib}_{\mathsf{lds}}(\mathcal{A})$.

What is more immediately apparent is that $\mathfrak{Q}$ is induced by the forgetful 2-functor $\mathcal{U}\colon \mathsf{Fib}_{\mathsf{lds}}(\mathcal{A}) \to 2\mathsf{Cat}/_{\mathsf{lds}}\mathcal{A}$. Because $\textstyle\int\colon [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}} \to 2\mathsf{Cat}/_{\mathsf{lds}}\mathcal{A}$ extends the 2-category of elements map $\int\colon [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}] \to \mathsf{Fib}_{\mathsf{lds}}(\mathcal{A})$, the pair of equivalences $(\int, \textstyle\int)$ define a map from the forgetful 2-functor $\mathsf{U}\colon [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}] \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}$ to the forgetful functor $\mathcal{U}\colon \mathsf{Fib}_{\mathsf{lds}}(\mathcal{A}) \to 2\mathsf{Cat}/_{\mathsf{lds}}\mathcal{A}$. It follows that $\mathcal{Q}\colon [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ is transported along $(\int, \textstyle\int)$ to a left adjoint to $\mathcal{U}$, and thus that the comonad $\mathfrak{Q}$ on $\mathsf{Fib}_{\mathsf{lds}}(\mathcal{A})$ is the one induced by a free-forgetful adjunction between $\mathsf{Fib}_{\mathsf{lds}}(\mathcal{A})$ and $2\mathsf{Cat}/_{\mathsf{lds}}\mathcal{A}$. We note that this adjunction is a coKleisli adjunction, as $\mathcal{U}$ is bijective on objects.

$$
\begin{tikzcd}
{[\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}]} \arrow[rr, bend right, "\mathsf{U}"'] \arrow[d, "\int"'] \arrow[rr, phantom, "\perp"] & & {[\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}]_{\mathrm{oplax}}} \arrow[ll, bend right, "\mathcal{Q}"'] \arrow[d, "\int"] \arrow[lld, phantom, ""] \\
\mathsf{Fib}_{\mathsf{lds}}(\mathcal{A}) \arrow[rr, bend right, "\mathcal{U}"'] \arrow[rr, phantom, "\perp"] & & \mathsf{2Cat}/_{\mathsf{lds}}\mathcal{A} \arrow[ll, bend right, dashed, "\mathcal{F}"']
\end{tikzcd}
$$

But we can show that $\mathfrak{Q}$ is also the comonad induced by a "larger" free-forgetful adjunction. The forgetful functor $\mathcal{U}$ is the first component of the bijective-on-objects/fully-faithful factorisation of the inclusion $\mathcal{R}\colon \mathsf{Fib}_{\mathsf{lds}}(\mathcal{A}) \hookrightarrow \mathsf{2Cat}/\mathcal{A}$. The inclusion $\mathcal{R}$ also factorises as $\mathsf{Fib}_{\mathsf{lds}}(\mathcal{A}) \hookrightarrow \mathsf{Fib}_{\mathsf{s}}(\mathcal{A}) \hookrightarrow \mathsf{2Cat}/\mathcal{A}$ and each 2-functor in that composite has a left 2-adjoint. The inclusion $\mathsf{Fib}_{\mathsf{s}}(\mathcal{A})$ of split 2-fibrations and split morphisms into $\mathsf{2Cat}/\mathcal{A}$ has a left 2-adjoint (in fact, a left 3-adjoint) given by the "free split 2-fibration" functor $\mathsf{L}_1\colon \mathsf{2Cat}/\mathcal{A} \to \mathsf{Fib}_{\mathsf{s}}(\mathcal{A})$ (Remark 4.2.6 in [Buc14]). The inclusion $\mathsf{Fib}_{\mathsf{lds}}(\mathcal{A}) \hookrightarrow \mathsf{Fib}_{\mathsf{s}}(\mathcal{A})$ of split *locally-discrete* 2-fibrations has a left 2-adjoint given by quotienting out vertical 2-cells. One can observe directly that any map in $\mathsf{2Cat}/\mathcal{A}$ into a split locally discrete 2-fibration must map vertical 2-cells to identities and that quotienting vertical 2-cells does indeed produce a split locally discrete 2-fibration from a split 2-fibration. Alternatively, observe that we have another equivalence of adjunctions mediated by the 2-category of elements and the 2-categorical Grothendieck construction, where $i\colon \mathsf{Cat} \to \mathsf{2Cat}$ sends a category to the corresponding locally-discrete 2-category, and $\pi_1\colon \mathsf{2Cat} \to \mathsf{Cat}$ replaces the hom-categories of a 2-category with the hom-set of its connected components, which is left adjoint to i:

$$
\begin{tikzcd}
{[\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}]} \arrow[rr, bend right, "{[\mathcal{A}, i]}"'] \arrow[d, "\int"'] \arrow[rr, phantom, "\perp"] & & {[\mathcal{A}^{\mathrm{op}}, \mathsf{2Cat}]} \arrow[ll, bend right, "{[\mathcal{A}, \pi_1]}"'] \arrow[d, "\int"] \\
\mathsf{Fib}_{\mathsf{lds}}(\mathcal{A}) \arrow[rr, bend right, "\mathsf{incl}"'] \arrow[rr, phantom, "\perp"] & & \mathsf{Fib}_{\mathsf{s}}(\mathcal{A}) \arrow[ll, bend right, dashed, "\mathsf{L}_2"']
\end{tikzcd}
$$

Taking $\mathsf{L}_2\colon \mathsf{Fib}_{\mathsf{s}}(\mathcal{A}) \to \mathsf{Fib}_{\mathsf{lds}}(\mathcal{A})$ to be the 2-functor which quotients out vertical 2-cells does indeed make the "right-to-left" square commute, and so defines a left adjoint to $\mathsf{incl}\colon \mathsf{Fib}_{\mathsf{lds}}(\mathcal{A}) \to \mathsf{Fib}_{\mathsf{s}}(\mathcal{A})$.

The composition $\mathcal{L} = \mathsf{L}_2\mathsf{L}_1$ of these two left adjoints therefore defines a left adjoint to $\mathcal{R}\colon \mathsf{Fib}_{\mathsf{lds}}(\mathcal{A}) \hookrightarrow \mathsf{2Cat}/\mathcal{A}$. We summarise these data in the following diagram, where R denotes the inclusion of the full subcategory $\mathsf{2Cat}/_{\mathsf{lds}}\mathcal{A}$ into $\mathsf{2Cat}/\mathcal{A}$:

$$
\begin{tikzcd}
\mathsf{Fib}_{\mathsf{lds}}(A) \arrow[r, hook] \arrow[d, "\mathcal{U}"'] \arrow[dr, hook, "\mathcal{R}"] \arrow[r, phantom, bend left, "\perp"] & \mathsf{Fib}_{\mathsf{s}}(\mathcal{A}) \arrow[l, bend right, "\mathsf{L}_2"'] \arrow[d, hook] \\
\mathsf{2Cat}/_{\mathsf{lds}}A \arrow[r, hook, "\mathsf{R}"'] & \mathsf{2Cat}/A \arrow[u, bend right, "\mathsf{L}_1"'] \arrow[u, phantom, "\vdash"]
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
\mathsf{Fib}_{\mathsf{lds}}(A) \arrow[d, "\mathcal{U}"'] \arrow[dr, hook, "\mathcal{R}"] & \\
\mathsf{2Cat}/_{\mathsf{lds}}A \arrow[r, hook, "\mathsf{R}"'] & \mathsf{2Cat}/A \arrow[ul, bend right, "\mathcal{L}"']
\end{tikzcd}
$$

Lemma 3.3.9. *The comonad* $\mathfrak{Q}\colon \mathsf{Fib}_{\mathsf{lds}}(\mathcal{A}) \to \mathsf{Fib}_{\mathsf{lds}}(\mathcal{A})$ *is isomorphic to the comonad induced by the free–forgetful adjunction* $\mathcal{L}\colon \mathsf{Fib}_{\mathsf{lds}}(\mathcal{A}) \rightleftharpoons \mathsf{2Cat}/\mathcal{A} : \mathcal{R}$.

Proof. Using the fact that R is 2-fully-faithful, we have that for $x \in \mathsf{2Cat}/_{\mathsf{lds}}\mathcal{A}$ and $y \in \mathsf{Fib}_{\mathsf{lds}}(\mathcal{A})$:

$$
\mathsf{2Cat}/_{\mathsf{lds}}\mathcal{A}(x, \mathcal{U}y) \cong \mathsf{2Cat}/\mathcal{A}(\mathsf{R}x, \mathsf{R}\mathcal{U}y) \cong \mathsf{Fib}_{\mathsf{lds}}(\mathcal{A})(\mathsf{LR}x, y)
$$

Thus LK is left adjoint to $\mathcal{U}$. The comonad $\mathfrak{Q}$ is defined up-to-isomorphism as being induced by the adjunction $\mathcal{U} \dashv \mathcal{L}\mathsf{R}$, which is equal to the comonad induced by $\mathcal{L} \dashv \mathcal{R}$ because R is a morphism between these adjunctions. □

Knowing that the comonad $\mathfrak{Q}$ is generated by the adjunction $\mathcal{L} \dashv \mathcal{R}$ will have applications in our description of $\mathfrak{Q}$-coalgebras, and thus $\mathcal{Q}$-coalgebras. Note that this adjunction is not as simple to describe from the indexed perspective; the indexed categories corresponding to arbitrary objects of $\mathsf{2Cat}/\mathcal{A}$ are some form of trihomomorphism from $\mathcal{A}$ to $\mathsf{Prof}(\mathsf{2Cat})$, and the appropriate morphisms of such indexed categories are rather complicated.

To proceed to our description of $\mathfrak{Q}$-coalgebras we will require an explicit definition for the action of the left adjoint $\mathcal{L}\colon \mathsf{2Cat}/\mathcal{A} \to \mathsf{Fib}_{\mathsf{lds}}(\mathcal{A})$. By abuse of notation, we will henceforth refer to this left adjoint by the same name as the comonad $\mathfrak{Q}$ it induces, so as to avoid having two notation for the same action on objects. The context should suffice to disambiguate the left adjoint from the comonad when necessary, as for $\mathcal{Q}$. We've described $\mathfrak{Q}\colon \mathsf{2Cat}/\mathcal{A} \to \mathsf{Fib}_{\mathsf{lds}}(\mathcal{A})$ as the composition of the free split 2-fibration map and the map which quotients out vertical 2-cells, so some understanding of the free split 2-fibration map will be necessary. The free split 2-fibration on a 2-functor $p\colon \mathcal{X} \to \mathcal{A}$ can be characterised as the *lax comma* 2-category of the co-span $\mathcal{A} = \mathcal{A} \xleftarrow{p} \mathcal{X}$, denoted $p \Downarrow \mathcal{A}$. That is, $p \Downarrow \mathcal{A}$ is endowed with maps $\mathsf{s}\colon p \Downarrow \mathcal{A} \to \mathcal{X}$, $\mathsf{t}\colon p \Downarrow \mathcal{A} \to \mathcal{A}$ and a universal *lax* transformation $\sigma\colon p\,\mathsf{s} \Rightarrow \mathsf{t}$; the map t being the split 2-fibration. The (oplax variant of) the lax comma is Construction 4.2.1 in [Buc14], but appeared earlier in [Kel74, §4.1] and in [Gra69, §6]. This construction is analogous to, and extends, the construction of the free split fibration on a morphism in Cat as a comma category (or lax limit of an arrow). Applying this lax comma construction and then quotienting the vertical 2-cells of the resulting split 2-fibration produces the following definition:

Definition 3.3.10 ($\mathfrak{Q}\colon \mathsf{2Cat}/\mathcal{A} \to \mathsf{Fib}_{\mathsf{lds}}(\mathcal{A})$)**.** The 2-functor $\mathfrak{Q}$ acts on objects by sending a 2-functor $p\colon \mathcal{X} \to \mathcal{A}$ to a locally discrete split 2-fibration denoted $\mathfrak{q}_p\colon \mathfrak{Q}_p \to \mathcal{A}$, where $\mathfrak{Q}_p$ is the 2-category with:

0-cells given by triples $(a \in \mathcal{A}, x \in \mathcal{X}, u\colon a \to px)$

1-cells from $(u\colon a \to px)$ to $(v\colon b \to py)$ given by equivalence classes of triples $a \xrightarrow{s} b$, $x \xrightarrow{f} y$ and $pf\ u \xRightarrow{\sigma} v\ s$ with the imposed relation that for any $\alpha\colon g \Rightarrow f$ in $\mathcal{X}$, $(s, f, \sigma) \sim (s, g, \sigma \circ p\alpha\, u)$:

$$
\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, "s"'] \arrow[dr, phantom, "\swArrow\sigma"] & px \arrow[d, "pf"] \\
b \arrow[r, "v"'] & py
\end{tikzcd}
\quad\sim\quad
\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, "s"'] \arrow[dr, phantom, "\swArrow\sigma"] & px \arrow[d, "pf"] \arrow[d, bend left=60, "pg"] \arrow[d, phantom, shift left=4, "\overset{p\alpha}{\Leftarrow}"] \\
b \arrow[r, "v"'] & py
\end{tikzcd}
\tag{3.16}
$$

The symbol $\sim$ will henceforth denote the *equivalence* relation generated by the relation described above.

2-cells from (s, f, σ) to (t, g, τ) are 2-cells $\kappa\colon s \Rightarrow t$ in $\mathcal{A}$ such that $(t, f, v\kappa\,\sigma) \sim (t, g, \tau)$:

$$
\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, "s"] \arrow[d, bend right=60, "t"'] \arrow[d, phantom, shift right=4, "\overset{\kappa}{\Leftarrow}"] \arrow[dr, phantom, "\swArrow\sigma"] & px \arrow[d, "pf"] \\
b \arrow[r, "v"'] & py
\end{tikzcd}
\quad\sim\quad
\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, "t"'] \arrow[dr, phantom, "\swArrow\tau"] & px \arrow[d, "pg"] \\
b \arrow[r, "v"'] & py
\end{tikzcd}
$$

The horizontal and vertical composition are given as in $\mathcal{A}$.

The 2-functor $\mathfrak{q}_p\colon \mathfrak{Q}_p \to \mathcal{A}$ maps each of the n-cells onto the first component, e.g. $\left(a \xrightarrow{u} px\right) \mapsto a$ and $(s, f, \sigma) \mapsto s$. The chosen cartesian lift of a morphism $s\colon a \to b$ in $\mathcal{A}$ with codomain $v\colon b \to py$ in $\mathfrak{Q}_p$ is the 1-cell represented by the square shown below on the left. The chosen opcartesian lift of (s, f, σ) via a 2-cell $\kappa\colon s \Rightarrow t$ is the unique lift shown on the right:

$$
\begin{tikzcd}
a \arrow[r, "vs"] \arrow[d, "s"'] \arrow[dr, phantom, "\circlearrowright"] & py \arrow[d, equal, "p1_y"] \\
b \arrow[r, "v"'] & py
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, "s"] \arrow[d, bend right=60, "t"'] \arrow[d, phantom, shift right=4, "\overset{\kappa}{\Leftarrow}"] \arrow[dr, phantom, "\overset{\sigma}{\Longleftarrow}"] & px \arrow[d, "pf"] \\
b \arrow[r, "v"'] & py
\end{tikzcd}
\tag{3.17}
$$

This describes the action of $\mathfrak{Q}$ on objects. The action of $\mathfrak{Q}$ on the 1-cells of $\mathsf{2Cat}$, i.e. 2-functors, is given by sending a 2-functor $J\colon \mathcal{X} \to Y$ over $\mathcal{A}$ between $p\colon \mathcal{X} \to \mathcal{A}$ and $q\colon Y \to \mathcal{A}$ to the 2-functor $\mathfrak{Q}_J\colon \mathfrak{Q}_{\mathcal{X}} \to \mathfrak{Q}_Y$ which acts by J on the "x" components. That is, $(u\colon a \to px) \mapsto (u\colon a \to qJx)$. A vertical 2-natural transformation $\rho\colon J \Rightarrow K\colon \mathcal{X} \to Y$

is sent to the natural transformation $\mathfrak{Q}_\rho \colon \mathfrak{Q}_J \to \mathfrak{Q}_K$ with component at $(u \colon a \to px)$ given by:

$$\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & qJx \arrow[d, "q\rho_x"] \\
a \arrow[r, "u"'] & qKx
\end{tikzcd}$$

◊

The counit for the adjunction $\mathfrak{Q} \dashv \mathcal{R}$ is given by the image of the counit for the free split 2-fibration adjunction under the "quotient vertical 2-cells" map, $\mathsf{L}_2 \colon \mathsf{Fib}_{\mathsf{s}}(\mathcal{A}) \to \mathsf{Fib}_{\mathsf{lds}}(\mathcal{A})$:

Definition 3.3.11 (The counit $\mathsf{c} \colon 1_{\mathsf{Fib}_{\mathsf{lds}}(\mathcal{A})} \Rightarrow \mathfrak{Q}$)**.** The counit c has component at a locally discrete split 2-fibration $p \colon \mathcal{X} \to \mathcal{A}$ given by a 2-functor $\mathsf{c}_p \colon \mathfrak{Q}_p \to \mathcal{X}$ defined as follows:

0-cells $(u \colon a \to px) \mapsto u^*x$ (where $\bar{u}x \colon u^*x \to x$ is the chosen cartesian lift of u with codomain x)

1-cells given a representative of a 1-cell, (s, f, σ), we can first push forward the 1-cell $f\,\bar{u}x$ by σ using the 2-fibration structure on p to obtain a 1-cell $\sigma_* \colon u^*x \to y$ over $v\ s$. This then factorises uniquely through $\bar{v}y$ via a map over s from $u^*x \to v^*y$ by the cartesianness of $\bar{v}y$. This map, which we denote $\sigma^\sharp f$, is the image of (s, f, σ) under c_p.[2]

$$\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, "s"'] \arrow[dr, phantom, "\swarrow\sigma"] & px \arrow[d, "pf"] \\
b \arrow[r, "v"'] & py
\end{tikzcd}
\quad\rightsquigarrow\quad
\begin{tikzcd}
u^*x \arrow[r, "\bar{u}x"] \arrow[dr, dashed, "\sigma_*"'] & x \arrow[d, "f"] \arrow[dl, phantom, "\swarrow\underline{\sigma}"] \\
v^*y \arrow[r, "\bar{v}y"'] & y
\end{tikzcd}
\quad\rightsquigarrow\quad
\begin{tikzcd}
u^*x \arrow[r, "\bar{u}x"] \arrow[d, dashed, "\sigma^\sharp f"'] \arrow[dr, "\sigma_*"] & x \arrow[d, "f"] \arrow[dl, phantom, "\swarrow\underline{\sigma}"] \\
v^*y \arrow[r, "\bar{v}y"'] \arrow[ur, phantom, "\circlearrowright"] & y
\end{tikzcd}$$

To see that this map respects the equivalence relation on 1-cells, imagine we have a 2-cell $\alpha \colon g \Rightarrow f$ in $\mathcal{A}$ and we attach the 2-cell $p\alpha$ to the right of the left-most square above to obtain a different representation for the same 1-cell. Then the opcartesian lift of $\sigma \circ p(\alpha)u$ from $g\ \bar{u}x$ is $\underline{\sigma} \circ \alpha\bar{u}x$, because every 2-cell in $\mathcal{X}$ is chosen opcartesian (the split 2-fibration p is locally discrete). So we obtain the same σ_*, and thus $\sigma^\sharp f = (\sigma \circ (p\alpha)\,u)^\sharp g$.

2-cells Assume we have a (necessarily unique) 2-cell from (s, f, σ) to (t, g, τ) whose underlying 2-cell in $\mathcal{A}$ is $\kappa \colon s \Rightarrow t$. The existence of such a 2-cell implies $(t, g, \tau) \sim (t, f, v\kappa\,\sigma)$, and thus $\tau^\sharp g = (v\kappa\,\sigma)^\sharp f$. By the opcartesianness of $\underline{\sigma}$, we obtain a unique factorisation of $\underline{v\kappa \circ \sigma}$ through $\underline{\sigma}$ via a 2-cell from σ_* to $(v\kappa \circ \sigma)_*$ above $\bar{v}y\,\kappa$. Then, by the cartesianness of $\bar{v}y$, we get a 2-cell from $\sigma^\sharp f$ to $(v\kappa \circ \sigma)^\sharp f$.

$$\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, "s"] \arrow[d, bend right=60, "t"', ""{name=T}] \arrow[dr, phantom, "\swarrow\sigma"] & px \arrow[d, "pf"] \\
b \arrow[r, "v"'] & py
\end{tikzcd}
\quad\rightsquigarrow\quad
\begin{tikzcd}
u^*x \arrow[r, "\bar{u}x"] \arrow[dr, dashed, "\sigma_*"] \arrow[dr, bend right=40, "(v\kappa\circ\sigma)_*"'] & x \arrow[d, "f"] \arrow[dl, phantom, "\swarrow\underline{\sigma}"] \\
v^*y \arrow[r, "\bar{v}y"'] & y
\end{tikzcd}
\quad\rightsquigarrow\quad
\begin{tikzcd}
u^*x \arrow[r, "\bar{u}x"] \arrow[d, dashed, "\sigma^\sharp f"] \arrow[d, bend right=60, "(v\kappa\circ\sigma)^\sharp f"'] \arrow[dr, "\sigma_*"] & x \arrow[d, "f"] \arrow[dl, phantom, "\swarrow\underline{\sigma}"] \\
v^*y \arrow[r, "\bar{v}y"'] \arrow[ur, phantom, "\circlearrowright"] & y
\end{tikzcd}$$

Note that this is the only possible mapping of 2-cells given the definition of the mapping of 1-cells as the domain and codomain are both locally discrete split 2-fibrations. ◊

It is straightforward to check that this is a well-defined map of split 2-fibrations, i.e. that it commutes with the split 2-fibrations down to $\mathcal{A}$ and restricts to a map of chosen cartesian 1-cells (opcartesianness of 2-cells is automatically preserved).

[2] The notation $\sigma^\sharp f$ only makes sense if we think of σ explicitly as a component of a 1-cell in $\mathfrak{Q}_p$, i.e. having a given factorisation of its domain and codomain. Though this notation should be unambiguous in the contexts in which it is used.

Compared to the counit, c, the definition of the comultiplication is relatively simple as it doesn't depend on the fibration structure of $p\colon \mathcal{X} \to \mathcal{A}$. Before we describe the comultiplication, let's first describe $\mathfrak{Q}_p^2$:

0-cells According to our definition of the comonad $\mathfrak{Q}$, these should be triples $a \in \mathcal{A}$, $y \in \mathfrak{Q}_p$, $u\colon a \to \mathfrak{q}_p y$. But y itself is a triple of the form $b \in \mathcal{A}$, $x \in \mathcal{X}$, $v\colon b \to px$, with $\mathfrak{q}_p y = b$. So we can represent 0-cells as pairs of composable arrows in $\mathcal{A}$ with a lift of the codomain of the second arrow, e.g. $a \xrightarrow{u} b \xrightarrow{v} px$.

1-cells from (u, v, x) to (u', v', x') are equivalence classes of a set with elements of the form:

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[d, "s"'] & b \arrow[r, "v"] \arrow[d, dashed, "t"] \arrow[dl, phantom, "\Leftarrow\alpha"] & px \arrow[d, "pf"] \arrow[dl, phantom, "\Leftarrow\beta"] \\ a' \arrow[r, "u'"'] & b' \arrow[r, "v'"'] & px' \end{tikzcd}$$

under two sorts of relations:

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[d, "s"'] & b \arrow[r, "v"] \arrow[d, dashed, "t"] \arrow[d, dotted, bend left, "t'"] & px \arrow[d, "pf"] \\ a' \arrow[r, "u'"'] & b' \arrow[r, "v'"'] & px' \end{tikzcd} \quad (\alpha,\ \gamma,\ \beta) \quad \sim \quad \begin{tikzcd} a \arrow[r, "u"] \arrow[d, "s"'] & b \arrow[r, "v"] \arrow[d, "t'"] \arrow[d, dotted, bend right, "t"'] & px \arrow[d, "pf"] \\ a' \arrow[r, "u'"'] & b' \arrow[r, "v'"'] & px' \end{tikzcd} \tag{3.18}$$

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[d, "s"'] & b \arrow[r, "v"] \arrow[d, dashed, "t"] \arrow[dl, phantom, "\Leftarrow\alpha"] & px \arrow[d, "pf"] \arrow[dl, phantom, "\Leftarrow\beta"] \\ a' \arrow[r, "u'"'] & b' \arrow[r, "v'"'] & px' \end{tikzcd} \quad \sim \quad \begin{tikzcd} a \arrow[r, "u"] \arrow[d, "s"'] & b \arrow[r, "v"] \arrow[d, dashed, "t"] \arrow[dl, phantom, "\Leftarrow\alpha"] & px \arrow[d, dotted, "pf"'] \arrow[d, bend left=60, "pg"] \arrow[d, phantom, shift left=3, "\Leftarrow p\delta"] \arrow[dl, phantom, "\Leftarrow\beta"] \\ a' \arrow[r, "u'"'] & b' \arrow[r, "v'"'] & px' \end{tikzcd} \tag{3.19}$$

2-cells a 2-cell $\kappa\colon s \Rightarrow s'$ in $\mathcal{A}$ is also a 2-cell between 1-cells in $\mathfrak{Q}_p^2$ between 1-cells of the form:

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[d, "s"'] & b \arrow[r, "v"] \arrow[d, dashed, "t"] \arrow[dl, phantom, "\Leftarrow\alpha"] & px \arrow[d, "pf"] \arrow[dl, phantom, "\Leftarrow\beta"] \\ a' \arrow[r, "u'"'] & b' \arrow[r, "v'"'] & px' \end{tikzcd} \quad \overset{\kappa}{\Rightarrow} \quad \begin{tikzcd} a \arrow[r, "u"] \arrow[d, dotted, "s"] \arrow[d, bend right=60, "s'"'] \arrow[d, phantom, shift right=3, "\Leftarrow\kappa"] & b \arrow[r, "v"] \arrow[d, dashed, "t"] \arrow[dl, phantom, "\Leftarrow\alpha"] & px \arrow[d, "pf"] \arrow[dl, phantom, "\Leftarrow\beta"] \\ a' \arrow[r, "u'"'] & b' \arrow[r, "v'"'] & px' \end{tikzcd}$$

Definition 3.3.12 ($\mathsf{w}\colon \mathfrak{Q} \to \mathfrak{Q}^2$)**.** For a locally discrete split 2-fibration $p\colon \mathcal{X} \to \mathcal{A}$, the component of the comultiplication for the comonad $\mathfrak{Q}$ at p is the 2-functor $\mathsf{w}_p\colon \mathfrak{Q}_p \to \mathfrak{Q}_p^2$ with the following action:

0-cells

$$\left(a \xrightarrow{u} px\right) \quad \mapsto \quad \left(a \xrightarrow{u} px \xrightarrow{1_{px}} px\right)$$

1-cells

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[d, "s"'] & px \arrow[d, "pf"] \arrow[dl, phantom, "\Leftarrow\alpha"] \\ b \arrow[r, "v"'] & py \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} a \arrow[r, "u"] \arrow[d, "s"'] & px \arrow[r, equal] \arrow[d, dashed, "pf"] \arrow[dl, phantom, "\Leftarrow\alpha"] \arrow[dr, phantom, "\circlearrowleft"] & px \arrow[d, "pf"] \\ b \arrow[r, "v"'] & py \arrow[r, equal] & py \end{tikzcd}$$

2-cells

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[d, dashed, "s"] \arrow[d, bend right=60, "s'"'] \arrow[d, phantom, shift right=3, "\Leftarrow\kappa"] & px \arrow[d, "pf"] \arrow[dl, phantom, "\Leftarrow\alpha"] \\ b \arrow[r, "v"'] & py \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} a \arrow[r, "u"] \arrow[d, dashed, "s"] \arrow[d, bend right=60, "s'"'] \arrow[d, phantom, shift right=3, "\Leftarrow\kappa"] & px \arrow[r, equal] \arrow[d, dashed, "pf"] \arrow[dl, phantom, "\Leftarrow\alpha"] \arrow[dr, phantom, "\circlearrowleft"] & px \arrow[d, "pf"] \\ b \arrow[r, "v"'] & py \arrow[r, equal] & py \end{tikzcd}$$

◊

3.4 Coalgebras

Our motivation for giving such explicit descriptions of the comonads $\mathscr{Q}$ and $\mathfrak{Q}$ is to facilitate our investigation into their coalgebras in this section. By the general theory of oplax-morphism classifiers we already know that both $\mathscr{Q}$ and $\mathfrak{Q}$ are oplax-idempotent, so a coalgebra structure map for $\mathscr{Q}$ (resp. $\mathfrak{Q}$) on a presheaf (resp. locally split 2-fibration) is unique up-to-isomorphism, if it exists. Thus, the categories of coalgebras define — and essentially are defined by — the subclasses of presheaves/fibrations which admit coalgebra structures. For $\mathscr{Q}$, as we shall see, the presheaves admitting $\mathscr{Q}$-coalgebra structures are precisely the oplax colimits of representables.

We obtain this result from the fibred-perspective starting with the fact that $\mathscr{Q}$ is the comonad induced by the forgetful map $\mathcal{R}\colon \mathsf{Fib}_{\mathsf{Ids}}(\mathscr{A}) \to \mathsf{2Cat}/\mathscr{A}$ (Lemma 3.3.9). It follows from the theory of comonads that there is a comparison functor $\mathsf{K}\colon \mathsf{2Cat}/\mathscr{A} \to \mathfrak{Q}\text{-}\mathsf{coalg}$, which has a right adjoint G by the fact that $\mathsf{2Cat}/\mathscr{A}$ is complete, and so in particular has $\mathfrak{Q}$-split equalisers:

$$
\begin{tikzcd}
\mathsf{2Cat}/\mathscr{A} \arrow[rr, bend right, "\mathsf{K}"'] \arrow[rr, phantom, "\top"] \arrow[dr, "\mathcal{R}"'] & & \mathfrak{Q}\text{-}\mathsf{coalg} \arrow[ll, bend right, "\mathsf{G}"'] \arrow[dl, "\mathsf{U}_{\mathfrak{Q}}"] \\
& \mathsf{Fib}_{\mathsf{Ids}}(\mathscr{A}) &
\end{tikzcd}
$$

We will show that the counit $\mathsf{K}\,\mathsf{G} \Rightarrow 1_{\mathfrak{Q}\text{-}\mathsf{coalg}}$ is invertible, and thus that $\mathfrak{Q}\text{-}\mathsf{coalg}$ is equivalent to its image under G. This image happens to be the (reflexive) full subcategory of 2-functors which locally are discrete opfibrations.

3.4.1 $\mathfrak{Q}$-coalgebras

A strict $\mathfrak{Q}$-coalgebra is a locally discrete split 2-fibration $p\colon \mathcal{X} \to \mathscr{A}$ along with a split cartesian map $H\colon \mathcal{X} \to \mathfrak{Q}_p$ making the following diagrams commute:

$$
\begin{tikzcd}
\mathcal{X} \arrow[r, "H"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & \mathfrak{Q}_p \arrow[d, "\mathsf{c}_p"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{X} \arrow[r, "H"] \arrow[d, "H"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{Q}_p \arrow[d, "\mathsf{w}_p"] \\
\mathfrak{Q}_p \arrow[r, "\mathfrak{Q}_H"'] & \mathfrak{Q}_p^2
\end{tikzcd}
\tag{3.20}
$$

Ignoring these conditions for now, let's consider what an arbitrary split cartesian map $H\colon \mathcal{X} \to \mathfrak{Q}_p$ looks like. The image of $x \in \mathcal{X}$ under H is determined by some object, $G_x \in \mathcal{X}$ and a morphism $h_x\colon px \to pG_x$. The codomain of h_x is forced to be px because H is a map in the slice over $\mathscr{A}$. For each morphism $f\colon x \to y$, H chooses an equivalence class whose represented by data of the following form:

$$
\begin{tikzcd}
px \arrow[r, "h_x"] \arrow[d, "pf"'] & pG_x \arrow[d, "pG_f"] \arrow[dl, phantom, "\overset{h_f}{\Longleftarrow}"] \\
py \arrow[r, "h_y"'] & pG_y
\end{tikzcd}
$$

To be a split cartesian map, G must additionally map chosen cartesian 1-cells to chosen cartesian 1-cells. Recalling the definition of the chosen cartesian lifts along $\mathfrak{q}_p\colon \mathfrak{Q}_p \to \mathscr{A}$ from (3.17) we obtain the following equality of 1-cells in $\mathfrak{Q}_p$ for $u\colon a \to b$ in $\mathscr{A}$ and $px = a$ for $x \in \mathcal{X}$:

$$
\begin{tikzcd}
u^*x \arrow[d, "\bar{u}x"'] \\
x
\end{tikzcd}
\quad \overset{H}{\longmapsto} \quad
\begin{tikzcd}
a \arrow[r, "h_{u^*x}"] \arrow[d, "u"'] & pG_{u^*x} \arrow[d, "pG_{\bar{u}x}"] \arrow[dl, phantom, "\overset{h_{\bar{u}x}}{\Longleftarrow}"] \\
b \arrow[r, "h_x"'] & pG_x
\end{tikzcd}
\quad \sim \quad
\begin{tikzcd}
a \arrow[r, "h_x u"] \arrow[d, "u"'] \arrow[dr, phantom, "\circlearrowleft"] & pG_x \arrow[d, equal, "p1_{Gx}"] \\
b \arrow[r, "h_x"'] & pG_x
\end{tikzcd}
$$

From which we conclude, in particular, that $G_{u^*x} = G_x$ and $h_{u^*x} = h_x\, u$. The preservation of opcartesian 2-cells is automatic, as $\mathfrak{Q}_p$ is a local discrete opfibration.

We now consider the additional properties H must satisfy to be a $\mathfrak{Q}$-coalgebra. Recall from Definition 3.3.11 that the counit c_p sends an object $a \xrightarrow{u} px$ to u^*x and maps (s, σ, f) to $\sigma^\sharp f$. So the co-identity condition for H says that $h_x^* G_x = x$ and $h_f^\sharp G_f = f$, for x a 0-cell and f a 1-cell of $\mathcal{X}$.

The co-associativity condition says that for an object $x \in \mathcal{X}$ we have the following equality of objects in $\mathfrak{Q}_p^2$:

$$\left(px \xrightarrow{h_x} pG_x = pG_x\right) = \left(px \xrightarrow{h_x} pG_x \xrightarrow{h_{G_x}} pG_{G_x}\right) \tag{3.21}$$

So $G_{G_x} = G_x$ and $h_{G_x} = 1_{G_x}$. And for a morphism $f\colon x \to y$ in $\mathcal{X}$ the co-associativity condition says the following pasting diagrams represent the same 1-cell in $\mathfrak{Q}_p^2$:

$$\begin{tikzcd} px \arrow[r, "h_x"] \arrow[d, "pf"'] & pG_x \arrow[r, equal] \arrow[d, dashed, "pG_f"] \arrow[dl, phantom, "\xLeftarrow{h_f}"] & pG_x \arrow[d, "pG_f"] \arrow[dl, phantom, "\circlearrowleft"] \\ py \arrow[r, "h_y"'] & pG_y \arrow[r, equal] & pG_y \end{tikzcd} \sim \begin{tikzcd} px \arrow[r, "h_x"] \arrow[d, "pf"'] & pG_x \arrow[r, equal] \arrow[d, dashed, "pG_f"] \arrow[dl, phantom, "\xLeftarrow{h_f}"] & pG_x \arrow[d, "pG_{pG_f}"] \arrow[dl, phantom, "\xLeftarrow{h_{pG_f}}"] \\ py \arrow[r, "h_y"'] & pG_y \arrow[r, equal] & pG_y \end{tikzcd}$$

Putting this all together, we see that a coalgebra structure H on $p\colon \mathcal{X} \to \mathcal{A}$ endows each $x \in \mathcal{X}$ with a chosen cartesian morphism $\zeta_x\colon \bar{h}_x x$ to some other object G_x along h_x. Because the map H is required to preserve chosen cartesian morphisms, if some other object u^*x is the pullback of x along a morphism $u\colon a \to px$ in $\mathcal{A}$, we have $h_{u^*x} = h_x\; u$, and so $\zeta_{u^*x} = \zeta_x\; \bar{u}x$ and $G_{u^*x} = G_x$. We also know that $\zeta_{Gx} = \overline{1_{pGx}} G_x = G_x$ for any x, by (3.21). It follows that $\zeta_x\colon x \to G_x$ is the *unique* chosen-cartesian 1-cell with that domain and codomain; if $f\colon x \to G_x$ were chosen-cartesian, then $\zeta_x = \zeta_{Gx}\, f = f$. Moreover, if there exists a chosen-cartesian morphism $f\colon G_x \to G_y$, then we have $G_x = G_{pf^*Gy} = G_{Gy} = G_y$. These observations motivate the following definitions:

Definition 3.4.1 (Generic object, generic core, cartesian core)**.** For a $\mathfrak{Q}$-coalgebra structure $H\colon \mathcal{X} \to \mathfrak{Q}_p$ on a locally discrete split 2-fibration $p\colon \mathcal{X} \to \mathcal{A}$, we call objects of the form $G_x \in \mathcal{X}$ *generic*. The restriction of $p\colon \mathcal{X} \to \mathcal{A}$ to the full sub-2-category on the generic objects will be referred to as the *generic core* of the coalgebra, denoted $p^\circ\colon \mathcal{X}^\circ \to \mathcal{A}$. The *wide* sub-1-category of chosen-cartesian morphisms and no 2-cells will be called the *cartesian* core, and the restriction of p to a discrete fibration form the cartesian core to the underlying 1-category of $\mathcal{A}$ is denoted $p^\mathfrak{c}\colon \mathcal{X}^\mathfrak{c} \to \mathcal{A}_0$. ◊

From our discussion above we conclude the following:

Lemma 3.4.2. *For $H\colon \mathcal{X} \to \mathfrak{Q}_p$ a $\mathfrak{Q}$-coalgebra on $p\colon \mathcal{X} \to \mathcal{A}$, each connected component of the cartesian core of $\mathcal{X}$ contains a unique generic object which is terminal in that component.*

Proof. We've already noted that every object $x \in \mathcal{X}$ admits a unique chosen-cartesian morphism to a generic object G_x, and any object connected to x in the cartesian core admits a morphism to the same generic object. Also, any two generic objects connected by a chosen-cartesian morphism are equal, so generic objects are unique in their connect components of the cartesian core. □

Corollary 3.4.3. *The discrete fibration $p^\mathfrak{c}\colon \mathcal{X}^\mathfrak{c} \to \mathcal{A}_0$ is isomorphic to $\coprod_{x \text{ generic}} \mathcal{A}_0/px \xrightarrow{\pi} \mathcal{A}_0$.*

Proof. Final functors and discrete fibrations form a factorisation system in Cat, (called the comprehensive factorisation system, [SW73]). The restriction of $p^\mathfrak{c}\colon \mathcal{X}^\mathfrak{c} \to \mathcal{A}_0$ to the discrete category of generic objects admits such a factorisation through both $p^{\mathfrak{c}}\colon \mathcal{X}^{\mathfrak{c}} \to \mathcal{A}_0$ and $\pi\colon \coprod_{x \text{ generic}} \mathcal{A}_0/px$, from which it follows that these two discrete fibrations are isomorphic. □

Remark 3.4.4. The result of Corollary 3.4.3 corresponds under the fibred-indexed equivalence to the fact that a presheaf $P\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ associated to a $\mathfrak{Q}$-coalgebra $\mathcal{X}$ has its underlying Set-presheaf $P_0\colon \mathcal{A}_0{}^{\mathsf{op}} \to \mathsf{Set}$ given by a coproduct of representables $P_0 \cong \coprod_{x \text{ generic}} \mathcal{A}(-, px)$. In [PR91, Cor. 3.3] this was shown to be equivalent to P being a PIE weight, so we now know that $\mathcal{Q}$-coalgebras are in particular PIE weights. ◊

Corollary 3.4.3 demonstrates that the underlying discrete fibration of a $\mathfrak{Q}$-coalgebra $p\colon \mathcal{X} \to \mathcal{A}$ can be recovered up-to-isomorphism from its restriction to generic objects. We now demonstrate that p itself can be reconstructed from its generic core:

Lemma 3.4.5. *If $p\colon \mathcal{X} \to \mathcal{A}$ is a $\mathfrak{Q}$-coalgebra with structure map $H\colon \mathcal{X} \to \mathfrak{Q}_p$, and $p^\circ\colon X^\circ \to \mathcal{A}$ is its generic core, then $\mathfrak{Q}_{p^\circ}$ and p are isomorphic as $\mathfrak{Q}$-coalgebras.*

Proof. The comonad $\mathfrak{Q}$ is oplax-idempotent, so any coalgebra $H\colon \mathcal{X} \to \mathfrak{Q}_p$ forms an adjunction $\mathsf{c}_p \dashv H$ with identity counit, from which it follows that H is fully-faithful. The objects in the image of H are precisely those $u\colon a \to px$ with x generic. The full sub-2-category of $\mathfrak{Q}_p$ on those objects is $\mathfrak{Q}_{p^\circ}$, and so by the fact that H is injective on objects (as a right inverse to c_p we conclude that H restricts to an isomorphism $\mathcal{X} \to \mathfrak{Q}_{p^\circ}$ of locally discrete split 2-fibrations. This restriction of H is moreover an isomorphism of coalgebras because $H\colon \mathcal{X} \to \mathfrak{Q}_p$ is a coalgebra map — being a coalgebra structure map — and $\mathfrak{Q}_{p^\circ} \subseteq \mathfrak{Q}_p$ is a sub-coalgebra. □

This isomorphism in fact describes the counit to the adjunction formed by the comparison functor $\mathsf{K}\colon \mathsf{2Cat}/\mathcal{A} \to \mathfrak{Q}\text{-}\mathsf{coalg}$, which we now describe.

The general theory of comonads tells us that for an adjunction $F\colon C \rightleftarrows D : G$ a right adjoint W to a comparison functor $K\colon C \to FG\text{-}\mathsf{coalg}$ exists whenever C has F-split equalisers.[3] The right adjoint is given by sending a coalgebra $\alpha\colon a \to FGa$ to the equaliser of the F-split pair (where η is the unit of the adjunction):

$$\begin{tikzcd} E \arrow[r, hook, dashed, "e"] & Ga \arrow[r, shift left, "G\alpha"] \arrow[r, shift right, "\eta G"'] & GFGa \end{tikzcd} \tag{3.22}$$

A FG-coalgebra morphism induces 1-cells between equalisers which defines the action of W on 1-cells. The counit of the adjunction $K\colon C \rightleftarrows FG\text{-}\mathsf{coalg} : W$ then has component at $\alpha\colon a \to FGa$ given by the composition:

$$FE \xrightarrow{Fe} FGa \xrightarrow{\epsilon_a} a \tag{3.23}$$

This theory applies to the adjunction $\mathfrak{Q} \dashv \mathcal{R}$ because $\mathsf{2Cat}/\mathcal{A}$ is complete and so in particular has $\mathfrak{Q}$-split equalisers. The right adjoint, G, sends a coalgebra $(\mathcal{X}, p, H)$ to the equaliser in $\mathsf{2Cat}/\mathcal{A}$ of $H\colon \mathcal{X} \to \mathfrak{Q}_p$ and η_p, where η is the unit of the $\mathfrak{Q} \dashv \mathcal{R}$ adjunction on $\mathsf{2Cat}/\mathcal{A}$. For an object $x \in \mathcal{X}$, the images of x under H and η_p are $h_x\colon px \to pG_x$ and $1_{px} : px = px$ respectively. Clearly these only coincide when $x = G_x$, i.e. when x is generic. Now, if $f\colon x \to y$ is a 1-cell in $\mathcal{X}$ between generic objects, the images of f under H and η_p are respectively represented by:

$$\begin{tikzcd} pG_x \arrow[r, equal] \arrow[d, "pf"'] & pG_x \arrow[d, "pG_f"] \arrow[dl, phantom, "\xLeftarrow{h_f}"] \\ pG_y \arrow[r, equal] & pG_y \end{tikzcd} \qquad \begin{tikzcd} pG_x \arrow[r, equal] \arrow[d, "pf"'] & pG_x \arrow[d, "pf"] \arrow[dl, phantom, "\circlearrowleft"] \\ pG_y \arrow[r, equal] & pG_y \end{tikzcd} \tag{3.24}$$

[3] This result is more familiar in the context of monads, where a left adjoint to the comparison functor from $U\colon \mathcal{D} \to C$ exists whenever $\mathcal{D}$ has U-split coequalisers.

which represent the same 1-cell, as we show in the following lemma:

Lemma 3.4.6. *For $f\colon G_x \to G_y$ a 1-cell between generic objects in a $\mathfrak{Q}$-coalgebra $H\colon \mathcal{X} \to \mathfrak{Q}_p$, the two squares in (3.24) represent the same 1-cell in $\mathfrak{Q}_p$.*

Proof. The 2-cell h_f lifts to an opcartesian 2-cell $\underline{h_f}\colon G_f \Rightarrow h_f^{\sharp} H_f = f$, which forms a relation between the squares in (3.24) as follows:

$$
\begin{tikzcd}
pG_x \arrow[r, equal] \arrow[d, "pf"'] \arrow[dr, phantom, "\xLeftarrow{h_f}"] & pG_x \arrow[d, "pG_f"] \\
pG_y \arrow[r, equal] & pG_y
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
pG_x \arrow[r, equal] \arrow[d, "pf"'] \arrow[dr, phantom, "\circlearrowright"] & pG_x \arrow[d, bend right=60, "pf"'] \arrow[d, phantom, "\xLeftarrow{p\underline{h_f}}"] \arrow[d, bend left=60, "pG_f"] \\
pG_y \arrow[r, equal] & pG_y
\end{tikzcd}
\quad \sim \quad
\begin{tikzcd}
pG_x \arrow[r, equal] \arrow[d, "pf"'] \arrow[dr, phantom, "\circlearrowright"] & pG_x \arrow[d, "pf"] \\
pG_y \arrow[r, equal] & pG_y
\end{tikzcd}
$$

□

So for a $\mathfrak{Q}$-coalgebra $H\colon \mathcal{X} \to \mathfrak{Q}_p$, the equaliser of H and η_p is the generic core, $p^\circ\colon \mathcal{X}^\circ \to \mathcal{A}$. The action of the right adjoint G on 1-cells and 2-cells is given by restriction to the generic core. For example, given a coalgebra map $f\colon \mathcal{X} \to \mathcal{Y}$, the image of f under G is the restriction of the 2-functor f to $\mathcal{X}^\circ$, which by the universal property of generic cores as equalisers lands in the generic core $\mathcal{Y}^\circ$.

From (3.23), the counit for the adjunction $\mathsf{K} \dashv \mathsf{G}$ has component at a $\mathfrak{Q}$-coalgebra $H\colon \mathcal{X} \to \mathfrak{Q}_p$ given by the composition (where $i\colon \mathcal{X}^\circ \hookrightarrow \mathcal{X}$ is inclusion):

$$\mathfrak{Q}_{p^\circ} \xrightarrow{\mathfrak{Q}(i)} \mathfrak{Q}_p \xrightarrow{\mathsf{c}_p} \mathcal{X}$$

But this map is just the restriction of the counit c_p to the generic core, which we observed in Lemma 3.4.5 to be the inverse to the coalgebra map $H\colon \mathcal{X} \to \mathfrak{Q}_p$ restricted to its codomain. Thus, the counit for the adjunction $\mathfrak{Q} \dashv \mathcal{R}$ is invertible, from which it follows:

Lemma 3.4.7. $\mathsf{G}\colon \mathfrak{Q}\text{-}\mathsf{coalg} \to \mathsf{2Cat}/\mathcal{A}$ *is 2-fully-faithful.*

This means that the category of $\mathfrak{Q}$-coalgebras is a reflective subcategory of $\mathsf{2Cat}/\mathcal{A}$ and in particular equivalent to the image of G. This image is given by those 2-functors which are generic cores of some $\mathfrak{Q}$-coalgebra. Such 2-functors are special, as generic cores (or indeed any restriction to a full sub-2-category) of $\mathfrak{Q}$-coalgebras are in particular discrete opfibration on hom-categories. We will see shortly that the converse is also true. First, we give a name and notation to 2-functors with this property:

Definition 3.4.8 (Local discrete opfibration, $\mathsf{2Cat}/_{\mathsf{ld}}\mathcal{A}$)**.** A *local discrete opfibration* is a 2-functor which is a discrete opfibration on hom-categories. We let $\mathsf{2Cat}/_{\mathsf{ld}}\mathcal{A}$ denote the full subcategory of $\mathsf{2Cat}/\mathcal{A}$ whose objects are local discrete opfibrations. ◊

Proposition 3.4.9. *The essential image of* G *is* $\mathsf{2Cat}/_{\mathsf{ld}}\mathcal{A} \subset \mathsf{2Cat}/\mathcal{A}$.

Proof. We already know that the essential image of G is contained in $\mathsf{2Cat}/_{\mathsf{ld}}\mathcal{A}$, so it suffices to show that the component of the unit η of the adjunction $\mathsf{K} \dashv \mathsf{G}$ at a local discrete opfibration is an isomorphism. The monad GK acts on $\mathsf{2Cat}/\mathcal{A}$ by sending a 2-functor $p\colon \mathcal{X} \to \mathcal{A}$ to the generic core of $\mathfrak{Q}_p$. The generic objects of $\mathfrak{Q}_p$ are the identities $px = px$, so $\mathsf{GK}p$ is the full sub-2-category of $\mathfrak{Q}_p$ with those identities as objects. The unit

$\eta_p \colon \mathcal{X} \to \mathfrak{Q}_p^\circ$ sends $x \in \mathcal{X}$ to $px = px$ and maps 1-cells $f \colon x \to y$ and 2-cells $\sigma \colon f \Rightarrow g$ to the 1-cells and 2-cells with representatives shown below:

$$
\begin{tikzcd}
px \arrow[r, equal] \arrow[d, "pf"'] \arrow[dr, phantom, "\cup"] & px \arrow[d, "pf"] \\
py \arrow[r, equal] & py
\end{tikzcd}
\qquad
\begin{tikzcd}
px \arrow[r, equal] \arrow[d, dashed, "pf"] \arrow[d, bend right=60, "pg"'] \arrow[d, phantom, shift right=4, "\xLeftarrow{p\sigma}"] \arrow[dr, phantom, "\cup"] & px \arrow[d, "pf"] \\
py \arrow[r, equal] & py
\end{tikzcd}
$$

It is easy to see that for any 2-functor $p \colon \mathcal{X} \to \mathcal{A}$, η_p is a bijection on objects, since each $px = px$ is the image of x (recall that the underlying data of $px = px$ is both the element $x \in \mathcal{X}$ and the morphism 1_{px} in $\mathcal{A}$). All this is true for a general 2-functor $p \colon \mathcal{X} \to \mathcal{A}$; if we additionally assume that $p \colon \mathcal{X} \to \mathcal{A}$ is a local discrete opfibration then we can show η_p is also an isomorphism on hom-categories. To that end, consider a 1-cell (s, σ, f) between two generic objects in $\mathfrak{Q}_p$, shown on the left:

$$
\begin{tikzcd}
px \arrow[r, equal] \arrow[d, "s"'] \arrow[dr, phantom, "\xLeftarrow{\sigma}"] & px \arrow[d, "pf"] \\
py \arrow[r, equal] & py
\end{tikzcd}
=
\begin{tikzcd}
px \arrow[r, equal] \arrow[d, "s"'] \arrow[dr, phantom, "\cup"] & px \arrow[d, dashed, "pf'"'] \arrow[d, bend left=60, "pf"] \arrow[d, phantom, shift left=4, "\xLeftarrow{p\sigma}"] \\
py \arrow[r, equal] & py
\end{tikzcd}
\sim
\begin{tikzcd}
px \arrow[r, equal] \arrow[d, "s"'] \arrow[dr, phantom, "\cup"] & px \arrow[d, "pf'"] \\
py \arrow[r, equal] & py
\end{tikzcd}
\tag{3.25}
$$

Using the fact that p is a discrete opfibration on hom-categories, we can lift σ to a 2-cell $\underline{\sigma} \colon f \Rightarrow f'$ which relates the representation for the 1-cell on the left in (3.25) to a representation with identity 2-cell component. This demonstrates that η_p is surjective on 1-cells, since the 1-cell shown in (3.25) is the image of $f' : x \to y$ under η_p. On the other hand, the image of two 1-cells $f \colon x \to y$ and $g \colon x \to y$ under η_p will coincide precisely if $pf = pg$ and f is connected to g in the fibre over $pf = pg$. But by the fact that p is locally a discrete opfibration, it has discrete fibres over 1-cells, and so η_p is injective on 1-cells. We need not check that η_p is a bijection on 2-cells, since any map in $\mathsf{2Cat}/\mathcal{A}$ between local discrete opfibrations which is also an isomorphism on underlying 1-categories is an isomorphism of 2-categories. □

Corollary 3.4.10. *The restricted adjunction* $\mathsf{K}_{\mathsf{Id}} \dashv \mathsf{G} \colon \mathsf{2Cat}/_{\mathsf{Id}}\mathcal{A} \to \mathfrak{Q}\text{-}\mathsf{coalg}$ *is an adjoint equivalence.*

Corollary 3.4.11. *The full subcategory* $\mathsf{2Cat}/_{\mathsf{Id}}\mathcal{A} \subset \mathsf{2Cat}/\mathcal{A}$ *is reflective.*

To understand the action of the reflector $\mathsf{GK} \colon \mathsf{2Cat}/\mathcal{A} \to \mathsf{2Cat}/_{\mathsf{Id}}\mathcal{A}$, let's first consider the form of the 1-cells of a 2-functor $p \colon \mathcal{X} \to \mathcal{A}$ under GK as shown on the left of (3.25). We've observed in the proof of Proposition 3.4.9 that the unit $\eta_p : \mathcal{X} \to \mathsf{GK}_p$ is bijective on objects, so we can identify $x \in \mathcal{X}$ with $px = px$ in GK_p. A morphism from x to y in GK_p is then given by a morphism $s \colon px \to py$ along with a connected component of $p_{x,y} \downarrow s$, where $p_{x,y}$ is the hom-functor $\mathcal{X}(x, y) \to \mathcal{A}(px, py)$. The projection down to $\mathcal{A}$ returns the 1-cell s. For a 1-category, C, the full inclusion $\mathsf{DOpfib}(C) \hookrightarrow \mathsf{Cat}/C$ of discrete opfibrations is also reflective, with the reflector given by sending $F \colon \mathcal{B} \to C$ to the discrete opfibration whose domain has objects pairs of $c \in C$ and a connected component of $F \downarrow c$. So we see that the reflection of 2-functors to local discrete opfibrations is given by fixing the action on objects and reflecting each of the hom-functors into discrete opfibrations. The unit of the reflector, η_p, is a bijection on objects, but on each of the hom-categories acts as the unit of the reflector $\mathsf{Cat}/\mathcal{A}(px, py) \to \mathsf{DOpfib}(\mathcal{A}(px, py))$.

3.4.2 Digression: a comprehensive factorisation system on $\mathsf{2Cat}$

The unit of the reflection of 1-functors into discrete opfibrations is an initial functor, and thus reflecting a 1-functor factorises it as an initial functor followed by a discrete opfibration. These two classes of morphisms are the left and right classes respectively of an orthogonal factorisation system called the *comprehensive factorisation system*

[4]on Cat [SW73], which we've already encountered in the proof for Corollary 3.4.5. There is a related orthogonal factorisation system on $\mathsf{2Cat}$, and the reflector $\mathsf{2Cat}/\mathcal{A} \to \mathsf{2Cat}/_{\mathsf{Id}}\mathcal{A}$ provides factorisations of this type.

Proposition 3.4.12. *Bijective-on-objects locally initial functors and local discrete opfibrations are the left and right classes respectively of an orthogonal factorisation system on* $\mathsf{2Cat}$.

Proof. For this proof we will use $\mathscr{L}$ and $\mathscr{R}$ to denote the classes of bijective-on-objects locally initial (i.e. initial on hom-categories) 2-functors and local discrete opfibrations respectively. Among the various equivalent conditions for two classes of morphisms to comprise an orthogonal factorisation system we choose to demonstrate the following:

(a) Every 2-functor factors as $r\ l$ for $r \in \mathscr{R}$ and $l \in \mathscr{L}$

(b) The classes $\mathscr{L}$ and $\mathscr{R}$ are replete in $\mathsf{2Cat}^{\to}$

(c) Morphisms in $\mathscr{L}$ are left-orthogonal to morphisms in $\mathscr{R}$

For (a), we simply note that the unit of the reflection $\mathsf{2Cat}/\mathcal{A} \to \mathsf{2Cat}/_{\mathsf{Id}}\mathcal{A}$ for any 2-category $\mathcal{A}$ is bijective-on-objects and locally initial by the fact that its action on hom-categories is given by the unit of the reflector $\mathsf{Cat}/\mathcal{A}(px, py) \to \mathsf{DOpfib}\mathcal{A}(px, py)$ which is initial. Thus, a suitable factorisation of a 2-functor $p\colon \mathcal{X} \to \mathcal{A}$ is given by $\mathsf{GK}_p\ \eta_p$.

(b) is trivial, as both classes $\mathscr{L}$ and $\mathscr{R}$ are closed under pre- and post-composition by isomorphisms. To prove (c), assume $l\colon \mathcal{A} \to \mathcal{B} \in \mathscr{L}$ and $r\colon \mathcal{C} \to \mathcal{D} \in \mathscr{R}$ and we have 2-functors $u\colon \mathcal{A} \to \mathcal{C}$, $v\colon \mathcal{B} \to \mathcal{D}$ satisfying $v\ l = r\ u$. Because l is b.o.o, there exists a unique map on objects $d_0 := u_0\ l^{-1}\colon \mathsf{ob}\mathcal{B} \to \mathsf{ob}\mathcal{C}$ satisfying $d_0\ l_0 = u_0$, and $r_0\ d_0 = v_0$, where $(-)_0$ is the forgetful map from $\mathsf{2Cat}$ to Set:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "u"] \arrow[d, "l"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{C} \arrow[d, "r"] \\ \mathcal{B} \arrow[r, "v"'] & \mathcal{D} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{A}_0 \arrow[r, "u_0"] \arrow[d, "l_0"'] & \mathcal{C}_0 \arrow[d, "r_0"] \\ \mathcal{B}_0 \arrow[r, "v_0"'] \arrow[ur, dashed, "\exists! d_0"] & \mathcal{D}_0 \end{tikzcd}$$

We can show that this map on objects has a unique extension to a 2-functor, by locally applying the fact that initial functors are left-orthogonal to discrete opfibrations. This property allows us to conclude that there is a unique possible action of d on hom-categories, because for any $x, y \in \mathcal{B}$ the functor $l_{l^{-1}x, l^{-1}y}\colon \mathcal{A}(l^{-1}x, l^{-1}y) \to \mathcal{B}(x, y)$ is initial and the functor $r_{dx,dy}\colon \mathcal{C}(dx, dy) \to \mathcal{D}(vx, vy)$ is a discrete opfibration:

$$\begin{tikzcd} \mathcal{A}(l^{-1}x, l^{-1}y) \arrow[r, "u_{l^{-1}x,l^{-1}y}"] \arrow[d, "l_{l^{-1}x,l^{-1}y}"'] & \mathcal{C}(dx, dy) \arrow[d, "r_{dx,dy}"] \\ \mathcal{B}(x, y) \arrow[r, "v_{x,y}"'] \arrow[ur, dashed, "\exists! d_{x,y}"] & \mathcal{D}(vx, vy) \end{tikzcd}$$

We now show that these hom-functors $d_{x,y}$ respect horizontal composition in $\mathcal{B}$. We will use the notation $\mathcal{B}_{x_1,x_2,x_3,\ldots} := \mathcal{B}(x_1, x_2) \times \mathcal{B}(x_2, x_3) \times \ldots$ so that unbiased composition in $\mathcal{B}$ is a functor $\mathsf{comp}_{\mathcal{B}}\colon \mathcal{B}_{\vec{x}} \to \mathcal{B}_{x_1,x_n}$. We extend this composition to include the identities by defining $\mathcal{B}_x := 1$ so that $\mathsf{comp}_{\mathcal{B}}\colon \mathcal{B}_x \to \mathcal{B}_{x,x}$ is the map which picks out the identity on x. Now consider the following diagram:

$$\begin{tikzcd} \mathcal{A}_{l^{-1}\vec{x}} \arrow[r, "u_{l^{-1}\vec{x}}"] \arrow[d, "l_{l^{-1}\vec{x}}"'] & \mathcal{C}_{d\vec{x}} \arrow[r, "\mathsf{comp}_{\mathcal{C}}"] & \mathcal{C}_{dx_1,dx_n} \arrow[d, "r_{dx_1,dx_n}"] \\ \mathcal{B}_{\vec{x}} \arrow[r, "\mathsf{comp}_{\mathcal{B}}"'] \arrow[ur, dashed, "d_{\vec{x}}"] & \mathcal{B}_{x_1,x_n} \arrow[r, "v_{x_1,x_n}"'] \arrow[ur, dashed, "d_{x_1,x_n}"] & \mathcal{D}_{vx_1,vx_n} \end{tikzcd}$$

[4]Actually, the comprehensive factorisation system usually refers to the final functor/discrete opfibration factorisation, whose image under $\mathsf{op}\colon \mathsf{Cat}^{\mathsf{co}} \to \mathsf{Cat}$ is the factorisation system described above.

Our objective is to show that $\mathsf{comp}_C\ d_{\vec{x}} = d_{x_1,x_n}\ \mathsf{comp}_{\mathcal{B}}$. Note that the outside square commutes by the functoriality of u and l. The functor r_{dx_1,dx_2} is a discrete opfibration, and the functor $l_{l^{-1}\vec{x}}$ is initial, by the fact that a finite product of initial functors (including the empty product) is initial. Both $\mathsf{comp}_C\ d_{\vec{x}}$ and $d_{x_1,x_2}\ \mathsf{comp}_{\mathcal{B}}$ are fillers for this square, and are therefore equal. □

Remark 3.4.13. The proof above demonstrates the more general fact that for any orthogonal factorisation system $(\mathscr{L},\mathscr{R})$ on a monoidal category $\mathcal{V}$ with $\mathscr{L}$ closed under monoidal products, one can obtain an orthogonal factorisation system on $\mathcal{V}\mathsf{Cat}$ whose left class is the b.o.o locally-$\mathscr{L}$ $\mathcal{V}$-functors, and whose right class is the locally-$\mathscr{R}$ $\mathcal{V}$-functors. ◊

3.4.3 $\mathcal{Q}$-coalgebras

We can now take what we've learned about $\mathfrak{Q}$-coalgebras and translate it to statements about the coalgebras of the comonad $\mathcal{Q}$ on $[\mathcal{A}^{\mathsf{op}},\mathsf{Cat}]$. Because the functors $\int\colon [\mathcal{A}^{\mathsf{op}},\mathsf{Cat}] \rightleftharpoons \mathsf{Fib}_{\mathsf{Ids}}(\mathcal{A}) : \mathsf{Ind}$ underlie an equivalence between the comonads $\mathcal{Q}$ and $\mathfrak{Q}$, they induce an equivalence of coalgebras for the two comonads. So we have a string of equivalences:

$$\mathsf{2Cat}/_{\mathsf{Id}}\mathcal{A} \underset{\mathsf{G}}{\overset{\mathsf{K}_{\mathsf{Id}}}{\rightleftarrows}} \mathfrak{Q}\text{-coalg} \underset{\int}{\overset{\mathsf{Ind}}{\rightleftarrows}} \mathcal{Q}\text{-coalg}$$

A presheaf $\mathcal{X}\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ therefore supports a $\mathcal{Q}$-coalgebra structure precisely if it is in the essential image of $\mathsf{Ind}\ \mathsf{K}_{\mathsf{Id}}$. Of course, it's equally true to say $\mathcal{X}$ admits a $\mathcal{Q}$-coalgebra structure precisely when it is in the essential image of $\mathsf{Ind}\ \mathsf{K}\colon \mathsf{2Cat}/\mathcal{A} \to [\mathcal{A}^{\mathsf{op}},\mathsf{Cat}]$, since K and K_{Id} have the same image. If $p\colon \mathcal{X} \to \mathcal{A}$ is a 2-functor, its image under $\mathsf{Ind}\ \mathsf{K}$ will be the presheaf $\mathsf{Ind}\ \mathfrak{Q}_p$, which sends $a \in \mathcal{A}$ to the category with objects given by pairs $(x \in \mathcal{X}, u\colon a \to px)$ and morphisms the vertical morphisms of $\mathfrak{Q}_p$ over a:

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[d, equal] & px \arrow[d, "pf"] \\ a \arrow[r, "v"'] \arrow[ur, phantom, "\overset{\sigma}{\Leftarrow}"] & py \end{tikzcd}$$

Another way to describe this presheaf is as the *oplax-image presheaf* of p, which we now define.

Definition 3.4.14 (Oplax-image presheaf, $\mathfrak{I}_F$)**.** Given a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$, the oplax-image presheaf $\mathfrak{I}_F : \mathcal{B}^{\mathsf{op}} \to \mathsf{Cat}$ is the oplax limit of the composite of F with the Yoneda embedding of $\mathcal{B}$, $よ_{\mathcal{B}}\ F : \mathcal{A} \to \mathcal{B} \to [\mathcal{B}^{\mathsf{op}},\mathsf{Cat}]$. ◊

This terminology is a reference to the *image presheaves* described in [PT22], which are defined analogously for 1-functors using ordinary colimits.

One can verify that $\mathsf{Ind}\ \mathfrak{Q}_p$ is isomorphic to $\mathfrak{I}_p$ by simply unravelling the definitions. For example, we observe that an object of $\mathfrak{I}_p(a)$ for $a \in \mathcal{A}$ is an object of the oplax colimit (i.e. the Grothendieck construction) of $\mathcal{A}(a,p-)$, since colimits of presheaves are computed point-wise. The objects of this oplax colimit are pairs $x \in \mathcal{X}, u \in \mathcal{A}(a,p-)$, which coincides with the description of the objects of $\mathsf{Ind}\ \mathfrak{Q}_p(a)$.

There is a simpler analogue of this fact for ordinary category theory: if $F\colon \mathcal{A} \to \mathcal{B}$ is a 1-functor then the Grothendieck construction of its image presheaf is the free discrete fibration over $\mathcal{B}$ induced by F.

Corollary 3.4.15. *A presheaf $\mathcal{X}\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ admits a $\mathcal{Q}$-coalgebra structure if and only if it is the oplax-image presheaf of a 2-functor.*

In other words the presheaves which admit $\mathcal{Q}$-coalgebra structures are precisely the conical oplax colimits of representables.

3.4.4 Recognising $\mathfrak{Q}$-coalgebras

In [Wal20, §3] Walker describes conditions on the category of elements for a pseudofunctor $P\colon \mathscr{A} \to \mathsf{Cat}$, with $\mathscr{A}$ a bicategory, which are equivalent to P being the *lax*-image presheaf of a pseudofunctor from a 1-category. This property differs from the property of being a $\mathscr{Q}$-coalgebra — equivalently an oplax-image presheaf of a strict 2-functor — in a few ways. In particular, it is a condition on presheaves in $[\mathscr{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{pseudo}}$ rather than $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$. But it's similar enough that we can adapt the recognition results given by Walker to obtain conditions which detect whether a locally discrete split 2-fibration $p\colon \mathcal{X} \to \mathcal{A}$ admits a $\mathfrak{Q}$-coalgebra structure.

It is helpful to first consider the related problem of detecting whether a discrete fibration in Cat can be expressed as the category of elements of a presheaf $P\colon A \to \mathsf{Set}$ which is the image presheaf of a functor from a discrete category (i.e. a coproduct of representables). Such discrete fibrations are precisely those with terminal objects in each connected component of the domain (for reasons given in our proof of Corollary 3.4.3). Labelling such objects *component-terminal*, the condition could equivalently be phrased as "every object in the domain admits a morphism to a component-terminal object". Both Walker's conditions for being a lax-image presheaf of a pseudofunctor from a 1-category and the conditions are about to give for being an oplax-image presheaf of a 2-functor resemble this "enough component-terminal objects" property. In both cases one defines a variant of "component-terminal object" — "generics" for Walker's case, "oplax-generics" for ours — and requires that all other objects be connected to such an object with by a suitably special kind of morphism — opcartesian and chosen cartesian respectively.

Our description of oplax-generic objects is deferred to Definition 3.4.18, after we've established the relevant terminology of lax coslice 2-categories and an alternative characterisation of the 1-cells in $\mathfrak{Q}_p$ for $p : \mathcal{X} \to \mathcal{A}$ is a locally discrete split 2-fibration.

Definition 3.4.16 (Lax coslice $(a \Downarrow \mathcal{A})$)**.** For an object a in a 2-category $\mathcal{A}$, the lax coslice $(a \Downarrow \mathcal{A})$ is the 2-category whose objects are morphisms $u\colon a \to x$ in $\mathcal{A}$, whose 1-cells are "lax commuting triangles" and whose 2-cells are 2-cells in $\mathcal{A}$ between the sides of the triangles opposite $\mathcal{A}$, which commute with the 2-cells filling the triangles. That is, $\kappa\colon g \Rightarrow f$ represents a 2-cell from (g, ψ) to (f, ϕ) if the following 2-cells are equal:

$$\begin{tikzcd} & x \arrow[d, "f"] \arrow[d, bend left=80, "g", ""{name=G}] \\ a \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[r, phantom, "\Downarrow\phi"] & y \end{tikzcd} \quad \kappa\colon g \Leftarrow f \quad = \quad \begin{tikzcd} & x \arrow[d, "g"] \\ a \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[r, phantom, "\Downarrow\psi"] & y \end{tikzcd} \tag{3.26}$$

◊

Lemma 3.4.17. *For $p\colon \mathcal{X} \to \mathcal{A}$ a locally discrete split 2-fibration, $u\colon a \to px$, $v\colon b \to py$ objects in $\mathfrak{Q}_p$, and $g\colon u^*x \to v^*y$ a 1-cell between their images under c_p in $\mathcal{X}$, the fibre of $(\mathsf{c}_p)_{u,v}\colon \mathfrak{Q}_p(u,v) \to \mathcal{X}(u^*x, y^*x)$ is bijective with the connected components of $(u^*x \Downarrow \mathcal{X})(\bar{u}x, \bar{v}y\ g)$*

Proof. Consider a 1-cell in $\mathfrak{Q}_p$ from (u, x) to (v, y) shown below on the left:

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[d, "s"'] & px \arrow[d, "pf"] \arrow[dl, phantom, "\overset{\sigma}{\Longleftarrow}"] \\ b \arrow[r, "v"'] & py \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} u^*x \arrow[r, "\bar{u}x"] \arrow[d, "g"'] & x \arrow[d, "f"] \arrow[dl, phantom, "\overset{\Sigma}{\Longleftarrow}"] \\ v^*y \arrow[r, "\bar{v}y"'] & y \end{tikzcd} \tag{3.27}$$

To give a 2-cell $\sigma\colon pf\ u \to v\ s$ in $\mathcal{A}$ is equivalent to giving a lift $\Sigma\colon f\ \bar{u}x \to k$ in $\mathcal{X}$ for some $k\colon u^*x \to y$ with $pk = v\ s$ by the fact that p is a local discrete opfibration. The morphism k has a unique factorisation $k = \bar{v}y\ g$ where g is $\sigma^\sharp f$. Thus, a *representation* for a 1-cell in $\mathfrak{Q}_p$ from (u, x) to (v, y) is equivalently a pair of a morphism $g\colon u^*x \to v^*y$and a 1-cell (f, Σ) in $(u^*x \Downarrow \mathcal{X})$ from $\bar{u}v$ to $\bar{v}y\ g$. The image of the 1-cell under c_p is simply the "g"

component. With representations for 1-cells given in this way, the equivalence classes of representations for 1-cells in $\mathfrak{Q}_p$ are just the connected components of the hom-categories of the lax-coslices of $\mathcal{X}$ (compare the 2-cells in (3.26) with the relations in (3.16)). So in particular, those 1-cells in $\mathfrak{Q}_p$ from (u,x) to (v,y) which lift $g\colon u^*x \to v^*y$ can be identified with the connected components of $(u^*x \Downarrow \mathcal{X})(\bar{u}x, \bar{v}y\ g)$. □

We now introduce the notion of "oplax-generic" objects used in the classification of $\mathfrak{Q}$-coalgebras and justify the similarity in terminology to generic objects of $\mathfrak{Q}$-coalgebras.

Definition 3.4.18 (Oplax-generic object)**.** An object x in the domain of a locally discrete split 2-fibration $p\colon \mathcal{X} \to \mathcal{A}$ is called *oplax-generic* if it satisfies:

(a) For any morphism $f\colon y \to x$ and a chosen-cartesian $g\colon y \to z$ the category $(y \Downarrow \mathcal{X})(g,f)$ has a single connected component.

(b) If $f\colon y \to x$ and $g\colon y \to z$ are both chosen-cartesian, then there exists a unique chosen-cartesian $H\colon z \to x$. ◊

Lemma 3.4.19. *If $p\colon \mathcal{X} \to \mathcal{A}$ is a $\mathfrak{Q}$-coalgebra with structure map $H\colon \mathcal{X} \to \mathfrak{Q}_p$, then each generic object G_x is oplax-generic.*

Proof. Condition (b) is equivalent to saying that oplax-generic objects are component-terminal in $\mathcal{X}^{\mathfrak{c}}$, which was shown to be true of the generic objects of a $\mathfrak{Q}$-coalgebra in Corollary 3.4.3. For (b), we first observe that any chosen-cartesian $y \to z$ is of the form $u\colon u^*z \to z$ for $u\colon a \to pz$ some morphism in $\mathcal{A}$, and a morphism from $f\colon u^*z \to G$, for G some generic object, factors through the chosen-cartesian $\overline{1_p G}(G) = 1_G\colon G = G$. So by Lemma 3.4.17, it suffices to show that there is a unique lift of $f\colon u^*z \to G$ along c_p to a morphism from $u\colon a \to pz$ to $=\colon pG \to pG$ for any such f. In other words, we must show that $(\mathsf{c}_p)_{u,1_{pG}}\colon \mathfrak{Q}_p(u, H(G)) \to \mathcal{X}(u^*x, G)$ is a bijection on objects for any $u\colon a \to pz$ and generic object G. This follows from the fact that the coalgebra map $H\colon \mathcal{X} \to \mathfrak{Q}_p$ is a right-adjoint right-inverse to c_p. □

Proposition 3.4.20. *A locally discrete split 2-fibration $p\colon \mathcal{X} \to \mathcal{A}$ admits a $\mathfrak{Q}$-coalgebra structure if and only if every object of $\mathcal{X}$ admits a chosen cartesian morphism to a oplax generic object.*

Proof. One direction is easy: every object x in a $\mathfrak{Q}$-coalgebra admits the chosen-cartesian ζ_x to the generic object G_x, which by Lemma 3.4.19 is oplax-generic.

For the other direction, since the image of $\mathfrak{Q}\colon \mathsf{2Cat}/\mathcal{A} \to \mathsf{Fib}_{\mathsf{lds}}(\mathcal{A})$ lands in $\mathfrak{Q}$-coalgebras it suffices to show that there exists some 2-functor $q\colon \mathcal{Y} \to \mathcal{A}$ and an isomorphism of locally discrete split 2-fibrations $\mathcal{X} \cong \mathfrak{Q}_q$. If every object of $\mathcal{X}$ admits a chosen-cartesian morphism to an oplax-generic, then every connected component of $\mathcal{X}^{\mathfrak{c}}$ contains an oplax-generic object, and such objects are component-terminal by Condition (b). We choose an oplax-generic for each connected component and let G_x denote the chosen oplax-generic in the component of $\mathcal{X}^{\mathfrak{c}}$ containing $x \in \mathcal{X}$. Then let $\mathcal{Y}$ denote the full-2-subcategory of $\mathcal{X}$ on the chosen oplax-generic objects, and let q denote the restriction of p to $\mathcal{Y}$. We obtain a map of locally discrete split 2-fibrations $T\colon \mathfrak{Q}_q \to \mathcal{X}$ by composing the inclusion $\mathfrak{Q}_q \hookrightarrow \mathfrak{Q}_p$ with the counit $\mathsf{c}_p\colon \mathfrak{Q}_p \to \mathcal{X}$. We will be done if we can show that this map is an isomorphism on the underlying 1-categories, as in particular this is a map between local discrete opfibrations.

The action of T on objects is given by sending $u\colon a \to pG$ to u^*G. For every $x \in \mathcal{X}$ there is by hypothesis a unique chosen-cartesian $c_x\colon x \to G_x$, so T has an inverse on objects given by $x \mapsto \left(px \xrightarrow{pc_x} pG_x\right)$.

To show that the hom-functors of T are also bijective on objects, we need to show that every 1-cell $g\colon u^*G \to v^*G'$ in $\mathcal{X}$ (with G, G' oplax-generic) has a unique lift along c_p to a 1-cell from (u,G) to (v,G') in $\mathfrak{Q}_q$. By Lemma 3.4.17 this is equivalent to the hom-category $(u^*G \Downarrow \mathcal{X})(\bar{u}G, \bar{v}G'\ g)$ having a unique connected component, which is true by Condition (a) of the definition of oplax-generic object. □

3.4.5 $\mathcal{Q}$-coalgebras as weights

Strict algebras for the *pseudo*-morphism classifier $\mathsf{Q}_{\mathsf{pseudo}}$ are precisely the weights in the class generated by products, inserters and equifiers, known as PIE weights (shown in [BG13; LS12]). A natural question to ask, therefore, is "what sorts of (co)limits are $\mathcal{Q}$-coalgebras weights for?" Perhaps unsurprisingly, the answer is *conical oplax (co)limits*, where the conical oplax colimit of $F\colon \mathcal{A} \to \mathcal{B}$ is defined as the representing object of the presheaf $\mathcal{B} \to \mathsf{Cat}$ given by:

$$b \mapsto [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}(\Delta\mathbb{1}, \mathcal{B}(F-, d)) \cong \oint_{a\in\mathcal{A}^{\mathsf{op}}} \mathcal{B}(Fa, b) \tag{3.28}$$

By the fact that representables preserve oplax ends it follows that the conical oplax colimit of F is isomorphic to $\oint^{a\in\mathcal{A}} Fa$.

Having expressed $\mathcal{Q}$ as a comonad on $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$, it is more natural to consider the coalgebras as weights for colimits, of 2-functors from $\mathcal{A}$, rather than as weights for limits of functors from $\mathcal{A}^{\mathsf{op}}$, though of course every statement we make about colimits for these weights dualises to a corresponding statement about the limits.

To see that all $\mathcal{Q}$-coalgebras are contained in the (saturated) class of weights for conical oplax colimits, we note that if $W\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ is a $\mathcal{Q}$-coalgebra, then it has an expression as an oplax-image presheaf of some functor $J\colon \mathcal{D} \to \mathcal{A}$:

$$W \cong \Im J = \oint^{d\in\mathcal{D}} \mathcal{A}(-, Jd)$$

Coend calculus then shows that the W-weighted colimit of $F\colon \mathcal{A} \to \mathcal{B}$ is the conical oplax colimit of FJ:

$$\begin{aligned}
[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}](W, \mathcal{B}(F-, b)) &\cong \int_{x\in\mathcal{A}} \left[\oint^{d\in\mathcal{D}} \mathcal{A}(x, Jd), \mathcal{B}(Fx, b)\right] \\
&\cong \oint_{d\in\mathcal{D}^{\mathsf{op}}} \int_{x\in\mathcal{A}} [\mathcal{A}(x, Jd), \mathcal{B}(Fx, b)] \\
&\cong \oint_{d\in\mathcal{D}^{\mathsf{op}}} \mathcal{B}(FJd, b) \\
&\cong \mathcal{A}\left(\oint^{d\in\mathcal{D}} FJd, b\right)
\end{aligned}$$

So $\mathcal{Q}$-coalgebras are in the saturation of the weights for conical oplax colimits of 2-functors which factor through $\mathcal{A}$. On the other hand, for a presheaf $P\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ the free $\mathcal{Q}$-coalgebra $\mathcal{Q}P$ is the weight for oplax P-weighted colimits — for $F\colon \mathcal{A} \to \mathcal{B}$ we have:

$$[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}](\mathcal{Q}P, \mathcal{B}(F-, b)) \cong [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}(P, \mathcal{B}(F-, b))$$

So every oplax colimit is a colimit for a $\mathcal{Q}$-coalgebra weight, which is a conical oplax colimit — all oplax colimits is conical. More precisely, if $F\colon \mathcal{A} \to \mathcal{B}$, and $W\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$, then the oplax W-weighted colimit of F is isomorphic to the oplax conical colimit of $\int W \xrightarrow{\pi_W} \mathcal{A} \xrightarrow{F} \mathcal{B}$ (recall from Section 3.3.2 that $\mathcal{Q}W \cong \Im\pi_W$). This resembles the property that weighted limits in ordinary categories admit constructions as conical limits. Of course, this isn't true for weighted limits in 2-categories. In particular, copowers in 2-categories are not typically conical colimits[5] in this way. For a 2-category $\mathcal{B}$, $b \in \mathcal{B}$ and a 1-category K, the copower $K \odot b$ is the colimit of $\langle b\rangle : \mathbb{1} \to \mathcal{B}$ weighted by $\langle K\rangle : \mathbb{1}^{\mathsf{op}} \to \mathsf{Cat}$, but for any 2-functor $F\colon \mathcal{X} \to \mathcal{B}$ which factors through $\langle b\rangle$ (i.e. is constant at $b \in \mathcal{B}$), the conical colimit of F is $\mathcal{X}_0 \odot b \cong \coprod_{x\in\mathsf{ob}\mathcal{X}} b$. However, the *oplax* conical colimit of $\Delta_b\colon K \to \mathcal{B}$ *is* isomorphic to $K \odot b$. This is related to the fact that Cat is generated by the terminal category $\mathbb{1}$ under *oplax* conical colimits, but not under strict conical colimits.

[5] By *conical colimit* here, we mean a weighted colimit whose weight is constant at the terminal 2-category $\mathbb{1}$, rather than the more abstract notion of conical colimits indexed by 1-categories described in [Kel05, §3.8].

3.4.6 Comments on size issues

Throughout the discussion of $\mathcal{Q}$-coalgebras we have assumed that $\mathcal{A}$ is itself a small 2-category. Our definition of $\mathcal{Q}$ in Section 3.2 relied on the existence of $\mathcal{A}$-indexed (oplax) colimits in $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$, which doesn't hold in general for non-small $\mathcal{A}$. Moreover, the Grothendieck construction for presheaves on a non-small category will not in general produce an object of $2\mathsf{Cat}/\mathcal{A}$. For example, the Grothendieck construction for $\Delta\mathbb{1} \colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ is isomorphic to $\mathcal{A} \Longrightarrow \mathcal{A}$, which only lies in $2\mathsf{Cat}/\mathcal{A}$ if $\mathcal{A}$ is small. The issue is fundamentally due to $\mathcal{A}$ failing to be an object in $2\mathsf{Cat}$. Depending on one's choice of foundations, this can be resolved by moving to a Grothendieck universe V which contains the set of arrows of $\mathcal{A}$. If we call the members of such a universe "moderate", then we can form the category SET of moderate sets, and thence the 2-category CAT of moderate Set-categories and the 3-category $2\mathsf{CAT}$ of moderate Cat-categories, which contains $\mathcal{A}$. The size-agnostic nature of $\mathcal{Q}$ means we can just as well construct an oplax morphism classifier comonad $\mathcal{Q}'$ on $[\mathcal{A}^{\mathsf{op}}, \mathsf{CAT}]$ which is induced by a moderate Grothendieck construction $[\mathcal{A}^{\mathsf{op}}, \mathsf{CAT}] \to 2\mathsf{CAT}/\mathcal{A}$ and a moderate oplax-image presheaf as its left adjoint. Then, even for large (but locally-small) $\mathcal{A}$, the restriction of the moderate oplax-image presheaf functor $\mathfrak{I} \colon 2\mathsf{CAT}/\mathcal{A} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{CAT}]$ to the full subcategory $2\mathsf{Cat}/\mathcal{A}$ of its domain has its image in $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$, since the Yoneda embedding for $\mathcal{A}$ lands in $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ and small oplax colimits of small categories are small. This fact will be relevant in the next chapter when we describe the free cocompletion under the class of weights given by $\mathcal{Q}$-coalgebras as the image of this restricted map $\mathfrak{I} \colon 2\mathsf{Cat}/\mathcal{A} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$. At some point we would like to consider the case $\mathcal{A} = \mathsf{Cat}$ and some other related 2-categories, which is why we address these size issues explicitly. Aside from the case $\mathcal{A} = \mathsf{Cat}$ and some other situations where maintaining the size distinction $\mathsf{Cat} < \mathsf{CAT}$ is truly necessary, we will typically avoid the notation CAT and instead allow the meaning of Cat to be "universe polymorphic", tacitly expanding to sufficient size to contain a given locally-small $\mathcal{A}$.

Chapter 4

The Oplax Fam Construction

From the previous Chapter we've learned that the coalgebras for the oplax-morphism classifier are, in a sense, the weights for oplax colimits. We will begin this chapter by making this notion precise, after reviewing the general theory of free cocompletions with respect to a class of weights for general $\mathcal{V}$-categories. Some basic results of this theory allow us to observe the free completion of a 2-category under oplax-colimits can be constructed from the Kleisli category of the monad for the free–forgetful adjunction between $\mathsf{CAT}/\mathcal{K}$ and $\mathsf{Fib}_{\mathsf{Ids}}(\mathcal{K})$. Our first 2-categorical Fam construction, $\mathsf{F}_\Omega\mathcal{K}$, is then equivalent to a full subcategory of this larger free cocompletion. It is not immediately obvious that this full subcategory is also a free cocompletion. Indeed, showing that it forms the free cocompletion with respect to the class of oplax conical colimits of oplax functors from 1-categories is our primary focus for most of this chapter, and one significant component of this proof is a combinatorial argument about certain diagram constructions in Section 4.5.

There are, of course, two ways to dualise the results of this section: changing "oplax" to "lax" and considering limits instead of colimits. We give explicit descriptions for some of these dual results in Section 4.6, as well as describing related free cocompletions. For example, the completion under oplax colimits for *normal* oplax functors from 1-categories, and the *coKleisli* completion of [LS02] are both full submonads of F_Ω. We conclude in Section 4.7 by providing some examples of F_Ω constructions and noting a connection to enrichments over strict monoidal categories and 2-categories.

4.1 The Free Completion under Oplax Colimits

For a given base of enrichment, $\mathcal{V}$, and a class of weights Φ, there exists a large 2-category $\Phi\text{-}\mathsf{coCts}$ whose objects are Φ-cocomplete $\mathcal{V}$-categories (with size in some Grothendieck universe we leave undetermined) and whose morphisms are Φ-cocontinuous $\mathcal{V}$-functors. The 2-cells are arbitrary $\mathcal{V}$-natural transformations. The *free completion* of a $\mathcal{V}$-category $\mathcal{K}$ under a class of weights Φ is a $\mathcal{V}$-functor $Z\colon \mathcal{K} \to \mathcal{K}'$ to a Φ-cocomplete $\mathcal{V}$-category such that for any other Φ-cocomplete $\mathcal{V}$-category $\mathcal{B}$, precomposition by Z induces an equivalence of categories: $\Phi\text{-}\mathsf{coCts}(\mathcal{K}', \mathcal{B}) \to \mathcal{V}\text{-}\mathsf{Cat}(\mathcal{K}, \mathcal{B})$.

This property characterises the free cocompletion $\mathcal{K}'$ up to equivalence. From the general theory of free cocompletions described in [Kel05, §5], $\mathcal{K}'$ can be constructed as the closure of the representables in the $\mathcal{V}$-category of presheaves $[\mathcal{K}^{\mathsf{op}}, \mathcal{V}]$ under Φ-colimits, assuming the weights in Φ are small. Let $\Phi\mathcal{K}$ denote this closure. The Yoneda embedding factors through $\Phi\mathcal{K}$ via functors which we name as follows:

$$\mathcal{K} \xhookrightarrow{\mathcal{Z}_\mathcal{K}} \Phi\mathcal{K} \xhookrightarrow{\mathcal{W}_\mathcal{K}} [\mathcal{K}^{\mathsf{op}}, \mathcal{V}]$$

It is the inclusion $\mathcal{Z}_{\mathcal{K}}$ which defines the Φ-cocompletion of $\mathcal{K}$. Moreover, it is shown in [Kel05] that the pseudo-inverse to precomposition $\mathcal{Z}_{\mathcal{K}}^*\colon \Phi\text{-}\mathsf{coCts}(\Phi\mathcal{K},\mathcal{B}) \to \mathcal{V}\text{-}\mathsf{Cat}(\mathcal{K},\mathcal{B})$ is given by the left Kan extension along $\mathcal{Z}_{\mathcal{K}}$.

The closure $\Phi\mathcal{K}$ for a given class of weights Φ and locally small $\mathcal{V}$-category $\mathcal{K}$ will contain in particular all Φ-colimits of representables, but must also contain all Φ-colimits of Φ-colimits of representables, and so on. For example, the closure of $[\mathcal{K}^{\mathsf{op}},\mathsf{Cat}]$ under binary coproducts for $\mathcal{K}$ a Cat-category contains not just the binary coproducts of representables, but all non-empty finite coproducts of representables. However, for certain classes of weights Φ, all "iterated" Φ-colimit constructions can be expressed a single Φ-colimit, so that every object of $\Phi\mathcal{K}$ is a Φ-colimit of representables. The class of finite coproducts for ordinary categories has this property, and so does the class of filtered colimits. Such classes of colimits are said to be *presaturated*. If we use $\Phi_1\mathcal{K}$ to denote the full subcategory of $[\mathcal{K}^{\mathsf{op}},\mathcal{V}]$ whose objects are Φ-colimits of representables — henceforth "Φ-presheaves" — then Φ is a presaturated class precisely if $\Phi_1\mathcal{K}$ contains all representables and is closed in $[\mathcal{K}^{\mathsf{op}},\mathcal{V}]$ under Φ-colimits for all $\mathcal{K}$. Equivalently, Φ is presaturated if $\Phi_1\mathcal{K}$ is equal to $\Phi\mathcal{K}$ for any $\mathcal{V}$-category $\mathcal{K}$.

The *saturation* of a class of weights Φ is the class containing the weights ψ such that all Φ-cocomplete $\mathcal{V}$-categories are ψ-cocomplete and all Φ-cocontinuous $\mathcal{V}$-functors are ψ-cocontinuous. We let Φ^* denote the saturation of Φ, and say a class Φ is *saturated* if $\Phi^* = \Phi$. Theorem 1.1 of [AK88] says that for a small 2-category $\mathcal{A}$, the weights in Φ^* with domain $\mathcal{A}$ — denoted $\Phi^*[\mathcal{A}]$ — are precisely those weights in $\Phi\mathcal{A}$. Now, since $\Phi[\mathcal{A}] \subseteq \Phi_1\mathcal{A}$ by the fact that any weight $\phi\colon \mathcal{A}^{\mathsf{op}} \to \mathcal{V}$ is itself a ϕ-colimit of representables as $\phi \cong \phi * \text{よ}_{\mathcal{A}}$, we have:

$$\Phi[\mathcal{A}] \subseteq \Phi_1\mathcal{A} \subseteq \Phi\mathcal{A} = \Phi^*[\mathcal{A}]$$

If Φ is saturated then $\Phi\mathcal{A} = \Phi[\mathcal{A}] \subseteq \Phi_1\mathcal{A}$, which means Φ must also be presaturated.[1]

The $\mathcal{Q}$-coalgebras of Chapter 3 which like in $[\mathcal{A}^{\mathsf{op}},\mathsf{Cat}]$ for some small 2-category $\mathcal{A}$ form a class Θ of weights for $\mathcal{V} = \mathsf{Cat}$. We can now more precisely state the observation made in Section 3.4.5 that "$\mathcal{Q}$-coalgebras are weights for conical oplax colimits" using the above terminology:

Proposition 4.1.1. *The saturation of the class of oplax conical colimits contains* Θ.

Proof. Let Ξ denote the class of weights for oplax conical colimits — those weights of the form $\mathcal{Q}\Delta\mathbb{1}$ for $\Delta\mathbb{1}\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$. The observation of Section 3.4.5 shows that whenever a 2-category has all Ξ-colimits it also has Θ-colimits. A similar argument shows that any 2-functor which preserves all Ξ-colimits preserves Θ-colimits. □

Remark 4.1.2. Alternatively, we can observe that for small $\mathcal{A}$, every $\mathcal{Q}$-coalgebra in $\Theta[\mathcal{A}]$ is a conical oplax colimit of representables, so $\Theta[\mathcal{A}] \subseteq \Xi_1\mathcal{A} \subseteq \Xi\mathcal{A} = \Xi^*[\mathcal{A}]$by [AK88, Thm. 1.1]. Thus, $\Theta \subseteq \Xi^*$. ◊

Proposition 4.1.3. Θ *is a saturated class of weights.*

Proof. By [Thm 1.1 AK88], Θ is saturated if and only if $\Theta[\mathcal{A}] = \Theta\mathcal{A}$ for all small $\mathcal{A}$. By definition $\Theta\mathcal{A}$ is the smallest full replete subcategory of $[\mathcal{K}^{\mathsf{op}},\mathsf{Cat}]$ containing the representables and closed in $[\mathcal{K}^{\mathsf{op}},\mathsf{Cat}]$ under Θ-colimits — equivalently conical oplax colimits by Proposition 4.1.1 — so to show that $\Theta[\mathcal{A}] = \Theta\mathcal{A}$ it suffices to show:

(a) $\Theta[\mathcal{A}]$ contains the representables

(b) The inclusion $z\colon \Theta[\mathcal{A}] \hookrightarrow [\mathcal{A}^{\mathsf{op}},\mathsf{Cat}]$ creates conical oplax colimits

For (a), we observe that $\mathcal{A}(-,a)$ is the oplax image presheaf of $a\colon \mathbb{1} \to \mathcal{A}$, thus an object of $\Theta[\mathcal{A}]$. For (b) we appeal to a general result about categories of coalgebras. Because $\mathcal{Q}$ is an oplax-idempotent comonad, the forgetful 2-functor $\mathsf{U}_{\mathcal{Q}}\colon \mathcal{Q}\text{-}\mathsf{coalg}_{\mathsf{oplax}} \to [\mathcal{A}^{\mathsf{op}},\mathsf{Cat}]$ is fully faithful, and thus equivalent to the inclusion $\Theta[\mathcal{A}] \hookrightarrow [\mathcal{A}^{\mathsf{op}},\mathsf{Cat}]$.

[1]This doesn't quite follow from what's written, since we've only shown $\Phi_1\mathcal{A} = \Phi\mathcal{A}$ for small $\mathcal{A}$. However, by [AK88, Prop. 7.2] the result extends from small $\mathcal{A}$ to locally small $\mathcal{A}$ when Φ is a class of small weights.

The 2-category $\mathcal{Q}\text{-}\mathsf{coalg}_{\mathsf{oplax}}$ has objects $\mathcal{Q}$-coalgebras and morphism oplax algebra maps given by a 1-cell $f\colon X \to Y$ between the underlying presheaves and a 2-cell, as shown below:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h_X"'] & Y \arrow[d, "h_Y"] \\ \mathcal{Q}X \arrow[r, "\mathcal{Q}f"'] \arrow[ur, phantom, "\overset{\phi}{\Nearrow}"] & \mathcal{Q}Y \end{tikzcd}$$

which satisfy certain coherence conditions. It is shown in [Thm. 4.8 Lac05] that for a 2-monad $T\colon C \to C$, the forgetful 2-functor $T\text{-}\mathsf{alg}_{\mathsf{lax}} \to C$ creates oplax limits. For a comonad, $W\colon C \to C$, W^{op} is a monad on C^{op} and thus $W^{\mathsf{op}}\text{-}\mathsf{alg}_{\mathsf{lax}} \to C^{\mathsf{op}}$ creates oplax limits. But $W^{\mathsf{op}}\text{-}\mathsf{alg}_{\mathsf{lax}}$ is isomorphic to ${W\text{-}\mathsf{coalg}_{\mathsf{oplax}}}^{\mathsf{op}}$, and the forgetful 2-functor ${W\text{-}\mathsf{coalg}_{\mathsf{oplax}}}^{\mathsf{op}} \to C^{\mathsf{op}}$ creates oplax limits precisely if $W\text{-}\mathsf{coalg}_{\mathsf{oplax}} \to C$ creates oplax colimits. Thus, in particular, the forgetful map $\mathcal{Q}\text{-}\mathsf{coalg}_{\mathsf{oplax}} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ creates conical oplax colimits and by equivalence so does the inclusion $\Theta[A] \hookrightarrow [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$. ◻

Corollary 4.1.4. *The saturation of the classes of weights for oplax conical colimits and for general oplax colimits are both equal to* Θ*.*

Proof. Let Υ denote the class of weights for oplax colimits, i.e. weights of the form $\mathcal{Q}P$ for $P\colon \mathcal{A} \to \mathsf{Cat}$ an arbitrary 2-functor. We clearly have inclusions $\Xi \subseteq \Upsilon \subseteq \Theta$, from which we deduce corresponding inclusions of their saturations $\Xi^* \subseteq \Upsilon^* \subseteq \Theta^*$. By Proposition 4.1.1 we have $\Theta \subseteq \Xi^*$, and by Proposition 4.1.3 we have $\Theta^* = \Theta$, so we obtain our result by "sandwiching" Ξ^* and Υ^* on both sides: $\Theta \subseteq \Xi^* \subseteq \Upsilon^* \subseteq \Theta^* = \Theta$. ◻

4.1.1 An alternative construction of $\Theta\mathcal{K}$

By Proposition 4.1.3 the class Θ is in particular presaturated, so the free Θ-cocompletion of a locally small 2-category $\mathcal{K}$ is $\Theta_1\mathcal{K}$ — the full 2-subcategory of $[\mathcal{K}^{\mathsf{op}}, \mathsf{Cat}]$ whose objects are oplax image presheaves of 2-functors from small 2-categories. This subcategory is equivalently the image of oplax-image presheaf functor $\mathfrak{I}\colon 2\mathsf{Cat}/\mathcal{K} \to [\mathcal{K}^{\mathsf{op}}, \mathsf{Cat}]$ which is the restriction of a large 2-functor $\mathfrak{I}'\colon 2\mathsf{CAT}/\mathcal{K} \to [\mathcal{K}^{\mathsf{op}}, \mathsf{CAT}]$ whose right adjoint given by a large version of the Grothendieck construction. The b.o.o/f.f factorisation of any left adjoint functor is given by the canonical map into the Kleisli category for the induced monad, followed by the comparison functor from the Kleisli category. In particular, the b.o.o/f.f factorisation of $\mathfrak{I}'$ goes via the Kleisli 2-category for the induced monad $\mathsf{T} = \int \circ \mathfrak{I}'$ on $2\mathsf{CAT}/\mathcal{K}$:

$$2\mathsf{CAT}/\mathcal{K} \xrightarrow{\text{b.o.o}} \mathsf{Kl}_{\mathsf{T}} \xrightarrow{\text{f.f}} [\mathcal{K}^{\mathsf{op}}, \mathsf{CAT}]$$

The b.o.o/f.f factorisation of the restriction $\mathfrak{I}\colon 2\mathsf{Cat}/\mathcal{K} \to [\mathcal{K}^{\mathsf{op}}, \mathsf{Cat}]$, and thus $\Theta_1\mathcal{K}$, is therefore given up to equivalence by the full sub-2-category of Kl_{T} whose objects are 2-functors from small 2-categories. We will denote this full sub-2-category $\mathsf{F}_\Theta\mathcal{K}$. The 1-cells of $\mathsf{F}_\Theta\mathcal{K}$ between a pair of 2-functors $F\colon \mathcal{A} \to \mathcal{K}$ and $G\colon \mathcal{B} \to \mathcal{K}$ are given by Kleisli 1-cells for the monad T, which means a 2-functor $\mathcal{A} \to \mathfrak{Q}_G$ which is a map in the slice $2\mathsf{Cat}/\mathcal{K}$ from F to $\mathfrak{q}_G\colon \mathfrak{Q}_G \to \mathcal{K}$.

Given that the free cocompletion is only characterised up to equivalence by its universal property, we can justify referring to $\mathsf{F}_\Theta\mathcal{K}$ as *the* free Θ-cocompletion of $\mathcal{K}$, though we reserve the notation $\Theta\mathcal{K}$ for the full sub-2-category of $[\mathcal{K}^{\mathsf{op}}, \mathsf{Cat}]$. Of course, the free Θ-cocompletion of $\mathcal{K}$ is not merely a 2-category; one must also specify an embedding of $\mathcal{K}$. We endow $\mathsf{F}_\Theta\mathcal{K}$ with such an embedding by composing $\mathcal{Z}_{\mathcal{K}}\colon \mathcal{K} \to \Theta\mathcal{K}$ with any pseudo-inverse to the equivalence $\mathsf{F}_\Theta\mathcal{K} \to \Theta\mathcal{K}$ which sends a 2-functor to its oplax-image presheaf. A pseudo-inverse to this 2-functor amounts to a choice for each presheaf X in $\Theta\mathcal{K}$ of a 2-functor $P\colon C \to \mathcal{K}$ from a small 2-category such that $\mathfrak{I}P \cong X$. For the representable presheaves $\mathcal{K}(-, k)$ we can choose the 2-functor $\langle k \rangle : \mathbb{1} \to \mathcal{K}$ which picks out the object k, so that our cocompletion embedding $\mathsf{Z}_{\mathcal{K}}\colon \mathcal{K} \to \mathsf{F}_\Theta\mathcal{K}$ is given on objects by $k \mapsto \mathcal{A}(-, k) \mapsto \langle k \rangle$. The action on morphisms is uniquely determined by the action on objects by the following string of isomorphisms:

$$\mathcal{K}(k,l) \xrightarrow{\cong} \Theta\mathcal{K}(\mathfrak{I}'\langle k\rangle, \mathfrak{I}'\langle l\rangle) \xrightarrow{\cong} \mathsf{2CAT}/\mathcal{K}(\langle k\rangle, \mathsf{T}\langle l\rangle) \xrightarrow{\cong} \mathsf{F}_\Theta\mathcal{K}(\langle k\rangle, \langle l\rangle)$$

Remark 4.1.5. We could just as well have described the free Θ-cocompletion of $\mathcal{K}$ as Kl_T restricted to local discrete opfibrations from small 2-categories into $\mathcal{K}$, since every oplax-image presheaf of a 2-functor is also the oplax-image presheaf of a local discrete opfibration. However, the embedding $\mathsf{Z}_\mathcal{K} : \mathcal{K} \to \mathsf{F}_\Theta\mathcal{K}$ doesn't factor through this full sub-2-category because $k\colon \mathbb{1} \to \mathcal{K}$ is not in general a local discrete opfibration. Instead, an embedding could be defined by sending $k \in \mathcal{K}$ to the free local discrete opfibration induced by $k\colon \mathbb{1} \to \mathcal{K}$ whose domain is the suspension of the coslice category of $\mathcal{K}(k,k)$ at 1_k. The local discrete opfibration from this 2-category to $\mathcal{K}$ sends the unique object of the suspension to k and on hom-categories is the codomain projection $1_k\backslash\mathcal{K}(k,k) \to \mathcal{K}(k,k)$. ◊

Example 4.1.6 ($\mathsf{F}_\Theta\mathbb{1}$)**.** Recall that the monad T on $\mathsf{2CAT}/\mathcal{K}$ sends a 2-functor $F\colon C \to \mathcal{K}$ to its induced free locally discrete split 2-fibration $\mathsf{q}_F\colon \mathfrak{Q}_F \to \mathcal{K}$ as described in Definition 3.3.10. When $\mathcal{K} = \mathbb{1}$, we have $\mathsf{2CAT}/\mathcal{K} \cong \mathsf{2CAT}$, and under this equivalence the monad $\mathsf{T}\colon \mathsf{2CAT} \to \mathsf{2CAT}$ is given by sending a 2-category $\mathcal{A}$ to the locally discrete 2-category $\pi_1(\mathcal{A})$ (i.e. 1-category) with the same objects as $\mathcal{A}$ but whose 1-cells are the connected components of the hom-categories of $\mathcal{A}$. For small 2-categories $\mathcal{A}$, $\mathcal{B}$ viewed as objects of $\mathsf{F}_\Theta\mathbb{1}$, we therefore have:

$$\mathsf{F}_\Theta\mathbb{1}(\mathcal{A},\mathcal{B}) \cong \mathsf{2Cat}(\mathcal{A},\pi_1(\mathcal{B})) \cong \mathsf{Cat}(\pi_1(\mathcal{A}),\pi_1(\mathcal{B}))$$

Moreover, composition is given by the ordinary composition of the underlying 1-functors in Cat, so $\mathsf{F}_\Theta\mathbb{1}$ is equivalent to Cat.

This fact can also be verified by considering the oplax-image presheaves in $[\mathbb{1}^{\mathsf{op}},\mathsf{Cat}] \cong \mathsf{Cat}$. The isomorphism $[\mathbb{1}^{\mathsf{op}},\mathsf{Cat}] \cong \mathsf{Cat}$ identifies the oplax-image presheaf of $!\colon \mathcal{A} \to \mathbb{1}$ with the oplax colimit of $\Delta\mathbb{1}\colon X \to \mathsf{Cat}$, which is given by the copower $\mathcal{A} \odot \mathbb{1} \cong \pi_1(\mathcal{A})$. ◊

Example 4.1.7 (The Θ-cocompletion of a 1-category)**.** For a locally-small 1-category[2], K, the objects of $\mathsf{F}_\Theta K$ can be identified with a pair of a small 2-category C and a 1-functor $P\colon \pi_1(C) \to K$, since any 2-functor $P' : C \to K$ factors uniquely through the projection $\mathsf{p}_1\colon C \to \pi_1(C)$ as $P' = P\,\mathsf{p}_1$. Since K has no 2-cells, the relations on 1-cells of $\mathfrak{Q}_{P'}$ become trivial, and so $\mathsf{q}_{P'}\colon \mathfrak{Q}_{P'} \to K$ is just the ordinary lax limit of the arrow $P\colon \pi_1(C) \to K$. A morphism from $(C, P\colon \pi_1(C) \to K)$ to $(\mathcal{D}, Q\colon \pi_1(\mathcal{D}) \to K)$ can therefore be identified with a 1-functor $F\colon \pi_1(C) \to \pi_1(\mathcal{D})$ and a natural transformation $\phi\colon P \to QF$. Similarly, a 2-cell $(F,\phi) \Rightarrow (G,\psi)$ can be identified with a natural transformation $\gamma\colon F \Rightarrow G$ satisfying $Q\gamma\;\phi = \psi$. In this 2-category, every object $(C, P\colon \pi_1(C) \to K)$ is isomorphic to $(\pi_1(C), P)$ — i.e. a 1-functor — so $\mathsf{F}_\Theta K$ is equivalent to the lax slice category[3] $\mathsf{Cat} \Downarrow K$. A 1-dimensional analogue of this result is the fact that the free cocompletion of a small 0-category X (i.e. a set) under small colimits is given by the ordinary slice category $\mathsf{Set} \downarrow X$ (which is equivalent to $[X^{\mathsf{op}},\mathsf{Set}]$). ◊

Remark 4.1.8. In [PT22] the lax slice $\mathsf{Cat} \Downarrow K$ is given the name $\mathsf{Diag}(K)$. By ignoring the 2-dimensional structure of $\mathsf{Diag}(K)$, one obtains a pseudomonad Diag on CAT described earlier in [Gui73] under the name *diagramme foncteur*. The pseudoalgebras for this pseudomonad include maps $\mathsf{Diag}(K) \to K$ for a cocomplete 1-category K which send a functor to its colimit in K, though the authors of [PT22] observe that not all Diag-pseudoalgebras have this form and conjecture that Diag-pseudoalgebras are related to oplax colimits, rather than ordinary colimits. We can make some comments supporting this conjecture in light of the equivalence $\mathsf{F}_\Theta K \simeq \mathsf{Diag}K$. Any pseudoalgebra for the free cocompletion pseudomonad F_Θ (which we describe in detail later) is determined by a choice of oplax

[2] We tacitly cast 1-categories as locally discrete 2-categories according to context.

[3] If K isn't small, this is really the comma from the inclusion $\mathsf{Cat} \hookrightarrow \mathsf{CAT}$ to the object K.

colimits for a locally-small 2-category $\mathcal{K}$. Up to equivalence, then, F_Θ-pseudoalgebras are simply locally-small Θ-cocomplete 2-categories. If we let $\mathsf{d}\colon \mathsf{CAT} \to \mathsf{2CAT}$ denote the 2-functor sending a 1-category to the corresponding locally-discrete 2-category, and $\mathsf{u}\colon \mathsf{2CAT} \to \mathsf{CAT}$ denote the underlying-1-category functor[4] , then $\mathsf{Diag}\colon \mathsf{CAT} \to \mathsf{CAT}$ is equivalent to $\mathsf{u}\,\mathsf{F}_\Theta\,\mathsf{d}$. It follows that $u\colon \mathsf{2CAT} \to \mathsf{CAT}$ underlies a morphism of monads $\mathsf{F}_\Theta \to \mathsf{Diag}$, whose 2-cell component $\phi\colon \mathsf{Diag}\,\mathsf{u} \Rightarrow \mathsf{u}\,\mathsf{F}_\Theta$ is given in terms of the counit ϵ of the adjunction $\mathsf{d} \dashv \mathsf{u}$ by:

$$\mathsf{Diag}\,\mathsf{u} \xrightarrow{\simeq} \mathsf{u}\,\mathsf{F}_\Theta\,\mathsf{d}\,\mathsf{u} \xrightarrow{\mathsf{u}\,\mathsf{F}_\Theta\,\epsilon} \mathsf{u}\,\mathsf{F}_\Theta$$

So, from any Θ-cocomplete 2-category $\mathcal{K}$ we obtain a F_Θ-pseudoalgebra $h\colon \mathsf{F}_\Theta\mathcal{K} \to \mathcal{K}$ and thence a Diag-pseudoalgebra $\mathsf{Diag}\,\mathcal{K}_0 \xrightarrow{\mathsf{u}\,\mathsf{F}_\Theta\,\epsilon_{\mathcal{K}}} (\mathsf{F}_\Theta\mathcal{K})_0 \xrightarrow{(h)_0} \mathcal{K}_0$. All Diag-pseudoalgebras which compute colimits in the underlying 1-category K are produced in this way from the Θ-cocomplete 2-category $\mathsf{d}K$, as Θ-cocompleteness and small-cocompleteness coincide for 1-categories. However, if ordinary and oplax colimits don't coincide for a given Θ-cocomplete 2-category $\mathcal{K}$ then the corresponding Diag-pseudoalgebra will not compute colimits in the underlying category. In particular, the Θ-cocomplete 2-category Cat induces a Diag-pseudoalgebra $\mathsf{Diag}\,\mathsf{Cat} \to \mathsf{Cat}$ which sends a functor $P\colon C \to \mathsf{Cat}$ to its Grothendieck construction, which is the oplax colimit of P, rather than the ordinary one. This is actually the free Diag-algebra on the terminal 1-category, and in general the free Diag-algebras on a 1-category K corresponds to the Θ-cocomplete 2-category $\mathsf{F}_\Theta K$. Though whether all Diag-pseudoalgebras come from Θ-cocomplete 2-categories in this way is unclear (to me). ◊

4.1.2 The pseudomonad F_Θ

The free cocompletion under a class of small colimits defines an action on objects of $\mathsf{2CAT}$ (or more generally $\mathcal{V}$-CAT) which naturally inherits the structure of a pseudomonad. One path to deriving this structure is to first observe that the free Φ-cocompletion embeddings $\mathsf{Z}_{\mathcal{K}}\colon \mathcal{K} \to \Phi\mathcal{K}$ define a fully faithful KZ-doctrine on the 2-category $\mathsf{2CAT}$, as described in [Wal18, Defn. 2][5] which in [MW12] is shown to be equivalent to a lax-idempotent pseudomonad on $\mathsf{2CAT}$ with unit components given by the arrows of the KZ-doctrine. The lax-idempotent pseudomonad structure is essentially given by left extensions along the embeddings $\mathsf{Z}_{\mathcal{K}}\colon \mathcal{K} \to \Phi\mathcal{K}$. Below we describe the induced pseudomonad below for the class Θ and show how it arises from extensions along $\mathsf{Z}_{\mathcal{K}}\colon \mathcal{K} \to \mathsf{F}_\Theta\mathcal{K}$, appealing to the general theory of such constructions to justify the soundness of the resulting pseudomonad rather than verifying the axioms explicitly.

For a 2-functor $F\colon \mathcal{K} \to \mathcal{L}$, we obtain a 2-functor $\mathsf{F}_\Theta F\colon \mathsf{F}_\Theta\mathcal{K} \to \mathsf{F}_\Theta\mathcal{K}$ by taking a left extension of $\mathsf{Z}_{\mathcal{L}}\,F$ along $\mathsf{Z}_{\mathcal{K}}$:

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "F"] \arrow[d, "\mathsf{Z}_{\mathcal{K}}"'] \arrow[dr, phantom, "\theta_F \Downarrow"] & \mathcal{L} \arrow[d, "\mathsf{Z}_{\mathcal{L}}"] \\
\mathsf{F}_\Theta\mathcal{K} \arrow[r, dashed, "\mathsf{F}_\Theta F"'] & \mathsf{F}_\Theta\mathcal{L}
\end{tikzcd}$$

Such an extension is only determined up to isomorphism, but having chosen $\mathsf{F}_\Theta\mathcal{K}$ as a representative for the free Θ-cocompletion we will see there is a natural choice for the action on objects of $\mathsf{F}_\Theta F$. First, we note the following:

(a) $\mathsf{Z}_{\mathcal{K}}\colon \mathcal{K} \to \mathsf{F}_\Theta\mathcal{K}$ is fully faithful, which means the 2-natural transformation θ_F is invertible and so $\mathsf{F}_\Theta F\,\mathsf{Z}_{\mathcal{K}} \cong \mathsf{Z}_{\mathcal{L}}\,F$.

(b) $\mathsf{F}_\Theta F$ is Θ-cocontinuous, because left-extension along $\mathsf{Z}_{\mathcal{K}}$ is a pseudo-inverse to $\mathsf{Z}_{\mathcal{K}}^*\colon \Theta\text{-}\mathsf{coCts}(\mathsf{F}_\Theta\mathcal{K}, \mathcal{B}) \to \mathsf{2CAT}(\mathcal{K}, \mathcal{B})$.

(c) Every object $P\colon C \to \mathcal{K}$ in $\mathsf{F}_\Theta\mathcal{K}$ is the conical oplax colimit of $\mathsf{Z}_{\mathcal{K}}P : C \to \mathsf{F}_\Theta\mathcal{K}$

[4]This functor ignores the 2-cells of a 2-category to obtain a 1-category. It is right adjoint to d.
[5]An earlier definition for the dual notion of coKZ-doctrine is given in [MW12].

From these three facts we conclude:

$$\mathsf{F}_\Theta F(P) \cong \mathsf{F}_\Theta F\left(\oint^{c\in C} \mathsf{Z}_{\mathcal{K}} Pc\right) \cong \oint^{c\in C} \mathsf{F}_\Theta F\ \mathsf{Z}_{\mathcal{K}}(Pc) \cong \oint^{c\in C} \mathsf{Z}_{\mathcal{L}} FPc \cong FP$$

We can therefore choose $\mathsf{F}_\Theta F$ to act on all objects of $\mathsf{F}_\Theta \mathcal{K}$ by post-composition with F. This choice of extension make the unit θ_F of the extension an identity. The unit for the pseudomonad F_Θ is the pseudonatural transformation $1_{\mathsf{2CAT}} \to \mathsf{F}_\Theta$ whose component at $\mathcal{K}$ is $\mathsf{Z}_{\mathcal{K}}$, and whose pseudo-naturality 2-cell at $F\colon \mathcal{K} \to \mathcal{L}$ is given by θ_F. So, an advantage of our choice of extension $\mathsf{F}_\Theta F$ aside from its simplicity is that it makes the unit for the pseudomonad F_Θ strictly 2-natural, and in fact, *3-natural* given that pointwise extensions in Cat also have a lifting property with respect to modifications.

We've only chosen the action of the 2-functor $\mathsf{F}_\Theta F$ on 0-cells; the action of $\mathsf{F}_\Theta F$ on 1-cells and 2-cells is uniquely determined by the property of being a left extension once the action on objects is fixed. It can be expressed in terms of the functoriality of the weighted colimit in its first argument as:

$$\mathsf{F}_\Theta\mathcal{K}(P,Q) \cong [\mathcal{K}^{\mathsf{op}}, \mathsf{Cat}](\mathfrak{I}P, \mathfrak{I}Q) \xrightarrow{-*\mathsf{Z}_{\mathcal{L}}F} \mathsf{F}_\Theta\mathcal{L}(FP, FQ) \tag{4.1}$$

though this isn't immediately enlightening. Unfortunately an explicit description of this action is rather complicated, owing to the fact that the Kleisli 1-cells in our description of $\mathsf{F}_\Theta\mathcal{K}$ are complicated. Though it turns out this action can be described in terms of the post-composition mapping between slice categories $\mathsf{2CAT}/F\colon \mathsf{2CAT}/\mathcal{K} \to \mathsf{2CAT}/L$ by the fact that this 2-functor underlies a monad morphism between the monads $\mathsf{T}_{\mathcal{K}}$ and $\mathsf{T}_{\mathcal{L}}$ whose Kleisli categories contain $\mathsf{F}_\Theta\mathcal{K}$ and $\mathsf{F}_\Theta\mathcal{L}$ as full subcategories.

Recall that the monad $\mathsf{T}_{\mathcal{K}}$ on $\mathsf{2CAT}/\mathcal{K}$ is given by sending $Q\colon \mathcal{D} \to \mathcal{K}$ to $\mathsf{q}_Q\colon \mathfrak{Q}_Q \to \mathcal{K}$, which is the free locally discrete split 2-fibration induced by Q. The free locally discrete split 2-fibration is equivalently the result of taking the free split 2-fibration — i.e. the lax comma category of $1_{\mathcal{K}}$ and Q, denoted $\mathcal{K} \Downarrow Q$ — and then quotienting by the vertical 2-cells. The lax comma $\mathcal{K} \Downarrow Q$ is equipped with a universal *lax* transformation $\mathsf{m}_Q\colon \mathsf{s}_Q \Rightarrow Q\mathsf{t}_Q\colon \mathcal{K}\Downarrow Q \to \mathcal{K}$ such that for any other lax transformation $\beta\colon a \Rightarrow Qb\colon \mathcal{X} \to \mathcal{K}$ there exists a unique 2-functor $\overline{\beta}\colon \mathcal{X} \to \mathcal{K} \Downarrow Q$ satisfying $\beta = \mathsf{m}_Q\,\overline{\beta}$:

$$\begin{tikzcd} & \mathcal{D} \arrow[d, "Q"] \\ \mathcal{X} \arrow[ur, "b"] \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] \arrow[dr, "a"'] & \mathcal{K} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & & \mathcal{D} \arrow[d, "Q"] \\ \mathcal{X} \arrow[r, "\overline{\beta}"] & \mathcal{K}\Downarrow Q \arrow[ur, "\mathsf{t}_Q"] \arrow[r, phantom, "\overset{\mathsf{m}_Q}{\Rightarrow}"] \arrow[dr, "\mathsf{s}_Q"'] & \mathcal{K} \end{tikzcd}$$

The projection $\kappa_Q\colon \mathcal{K}\Downarrow Q \to \mathfrak{Q}_Q$ from the free split 2-fibration to the free split locally discrete 2-fibration then has a universal property with respect to 2-functors out of $\mathcal{K} \Downarrow Q$ which map q_Q-vertical 2-cells to identities.

$$\begin{tikzcd} \mathcal{K}\Downarrow Q \arrow[r, dashed, "\kappa_Q"] \arrow[d, "\mathsf{t}_Q"'] \arrow[dr, "\mathsf{s}_Q"] \arrow[d, phantom, shift left=4ex, "\Leftarrow \mathsf{m}_Q"] & \mathfrak{Q}_Q \arrow[d, "\mathsf{q}_Q"] \\ \mathcal{D} \arrow[r, "Q"'] & \mathcal{K} \end{tikzcd}$$

We can show that $\mathsf{2CAT}/F$ extends to a morphism of 2-monads by appealing to both of these universal properties.

First, we produce a 2-functor $\mathsf{j}_F(Q)\colon \mathcal{K} \Downarrow Q \to \mathcal{L} \Downarrow FQ$ from the lax transformation $F\mathsf{m}_Q\colon F\mathsf{s}_Q \to FQ\mathsf{t}_Q$ according to the universal property of $\mathcal{L} \Downarrow FQ$:

$$\begin{tikzcd} & \mathcal{D} \arrow[d, "FQ"] \\ \mathcal{K}\Downarrow Q \arrow[ur, "\mathsf{t}_Q"] \arrow[r, phantom, "\overset{F\mathsf{m}_Q}{\Longrightarrow}"] \arrow[dr, "F\mathsf{s}_Q"'] & \mathcal{L} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & & \mathcal{D} \arrow[d, "FQ"] \\ \mathcal{K}\Downarrow Q \arrow[r, "\mathsf{j}_F(Q)"] & \mathcal{K}\Downarrow FQ \arrow[ur, "\mathsf{t}_{FQ}"] \arrow[r, phantom, "\overset{\mathsf{m}_{FQ}}{\Rightarrow}"] \arrow[dr, "\mathsf{s}_{FQ}"'] & \mathcal{L} \end{tikzcd}$$

This functor satisfies $\mathsf{s}_{FQ}\ \mathsf{j}_F(Q) = F\mathsf{s}_Q$ and $\mathsf{t}_{FQ}\ \mathsf{j}_F(Q) = F\mathsf{t}_Q$, so any s_Q-vertical 2-cell in $\mathcal{K}$ is mapped by $\mathsf{j}_F(Q)$ to a s_{FQ}-vertical morphism in $\mathcal{L}$, which is then mapped by κ_{FQ} to an identity. The 2-functor $\kappa_{FQ}\ \mathsf{j}_F(Q)\colon \mathcal{K} \Downarrow Q$ therefore factorises as $\mathsf{J}_F(Q)\ \kappa_Q$ for some 2-functor $\mathsf{J}_F(Q)\colon \mathfrak{Q}_Q \to \mathfrak{Q}_{FQ}$, which is moreover a map in $\mathsf{2CAT}$ from $F\mathfrak{q}_Q$ to $\mathfrak{q}_{FQ}$:

$$\begin{tikzcd}
\mathcal{K} \Downarrow Q \arrow[r, "\mathsf{j}_F(Q)"] \arrow[d, "\kappa_Q"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{L} \Downarrow FQ \arrow[d, "\kappa_{FQ}"] \\
\mathfrak{Q}_Q \arrow[r, "\mathsf{J}_F(Q)"'] & \mathfrak{Q}_{FQ}
\end{tikzcd}$$

Explicitly, $\mathsf{J}_F(Q)$ acts on $\mathfrak{Q}_Q$ by the below mapping of (representatives of) 1-cells. The mapping of 2-cells is uniquely determined so that this becomes a map of split locally discrete 2-fibrations.

$$\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, "s"'] & Qx \arrow[d, "Qf"] \arrow[dl, phantom, "\Leftarrow\sigma"] \\
b \arrow[r, "v"'] & Qy
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
Fa \arrow[r, "Fu"] \arrow[d, "Fs"'] & FQx \arrow[d, "FQf"] \arrow[dl, phantom, "\Leftarrow F\sigma"] \\
Fb \arrow[r, "Fv"'] & FQy
\end{tikzcd} \tag{4.2}$$

So for each $Q\colon \mathcal{D} \to \mathcal{K}$ there is a map $\mathsf{J}_F(Q)$ from $F\ \mathfrak{q}_Q = (\mathsf{2CAT}/F\ \mathsf{T}_{\mathcal{K}})\,Q$ to $\mathfrak{q}_{FQ} = (\mathsf{T}_{\mathcal{L}}\ \mathsf{2CAT}/F)\,Q$. These are indeed the components of a 2-natural transformation $\mathsf{2CAT}/F\ \mathsf{T}_{\mathcal{K}} \Rightarrow \mathsf{T}_{\mathcal{L}}\ \mathsf{2CAT}/F$; the naturality follows from the universal properties of both the lax comma categories and the associated locally discrete split 2-fibrations. For example, for $f\colon (C \xrightarrow{P} \mathcal{K}) \to (D \xrightarrow{Q} \mathcal{K})$, the maps $\mathsf{j}_Q(F) \circ \mathcal{K} \Downarrow f$ and $\mathcal{L} \Downarrow f \circ \mathsf{j}_P(F)$ are equal by the universal property of $\mathcal{L} \Downarrow FQ$, and then the lifts to $\mathsf{J}_Q(F) \circ \mathfrak{Q}_{\mathcal{K}} f$ and $\mathfrak{Q}_{\mathcal{L}} f \circ \mathsf{J}_P(F)$ are equal by the universal property of $\mathfrak{Q}_P$. The multiplication and unit laws for this 2-natural transformation are easily observed from the description in (4.2) to correspond to the commutativity of diagrams of the forms below (shown only for objects):

$$\begin{tikzcd}
\left(a \xrightarrow{u} b \xrightarrow{v} Qx\right) \arrow[r, mapsto, "\mu_{\mathcal{K}}"] \arrow[d, mapsto, "\mathsf{2CAT}/F"'] \arrow[dr, phantom, "\circlearrowleft"] & \left(a \xrightarrow{vu} Qx\right) \arrow[d, mapsto, "\mathsf{2CAT}/F"] \\
\left(Fa \xrightarrow{Fu} Fb \xrightarrow{Fv} FQx\right) \arrow[r, mapsto, "\mu_{\mathcal{L}}"'] & \left(Fa \xrightarrow{F(vu)} FQx\right)
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, mapsto, "\eta_{\mathcal{K}}"] \arrow[d, mapsto, "\mathsf{2CAT}/F"'] \arrow[dr, phantom, "\circlearrowleft"] & (Qx = Qx) \arrow[d, mapsto, "\mathsf{2CAT}/F"] \\
Fx \arrow[r, mapsto, "\eta_{\mathcal{L}}"'] & (FQx = FQx)
\end{tikzcd}$$

The induced action on Kleisli arrows then maps a 1-cell $f\colon P \to \mathfrak{q}_Q$ in $\mathsf{F}_\Theta\mathcal{K}$ to:

$$FP \xrightarrow{\mathsf{2CAT}/F\ f} F\mathfrak{q}_Q \xrightarrow{\mathsf{J}_F(Q)} \mathfrak{q}_{FQ} \tag{4.3}$$

Because $\mathsf{2CAT}/F$ sends a 1-cell $f\colon (C,P) \to (\mathcal{D},Q)$ — defined by functor $f\colon C \to \mathcal{D}$ — to the 1-cell $f\colon (C,FP) \to (\mathcal{D},FQ)$ defined by that same functor, the action of $\mathsf{2CAT}/F$ on Kleisli arrows is essentially just post-composition of the underlying functor by $\mathsf{J}_F(Q)$. Similarly, a 2-cell $\alpha\colon f \Rightarrow g\colon P \to \mathfrak{q}_Q$ is sent to $\mathsf{J}_F(Q)\ \alpha$. This action on the hom-categories of $\mathsf{Kl}_{\mathsf{T}_{\mathcal{K}}}$ when restricted to the full subcategories $\mathsf{F}_\Theta\mathcal{K}$ and $\mathsf{F}_\Theta\mathcal{L}$ then gives the action of $\mathsf{F}_\Theta F$ on hom-categories. This can be verified by observing that does indeed correspond by the universal property of the extensions as in (4.1). We can express the universal construction of $\mathsf{F}_\Theta F(f)$ for f a 1-cell $(C,P) \to (\mathcal{D},Q)$ as follows. For $c \in C$, let $\vec{f}_c\colon Pc \to Q\hat{f}_c$ denote the image of c under the functor $f\colon C \to \mathfrak{Q}_Q$ defining the 1-cell f in $\mathsf{F}_\Theta\mathcal{K}$, and notice that maps of the form $\langle k\rangle \to (C,P)$ for $k \in \mathcal{K}$ can be identified with a choice of $c \in C$ and a morphism $\phi\colon k \to Pc$ in $\mathcal{K}$. For any $k \in \mathcal{K}$, $(C,P) \in \mathsf{F}_\Theta\mathcal{K}$, and some map $(c,\phi)\colon \langle k\rangle \to (C,P)$, there is a canonically induced map $\left(\hat{f}_c, F\vec{f}_c\ F\phi\right)\colon \langle Fk\rangle \to (\mathcal{D},Q)$ associated to the object $\hat{f}_c \in \mathcal{D}$ and morphism $F\vec{f}_c\ F\phi\colon Fk \to FQ\hat{f}_c$ in $\mathcal{L}$. The map $\mathsf{F}_\Theta F(f)\colon (C,FP) \to (\mathcal{D},FQ)$ is then characterised as the unique map such that $\left(\hat{f}_c, F\vec{f}_c\ F\phi\right)$ is the composite:

$$\langle Fk\rangle \xrightarrow{(c,F\phi)} (C,FP) \xrightarrow{\mathsf{F}_\Theta F(f)} (\mathcal{D},FQ)$$

for all such 1-cells (c, ϕ), as well as a similar factorisation result for 2-cells $(c, \phi) \Rightarrow (c', \phi') \colon \langle k \rangle \to (C, P)$. These two properties are easily verified for the action described in (4.3) and (4.2).

Similar arguments show that for $\Gamma \colon F \Rightarrow G \colon \mathcal{K} \to \mathcal{L}$ a 2-natural transformation we obtain a 2-natural transformation $\mathsf{F}_\Theta \Gamma \colon \mathsf{F}_\Theta F \Rightarrow \mathsf{F}_\Theta G$ whose component at $P \colon C \to \mathcal{K}$ is the 2-functor $\overline{\Gamma}_P \colon C \to \mathfrak{Q}_{GP}$ given by composing the canonical map $C \to \mathcal{L} \Downarrow GP$ induced by ΓP with the quotient $\mathcal{L} \Downarrow GP \to \mathfrak{Q}_{GP}$. Explicitly, this 2-functor maps object $c \in C$ to $(c, \Gamma P_c \colon FPc \to GPc) \in \mathfrak{Q}_{GP}$ and acts on 1-cells as follows:

$$\begin{tikzcd} c \arrow[d, "u"'] & & FPc \arrow[r, "\Gamma P_c"] \arrow[d, "FPu"'] \arrow[dr, phantom, "\circlearrowleft"] & GPc \arrow[d, "GPu"] \\ c' & \mapsto & FPc' \arrow[r, "\Gamma P_{c'}"'] & GPc' \end{tikzcd} \tag{4.4}$$

We also obtain for any modification $\mathfrak{M} \colon \Gamma \Rightarrow \Delta \colon F \Rightarrow G \colon \mathcal{K} \to \mathcal{L}$ the modification $\mathsf{F}_\Theta \mathfrak{M} \colon \mathsf{F}_\Theta \Gamma \Rightarrow \mathsf{F}_\Theta \Delta$ whose component at $P \colon C \to \mathcal{K}$ is the 2-natural transformation $\overline{\mathfrak{M}}_P \colon \overline{\Gamma}_P \Rightarrow \overline{\Delta}_P \colon C \to \mathfrak{Q}_Q$ whose component at $c \in C$ is given by the following 1-cell in $\mathfrak{Q}_Q$:

$$\begin{tikzcd} FPc \arrow[r, "\Gamma P_c"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathfrak{M}_{Pc}"] & GPc \arrow[d, equal] \\ FPc \arrow[r, "\Delta P_c"'] & GPc \end{tikzcd}$$

This concludes the description of the underlying endo-3-functor of the free-cocompletion F_Θ, though we will largely ignore the 3-categorical structure for now, as the underlying 2-functor is all one obtains from free cocompletions for general $\mathcal{V}$. Free-cocompletions are generally only pseudo-functorial, which is appropriate given the free-cocompletion of a $\mathcal{V}$-category is only defined up to equivalence. However, by choosing $\mathsf{F}_\Theta \mathcal{K}$ as our representative for the Θ-cocompletion and postcomposition by F as the action on objects of the left-extension of $\mathsf{Z}_\mathcal{L}\, F$ along $\mathsf{Z}_\mathcal{K}$, we've managed to make our cocompletion pseudofunctor strictly 2-functorial (and moreover 3-functorial). Notice that $\mathsf{j}_G(Q)\mathsf{j}_F(Q) = \mathsf{j}_{GF}(Q)$ for suitable 2-functors F and G, so F_Θ is 1-functorial. Similarly, the universal construction of the 2-functor in (4.4) ensures that $\mathsf{F}_\Theta(G * \Gamma) = \mathsf{F}_\Theta G * \mathsf{F}_\Theta \Gamma$ and $\mathsf{F}_\Theta(\Gamma * F) = \mathsf{F}_\Theta \Gamma * \mathsf{F}_\Theta F$ for appropriate values of F, G and Γ. And, as we've already observed, the *a priori* pseudonatural unit for F_Θ is in fact strictly 3-natural. Though we will not explicitly describe the multiplication for F_Θ, we observe that it is strictly 3-natural as well. The components $\mu_\mathcal{K} \colon \mathsf{F}^2_\Theta \mathcal{K} \to \mathsf{F}_\Theta \mathcal{K}$ of the multiplication are given by taking left extensions of the identity $\mathsf{F}_\Theta \mathcal{K} = \mathsf{F}_\Theta \mathcal{K}$ along $\mathsf{Z}_\mathcal{K} \colon \mathsf{F}_\Theta \mathcal{K} \to \mathsf{F}^2_\Theta \mathcal{K}$. The pseudo-naturality 2-cell μ_F is then the unique 2-cell shown on the right such that the two pastings below are equal, from which it follows that μ_F is an identity, and thus μ is strictly 2-natural (and 3-natural).

$$\begin{tikzcd} \mathsf{F}_\Theta \mathcal{K} \arrow[r, equal] \arrow[d, "\mathsf{Z}_\mathcal{K}"'] \arrow[dr, "\mathsf{F}_\Theta F"] & \mathsf{F}_\Theta \mathcal{K} \arrow[dr, "\mathsf{F}_\Theta F"] \arrow[d, phantom, "\circlearrowleft"] & \\ \mathsf{F}^2_\Theta \mathcal{K} \arrow[r, phantom, "\circlearrowleft"] \arrow[dr, "\mathsf{F}^2_\Theta F"'] & \mathsf{F}_\Theta \mathcal{L} \arrow[r, equal] \arrow[d, "\mathsf{Z}_\mathcal{L}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathsf{F}_\Theta \mathcal{L} \\ & \mathsf{F}^2_\Theta \mathcal{L} \arrow[ur, "\mu_\mathcal{L}"'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathsf{F}_\Theta \mathcal{K} \arrow[r, equal] \arrow[d, "\mathsf{Z}_\mathcal{K}"'] & \mathsf{F}_\Theta \mathcal{K} \arrow[dr, "\mathsf{F}_\Theta F"] & \\ \mathsf{F}^2_\Theta \mathcal{K} \arrow[ur, "\mu_\mathcal{K}"'] \arrow[dr, "\mathsf{F}^2_\Theta F"'] \arrow[rr, phantom, "\Downarrow \mu_F"] & & \mathsf{F}_\Theta \mathcal{L} \\ & \mathsf{F}^2_\Theta \mathcal{L} \arrow[ur, "\mu_\mathcal{L}"'] & \end{tikzcd}$$

Example 4.1.9 ($\varpi \colon \mathsf{F}_\Theta \mathcal{K} \to \mathsf{Cat}$)**.** For any 2-category $\mathcal{K}$ there is a unique 2-functor $! \colon \mathcal{K} \to \mathbb{1}$, and thus a 2-functor $\mathsf{F}_\Theta ! \colon \mathsf{F}_\Theta \mathcal{K} \to_{\simeq} \mathsf{F}_\Theta \mathbb{1}$ which when composed with the equivalence $\mathsf{F}_\Theta \mathbb{1} \simeq \mathsf{Cat}$ gives a 2-functor $\varpi \colon \mathsf{F}_\Theta \mathcal{K} \to \mathsf{Cat}$. The action of ϖ on objects sends a 2-functor $P \colon C \to \mathcal{K}$ to $\pi_1(C)$. A 1-cell $P \to Q$, given by a 2-functor $f \colon C \to \mathfrak{Q}_Q$, is sent by $\mathsf{F}_\Theta !$ to its postcomposition by $\mathsf{J}_!(Q) \colon \mathfrak{Q}_Q \to \mathfrak{Q}_! \cong \pi_1(\mathcal{D})$, where $\mathcal{D}$ is the domain of Q. Under the equivalence $\mathsf{F}_\Theta \mathbb{1} \simeq \mathsf{Cat}$, this composite is mapped to the 1-functor $f_1 \colon \pi_1(C) \to \pi_1(\mathcal{D})$ induced by $\mathsf{F}_\Theta ! f \colon C \to \pi_1(\mathcal{D})$. $\Diamond$

This description of the Θ-cocompletion of a 2-category provides a satisfying description of the objects, such that the "freely added" conical oplax colimit of $P \colon C \to \mathcal{K}$ is represented by P itself and thus the Θ-cocontinuous extensions of maps $F : \mathcal{K} \to \mathcal{L}$ are given by $P \mapsto \oint^{C} FP$. However, the morphisms are quite complicated; a map in $2\mathsf{CAT}/\mathcal{K}$ from P to $\mathfrak{q}_Q$ is not obviously more pleasant to work with than a 2-natural transformation between the corresponding presheaves on $\mathcal{K}$, to say nothing of how composition is defined. If our Kleisli maps in $\mathsf{F}_\Theta\mathcal{K}$ were instead morphisms in $2\mathsf{CAT}/\mathcal{K}$ from $P\colon C \to \mathcal{K}$ to $\mathsf{s}_Q\colon \mathcal{K} \Downarrow Q \to \mathcal{K}$, rather than $\mathfrak{q}_Q$, we could identify the 1-cells with a pair of a 2-functor $f\colon C \to \mathcal{D}$ and a lax transformation $P \to Q\,f$ by the universal property of $\mathcal{K} \Downarrow Q$. As it turns out, we can achieve a similar description of the 1-cells of $\mathsf{F}_\Theta\mathcal{K}$ by restricting to a subclass of its objects. For example, if C is locally discrete (i.e. a 1-category) then for any $P\colon C \to \mathcal{K}$, $\mathcal{K} \Downarrow P$ and $\mathfrak{Q}_P$ coincide. It follows that the full sub-2-category of $\mathsf{F}_\Theta\mathcal{K}$ whose objects are functors from 1-categories is equivalent to the lax slice 2-category $\mathsf{Cat} \Downarrow \mathcal{K}$.

In the next section we show that something similar happens when taking the full sub-2-category of $\mathsf{F}_\Theta\mathcal{K}$ whose objects are 2-functors from a 2-category which has initial objects in each connected component of each hom-category. This subcategory contains the subcategory of functors from 1-categories but is not equivalent to it. In later sections we will show that this larger full subcategory of $\mathsf{F}_\Theta\mathcal{K}$ forms another free cocompletion of $\mathcal{K}$ for a different class of weights, which is not true of the class of functors from 1-categories in $\mathsf{F}_\Theta\mathcal{K}$.

4.2 Ω Colimits

Recall that 1-cells in $\mathfrak{Q}_Q$ are represented by certain squares in $\mathcal{K}$, and that the following representations are considered equivalent:

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[d, "s"'] & Px \arrow[d, "Pf"] \arrow[dl, phantom, "\xLeftarrow{\sigma}"] \\ b \arrow[r, "v"'] & Py \end{tikzcd} \quad\sim\quad \begin{tikzcd} a \arrow[r, "u"] \arrow[d, "s"'] & Px \arrow[d, "Pf"] \arrow[dl, phantom, "\xLeftarrow{\sigma}"] \arrow[d, bend left=80, "Pg"] \arrow[d, phantom, shift left=4ex, "\xLeftarrow{P\alpha}"] \\ b \arrow[r, "v"'] & Py \end{tikzcd} \tag{4.5}$$

If C is locally discrete the equivalence relations on 1-cells become trivial, so that each 1-cell of $\mathfrak{Q}_P$ has a unique representative and is isomorphic to $\mathcal{K} \Downarrow P$. A weaker condition is that C merely has initial objects in each connected component of each hom-category. For such a 2-category C we can still obtain unique representatives for 1-cells by requiring that the morphism $f\colon x \to y$ in the representation on the left of (4.5) be initial in its connected component — henceforth "component-initial". This property of C, as it turns out, is enough for 1-cells into $\mathfrak{Q}_P$ to classify certain lax transformations. Moreover, any 2-functor $P\colon C \to \mathcal{K}$ from such a C is determined up to isomorphism by an oplax functor $P'\colon \pi_1(C) \to \mathcal{K}$, as we now explain.

For a 1-category C with component-initial objects in every connected component — henceforth "enough component-initial objects" — there exists a left-adjoint section $\lambda_C\colon \pi_0(C) \to C$ to the projection from the category onto its set of connected components $\mathsf{p}_0\colon C \to \pi_0(C)$. Because $\pi_0\colon \mathsf{Cat} \to \mathsf{Set}$ is strong-monoidal (the connected components of a product are the products of the connected components) it induces the functor $\pi_1\colon 2\mathsf{Cat} \to \mathsf{Cat}$ which acts on 2-categories by preserving the underlying set of objects and mapping hom-categories by π_0. If every hom-category of a 2-category C has enough component-initial objects — henceforth "enough component-initial 1-cells" — then there exists a section $\lambda_C\colon \pi_1(C) \to C$ to the bijective-on-objects projection $\mathsf{p}_1\colon C \to \pi_1(C)$ given by the sections on the hom-categories. Composition in C only weakly preserves component-initial objects, in the sense that for component-initial 1-cells $f\colon a \to b$ and $g\colon b \to c$ there is a component-initial object $h\colon a \to c$ and a unique 2-cell $h \Rightarrow g\,f$, as well as a unique 2-cell from a component-initial 1-cell in $C(a,a)$ to 1_a. The map $\lambda_C\colon \pi_1(C) \to C$ is therefore merely *oplax* functorial. In fact, it is a left adjoint section to p_1 in the 2-category of 2-categories, oplax functors and icons. In particular, it is bijective on objects and locally-initial, and thus so is the induced strict 2-functor from the oplax functor classifier, $\lambda'_C\colon \pi_1(C)^\dagger \to C$ (cf. 2.2.4).

Lemma 4.2.1. *If a 2-category C has enough component-initial 1-cells, then any 2-functor $P\colon C \to \mathcal{K}$ is isomorphic to $P\,\lambda'_C\colon \pi_1(C)^\dagger \to \mathcal{K}$ in $\mathsf{F}_\Theta\mathcal{K}$.*

Proof. From Proposition 3.4.12 we know that for $P\colon C \to \mathcal{K}$ a 2-functor and $i\colon \mathcal{D} \to C$ locally-initial and bijective on objects, the oplax image presheaves of P and Pi are isomorphic, and thus P and Pi are isomorphic as objects in $\mathsf{F}_\Theta\mathcal{K}$. From our discussion above, any 2-category C with enough component-initial 1-cells admits a locally initial b.o.o 2-functor $\lambda'_C\colon \pi_1(C)^\dagger \to C$. □

It follows that the full subcategory $\mathsf{F}_\Omega\mathcal{K}$ of $\mathsf{F}_\Theta\mathcal{K}$ of 2-functors indexed by oplax-functor classifiers of 1-categories is equivalent to the full subcategory of 2-functors indexed by 2-categories with enough component-initial 1-cells. However, this is still not a *replete* subcategory — there exist 2-functors $F\colon \mathcal{A} \to \mathcal{K}$ where $\mathcal{A}$ does not have component-initial 1-cells which are nevertheless isomorphic in $\mathsf{F}_\Theta\mathcal{K}$ to a 2-functor of the form $G\colon C^\dagger \to \mathcal{K}$. For example, the terminal 2-category $\mathbb{1}$ has component-initial 1-cells so any $\langle k\rangle : \mathbb{1} \to \mathcal{K}$ is in the full sub-2-category described above; and so is its precomposition by the unique 2-functor from the suspension of the strict-monoidal poset $\mathbb{Z}$ which is both bijective on objects and locally initial, yet the suspension of $\mathbb{Z}$ does not have component-initial 1-cells.

More generally, precomposing a 2-functor by a bijective-on-objects locally-initial 2-functor produces a new 2-functor which is isomorphic in $\mathsf{F}_\Theta\mathcal{K}$, and there exist b.o.o locally-initial 2-functors whose codomain has component-initial 1-cells but whose domain does not. However, if the domain of a b.o.o locally-initial 2-functor has component-initial 1-cells then the codomain must also. This is because having component-initial objects is equivalent to admitting an initial functor from a discrete category, and initial functors compose. So, if some 2-functor $F\colon \mathcal{A} \to \mathcal{K}$ is isomorphic in $\mathsf{F}_\Theta\mathcal{K}$ to a 2-functor $G\colon C^\dagger \to \mathcal{K}$, then the free local discrete opfibration induced by F is isomorphic to the one induced by G, which necessarily has component-initial 1-cells. So F being isomorphic to a $C^\dagger \to \mathcal{K}$ functor is equivalent to the local discrete opfibration of F having component-initial 1-cells.

Corollary 4.2.2. *The repletion of $\mathsf{F}_\Omega\mathcal{K} \subseteq \mathsf{F}_\Theta\mathcal{K}$ is the full subcategory of $\mathsf{F}_\Theta\mathcal{K}$ on 2-functors $F\colon C \to \mathcal{K}$ for which $\mathfrak{Q}_F$ has enough component-initial 1-cells. This class includes, in particular, all 2-functors from 2-categories with component initial 1-cells.*

Remark 4.2.3. If for a 2-category C with enough component-initial 1-cells we additionally require that the component-initial objects in the hom-categories of C are preserved by composition up to isomorphism, then $\lambda_C\colon \pi_1(C) \to C$ will be a pseudo functor. Any 2-functor to $\mathcal{K}$ from a 2-category C with this property is therefore isomorphic in $\mathsf{F}_\Theta\mathcal{K}$ to a 2-functor from the *pseudo* functor classifier $P\colon \pi_1(C)^\natural \to \mathcal{K}$, and thus determined by a pseudofunctor $P'\colon \pi_1(C) \to \mathcal{K}$. ◊

Let Ω denote the class of weights $@\Delta_{\mathbb{1}}\colon C^\dagger \to \mathsf{Cat}$, where C is a 1-category. We can think of this class as the weights for conical oplax colimits of oplax functors from 1-categories. Given $P\colon C^\dagger \to \mathcal{K}$, the corresponding oplax colimit of "representables" $\oint^C Z_{\mathcal{K}}P$ is just P, so $\Omega_1\mathcal{K}$ is equivalent to the full sub-2-category of $\mathsf{F}_\Omega\mathcal{K} \subseteq \mathsf{F}_\Theta\mathcal{K}$ whose objects are 2-functors out of the oplax-functor classifier of a 1-category. Objects of $\mathsf{F}_\Omega\mathcal{K}$ can be identified with an oplax functor from a 1-category $C \to \mathcal{K}$ and this is how we shall refer to them. Thus, an object of $\mathsf{F}_\Omega\mathcal{K}$ is explicitly a pair of a 1-category C and an oplax functor $P\colon C \to \mathcal{K}$. When we wish to refer to the corresponding strict 2-functor we will denote it by $P'\colon C^\dagger \to \mathcal{K}$.

For $P\colon C \to \mathcal{K}$ oplax, a 1-cell in $\mathfrak{Q}_{P'}$ from $u\colon k \to Pc$ to $u'\colon k' \to Pc'$ can be described as data $s\colon k \to k'$, $f\colon c \to c' \in C$, $\phi\colon Pf\, u \to u'\, s$ without the need for any equivalence relation by the fact that the 1-cells of $C^\dagger$ which are "in C" (i.e. *atomic*) are component-initial. Composition in $\mathfrak{Q}_{P'}$ is given as follows:

$$
\begin{tikzcd}
k' \arrow[r, "t"] \arrow[d, "u'"'] & k'' \arrow[d, "u''"] \\
Pc' \arrow[r, "Pg"'] \arrow[ur, phantom, "\psi\Uparrow"] & Pc''
\end{tikzcd}
\circ
\begin{tikzcd}
k \arrow[r, "s"] \arrow[d, "u"'] & k' \arrow[d, "u'"] \\
Pc \arrow[r, "Pf"'] \arrow[ur, phantom, "\phi\Uparrow"] & Pc'
\end{tikzcd}
=
\begin{tikzcd}
k \arrow[r, "s"] \arrow[d, "u"'] & k' \arrow[r, "t"] \arrow[d, dashed, "u'"] & k'' \arrow[d, "u''"] \\
Pc \arrow[r, dashed, "Pf"] \arrow[ur, phantom, "\phi\Uparrow"] \arrow[rr, bend right, "P(gf)"'] & Pc' \arrow[r, dashed, "Pg"] \arrow[ur, phantom, "\psi\Uparrow"] \arrow[d, phantom, "P_2\Uparrow"] & Pc'' \\
& {} &
\end{tikzcd}
$$

Note that there is a 2-functor from $\mathfrak{Q}_{P'}$ to C sending $u\colon k \to Pc$ to $c \in C$. In fact, this is just the 2-functor $\mathsf{J}_!(P')$ from $\mathfrak{Q}_{P'}$ to $\mathfrak{Q}_{!P'} \cong \pi_1(C^\dagger) \cong C$. There is moreover a universal lax transformation $\tau_P\colon \mathfrak{q}_{P'} \Rightarrow P\,\mathsf{J}_!(P')\colon \mathfrak{Q}_{P'} \to \mathcal{K}$ whose component at the object $u\colon k \to Pc \in \mathfrak{Q}_{P'}$ is the morphism $u\colon k \to Pc$. For any other 2-category $\mathcal{A}$ with 2-functors $F\colon \mathcal{A} \to C$, $S\colon \mathcal{A} \to \mathcal{K}$ and a lax transformation $\gamma\colon S \Rightarrow PF$ there exists a unique factorisation of this data through the universal such data from $\mathfrak{Q}_{P'}$ via a 2-functor $\overline{\gamma}\colon \mathcal{A} \to \mathfrak{Q}_{P'}$ which maps $a \in \mathcal{A}$ to $\gamma_a\colon Sa \to PFa$ and maps $u\colon a \to b$ to:

$$
\begin{tikzcd}
Sa \arrow[r, "\gamma_a"] \arrow[d, "Su"'] & PFa \arrow[d, "PFu"] \\
Sb \arrow[r, "\gamma_b"'] \arrow[ur, phantom, "\overset{\Longleftarrow}{\gamma_u}"] & PFb
\end{tikzcd}
$$

where γ_u is the oplax-naturality 2-cell for $\gamma\colon S \Rightarrow PF$ corresponding to u in $\mathcal{A}$.

Remark 4.2.4. The existence of this universal lax transformation from $\mathfrak{Q}_{P'}$ seems to identify $\mathfrak{Q}_{P'}$ as the lax-comma construction $\mathcal{K} \Downarrow P$, though the fact that P is oplax makes such a notion ill-defined. In particular, any attempt to extend the universality of $\mathfrak{Q}_{P'}$ from a bijection of sets to an isomorphism on hom-categories must contend with the impossibility of whiskering 2-natural transformations with oplax functors. This fact manifests in the slightly unusual description of the 2-cells of $\mathsf{F}_\Omega\mathcal{K}$, which involve comodules and modulations where we might expect lax transformations and modifications. ◊

A 1-cell between (the strict 2-functors which classify) oplax functors $P\colon C \to \mathcal{K}$ and $Q\colon D \to \mathcal{K}$ in $\mathsf{F}_\Omega\mathcal{K}$ is a 2-functor $f\colon C^\dagger \to \mathfrak{Q}_{Q'}$ satisfying $\mathfrak{q}_{Q'}\, f = P'$, which by the (1-dimensional) universal property of $\mathfrak{Q}_{Q'}$ is equivalent to a 2-functor $F'\colon C^\dagger \to D$ and a lax transformation $\phi'\colon P' \Rightarrow QF'$. This in turn is equivalent to giving a 1-functor $F\colon C \to D$ and a lax transformation $\phi\colon P \Rightarrow QF$, which will be our preferred description for the 1-cells in $\mathsf{F}_\Omega\mathcal{K}$.

The composition of such 1-cells in $\mathsf{F}_\Omega\mathcal{K}$ is given by the obvious pasting. To show this, we translate from the "lax triangle" representation of 1-cells to the Kleisli representation, perform composition of Kleisli arrows, then translate back to a lax triangle. The Kleisli arrow $C^\dagger \to \mathfrak{Q}_{Q'}$ corresponding to a lax triangle $F\colon C \to D$, $\phi\colon P \to QF$ will be denoted $\overline{\phi}$. Now, assume $P\colon C \to \mathcal{K}$, $Q\colon D \to \mathcal{K}$, $R\colon E \to \mathcal{K}$ are oplax functors from 1-categories with 1-cells between them in $\mathsf{F}_\Omega\mathcal{K}$ as shown below:

$$
\begin{tikzcd}
C \arrow[r, "F"] \arrow[dr, "P"'] & D \arrow[r, "G"] \arrow[d, "Q"] & E \arrow[dl, "R"] \\
& \mathcal{K} &
\end{tikzcd}
\qquad \overset{\phi}{\Rightarrow}\ \overset{\psi}{\Rightarrow}
$$

The corresponding Kleisli arrows $\overline{\phi}\colon C^\dagger \to \mathfrak{Q}_{Q'}$ and $\overline{\psi}\colon D^\dagger \to \mathfrak{Q}_{R'}$ act on 0-cells and 1-cells as:

$$
\begin{tikzcd}
c \arrow[d, "u"'] \\
c'
\end{tikzcd}
\overset{\overline{\phi}}{\mapsto}
\begin{tikzcd}
Pc \arrow[r, "\phi_c"] \arrow[d, "Pu"'] & QFc \arrow[d, "QFu"] \\
Pc' \arrow[r, "\phi_{c'}"'] \arrow[ur, phantom, "\overset{\Longleftarrow}{\phi_u}"] & QFc'
\end{tikzcd}
\qquad
\begin{tikzcd}
d \arrow[d, "u"'] \\
d'
\end{tikzcd}
\overset{\overline{\psi}}{\mapsto}
\begin{tikzcd}
Qd \arrow[r, "\psi_d"] \arrow[d, "Qu"'] & RGd \arrow[d, "RGu"] \\
Qd' \arrow[r, "\psi_{d'}"'] \arrow[ur, phantom, "\overset{\Longleftarrow}{\psi_u}"] & RGd'
\end{tikzcd}
\tag{4.6}
$$

The composition of these Kleisli arrows — denoted $(-\star-)$ — is given by $\overline{\psi}\star\overline{\phi} = \mu_{R'}\ T\overline{\psi}\ \overline{\phi}$. The map $T\overline{\psi}\colon \mathfrak{Q}_{Q'} \to \mathfrak{Q}_{\mathfrak{q}_{R'}}$ acts on 0-cells and 1-cells of $\mathfrak{Q}_{Q'}$ as:

$$
\begin{tikzcd}
k \arrow[r, "u"] \arrow[d, "s"'] \arrow[dr, phantom, "\overset{\sigma}{\Leftarrow}"] & Qd \arrow[d, "Qf"] \\
k' \arrow[r, "u'"'] & Qd'
\end{tikzcd}
\quad\mapsto\quad
\begin{tikzcd}
k \arrow[r, "u"] \arrow[d, "s"'] \arrow[dr, phantom, "\overset{\sigma}{\Leftarrow}"] & Qd \arrow[r, "\psi_d"] \arrow[d, dashed, "Qf"] \arrow[dr, phantom, "\overset{\psi_f}{\Leftarrow}"] & RGd \arrow[d, dashed, "RGf"] \\
k' \arrow[r, "u'"'] & Qd' \arrow[r, "\psi_{d'}"'] & RGd'
\end{tikzcd}
$$

The post-composition of this with the multiplication for the monad, $\mu_{R'}$, gives the same map with the 1-cell on the right viewed as a 1-cell in $\mathfrak{Q}_{R'}$ by pasting together the two squares. Thus, the Kleisli composite acts on a 1-cell in $C^\dagger$ as follows:

$$
\begin{tikzcd}
c \arrow[d, "u"'] \\
c'
\end{tikzcd}
\quad\mapsto\quad
\begin{tikzcd}
Pc \arrow[r, "\phi_c"] \arrow[d, "Pu"'] \arrow[dr, phantom, "\overset{\phi_u}{\Leftarrow}"] & QFc \arrow[d, "QFu"] \\
Pc' \arrow[r, "\phi_{c'}"'] & QFc'
\end{tikzcd}
\quad\mapsto\quad
\begin{tikzcd}
Pc \arrow[r, "\phi_c"] \arrow[d, "Pu"'] \arrow[dr, phantom, "\overset{\phi_u}{\Leftarrow}"] & QFc \arrow[r, "\psi F_c"] \arrow[d, dashed, "QFu"] \arrow[dr, phantom, "\overset{\psi F_u}{\Leftarrow}"] & RGFc \arrow[d, dashed, "RGFu"] \\
Pc' \arrow[r, "\phi_{c'}"'] & QFc' \arrow[r, "\psi F_{c'}"'] & RGFc'
\end{tikzcd}
$$

Translating back to the lax triangle representation for 1-cells gives the pasting of the original 1-cells:

$$
\begin{tikzcd}
C \arrow[rr, "GF"] \arrow[dr, "P"'] \arrow[rr, phantom, shift right=3, "\overset{\psi F\ \phi}{\Longrightarrow}"] & & E \arrow[dl, "R"] \\
& \mathcal{K} &
\end{tikzcd}
$$

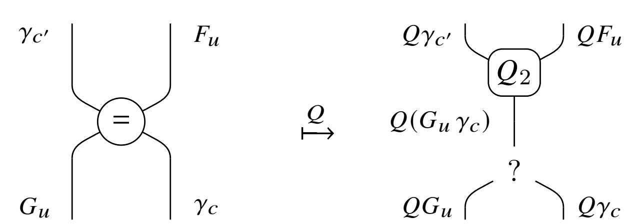
A similar argument reveals that the identity 1-cell on $P\colon C \to \mathcal{K}$ is given by the identity functor on C, and the identity lax transformation on P.

Now consider a 2-cell in $\mathsf{F}_{\Omega}\mathcal{K}$ given by a 2-natural transformation between Kleisli arrows $\overline{\gamma}\colon \overline{\phi} \Rightarrow \overline{\psi}\colon C^\dagger \to \mathfrak{Q}_{Q'}$ satisfying $\mathfrak{q}_{Q'}\ \overline{\gamma} = 1_P$. We would like to reinterpret such a 2-cell as some reasonable sort of morphism between lax-triangles (F,ϕ) and (G,ψ). Consider the underlying data for $\overline{\gamma}$, which are a family of 1-cells in $\mathfrak{Q}_{Q'}$ indexed by objects $c \in C$:

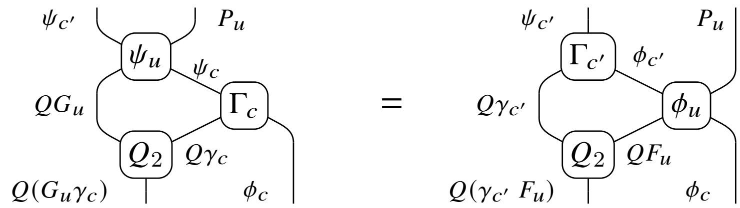
$$
\begin{tikzcd}
Pc \arrow[r, "\phi_c"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\Gamma_c}{\Leftarrow}"] & QFc \arrow[d, "Q\gamma_c"] \\
Pc \arrow[r, "\psi_{c'}"'] & QGc
\end{tikzcd}
\tag{4.7}
$$

The components γ_c are those of the 2-natural transformation $F \Rightarrow G\colon C \to D$ obtained from postcomposing $\overline{\gamma}$ by the projection $\mathsf{J}_!(Q')$ from $\mathfrak{Q}_{Q'}$ to D.

The Γ_c data might be expected to comprise a modification $Q\gamma\ \phi \to \phi$. This cannot possibly be true in a strict sense as $Q\gamma$ — and thus $Q\gamma\ \phi$ — fail to be oplax transformations. As alluded to in Remark 4.2.4 the problem is that postcomposition of a 2-natural transformation (whether strict, pseudo or op/lax) by a lax or oplax functor does not produce another 2-natural transformation. One can define components on objects $c \in C$ for $Q\gamma$ as $(Q\gamma)_c = Q(\gamma_c)\colon QF_c \to QG_c$, but for a morphism $u\colon c \to c'$ there is no suitable way to construct $(Q\gamma)_u$. The obstruction to adapting the definition that works in the case where Q is strict or pseudo is that there is no canonical 2-cell from $QG_u\ Q\gamma_c$ to $Q(G_u\gamma_c)$:



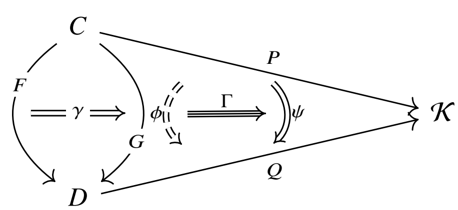
There is, however, a weaker notion of transformation between oplax functors known as a *comodule*, due to [Coc+03]. These *do* admit whiskering on the right by oplax functors (whereas *modules* admit right-whiskering by lax functors) and they also admit a notion of morphism between them known as a *modulation*. A general description of modules and modulations is given in Section 9.1 and we show in particular that a modulation from the comodule[6] $Q\gamma\ \phi$ to ψ is given by a collection of 2-cells $\Gamma_c \colon Q\gamma_c\phi_c \Rightarrow \psi_c$ for each $c \in C$ satisfying the following axiom for each $u \colon c \to c' \in C$:



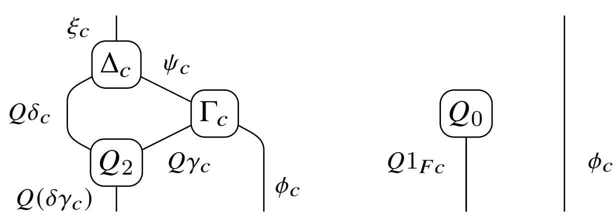
(4.8)

Remark 4.2.5. If Q were a pseudofunctor then each Q_2 would be invertible and the condition above is equivalent to Γ defining a modification $Q\gamma\ \phi \to \psi$. ◊

The condition (4.8) corresponds precisely to the condition that $\overline{\gamma}$ be 2-natural in $C^\dagger$, so we can identify the 2-cells in $\mathsf{F}_\Omega\mathcal{K}$ from (F, ϕ) to (G, ψ) with a pair of a natural transformation $\gamma \colon F \Rightarrow G$ and a modulation $\Gamma \colon Q\gamma\ \phi \to \psi$. We might draw such a 2-cell in $\mathsf{F}_\Omega\mathcal{K}$ as follows:



Translating back and forth between the Kleisli and lax triangle representation reveals that the vertical and horizontal composition of 2-cells are given by pasting together the component data of the modulations in the inevitable way. We omit the details of this translation and simply state the result. First, given 1-cells $(F, \phi), (G, \psi), (H, \xi) \colon P \to Q$, and 2-cells $(\gamma, \Gamma) \colon (F, \phi) \Rightarrow (G, \psi)$, $(\delta, \Delta) \colon (G, \psi) \Rightarrow (H, \xi)$, the underlying natural transformation $F \Rightarrow H$ of the vertical composite $(\delta, \Delta) \circ (\gamma, \Gamma)$ is $\delta \circ \gamma$ and the modulation $Q(\delta\ \gamma)\ \phi \Rightarrow \xi$ has component at c given by the string diagram on the left in (4.9). The component at c of the identity 2-cell on (F, ϕ) is given on the right.



(4.9)

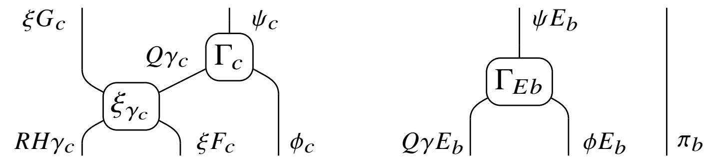
To describe the whiskering, assume we have the following diagram in $\mathsf{F}_\Omega\mathcal{K}$:

$$\begin{tikzcd} O \arrow[r, "{(E,\pi)}"] & P \arrow[r, bend left=50, "{(F,\phi)}"] \arrow[r, bend right=50, "{(G,\psi)}"'] \arrow[r, phantom, "\Downarrow (\gamma,\Gamma)"] & Q \arrow[r, "{(H,\xi)}"] & R \end{tikzcd}$$

The underlying natural transformations of the whiskerings $(H, \xi)(\gamma, \Gamma)$ and $(\gamma, \Gamma)(E, \pi)$ are $H\gamma$ and γE respectively.

[6]Lax transformations can be viewed as a special sort of comodule, cf. Section 9.1

The components of the corresponding modulations at objects c and b in the domains of P and O respectively are:



(4.10)

Because post-composition by a 2-functor preserves the property of having domain of the form $C^\dagger$, the image of the full sub-category$\mathsf{F}_\Omega\mathcal{K} \subseteq \mathsf{F}_\Theta\mathcal{K}$ under $\mathsf{F}_\Theta H \colon \mathsf{F}_\Theta\mathcal{K} \to \mathsf{F}_\Theta\mathcal{L}$ — $H \colon \mathcal{K} \to \mathcal{L}$ a 2-functor — is contained in $\mathsf{F}_\Omega\mathcal{L}$. It follows that $\mathsf{F}_\Theta \colon \mathsf{2CAT} \to \mathsf{2CAT}$ restricts to a sub-functor $\mathsf{F}_\Omega \colon \mathsf{2CAT} \to \mathsf{2CAT}$ with object map given by $\mathcal{K} \mapsto \mathsf{F}_\Omega\mathcal{K}$. The action of $\mathsf{F}_\Omega H$ on 1-cells and 2-cells is more satisfying when expressed in terms of lax-triangles rather than Kleisli maps. For a lax triangle $(F, \phi) \colon (C, P) \to (D, Q)$ in $\mathsf{F}_\Omega\mathcal{K}$, $\mathsf{F}_\Theta F$ acts on the corresponding Kleisli map $\overline{\phi} \colon C^\dagger \to \mathfrak{Q}_{Q'}$ by post-composing it with $\mathsf{J}_H(Q')$ (c.f. (4.2)). The result is the Kleisli map in $\mathsf{F}_\Theta\mathcal{L}$ from $C^\dagger \to \mathfrak{Q}_{HQ'}$ which acts on 0-cells and atomic 1-cells as follows:

$$\begin{tikzcd} c \arrow[d, "u"'] \\ c' \end{tikzcd} \qquad \overset{\mathsf{J}_H(Q')\,\overline{\phi}}{\longmapsto} \qquad \begin{tikzcd} Pc \arrow[r, "H\phi_c"] \arrow[d, "HPu"'] & QFc \arrow[d, "HQFu"] \arrow[dl, phantom, "\overset{H\phi_u}{\Longleftarrow}"] \\ Pc' \arrow[r, "H\phi_{c'}"'] & QFc' \end{tikzcd}$$

The corresponding lax triangle representation as a 1-cell in $\mathsf{F}_\Omega\mathcal{L}$ is simply given by the post-composition of ϕ with H — $(F, H\phi) \colon (C, HP) \to (D, HQ)$. A similar argument reveals that $\mathsf{F}_\Omega H$ acts on a 2-cell $(\gamma, \Gamma) \colon (F, \phi) \Rightarrow (G, \psi)$ by sending it to $(\gamma, H\Gamma)$, where $H\Gamma$ is the modulation whose component at $c \in C$ is given by $H(\Gamma_c)$.

The 2-natural transformation $\mathsf{F}_\Omega\rho \colon \mathsf{F}_\Omega H \Rightarrow \mathsf{F}_\Omega W \colon \mathsf{F}_\Omega\mathcal{K} \to \mathsf{F}_\Omega\mathcal{L}$ induced by a $\rho \colon H \Rightarrow W \colon \mathcal{K} \to \mathcal{L}$ has component at (C, P) given by[7] $(1_C, \rho P) \colon (C, HP) \to (C, WP)$, (c.f. (4.4)).

For convenience, we now gather the above description of F_Ω into a definition with a name:

Definition 4.2.6 (The oplax fam-construction)**.** For a locally small 2-category $\mathcal{K}$, the 2-category $\mathsf{F}_\Omega\mathcal{K}$ has:

0-cells oplax functors $P \colon C \to \mathcal{K}$ from 1-categories

1-cells from $P \colon C \to \mathcal{K}$ to $Q \colon D \to \mathcal{K}$ given by a functor $F \colon C \to D$ and a lax transformation $\phi \colon P \Rightarrow QF$

2-cells from (F, ϕ) to (G, ψ) given by a natural transformation $\gamma \colon F \Rightarrow G$ and a *modulation* $\Gamma \colon Q\gamma\ \phi \to \psi$ given by 2-cells $\Gamma_c \colon Q\gamma_c\ \phi \Rightarrow \psi$ satisfying the naturality condition described in (4.8).

Composition of 1-cells is given by pasting of lax triangles; horizontal and vertical composition of 2-cells is given on natural-transformation components by horizontal and vertical composition in Cat, and on modification components by (4.9) and (4.10). The mapping $\mathcal{K} \mapsto \mathsf{F}_\Omega\mathcal{K}$ extends to a 3-functor $\mathsf{F}_\Omega \colon \mathsf{2CAT} \to \mathsf{2CAT}$ by post-composition and post-whiskering. That is, for $S \colon \mathcal{K} \to \mathcal{L}$, the map $\mathsf{F}_\Omega S \colon \mathsf{F}_\Omega\mathcal{K} \to \mathsf{F}_\Omega\mathcal{L}$ acts on the 0-cells, 1-cells and 2-cells of $\mathsf{F}_\Omega\mathcal{K}$ respectively as:

$$(C, P) \mapsto (C, SP) \qquad (F, \phi) \mapsto (F, S\phi) \qquad (\gamma, \Gamma) \mapsto (\gamma, F\Gamma)$$

For $\alpha \colon S \Rightarrow T \colon \mathcal{K} \to \mathcal{L}$ a 2-natural transformation, the component of the image $\mathsf{F}_\Omega\alpha \colon \mathsf{F}_\Omega S \Rightarrow \mathsf{F}_\Omega T$ at an object (C, P) is $(1_C, \alpha P) \colon (C, SP) \to (C, TP)$. ◊

Definition 4.2.6 is the *oplax fam-construction* mentioned at the beginning of this chapter. Aside from the form of the 2-cells, perhaps, this construction is a natural one to consider. As evidence: Buckley gave essentially the same definition under the name $\mathsf{Fam}(\mathcal{K})$ in [Buc14, p. 2.3.1] but with *pseudo* functors, pseudo-natural transformations

[7] Note that whiskering a 2-natural transformation with an oplax functor on the *left* gives a well-defined 2-natural transformation.

and modifications defining the 0-cells, 1-cells and 2-cells. We will say more about this related construction in Chapter 7. As further evidence, this definition for $\mathsf{F}_\Omega\mathcal{K}$ was first conceived as a natural example when investigating semantics for the 2-dimensional type theory of [Gar09], which we address briefly in Section 7.9. The fact that $\mathsf{F}_\Omega\mathcal{K}$ is a free cocompletion only became apparent later.

4.3 F_Ω as a Free Cocompletion

The rest of this chapter is devoted to showing that Ω is a presaturated class of colimits, from which it follows that $\mathsf{F}_\Omega\mathcal{K} \simeq \Omega_1\mathcal{K}$ is equivalent to $\Omega\mathcal{K}$ — the free Ω-cocompletion of $\mathcal{K}$.

Being Ω-cocomplete is equivalent to having all oplax colimits of oplax functors from 1-categories. To show that Ω is presaturated it therefore suffices to demonstrate that:

(a) $\mathsf{F}_\Omega\mathcal{K}$ contains the image of $\mathsf{Z}_\mathcal{K}\colon \mathcal{K} \to \mathsf{F}_\Theta\mathcal{K}$ (i.e. $\mathsf{F}_\Omega\mathcal{K}$ contains the "representables").

(b) The inclusion $\mathfrak{i}_\mathcal{K} : \mathsf{F}_\Omega\mathcal{K} \to \mathsf{F}_\Theta\mathcal{K}$ lifts oplax colimits of oplax functors from 1-categories (i.e. $\mathsf{F}_\Omega\mathcal{K}$ is closed in $\mathsf{F}_\Omega\mathcal{K}$ under Ω-colimits).[8]

Condition (b) is *a priori* weaker than the necessary condition that $\mathfrak{I}\,\mathfrak{i}_\mathcal{K}\colon \mathsf{F}_\Omega\mathcal{K} \to [\mathcal{K}^{\mathsf{op}}, \mathsf{Cat}]$ lift Ω-colimits, but is in fact equivalent by the fact that $\mathfrak{I}\colon \mathsf{F}_\Theta\mathcal{K} \to [\mathcal{K}^{\mathsf{op}}, \mathsf{Cat}]$ lifts *all* oplax colimits, and Ω-colimits in particular.

Condition (a) is clearly true: the image of $k \in \mathcal{K}$ under $\mathsf{Z}_\mathcal{K}$ is $k\colon \mathbb{1} \to \mathcal{K}$, which is an oplax functor from a 1-category (which happens to be strict). To demonstrate Condition (b), we must show that for any oplax functor $B\colon C \to \mathsf{F}_\Omega\mathcal{K}$ from a 1-category, there exists a lift (up to equivalence) of the oplax colimit of $\mathfrak{i}_\mathcal{K} B\colon C \to \mathsf{F}_\Theta\mathcal{K}$ along $\mathfrak{i}_\mathcal{K}$ to an object $B'\colon X_B \to \mathcal{K}$ in $\mathsf{F}_\Omega\mathcal{K}$. Our procedure for this is as follows:

(a) assume $B\colon C \to \mathsf{F}_\Omega\mathcal{K}$ is an oplax functor from a 1-category

(b) construct from B a 2-category $\mathscr{E}(\varpi_\mathcal{K} B)$ and a 2-functor $B^\flat\colon \mathscr{E}(\varpi_\mathcal{K} B) \to \mathcal{K}$

(c) show that $B^\flat$ is the oplax colimit of $\mathfrak{i}_\mathcal{K} B$ in $\mathsf{F}_\Theta\mathcal{K}$

(d) show that $\mathscr{E}(\varpi_\mathcal{K} B)$ has component-initial objects in hom-categories, and thus by Corollary 4.2.2 lifts along $\mathfrak{i}_\mathcal{K}$ to $\mathsf{F}_\Omega\mathcal{K}$.

For the construction of $\mathscr{E}(\varpi_\mathcal{K} B)$ and $B^\flat$ we will appeal to a universal construction which we call the *extralax colimit*.

4.3.1 Extralax cocones

For 1-categories C and D, and $d \in D$, a lift of a 1-functor $F\colon C \to D$ along the projection from the slice category $\pi\colon D/d \to D$ is equivalent to a cocone from F to d. The 2-category $\mathsf{F}_\Omega\mathcal{K}$ in some ways resembles a slice over $\mathcal{K}$, and a lift of a 2-functor $A\colon C \to \mathsf{Cat}$ along $\varpi\colon \mathsf{F}_\Omega\mathcal{K} \to \mathsf{Cat}$ is equivalent to a particular sort of cocone from A to $\mathcal{K}$:

Definition 4.3.1 (Extralax cocone)**.** Given a strict 2-functor $A\colon C \to \mathsf{Cat}$, an extralax cocone $B : A \rhd \mathcal{K}$, from A to a 2-category $\mathcal{K}$ is given by the following data:

(a) For each object $c \in C$, an oplax functor $B_c\colon A_c \to \mathcal{K}$

(b) For each $u\colon c \to c'$ a lax transformation $B_u\colon B_c \Rightarrow B_{c'} A_u$

(c) For each $\alpha\colon u \Rightarrow v\colon c \to c'$ a modulation $B_\alpha\colon B_{c'} A_\alpha \circ B_u \to B_v$

[8] "lifts limits in" is the appropriate variation of "is closed under limits in" when the canonical functor $\mathsf{F}_\Omega\mathcal{K} \to \mathsf{F}_\Theta\mathcal{K}$ is merely fully faithful, rather than a strict inclusion of a full sub-2-category. The existence of a lift of a universal (co)cone (or *(co)cylinder*, in the case of weighted limits) implies the functor creates (co)limits that exist in the codomain, since fully faithful functors automatically reflect them.

$$\begin{tikzcd}[column sep=large] A_c \arrow[d, bend right=60, "A_u"'] \arrow[d, bend left=60, "A_v"] \arrow[d, phantom, "A_\alpha \Rightarrow"] \arrow[rr, "B_c"] & {} \arrow[d, phantom, "B_u \Downarrow \overset{B_\alpha}{\Rrightarrow} \Downarrow B_v"] & \mathcal{K} \\ A_{c'} \arrow[urr, "B_{c'}"'] & {} & \end{tikzcd}$$

These data are required to be functorial with respect to the 2-categorical structure of $\mathsf{F}_\Omega\mathcal{K}$, in the sense that for the following data in $\mathcal{C}$:

$$\begin{tikzcd} c \arrow[r, bend left=60, "v"] \arrow[r, "w" description] \arrow[r, bend right=60, "z"'] \arrow[r, phantom, shift left=3, "\Downarrow\beta"] \arrow[r, phantom, shift right=3, "\Downarrow\gamma"] & d \arrow[r, bend left=50, "u"] \arrow[r, bend right=50, "y"'] \arrow[r, phantom, "\Downarrow\alpha"] & e \end{tikzcd}$$

we have:

(a) $B_{1_c} = 1_{B_c}$

(b) $B_{uv} = B_u A_v \circ B_v$

(c) $B_{1_u} = 1_{B_u}$

(d) $B_{\gamma\circ\beta} = B_\gamma \circ \left(B_d A_\gamma\, B_\beta\right) \circ (B_d)_2\, B_v$

(e) $B_{\alpha\beta} = B_{\alpha\, w} \circ \left(B_e A_{\alpha\, w}\, B_{u\beta}\right) \circ (B_e)_2\, B_{uv}$ ◊

Conditions (d) and (e) are best understood by comparing with the definition of horizontal and vertical composition of 2-cells in $\mathsf{F}_\Omega\mathcal{K}$ in terms of string diagrams as shown in (4.9) and (4.10). These functoriality conditions correspond precisely to the condition that an extralax cocone $A \rhd \mathcal{K}$ is equivalent to a lift of A along $\pi\colon \mathsf{F}_\Omega\mathcal{K} \to \mathsf{Cat}$.

Remark 4.3.2. Cocones $F \rhd b$ can usually be described as transformations $F \Rightarrow \Delta b$ for some notion of transformation appropriate to the variety of cocone. But giving a notion of transformation corresponding to an extralax cocone is tricky. The component 1-cells of an extralax cocone are oplax functors, which typically don't exist as 1-cells in a 2-category whose 2-cells include the lax transformations involved in extralax cocones. One could replace the oplax functors $B_c\colon A_c \to \mathcal{K}$ with the corresponding 2-functors $B'_c\colon {A_c}^\dagger \to \mathcal{K}$ to obtain what looks like a lax cocone $(-)^\dagger\ A \Rightarrow \Delta\mathcal{K}$ in $\mathsf{Gray}_{\mathsf{lax}}$, but $(-)^\dagger$ doesn't form a 2-functor from Cat to Gray. The image of a natural transformation in Cat under $\left(-^\dagger\right)$ is (if anything) a comodule, rather than a lax natural transformation. It seems, therefore, that any notion of transformation used to describe an extralax colimit would need to take place in a 3-dimensional category of 2-categories with comodules as 2-cells. ◊

For the purpose of proving that Ω is presaturated we are particularly interested in extralax cocones from $A\colon \mathcal{C} \to \mathsf{Cat}$ where A is of the form $H'\colon \mathcal{C}^\dagger \to \mathsf{Cat}$ for $\mathcal{C}$ a 1-category and H an oplax functor, because these correspond to oplax functors from 1-categories into $\mathsf{F}_\Omega\mathcal{K}$. However, there will be other uses of the notion of extralax cocone for $A\colon \mathcal{C} \to \mathsf{Cat}$ an arbitrary 2-functor, which is why we choose to preserve this level of generality for the moment.

4.3.2 The extralax colimit

Given a strict 2-functor $A\colon \mathcal{C} \to \mathsf{Cat}$ and a 2-category $\mathcal{K}$ there exists a set $\mathsf{ELC}(A, \mathcal{K})$ of extralax cocones under A with vertex $\mathcal{K}$. A 2-functor $\mathcal{F}\colon \mathcal{K} \to \mathcal{L}$ induces a function from $\mathsf{ELC}(A, \mathcal{K})$ to $\mathsf{ELC}(A, \mathcal{L})$ by postcomposition, and this action is 1-functorial in $\mathcal{K}$. We therefore have a functor $\mathsf{ELC}(A, -)$ from the underlying 1-category $\mathsf{2CAT}_0$ to SET. The *extralax colimit* of A is a representation for this functor. In other words, it is a 2-category $\mathscr{E}A$ and an extralax cocone $\delta\colon A \rhd \mathscr{E}A$ through which all other extralax cocones under A factor via a unique 2-functor from $\mathscr{E}A$.

Before we give the general construction for the extralax colimit — which exists for all $A\colon C \to \mathsf{Cat}$ — we consider the special case where A is a constant functor $\Delta_X\colon C \to \mathsf{Cat}$ for C a 1-category. An extralax cocone from Δ_X to a 2-category $\mathcal{K}$ is given by a choice of an oplax functor $B_c\colon X \to \mathcal{K}$ for each $c \in C$ and for each $u\colon c \to c'$ in C a lax natural transformation $B_u\colon B_c \to B_{c'}$ assigned in a functorial way (i.e. $B_{uv} = B_u B_v$, $B_{1_c} = 1_{B_c}$). Specifying an oplax functor $B_c\colon X \to \mathcal{K}$ is equivalent to specifying a strict 2-functor $B'_c\colon X^\dagger \to \mathcal{K}$, and a lax transformation $B_c \Rightarrow B_{c'}$ is equivalent to a lax transformation $B'_c \Rightarrow B'_{c'}$. An extralax cocone from Δ_X to $\mathcal{K}$ can therefore be identified with a functor from C to the category $\left[X^\dagger, \mathcal{K}\right]_{\mathsf{lax}}$ of strict 2-functors and lax transformations. So we have a natural isomorphism of functors from $2\mathsf{CAT}_0$ to SET:

$$\mathsf{ELC}(A, -) \cong \mathsf{CAT}\Big(C, \left[X^\dagger, -\right]_{\mathsf{lax}}\Big)$$

The functor on the right is known to be represented by the lax Gray tensor product $C \boxminus X^\dagger$ (cf. Definition 2.2.10), which is therefore the extralax colimit of Δ_X. The component of the universal oplax cocone $\delta_{\Delta_X}\colon \Delta_X \rhd C \boxminus X^\dagger$ at $c \in C$ is given in terms of the canonical oplax inclusion $\lambda_X\colon X \to X^\dagger$ as:

$$X \xrightarrow{\cong} \mathbb{1} \boxminus X \xrightarrow{\langle c\rangle \boxminus \lambda_X} C \boxminus X^\dagger$$

It will assist our intuition to think of the general extralax colimit of $A\colon C \to \mathsf{Cat}$ as a sort of "dependent Gray tensor product" $C \boxminus A^\dagger$ where the type of the second component depends on the value of the first, similar to how one might view the category of elements of a functor $F\colon C \to \mathsf{Set}$ as a product $C \times Fc$ where the second component depends on the first.[9]

Construction 4.3.3 (Extralax colimit)**.** Given a strict 2-functor $A\colon C \to \mathsf{Cat}$, the extralax colimit $\mathscr{E}A$ is given by the following presentation:

0-cells: pairs of the form $(c \in C, a \in A_c)$

1-cells: are generated by morphisms of two types:

(a) $(c, a) \xrightarrow{[u]_a} (d, A_u a)$ for each $c \xrightarrow{u} d$ in C, $a \in A_c$

(b) $(c, a) \xrightarrow{[\alpha]} (c, b)$ for each $c \in C$ and $a \xrightarrow{\alpha} b$ in A_c

A general 1-cell is a composition of generators written in usual right-to-left order. Note, for example, that for $c \in C$, $\beta\colon x \to y$, $\alpha\colon y \to z$ in A_c, there are two distinct morphisms $[\alpha][\beta], [\alpha\beta]\colon (c, x) \to (c, z)$.

The only relations are those on the type-one generators which hold in C. That is, we have relations of the form:

$$[u]_{A_u a}[v_a] = (uv)_a\colon (c, a) \to (e, A_{uv} a) \qquad 1_{(c,a)} = [1_c]_a\colon (c, a) \to (c, a) \tag{4.11}$$

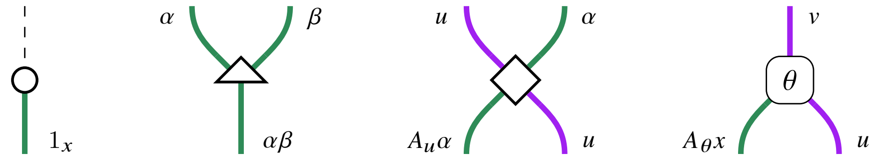
2-cells are generated by 2-cells of four types. For arbitrary $c \in C$, $\beta\colon x \to y$, $\alpha\colon y \to z$ in A_c, $u, v\colon c \to d$ in C and $\theta\colon u \Rightarrow v$, we have generating 2-cells of the following forms:

$$
\begin{tikzcd}
(c,x) \arrow[r, phantom, "\Uparrow"] & (c,x) \arrow[l, bend right, "1_{(c,x)}"'] \arrow[l, bend left, "{[1_x]}"]
\end{tikzcd}
\quad
\begin{tikzcd}
& (c,y) \arrow[dl, "{[\alpha]}"'] \arrow[d, phantom, "\Uparrow"] & \\
(c,z) & {} & (c,x) \arrow[ul, "{[\beta]}"'] \arrow[ll, bend left, "{[\alpha\beta]}"]
\end{tikzcd}
\quad
\begin{tikzcd}
& (c,y) \arrow[dl, "{[u]_y}"'] & \\
(d,A_u y) & \Uparrow & (c,x) \arrow[ul, "{[\alpha]}"'] \arrow[dl, "{[u]_x}"] \\
& (d,A_u x) \arrow[ul, "{[A_u\alpha]}"] &
\end{tikzcd}
\quad
\begin{tikzcd}
(d,A_v x) & \Uparrow{[\theta]_x} & (c,x) \arrow[ll, bend right, "{[v]_x}"'] \arrow[dl, "{[u]_x}"] \\
& (d,A_u x) \arrow[ul, "{[A_\theta x]}"] &
\end{tikzcd}
\tag{4.12}
$$

Only the generators of the fourth kind are labelled, as the rest can be determined from their domain and

[9]This actually corresponds to the type-theoretic notion of 'dependent sum'. "Dependent product" means something very different in type theory.

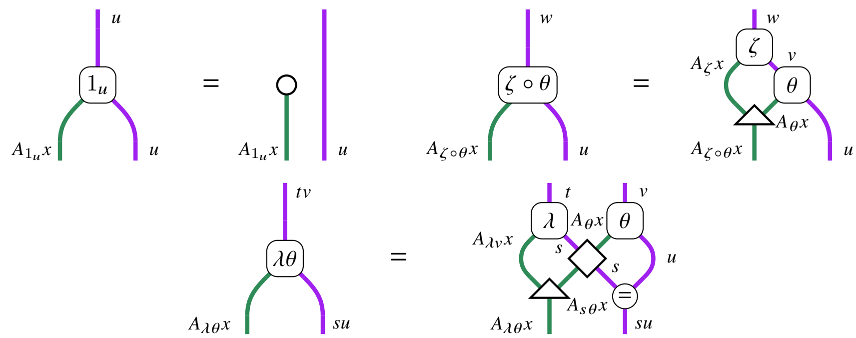
codomain. We will find it convenient to represent these generators by string diagrams, shown respectively in (4.13):



(4.13)

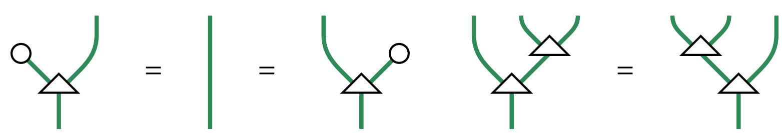
Each string labelled X represents the 1-cell generator [X], and coloured purple or green to indicate its type — purple for type-one, green for type-two. The subscripts for the type-one generators are usually suppressed for readability, as in most cases the subscript can be inferred by the presence of a type-two generator it is composed with, or otherwise it is irrelevant.

The generators satisfy relations which we group into four classes. First, there are the relations which preserve the horizontal and vertical composition of 2-cells in $\boldsymbol{C}$, shown here for arbitrary $\lambda\colon s \Rightarrow t$, $\theta\colon u \Rightarrow v$, $\zeta\colon v \Rightarrow w$ in $\boldsymbol{C}$:

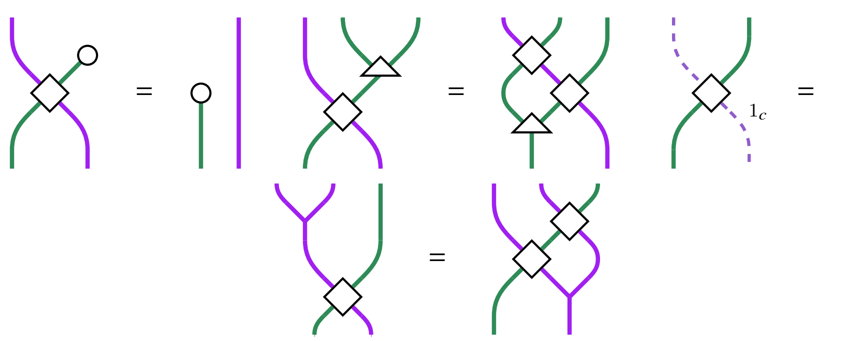


(4.14)

There are the usual coherence relations for the generators of the first two sorts (cf. Section 2.2.4) for any sound labelling of the type-two 1-cell generators:

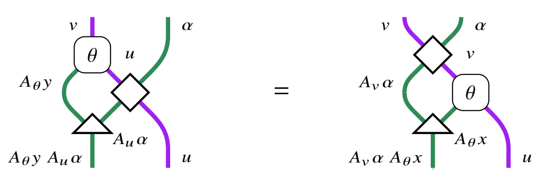

(4.15)

There are relations showing that generators of the third sort respect composition in $\boldsymbol{C}$ and A_c:



(4.16)

And finally, there are relations showing generators of the third sort preserve the 2-cells of $\boldsymbol{C}$:



(4.17)

◊

Example 4.3.4 ($\mathscr{E}\Delta_X$)**.** We've already observed that for 1-categories C and X, the extralax colimit of $\Delta_X\colon C \to \mathsf{Cat}$ is given by the lax Gray tensor product $C \boxminus X^\dagger$. We should therefore find that the presentation for $\mathscr{E}\Delta_X$ gives a presentation for this lax Gray tensor product. The $\mathscr{E}\Delta_X$ presentation differs slightly from the general presentation for a lax Gray tensor product described in Definition 2.2.10 because the type-two generators of $\mathscr{E}\Delta_X$ are the generators of $X^\dagger$, rather than arbitrary morphisms. Similarly, rather than including arbitrary 2-cells of $X^\dagger$ as generators in $\mathscr{E}\Delta_X$ we have only the generating X_0 and X_2 2-cells, which produce the generators $\bigcirc$ and $\triangle$. The $\Diamond$ generators of the presentation for $\mathscr{E}\Delta_X$ then clearly correspond to the $\Diamond$ generators for $C \boxminus X^\dagger$, since $(\Delta_X)_u = 1_X$. With C a 1-category there are no non-trivial 2-cell generators of the fourth sort for $\mathscr{E}\Delta_X$, or of the first sort for $C \boxminus X^\dagger$, so the presentation for $\mathscr{E}\Delta_X$ does provide a presentation for $C \boxminus X^\dagger$. However, if C has non-trivial 2-cells then this is no longer true. For even though $(\Delta_X)_\theta x = 1_x$ for all $\theta\colon u \to v$, $x \in X$, the 1-cell $[1_x]$ is not an identity in $\mathscr{E}\Delta_X$ so the generators of the fourth kind in $\mathscr{E}\Delta_X$ don't coincide with generators of the first kind in the presentation for $C \boxminus X^\dagger$. This is essentially because the lax hom-functor $[A, -]_{\mathsf{lax}}$ — to which $- \boxminus A$ is left-adjoint — has modifications rather than modulations as 2-cells. In Appendix 9.2 we describe a modified version of the Gray tensor product $(- \boxtimes -)$ in terms of modulations, which satisfies $C \boxtimes X \cong \mathscr{E}\Delta_X$ for C a 2-category and X a 1-category. When C is locally discrete we have $C \boxtimes X \cong C \boxminus X^\dagger$. ◊

Construction 4.3.5 ($\delta_A\colon A \rhd \mathscr{E}A$)**.** The universal extralax cocone δ_A from a 2-functor $A\colon C \to \mathsf{Cat}$ to $\mathscr{E}A$ is given as follows:

(a) For each $c \in C$ the oplax functor $\delta_c\colon A_c \to \mathscr{E}A$ sends $x \xrightarrow{\alpha} y \in A_c$ to $(c,x) \xrightarrow{\alpha} (c,y) \in \mathscr{E}A$. The oplax coherence data is given by the 2-cell generators $\bigcirc$ and $\triangle$ in $\mathscr{E}A$.

(b) For each $u\colon c \to d \in C$ the lax transformation $\delta_u\colon \delta_c \to \delta_d A_u$ has component at $x \in A_c$ given by $u_x : (c,x) \to (d, A_u x)$. The lax naturality data at a morphism $\alpha\colon x \to y$ in A_c is given by the 2-cell generator of the third kind, $\Diamond\colon [A_u\alpha][u]_x \Rightarrow [u]_y[\alpha]$.

(c) For each $\theta\colon u \Rightarrow v\colon c \to d \in C$ the modulation $\delta_\theta\colon \delta_d A_\theta \circ \delta_u \to \delta_v$ has component at $x \in A_c$ given by the generator of the fourth kind $\theta\colon [A_\theta x][u]_x \Rightarrow [v]_x$. ◊

That these data define an extralax cocone follows immediately from relations on the 2-cell generators. For example, comparing the relations for generators of the first two kinds, $\bigcirc$, $\triangle$, to the coherence conditions for an oplax functor makes it clear that the functors δ_c are oplax-functorial. Similarly, the "preservation of 2-cells in C by $\Diamond$" relations correspond to the weak naturality condition for the modulations δ_θ, and the inherited relations on the 2-cells of C correspond to the definitions of horizontal and vertical composition of the 2-cells in $\mathsf{F}_\Omega\mathcal{K}$.

Construction 4.3.6 (The classifying map for an extralax cocone)**.** Given a 2-functor $A\colon C \to \mathsf{Cat}$ and an extralax cocone $B\colon A \rhd \mathcal{K}$, the canonical 2-functor $B^\flat\colon \mathscr{E}A \to \mathcal{K}$ is defined as follows:

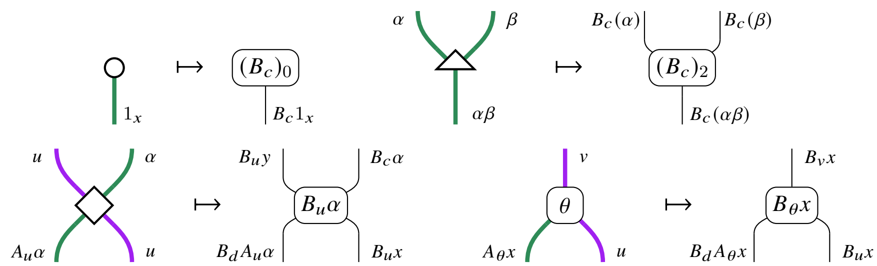
0-cells: $(c,x) \in \mathscr{E}A$ maps to $B_c x$

1-cells: The generators of each sort are mapped as follows:

(a) $(c,x) \xrightarrow{[u]_x} (d, A_u x)$ maps to $B_c x \xrightarrow{B_u x} B_d A_u x$

(b) $(c,x) \xrightarrow{[\alpha]} (c,y)$ maps to $B_c x \xrightarrow{B_c\,\alpha} B_c y$

2-cells: The four sorts of generators are mapped as follows:



The coherence conditions on the component data of the extralax cocone B and the functoriality of B ensure that all the relations on the generators of $\mathscr{E}A$ are satisfied. ◊

Proposition 4.3.7. *For a 2-functor $A\colon C \to \mathsf{Cat}$, precomposition by the extralax cocone $\delta_A\colon A \rhd \mathscr{E}A$ induces a bijection $\mathsf{2CAT}(\mathscr{E}A, \mathcal{K}) \cong \mathsf{ELC}(A, \mathcal{K})$ which is 1-natural in $\mathcal{K}$, whose inverse sends an extralax cocone $B\colon A \rhd \mathcal{K}$ to $B^\flat\colon \mathscr{E}A \to \mathcal{K}$.*

Proof. The image of each generating 2-cell of $\mathscr{E}A$ under F is a component of the data for the extralax cocone $F\delta_A$, and so $F\delta_A = G\delta_A$ only when $F, G\colon \mathscr{E}A \to \mathcal{K}$ agree on all the generating 2-cells of $\mathscr{E}A$ and are thus equal as 2-functors. To see that precomposition by δ_A is also surjective, we simply observe from Constructions 4.3.5 and 4.3.6 that any extralax cocone $B\colon A \rhd \mathcal{K}$ is equal to the precomposition of $B^\flat\colon \mathscr{E}A \to \mathcal{K}$ by δ_A. □

It follows that the underlying 1-functor of F_Θ is a *parametric right adjoint* in the following sense:

Proposition 4.3.8. *The map $\mathsf{F}_\Omega/\mathbb{1}\colon \mathsf{2CAT} \to \mathsf{2CAT}/\mathsf{Cat}$ which sends $\mathcal{K}$ to $\varpi_{\mathcal{K}}\colon \mathsf{F}_\Omega\mathcal{K} \to \mathsf{Cat}$ has a left 1-adjoint which sends $A\colon C \to \mathsf{Cat}$ to $\mathscr{E}A$.*

Proof. It suffices to show that for any 2-functor $A\colon C \to \mathsf{Cat}$, there is a bijection between the sets $\mathsf{2Cat}/\mathsf{Cat}(A, \varpi_{\mathcal{K}})$ and $\mathsf{2Cat}(\mathscr{E}\mathcal{A}, \mathcal{K})$ which is 1-natural in $\mathcal{K}$. Maps $B\colon C \to \mathsf{F}_\Omega\mathcal{K}$ in $\mathsf{2Cat}/\mathsf{Cat}$ from A to $\varpi_{\mathcal{K}}\colon \mathsf{F}_\Omega\mathcal{K} \to \mathsf{Cat}$ comprise the same data as an extralax cocone $A \rhd \mathcal{K}$ by definition, so this follows from Proposition 4.3.7. □

Corollary 4.3.9. *For any 2-category $\mathcal{K}$, the map $\mathsf{F}_\Omega/\mathcal{K}\colon \mathsf{2CAT}/\mathcal{K} \to \mathsf{2CAT}/\mathsf{F}_\Omega\mathcal{K}$ which sends $F\colon \mathcal{A} \to \mathcal{K}$ to $\mathsf{F}_\Omega F\colon \mathsf{F}_\Omega A \to \mathsf{F}_\Omega\mathcal{K}$ has a left 1-adjoint whose action on objects is given by sending $B\colon C \to \mathsf{F}_\Omega\mathcal{K}$ to $B^\flat\colon \mathscr{E}(\varpi_{\mathcal{K}} B) \to \mathcal{K}$.*

Proof. This follows from the general theory of parametric right adjoints (for example, [Web07, Proposition 2.6]). In general, for a functor $F\colon C \to D$ from a category with a terminal object, 1, if $F/1\colon C \to D/F1$ has a left adjoint L, then so will $F/c\colon C/c \to D/Fc$ for all c, with the left adjoint given by sending $f\colon d \to Fc$ to the transpose of the map f from $d \xrightarrow{f} Fc \xrightarrow{F!} F1$ to $F/1(c)$ under the $L \dashv F/1$ adjunction. In this case $F = \mathsf{F}_\Omega\mathcal{K}$, L is given by $\mathscr{E}$, and the transpose of $B\colon C \to \mathsf{F}_\Omega\mathcal{K}$ under this adjunction is the map $B^\flat\colon \mathscr{E}\varpi_{\mathcal{K}} B \to \mathcal{K}$ (where we have extended the notation $B^\flat$ to $B\colon C \to \mathsf{F}_\Omega\mathcal{K}$ by identifying such a 2-functor with its extralax cocone $B\colon A \rhd \mathcal{K}$). □

We will say more about the parametric right adjoint property of F_Ω in Chapter 7. For now, we simply observe that it allows us to produce for any 2-functor $B\colon C \to \mathsf{F}_\Omega\mathcal{K}$ another 2-functor $B^\flat\colon \mathscr{E}(\varpi_{\mathcal{K}} B) \to \mathcal{K}$ in a canonical way. Restricting to the case where B is of the form $B'\colon C^\dagger \to \mathsf{Cat}$ for C a 1-category gives us an association between oplax functors from a 1-category C to $\mathsf{F}_\Omega\mathcal{K}$ and objects of $\mathsf{F}_\Theta\mathcal{K}$, which is stage (b) of our proof that $\mathsf{F}_\Omega\mathcal{K} \to \mathsf{F}_\Theta\mathcal{K}$ lifts Ω-colimits. Stage (c) requires that we show $B^\flat$ is, in fact, the oplax colimit of $\mathfrak{i}_{\mathcal{K}} B\colon C \to \mathsf{F}_\Theta\mathcal{K}$. In fact, this is true even if we allow B to be an arbitrary 2-functor $C \to \mathsf{F}_\Omega\mathcal{K}$, so for now we maintain this level of generality.

4.3.3 Oplax colimits in $\mathsf{F}_\Theta\mathcal{K}$

In this section we achieve stage (c) of our argument that $\mathbf{i}_\mathcal{K}\colon \mathsf{F}_\Omega\mathcal{K} \to \mathsf{F}_\Theta\mathcal{K}$ lifts Ω-colimits by proving the following result

Lemma 4.3.10. *For $\mathcal{K}$ a 2-category, $B : C \to \mathsf{F}_\Omega\mathcal{K}$ a 2-functor, the oplax colimit of $\mathbf{i}_\mathcal{K}B\colon C \to \mathsf{F}_\Theta\mathcal{K}$ is given by $B^\flat\colon \mathscr{E}\varpi_\mathcal{K}B \to \mathcal{K}$, viewed as an object of $\mathsf{F}_\Theta\mathcal{K}$.*

Our first step is to show that it suffices to consider the case where B is the unit for the 1-adjunction $\mathscr{E}\colon \mathsf{2Cat}/\mathsf{Cat} \rightleftharpoons \mathsf{2Cat} : \mathsf{F}_\Omega/\mathbb{1}$ of Proposition 4.3.8. The component at $A\colon C \to \mathsf{Cat}$ of this unit is given by the 2-functor $\mathsf{G}_A\colon C \to \mathsf{F}_\Omega(\mathscr{E}A)$ corresponding to the universal extralax cocone $\delta_A\colon A \rhd \mathscr{E}A$:

$$\begin{tikzcd} C \arrow[rr, "\mathsf{G}_A"] \arrow[dr, "A"'] & \circlearrowright & \mathsf{F}_\Omega\mathscr{E}A \arrow[dl, "\varpi_{\mathscr{E}A}"] \\ & \mathsf{Cat} & \end{tikzcd} \tag{4.18}$$

The component at $B\colon C \to \mathsf{F}_\Omega\mathcal{K}$ of the unit for the 1-adjunction $\mathscr{E}\colon \mathsf{2Cat}/\mathsf{F}_\Omega\mathcal{K} \rightleftharpoons \mathsf{2Cat} : \mathsf{F}_\Omega/\mathcal{K}$ can then be expressed in terms of the unit (4.18) as follows, where $A \coloneqq \varpi_\mathcal{K}B\colon C \to \mathsf{Cat}$:

$$\begin{tikzcd} C \arrow[rr, "\mathsf{G}_A"] \arrow[dr, "B"'] & \circlearrowright & \mathsf{F}_\Omega\mathscr{E}A \arrow[dl, "\mathsf{F}_\Omega(B^\flat)"] \\ & \mathsf{F}_\Omega\mathcal{K} & \end{tikzcd}$$

By the naturality of $\mathbf{i}\colon \mathsf{F}_\Omega \Rightarrow \mathsf{F}_\Theta$ we have that $\mathbf{i}_\mathcal{K}B$ is equal to the composite:

$$C \xrightarrow{\mathsf{G}_A} \mathsf{F}_\Omega(\mathscr{E}A) \xrightarrow{\mathbf{i}_{\mathscr{E}A}} \mathsf{F}_\Theta(\mathscr{E}A) \xrightarrow{\mathsf{F}_\Theta B^\flat} \mathsf{F}_\Theta\mathcal{K}$$

Given that $\mathsf{F}_\Theta B^\flat$ is Θ-cocontinuous, and thus Ω-cocontinuous, we can show that the oplax colimit of $\mathbf{i}_\mathcal{K}B\colon C \to \mathsf{F}_\Theta\mathcal{K}$ is given by $B^\flat : \mathscr{E}A \to \mathcal{K}$ — seen as an object of $\mathsf{F}_\Theta(\mathscr{E}A)$ — by showing that the oplax colimit of $\mathbf{i}_\mathcal{K}\mathsf{G}_A\colon C \to \mathsf{F}_\Theta(\mathscr{E}A)$ has oplax colimit given by ${\mathsf{G}_A}^\flat = 1_{\mathscr{E}A}$ (the transpose of the unit is the identity), since the more general result would then follow by a coend argument:

$$\oint^C \mathbf{i}_\mathcal{K}B = \oint^C \mathsf{F}_\Theta B^\flat\ \mathbf{i}_{\mathscr{E}A}\ \mathsf{G}_A \cong \mathsf{F}_\Theta B^\flat\Big(\oint^C \mathbf{i}_{\mathscr{E}A}\ \mathsf{G}_A\Big) \cong \mathsf{F}_\Theta B^\flat(1_{\mathscr{E}A}) \cong B^\flat$$

So we prove Lemma 4.3.10 by proving the following simpler result:

Lemma 4.3.11. *For $A\colon C \to \mathsf{Cat}$, the oplax colimit of $\mathbf{i}_{\mathscr{E}A}\mathsf{G}_A\colon C \to \mathsf{F}_\Theta(\mathscr{E}A)$ is $1_{\mathscr{E}A}\colon \mathscr{E}A \to \mathscr{E}A$, viewed as an object of $\mathsf{F}_\Theta(\mathscr{E}A)$.*

To show that $1_{\mathscr{E}A}$ is the oplax colimit of $\mathbf{i}_\mathcal{K}\ \mathsf{G}_A$ we will describe the lax cocones in $\mathsf{F}_\Theta\mathcal{K}$ (recall that oplax colimits are universal lax cocones) and show that they correspond to certain *extralax* cocones, from which it will follow that the oplax colimit in $\mathsf{F}_\Theta\mathcal{K}$ is constructed from the extralax colimit $\mathscr{E}A$.

For a general 2-functor $B\colon C \to \mathsf{F}_\Omega\mathcal{K}$, a lax cocone from $\mathbf{i}_\mathcal{K}\,B$ to an object $p\colon \mathcal{X} \to \mathcal{K}$ in $\mathsf{F}_\Theta\mathcal{K}$ can be expressed as a lift of $\mathbf{i}_\mathcal{K}B$ along the projection from the lax slice category $\pi_1\colon \mathsf{F}_\Theta\mathcal{K} \Downarrow p \to \mathsf{F}_\Theta\mathcal{K}$. By the fact that $\mathbf{i}_\mathcal{K}\colon \mathsf{F}_\Omega\mathcal{K} \to \mathsf{F}_\Theta\mathcal{K}$ is fully faithful, such a lift is equivalent to a lift of $B\colon C \to \mathsf{F}_\Omega\mathcal{K}$ along the projection from the lax comma category $\mathbf{i}_\mathcal{K} \Downarrow \mathcal{K}$. The most important step in showing that $1_{\mathscr{E}A}$ is the oplax colimit of $\mathbf{i}_{\mathscr{E}A}\,\mathsf{G}_A$ is the observation that the projection $\pi_1\colon \mathbf{i}_\mathcal{K} \Downarrow p \to \mathsf{F}_\Omega\mathcal{K}$ is isomorphic in $\mathsf{2CAT}/\mathsf{F}_\Omega\mathcal{K}$ to $\mathsf{F}_\Omega\mathfrak{q}_p\colon \mathsf{F}_\Omega\mathfrak{Q}_p \to \mathsf{F}_\Omega\mathcal{K}$, where $\mathfrak{q}_p\colon \mathfrak{Q}_p \to \mathcal{K}$ is the free locally discrete split 2-fibration described in Definition 3.3.10.

Lemma 4.3.12. *For $\mathcal{K}$ a locally small 2-category and $p\colon \mathcal{X} \to \mathcal{K}$ a 2-functor from a small 2-category, there exists an isomorphism $\rho_p\colon \mathfrak{i}_{\mathcal{K}} \Downarrow p \cong \mathsf{F}_\Omega \mathfrak{Q}_p$ which makes the following diagram commute:*

$$\begin{tikzcd} \mathfrak{i}_{\mathcal{K}} \Downarrow p \arrow[r, "\rho_p"] \arrow[dr, "\pi_1"'] & \mathsf{F}_\Omega \mathfrak{Q}_p \arrow[d, "\mathsf{F}_\Omega \mathfrak{q}_p"] \arrow[dl, phantom, "\circlearrowright"] \\ & \mathsf{F}_\Omega \mathcal{K} \end{tikzcd} \tag{4.19}$$

Proof. We simultaneously define the action of ρ_p and observe that it forms an isomorphism in the slice over $\mathsf{F}_\Omega\mathcal{K}$. Essentially, the isomorphism is obtained by a rearrangement of data defining the n-cells in each 2-category in a way similar to how the definition of $\mathsf{F}_\Omega\mathcal{K}$ was obtained in the first place.

0-cells of $\mathfrak{i}_{\mathcal{K}} \Downarrow p$ are given by a 1-category C, an oplax functor $P\colon C \to \mathcal{K}$ and a morphism in $\mathsf{F}_\Theta\mathcal{K}$ from $P'\colon C^\dagger \to \mathcal{K}$ to $p : \mathcal{X} \to \mathcal{K}$. Such a morphism is equivalently an oplax functor $R\colon C \to \mathfrak{Q}_p$ satisfying $\mathfrak{q}_p\, R = P$. This last condition means the data P can be determined from R, so we can identify each 0-cell of $\mathfrak{i}_{\mathcal{K}} \Downarrow p$ with the tuple $(C, R\colon C \to \mathfrak{Q}_p)$ which — viewed as a 0-cell in $\mathsf{F}_\Omega\mathfrak{Q}_p$ — is the image of $(C, P, R) \in \mathfrak{i}_{\mathcal{K}} \Downarrow p$ under ρ_p.

1-cells in $\mathfrak{i}_{\mathcal{K}} \Downarrow p$ from ${\rho_p}^{-1}(C, R)$ to ${\rho_p}^{-1}(D, S)$ are given by:

(a) a 1-cell in $\mathsf{F}_\Omega\mathcal{K}$ from $P = \mathfrak{q}_p R$ to $Q = \mathfrak{q}_p S$ which is a functor $F\colon C \to D$ and lax transformation $\phi\colon P \to QF$

(b) a strict 2-natural transformation Ξ from R to $S \star \overline{\phi}$, where $\star$ indicates the composition of Kleisli arrows in $\mathsf{F}_\Theta\mathcal{K}$ and $\overline{\phi}$ is the oplax functor $C \to \mathfrak{Q}_{Q'}$ corresponding to ϕ (cf. (4.6)). The 2-cell Ξ must satisfy $\mathfrak{q}_p\ \Xi = 1_P$.

We can dissect the data of item (b) to obtain a simpler description. First, note that the composition $S \star \overline{\phi}$ has the following action on objects and morphisms of C (modulo a choice of representatives for equivalence classes in $\mathfrak{Q}_p$):

$$\begin{tikzcd} c \arrow[d, "u"'] & & Pc \arrow[r, "\phi_c"] \arrow[d, "Pu"'] & QFc \arrow[d, "QFu"] \arrow[dl, phantom, "\overset{\phi_u}{\Longleftarrow}"] & & Pc \arrow[r, "\phi_c"] \arrow[d, "Pu"'] & QFc \arrow[r, "sF_c"] \arrow[d, dashed, "QFu"] \arrow[dl, phantom, "\overset{\phi_u}{\Longleftarrow}"] & p\hat{s}Fc \arrow[d, dashed, "p\hat{s}Fu"] \arrow[dl, phantom, "\overset{sF_u}{\Longleftarrow}"] \\ c' \arrow[r, phantom, "\overset{\overline{\phi}}{\longmapsto}"] & {} & Pc' \arrow[r, "\phi_{c'}"'] & QFc' \arrow[r, phantom, "\overset{\mu_p\,\mathfrak{Q}_S}{\longmapsto}"] & {} & Pc' \arrow[r, "\phi_{c'}"'] & QFc' \arrow[r, "sF_{c'}"'] & p\hat{s}Fc' \end{tikzcd}$$

Where we use $sd\colon Qd \to p(\hat{s}d)$ to denote the image of d under $S\colon D \to \mathfrak{Q}_p$ as a morphism of $\mathcal{K}$. A vertical natural transformation $\Xi\colon R \Rightarrow S \star \overline{\phi}$ therefore has component at $c \in C$ as shown below on the left:

$$\begin{tikzcd} Pc \arrow[rr, "rc"] \arrow[d, equal] & & p\hat{r}c \arrow[d, "p\Xi_c"] \arrow[dll, phantom, "\overset{\xi_c}{\Longleftarrow}"] & & Pc \arrow[r, "rc"] \arrow[d, "\phi_c"'] & p\hat{r}c \arrow[d, "p\Xi_c"] \arrow[dl, phantom, "\overset{\xi_c}{\Longleftarrow}"] \\ Pc \arrow[r, "\phi_c"'] & QFc \arrow[r, "sF_c"'] & p\hat{s}Fc \arrow[r, phantom, "\rightsquigarrow"] & {} & QFc \arrow[r, "sF_c"'] & p\hat{s}Fc \end{tikzcd}$$

Such vertical 1-cell $\Xi_c\colon Rc \to S \star \overline{\phi}c$ is equivalent to a 1-cell $\Xi'_c\colon Rc \to SFc$ that lies over ϕ_c, shown on the right above. The strict naturality of Ξ corresponds to the following equality of morphisms in $\mathfrak{Q}_p$ for each $u\colon c \to c'$:

$$\begin{tikzcd} Pc \arrow[rr, "rc"] \arrow[d, equal] & & p\hat{r}c \arrow[d, "p\Xi_c"] \arrow[dll, phantom, "\overset{\xi_c}{\Longleftarrow}"] & & Pc \arrow[rr, "rc"] \arrow[d, "Pu"'] & & p\hat{r}c \arrow[d, "p\hat{r}u"] \arrow[dll, phantom, "\overset{su}{\Longleftarrow}"] \\ Pc \arrow[r, dashed, "\phi_c"] \arrow[d, "Pu"'] & QFc \arrow[r, dashed, "sF_c"] \arrow[d, dashed, "QFu"] \arrow[dl, phantom, "\overset{\phi_u}{\Longleftarrow}"] & p\hat{s}Fc \arrow[d, dashed, "p\hat{s}Fu"] \arrow[dl, phantom, "\overset{sF_u}{\Longleftarrow}"] \arrow[r, phantom, "="] & {} & Pc' \arrow[rr, "rc'"] \arrow[d, equal] & & p\hat{r}c' \arrow[d, "p\Xi_{c'}"] \arrow[dll, phantom, "\overset{\xi_{c'}}{\Longleftarrow}"] \\ Pc' \arrow[r, "\phi_{c'}"'] & QFc' \arrow[r, "sF_{c'}"'] & p\hat{s}Fc' & & Pc' \arrow[r, "\phi_{c'}"'] & QFc' \arrow[r, "sF_{c'}"'] & p\hat{s}Fc' \end{tikzcd}$$

From this equality we observe that the 1-cells $\Xi'_c\colon Rc \to SFc$ form a lax transformation whose lax naturality 2-cell components are given by the unique opcartesian lifts of the components of ϕ. This may be easier to detect if we rearrange the data in the diagrams above slightly, (cf. the definition of 2-cells in $\mathfrak{Q}_p$ from Definition 3.3.10):

$$
\begin{tikzcd}
& Pc \arrow[rr, "rc"] \arrow[dl, "Pu"'] \arrow[dr, "\phi_c"] & \arrow[d, phantom, "\Leftarrow \xi_c"] & p\hat{r}c \arrow[dr, "p\Xi_c"] & \\
Pc' \arrow[dr, "\phi_{c'}"'] \arrow[r, phantom, "\Leftarrow \phi_u"] & & QFc \arrow[rr, "sF_c"] \arrow[dl, "QF_u"'] \arrow[d, phantom, "\Leftarrow sF_u"] & & P\hat{s}c \arrow[dl, "p\hat{s}F_u"] \\
& QF_{c'} \arrow[rr, "sF'_c"'] & {} & p\hat{s}F_{c'} &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
Pc \arrow[r, "rc"] \arrow[d, "Pu"'] & p\hat{r}c \arrow[d, "p\hat{r}u"] \arrow[dl, phantom, "\Leftarrow su"] \\
Pc' \arrow[r, "rc'"] \arrow[d, "\phi_{c'}"'] & p\hat{r}c' \arrow[d, "p\Xi_{c'}"] \arrow[dl, phantom, "\Leftarrow \xi_{c'}"] \\
QFc' \arrow[r, "sF_{c'}"'] & p\hat{s}Fc'
\end{tikzcd}
$$

The uniqueness of lifts of 2-cells along $\mathfrak{q}_p$ also lifts the coherence conditions of ϕ to those for Ξ'. Thus, to each 1-cell from ${\rho_p}^{-1}(C,R)$ to ${\rho_p}^{-1}(D,S)$ we uniquely associate:

(a) A functor $F\colon C \to D$

(b) A lax transformation $\Xi\colon R \Rightarrow SF$

which is a 1-cell from (C,R) to (D,S) in $\mathsf{F}_\Omega\mathfrak{Q}_p$. The lax transformation $\phi\colon P \to QF$ can be recovered from Ξ by post-composing with $\mathfrak{q}_p$. The functoriality of this mapping can be directly verified by elaborating the composition of 1-cells in $\mathfrak{i}_\mathcal{K} \Downarrow p$.

2-cells in $\mathfrak{i}_\mathcal{K} \Downarrow \mathfrak{Q}_p$ of type ${\rho_p}^{-1}(F,\Xi) \Rightarrow {\rho_p}^{-1}(G,\Pi)\colon {\rho_p}^{-1}(C,R) \to {\rho_p}^{-1}(D,S)$ are given by a 2-cell in $\mathsf{F}_\Omega\mathcal{K}$ from $\big(F, \phi := \mathfrak{q}_p\Xi\big)$ to $\big(G, \psi := \mathfrak{q}_p\Pi\big)$, which we know consists of a natural transformation $\gamma\colon F \Rightarrow G$ and a modulation $\Gamma\colon Q\gamma\ \phi \to \psi$. This 2-cell must satisfy the property that $R\overline{\Gamma}\ \Xi = \Pi$, where $\overline{\Gamma}$ denotes the 2-cell in $\mathsf{F}_\Theta\mathcal{K}$ that is the image of (γ,Γ) under $\mathfrak{i}_\mathcal{K}$. Equating the components of each of these 2-cells at $c \in C$ gives:

$$
\begin{tikzcd}
Pc \arrow[rr, "rc"] \arrow[d, equal] & \arrow[d, phantom, "\Leftarrow \xi_c"] & p\hat{r}c \arrow[d, "p\Xi_c"] \\
Pc \arrow[r, dashed, "\phi_c"] \arrow[d, equal] & QFc \arrow[r, dashed, "sF_c"] \arrow[d, dashed, "Q\gamma_c"] \arrow[dl, phantom, "\Leftarrow \Gamma_c"] & p\hat{s}Fc \arrow[d, dashed, "pS\gamma_c"] \arrow[dl, phantom, "\Leftarrow s\gamma_c"] \\
Pc \arrow[r, "\psi_c"'] & QGc \arrow[r, "sG_c"'] & p\hat{s}Gc
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
Pc \arrow[rr, "rc"] \arrow[dd, equal] & & p\hat{r}c \arrow[dd, "p\Pi_c"] \\
& \Leftarrow \pi_c & \\
Pc \arrow[r, "\psi_c"'] & QGc \arrow[r, "sG_c"'] & p\hat{s}Gc
\end{tikzcd}
$$

The 2-cells Γ_c therefore lift from $\mathcal{K}$ to define a modulation $\Gamma'\colon S\gamma\ \Xi \to \Pi$. The coherence conditions of Γ lift to ensure the coherence conditions of Γ' by the fact that $\mathfrak{q}_p$ is a discrete opfibration on hom-categories. Each 2-cell in $\mathfrak{i}_\mathcal{K} \Downarrow p$ from ${\rho_p}^{-1}(F,\Xi)$ to ${\rho_p}^{-1}(G,\Pi)$ can thus be identified with:

(a) a natural transformation $\gamma\colon F \Rightarrow G$

(b) a modulation $\Gamma\colon S\gamma\ \Xi \to \Pi$

which is equivalently a 2-cell in $\mathsf{F}_\Omega\mathfrak{Q}_p$ from (F,Ξ) to (G,Π). Inspecting the definition of 2-cells in $\mathfrak{i}_\mathcal{K} \Downarrow p$ reveals that this bijection is moreover 2-functorial. □

From this result we obtain our proof for Lemma 4.3.11:

Proof of Lemma 4.3.11. For any $A\colon C \to \mathsf{Cat}$, we know that $\mathsf{F}_\Theta\mathscr{E}A$ has all small oplax colimits because $[\mathscr{E}A^{\mathsf{op}}, \mathsf{Cat}]$ does and $\mathsf{W}_{\mathscr{E}A}\colon \mathsf{F}_\Theta\mathscr{E}A \to [\mathscr{E}A^{\mathsf{op}}, \mathsf{Cat}]$ creates oplax colimits. This means we only need to show that $1_{\mathscr{E}\mathcal{A}} \in \mathsf{F}_\Theta\mathcal{K}$ satisfies the 1-dimensional aspect of the universal property for the oplax colimit, i.e. that there is a *bijection* between the set of lax cocones from $\mathfrak{i}_\mathcal{K}\mathsf{G}_A$ to p and the underlying set of $\mathsf{F}_\Theta\mathscr{E}A(1_{\mathscr{E}A}, p)$ which is natural over $p\colon \mathcal{X} \to \mathscr{E}A$. As we've already observed, the set of lax cocones $\mathfrak{i}_\mathcal{K}\mathsf{G}_A$ is bijective with the set of lifts of G_a along $\pi_1\colon \mathfrak{i}_\mathcal{K} \Downarrow p \to \mathsf{F}_\Omega(\mathscr{E}\mathcal{A})$, which is just the underlying set of $2\mathsf{CAT}/\mathsf{F}_\Omega(\mathscr{E}A)(\mathsf{G}_A, \pi_1)$. The universal property of $1_{\mathscr{E}A}$ then follows from Lemma

4.3.12 and Corollary 4.3.9 (recall that ${\mathsf{G}_A}^\flat = 1_{\mathscr{E}A}$):

$$\mathsf{2CAT}/\mathsf{F}_\Omega(\mathscr{E}A)(\mathsf{G}_A,\pi_1) \cong \mathsf{2CAT}/\mathsf{F}_\Omega(\mathscr{E}A)\big(\mathsf{G}_A,\mathsf{F}_\Omega\mathfrak{q}_p\big) \cong \mathsf{2CAT}/\mathscr{E}A\big(1_{\mathscr{E}A},\mathfrak{q}_p\big) \cong \mathsf{F}_\Theta(\mathscr{E}A)(1_{\mathscr{E}A},p) \qquad \square$$

4.3.4 The Extralax Colimit of an Oplax 1-Functor

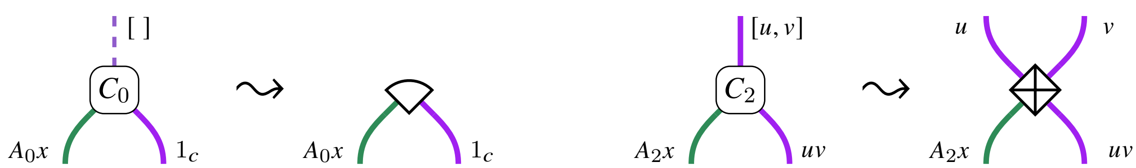
Stage (d) of our proof that $\mathfrak{i}_\mathcal{K}\colon \mathsf{F}_\Omega\mathcal{K} \to \mathsf{F}_\Theta\mathcal{K}$ lifts Ω-colimits requires that we show $\mathscr{E}A$ has component-initial 1-cells, and is thus in the essential image of $\mathfrak{i}_\mathcal{K}$. This property will not hold for general 2-functors $A\colon C \to \mathsf{Cat}$, so at this point we need to restrict to the case where A has domain $C^\dagger$ for some 1-category C. We will use $\mathscr{D}A$ to denote $\mathscr{E}A'$, where $A'\colon C^\dagger \to \mathcal{K}$ is the strict 2-functor classifying an oplax functor $A\colon C \to \mathcal{K}$.

Extralax colimits of the form $\mathscr{D}A$ admit a simplified presentation by exploitation of the canonical presentation of the oplax functor classifier $C^\dagger$ in terms of data in C (cf. Section 2.2.4). In particular, rather than having the type-one generators of the 1-cells in $\mathscr{D}A$ be arbitrary 1-cells of $C^\dagger$, we can choose to include only the "atomic" 1-cells, i.e. those of the underlying 1-category C. Since the 1-cells of $C^\dagger$ are freely generated by those of C, there are no relations on the generating 1-cells of $\mathscr{D}A$ when chosen in this way.

Similarly, all 2-cells of $C^\dagger$ are generated by the C_0 and C_2 2-cells. So, rather than having a generating 2-cell in $\mathscr{D}A$ for every 2-cell θ in $C^\dagger$ it suffices to include only those corresponding to the oplax-functoriality 2-cells C_0 and C_2. The type-four generators (cf. (4.12)) for these 2-cells have the following form:

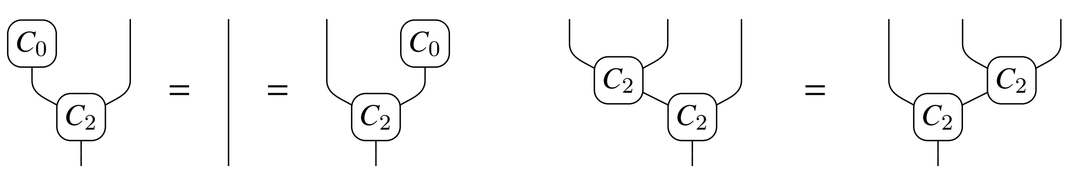
$$\begin{tikzcd}
(c,x) \arrow[rr, equal] & & (c,x) \arrow[dl, "{[1_c]_x}"] \\
& (c,A_{1_c}x) \arrow[ul, "{[A_0x]}"] \arrow[u, phantom, "\Uparrow{[C_0]_x}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& (d,A_vx) \arrow[dl, "{[u]_x}"'] & \\
(e,A_uA_vx) & \Uparrow{[C_2]_x} & (c,x) \arrow[ul, "{[v]_x}"'] \arrow[dl, "{[uv]_x}"] \\
& (e,A_{uv}x) \arrow[ul, "{[A_2x]}"] &
\end{tikzcd} \tag{4.20}$$

Because these 2-cells are determined by their codomain, we can suppress the $[C_0]$ and $[C_2]$ labels our string diagram representations, though we use symbols to indicate their type:



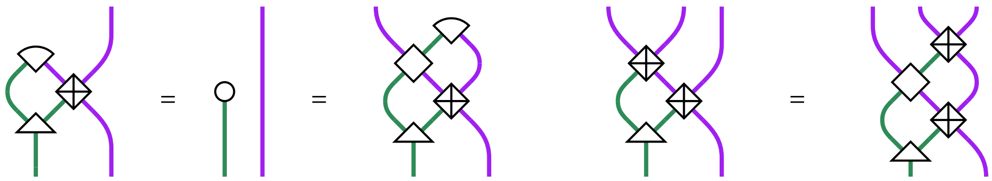
Note that the A_0 and A_2 in these string diagrams denote respectively the natural transformations $A_0\colon A_{1_c} \Rightarrow 1_{A_c}$, $A_2\colon A_{uv} \Rightarrow A_uA_v$ which define the oplax functoriality of A, or equivalently are the images of C_0 and C_2 under $A'\colon C^\dagger \to \mathsf{Cat}$. The appropriate domain and codomain of such 2-cells can always be inferred from context, which allows us to avoid notations such as $\big((A_2)_{u,v}\big)x$.

Having whittled away all $\boxed{\theta}$ generators except $\oplus$'s and $\triangledown$'s we can also remove all relations on $\boxed{\theta}$ generators other than those corresponding to the relations on C_0 and C_2 cells which give the presentation for $C^\dagger$. Those relations on C_0 and C_2 being:



(4.21)

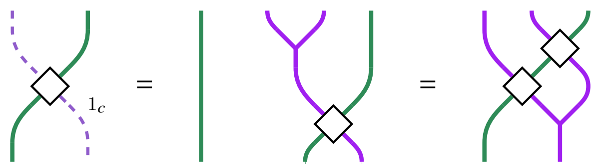
Expressing the $\boxed{\theta}$ 2-cell in $\mathscr{D}A$ for each of the 2-cells in the relations above in terms of $\oplus$ and $\triangledown$ by repeated use of the relations in (4.14) we obtain the necessary relations in $\mathscr{D}A$:



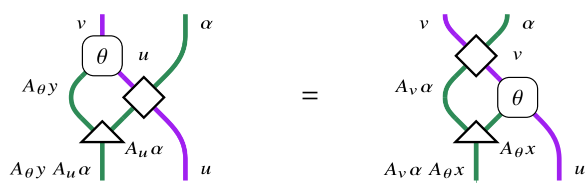
(4.22)

Remark 4.3.13. Erasing the green strings from these relations reveals the connection to the relations in (4.21). ◊

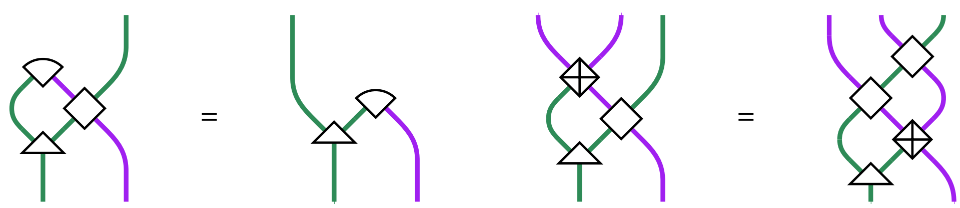
Generators of the third sort, ◇, now need only be defined for generating 1-cells in $C^\dagger$, rather than arbitrary 1-cells, and the relations requiring that these generators respect composition in C (shown below) become redundant:



Finally, we must replace the relations showing that ◇ commutes with 2-cells in $C^\dagger$:



These should be replaced with the specific instances of this relation for $\theta = C_0$ and $\theta = C_2$, which produces the following new relations:



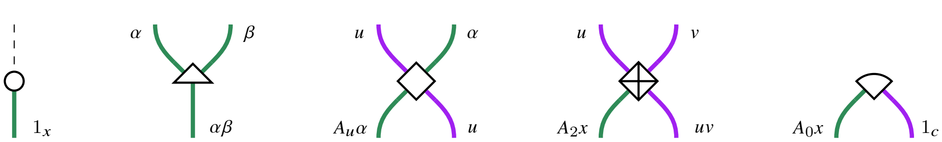
We summarise these alterations in the following definition:

Definition 4.3.14 (Extralax colimit of an oplax functor, $\mathscr{D}A$)**.** For an oplax functor from a 1-category, $A\colon C \to \mathsf{Cat}$ the extralax colimit $\mathscr{D}A$ admits the following presentation:

0-cells: pairs of the form $(c \in C, x \in A_c)$

1-cells: freely generated by $(c,x) \xrightarrow{[u]_x} (d, A_u x)$ for $u\colon c \to d \in C$ and $(c,x) \xrightarrow{[\alpha]} (c,y)$ for $\alpha\colon x \to y \in A_c$

2-cells: generated by the following five sorts of generators, indexed by arbitrary $v\colon c \to d$, $u\colon d \to e$ in C and $\beta\colon x \to y$, $\alpha\colon y \to z$ in A_c:



subject to the relations shown in Figure 4.1 (for any compatible labelling of the strings).

◊

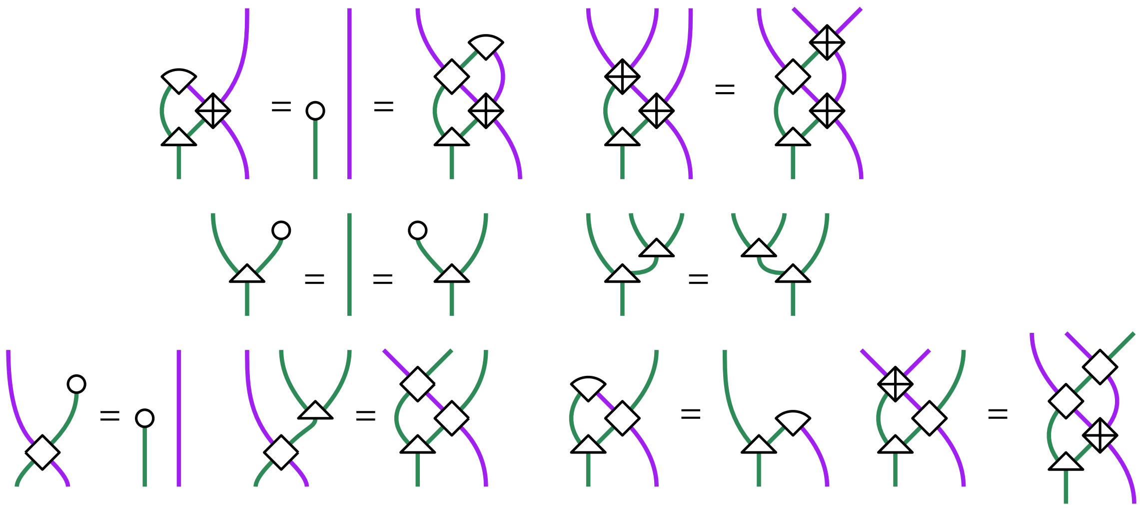


Figure 4.1: $\mathscr{D}A$ relations

Example 4.3.15 ($\mathscr{D}\tau$)**.** The simplest possible example of a $\mathscr{D}A$ category is where A is the functor from the terminal 1-category which picks out the terminal category in Cat. We denote this functor $\tau\colon \mathbb{1} \to \mathsf{Cat}$. The 2-category $\mathscr{D}\tau$ has a single object and 1-cells freely generated by a single type-one generator and a single type-two generator corresponding to the identities on $\mathbb{1}$ as the domain of τ and as the image of τ. We will call the two 1-cell generators purple and green respectively in allusion to the string diagrams. The 2-cells of $\mathscr{D}\tau$ are simply the "unlabelled" (i.e. labelled only by the colouring of strings) diagrams constructable from the generators in Definition 4.3.14 subject to the accompanying relations. ◊

4.3.5 Locally-initial 1-cells for extralax colimits

We are at stage (d) of the proof that $\mathbf{i}_{\mathcal{K}}\colon \mathsf{F}_\Omega\mathcal{K} \to \mathsf{F}_\Theta\mathcal{K}$ lifts Ω-colimits, where we must show:

Lemma 4.3.16. *For any oplax functor $A\colon C \to \mathsf{Cat}$, $\mathscr{D}A$ has enough component-initial 1-cells*

By Corollary 4.2.2, it will then follow that 2-functors $\mathscr{D}A \to \mathcal{K}$ viewed as objects of $\mathsf{F}_\Theta\mathcal{K}$ lie within the repletion of $\mathsf{F}_\Omega\mathcal{K}$.

We first observe that it suffices to demonstrate this property for $\mathscr{D}\tau$ of Example 4.3.15.

Construction 4.3.17 ($\partial\colon \mathscr{D}A \to \mathscr{D}\tau$)**.** For $A\colon C \to \mathsf{Cat}$ oplax from a 1-category there is a canonical map $\partial\colon \mathscr{D}A \to \mathscr{D}\tau$ given by sending all type-one generators to purple and all type-two generators to green. The map on 2-cells is given by forgetting the labels of the 1-cells in string diagrams. ◊

Lemma 4.3.18. *For any oplax functor $A\colon C \to \mathsf{Cat}$ from a 1-category, $\partial\colon \mathscr{D}A \to \mathscr{D}\tau$ is a discrete fibration on hom-categories.*

Proof. We need to show that for any 2-cell θ in $\mathscr{D}\tau$ and any lift $\overline{f}$ of the codomain f of θ to $\mathscr{D}A$ there exists a unique lift of θ to a 2-cell in $\mathscr{D}A$ with codomain $\overline{f}$. The map ∂ acts on 2-cells essentially by "unlabelling" all the strings in a given string diagram for the 2-cell. First, observe that given any unlabelled string diagram for $\mathscr{D}\tau$, a labelling of the strings along the top of the diagram with generators in $\mathscr{D}A$ determines a unique compatible labelling for the entire diagram. This is true for general string diagrams because it holds for each of the generators. Of course, there may be multiple string diagram representations for a given 2-cell in $\mathscr{D}\tau$, but because the relations on string diagrams for $\mathscr{D}A$ don't depend on the labelling of the 1-cells it must be the case that induced labellings of equivalent string diagrams in $\mathscr{D}\tau$ produce equivalent string diagrams in $\mathscr{D}A$. The unique lifting property for string diagrams therefore implies the lifting property for 2-cells. □

The proof that connected components of the hom-categories in $\mathscr{D}A$ have initial objects will refer to 1-cells in $\mathscr{D}A$ of the form $[\alpha, u]$, i.e. a type-one generator to the right of a type-two generator. We will call such 1-cells *normal*. If we think of each generating 2-cell of $\mathscr{D}A$ as a rewriting rule for the 1-cells from its codomain to its domain, for example the ◇ rule allows us to make the reduction $[u, \alpha, v] \to [A_u\alpha, u, v]$, then these normal 1-cells are indeed normal forms with respect to this reduction system. The following lemma states that any two "reductions" from a 1-cell to its corresponding normal 1-cell are equal as 2-cells:

Lemma 4.3.19. *For an oplax functor from a 1-category $A\colon C \to \mathsf{Cat}$, every 1-cell admits a unique 2-cell from a normal 1-cell.*

Part of the proof for Lemma 4.3.19 involves lengthy combinatorial arguments, which we will defer until Section 4.5. Below we give a sketch of the proof, and point out where the combinatorial justification becomes necessary.

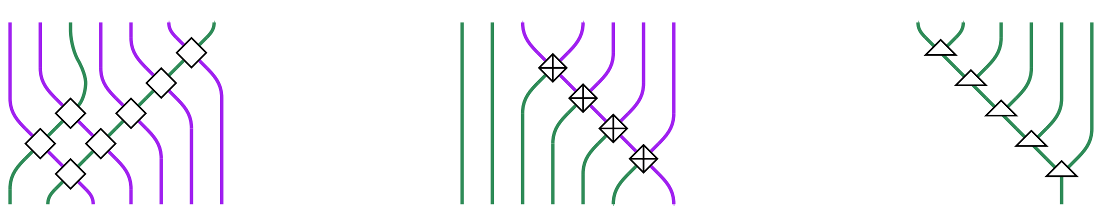
Sketch of proof. Observe that the canonical functor ∂ of Example 4.3.17 preserves and reflects normal 1-cells. Therefore, by Lemma 4.3.18 it suffices to show that the result holds for the case $\mathscr{D}\tau$. A 1-cell in $\mathscr{D}\tau$ is just a sequence of green's and purple's. We can construct a 2-cell from the unique normal 1-cell in $\mathscr{D}\tau$ to any given 1-cell u by building a string diagram below u according to the following procedure:

(a) Bring all green's to the left of all the purple's by attaching ◇'s.

(b) Convert all but the left-most purple to green's moving left to right by attaching ◈'s (travelling downwards, it looks like ◈ takes in a purple on the right and outputs a green on the left). If there are no purple's, instead use a ▽ to create one.

(c) Combine all green's into a single green by attaching △'s moving left to right. If there are no green's, instead use a ○ to create one.

These three stages are shown in Figure 4.2 for the 1-cell ●●●●●●●, which produces the 2-cell from a normal 1-cell shown in Figure 4.3.



Stage (a) Stage (b) Stage (c)

Figure 4.2: Constructing a 2-cell from a normal 1-cell

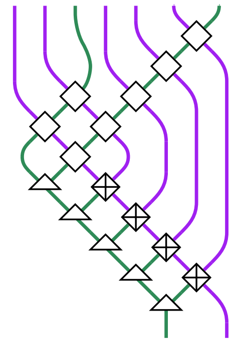


Figure 4.3: The constructed 2-cell from a normal 1-cell into

All 1-cells in $\mathscr{D}\tau$ admit a 2-cell from a normal 1-cell given by this construction. While the string diagram in Figure 4.2 is not unique among string diagrams from a normal 1-cell with that codomain (we could, for example, change the order we combine the green's at the end using $\triangle$'s), all string diagrams from normal 1-cells are equivalent modulo the relations on the generators. To show this we can establish a normal form for string diagrams from a normal 1-cell such that every 2-cell has a unique representation by a normal string diagram. We then show that a normal diagram with domain a normal 1-cell is uniquely determined by its codomain. The definition of normal diagram and a rigorous proof that they are determined uniquely by their codomain is what we defer to Section 4.5. □

Corollary 4.3.20. *The 2-category $\mathscr{D}A$ has enough component-initial 1-cells.*

This completes the last stage of the proof that $\mathsf{F}_\Omega\mathcal{K} \to \mathsf{F}_\Theta\mathcal{K}$ lifts Ω-colimits. In summary, we have shown that for any oplax functor from a 1-category $B\colon C \to \mathsf{F}_\Omega\mathcal{K}$:

(a) The transpose $B^\flat\colon \mathscr{D}A = \mathscr{E}A' \to \mathcal{K}$ under the adjunction $\mathscr{E}/\mathcal{K} \dashv \mathsf{F}_\Omega\mathcal{K}$ viewed as an object of $\mathsf{F}_\Theta\mathcal{K}$ is the oplax colimit of the oplax functor $\mathfrak{i}_\mathcal{K}B$.[10]

(b) $\mathscr{D}A$ has enough component-initial 1-cells, and so by Corollary 4.2.2, $B^\flat$ is isomorphic in $\mathsf{F}_\Theta\mathcal{K}$ to the image of the oplax functor $B^\flat\lambda$ under $\mathfrak{i}_\mathcal{K}\colon \mathsf{F}_\Omega\mathcal{K} \to \mathsf{F}_\Theta\mathcal{K}$, where $\lambda\colon \pi_1(\mathscr{D}A) \to \mathscr{D}A$ is the canonical oplax right adjoint to $\mathsf{p}_1\colon \mathscr{D}A \to \pi_1(\mathscr{D}A)$.

Corollary 4.3.21. *The class Ω of weights for oplax colimits of oplax colimits from 1-categories is presaturated.*

Corollary 4.3.22. *For any locally small 2-category $\mathcal{K}$, the oplax-image presheaf map $\mathfrak{I}\colon \mathsf{F}_\Theta\mathcal{K} \to [\mathcal{K}^{\mathsf{op}}, \mathsf{Cat}]$ restricts to an equivalence $\mathsf{F}_\Omega\mathcal{K} \simeq \Omega\mathcal{K}$, and the inclusion $\mathcal{K} \to \mathsf{F}_\Omega\mathcal{K}$ sending $k \in \mathcal{K}$ to $k\colon \mathbb{1} \to \mathcal{K}$ endows $\mathsf{F}_\Omega\mathcal{K}$ with the structure of the free Ω-cocompletion of $\mathcal{K}$.*

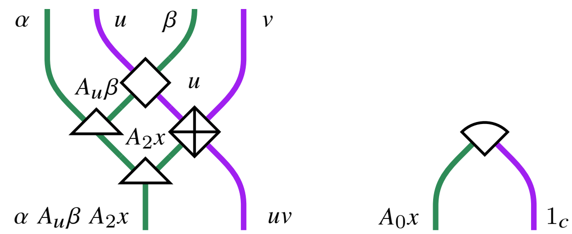
4.4 Ω-Colimits in $\mathsf{F}_\Omega\mathcal{K}$

We now know that the oplax colimit of an oplax 1-functor $B\colon C \to \mathsf{F}_\Omega\mathcal{K}$ is given by $B^\flat\lambda\colon \pi_1(\mathscr{D}A) \to \mathscr{D}A \to \mathcal{K}$. We can give a more elementary definition by the following observation:

Proposition 4.4.1. *For $A\colon C \to \mathsf{Cat}$ oplax from a 1-category, $\pi_1(\mathscr{D}A)$ is isomorphic to the ordinary Grothendieck construction $\int A$.*

[10]When $B\colon C \to \mathsf{F}_\Omega\mathcal{K}$ is an oplax functor, we will interpret $B^\flat$ to mean $(B')^\flat$.

Proof. The 1-cells of $\pi_1(\mathscr{D}A)$ can be identified with the normal 1-cells of $\mathscr{D}A$. Recall a normal 1-cell from (c,x) to (d,y) in $\mathscr{D}A$ is a pair of a type-one generator $u\colon c \to d$ and a type-two generator $\alpha\colon A_u x \to y$, which coincides with the usual definition of a 1-cell $(u,\alpha)\colon (c,x) \to (d,y)$ in $\int A$. Composition of 1-cells in $(\mathscr{D}A)_1$ is given by concatenating normal 1-cells and then reducing the result to a new normal 1-cell (shown on the left below):



(4.23)

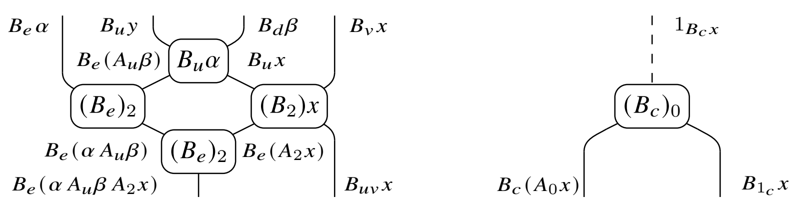
The normal 1-cell in the domain gives the usual composition in $\int A$. In fact, the 2-cell in (4.23) is the oplax-functoriality 2-cell λ_2 for the composition of (u,α) and (v,β). Similarly, the identity at (c,x) in $\pi_1(\mathscr{D}A)$ is the normal 1-cell in the component of the identity in $\mathscr{D}A$ given by the domain of the normal string diagram on the right of (4.23), which coincides with the identity on (c,x) in $\int A$. □

Remark 4.4.2. Another way to see that $\int A$ and $\pi_1(\mathscr{D}A)$ coincide is to observe that $\mathsf{F}_\Omega!\colon \mathsf{F}_\Omega\mathcal{K} \to \mathsf{Cat}$ is given by sending an oplax functor to its domain, and this 2-functor preserves $\boldsymbol{\Omega}$-colimits. So the domain of the oplax colimit of $B\colon C \to \mathsf{F}_\Omega\mathcal{K}$ — which we know to be $\pi_1(\mathscr{D}A)$ — must also be the oplax colimit in Cat of $A\colon C \to \mathsf{Cat}$, which is $\int A$. ◊

The oplax functor $B^\flat\lambda$ is just the restriction of $B^\flat$ to the normal 1-cells, which we recall sends a morphism $(u,\alpha)\colon (c,x) \to (d,y)$ to the 1-cell in $\mathcal{K}$:

$$B_c x \xrightarrow{B_u x} B_d A_u x \xrightarrow{B_d \alpha} B_d y$$

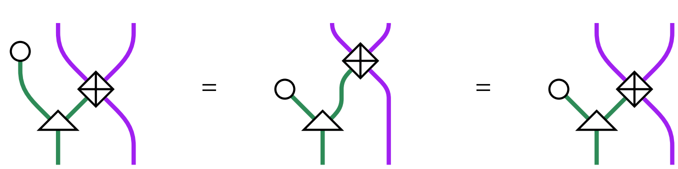
The oplax functoriality 2-cells are the images of the 2-cells in (4.23) under $B^\flat$ (cf. Construction 4.3.6):



(4.24)

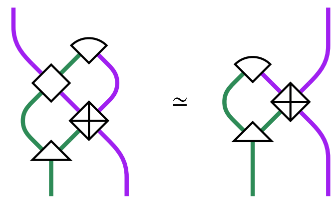
4.5 Normal diagrams in $\mathscr{D}\tau$

This section gives a rigorous proof that each 1-cell in $\mathscr{D}\tau$ has a unique 2-cell from a normal 1-cell. If this property holds for $\mathscr{D}\tau$, then it holds for $\mathscr{D}A$ for any oplax functor $A\colon C \to \mathsf{Cat}$ from a 1-category by Lemma 4.3.18. Our method is to describe a normal form for a string diagram of 2-cells in $\mathscr{D}\tau$ and then show both that there is a unique normal form string diagram from a normal 1-cell to any given 1-cell, and that every 2-cell from a normal 1-cell is represented by a normal form diagram. We first establish some terminology for talking about string diagrams of $\mathscr{D}\tau$. We will call two string diagrams *equal* if one can be deformed into the other, i.e. they represent the same 2-cell in the 2-category freely generated by the generators. For example, the following strings are considered equal:



(4.25)

If two string diagrams represent the same 2-cell in $\mathscr{D}\tau$, we shall call them *equivalent*. So a diagram D_1 for $\mathscr{D}\tau$ is equivalent to another diagram D_2 precisely if D_2 can be obtained from D_1 (up to string equality) through a sequence of substitutions by the relations of $\mathscr{D}\tau$ described in Definition 4.3.14. For example, the following two diagrams are equivalent and not equal:


(4.26)

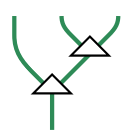
We can establish a partial ordering on the generators of a $\mathscr{D}\tau$ string diagram by declaring a generator X to be greater than another generator Y if one of the "outputs" (strings in the codomain) of Y is an "input" (string in the domain) of X, then taking the reflexive transitive extension. For example, the generators in the diagram on the left of (4.26) are totally ordered as $\triangle < \diamondplus < \Diamond < \bigtriangledown$, whereas the $\circ$ and $\diamondplus$ of the diagram in (4.25) are incomparable, but both greater than the $\triangle$. If in a string diagram generator X is greater than another generator Y according to this partial order, we will say X is *above* Y, even though this doesn't correspond precisely to the relative vertical positions of the generators, which is in any case not stable under string deformations. We say a generator has higher *rank* than another if the *type* of the first generator occurs to the right of the type of the second in the sequence:

$$\circ < \triangle < \bigtriangledown < \diamondplus < \Diamond$$

Definition 4.5.1 (Normal diagram)**.** A string diagram for $\mathscr{D}\tau$ is *normal* if it satisfies the following two conditions:

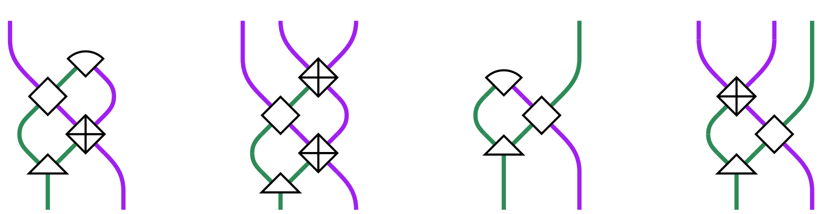
(a) No generator occurs above a generator of higher rank.

(b) The input of a $\triangle$ is never the right-hand output of another $\triangle$. In other words, there is no instance of the following subdiagram:

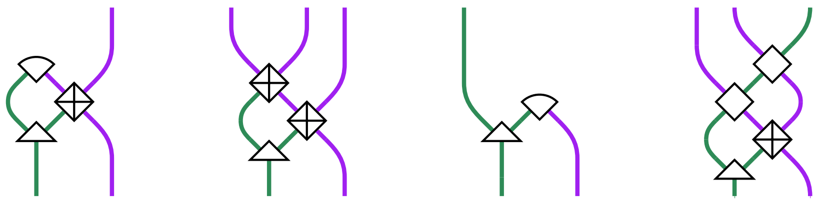


$\Diamond$

The string diagram in (4.25) is not normal, for example, because there is a $\circ$ above a $\triangle$, which has higher rank. Similarly, all of the following diagrams violate the rank condition, and so are not normal:



whereas the following diagrams are all normal (and respectively equivalent to the diagrams above):



The purpose of condition (b) is to fix the string diagram expressions for the unique 2-cells $\alpha_1 \ldots \alpha_n \Rightarrow [\alpha_1, \ldots, \alpha_n]$ in $A_c^\dagger$. We could instead require that the △'s are never attached to the left of another △ without significant alteration.

We will first show that any 1-cell in $\mathscr{D}\tau$ admits a unique normal string diagram from the unique normal 1-cell in $\mathscr{D}\tau$: ●●. Recall that a normal 1-cell in $\mathscr{D}A$ is any 1-cell which is expressed as a type-two generator to the left of a type-one generator. As an intermediate step, we will also refer to *separated* 1-cells in $\mathscr{D}\tau$, which are the 1-cells where all green's occur to the left of purple's. For example, ●●●, ●●, ● are all separated, but ●●● is not.

Lemma 4.5.2. *Any normal form string diagram from a normal 1-cell can be cut uniquely into diagrams A followed by B where:*

(a) A is a normal diagram from a normal 1-cell containing no ◇'s

(b) B is a normal diagram from a 1-cell containing only ◇'s

Proof. From the definition of a normal diagram, the only generator that can be above a ◇ is another ◇. It follows that if we cut every string that comes from either the domain or a non-◇ generator and goes to either the codomain or a ◇ then we will have horizontally bisected the string diagram, since any path from the bottom to the top of the diagram contains a unique such string. An example of this bisection is shown in Figure 4.4. □

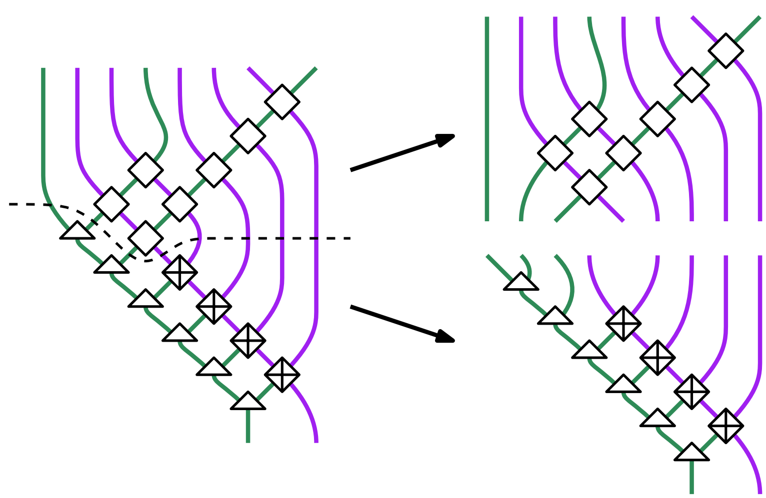


Figure 4.4: Bisecting a normal diagram

Lemma 4.5.3. *Any normal diagram from a normal 1-cell has no ◇'s if and only if its codomain is separated.*

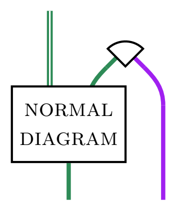
Proof. In a normal diagram from a normal 1-cell, every purple either comes from the domain or is the output of a ⟐. If the purple input to a ⟐ is to the right of all green's, then so are the outputs. So the result follows by induction, noting that the purple in the domain is to the right of all green's. On the other hand, if a normal diagram contained ◇'s, these would necessarily occur above all other generator types by the rank requirement. A ◇ would necessarily move a purple to the left of a green which cannot be undone by the application of other ◇'s and would thus produce a non-separated codomain. □

Lemma 4.5.4. *For any 1-cell f in $\mathscr{D}\tau$ there is a unique separated 1-cell g and diagram from g to f consisting only of ◇'s.*

Proof. The ◇ generators only allow a purple to be exchanged with a green to its right (moving from top to bottom). There is a unique way to produce a separated 1-cell by a sequence of these moves (up to string deformations).[11] □

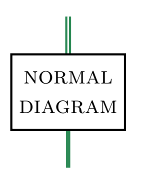
Lemma 4.5.5. *For any separated 1-cell f there is a unique normal diagram from a normal 1-cell to f.*

Proof. We perform iterated induction on the number of purple's in f. First, consider the case where f contains no purple's. The only generator that decreases the number of purple's (travelling upward) is ▽, so the normal diagram must contain this generator. Moreover, the purple input to ▽ must be the purple in ●●, because ▽ cannot follow a ⟐. Thus, the diagram must have the following form:



where ═ indicates an arbitrary number (including zero) of green's. Later, we will also use ═ to indicate a tuple of purple's, and ═ to indicate an arbitrary 1-cell in $\mathscr{D}\tau$ (i.e. tuple of green's and purple's). The normal subdiagram with a single green as input is unique. It is essentially a string diagram for a 2-cell in ${A_c}^\dagger$ from an atomic 1-cell, which is unique once we require that no ○ follows a △ and that the △'s are left-aligned, as per Conditions (a) and (b) of the definition of a normal diagram.

The case where f contains a single purple is similar. The normal diagram for such a 2-cell cannot contain a ▽, since this would necessarily absorb the purple in ●● leaving no way to produce the purple in f. The diagram also cannot contain any ⟐'s — the presence of these generators in a normal diagram always increases the number of purple's in the codomain as they cannot be followed by a ▽. We conclude that the purple in the normal 1-cell ●● is connected to the purple in f directly, and the normal diagram to f is of the following form:

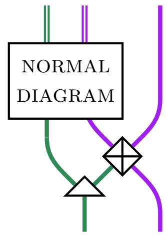


and is therefore unique.

If f has two or more purple's the normal diagram must contain a ⟐, and in particular the purple in the domain must be the right input to a ⟐. The right output purple of this ⟐ cannot be the right input to some other ⟐, as this would require a green to cross over purple's to arrive as the left input, which cannot happen in the absence of ◇'s. Any normal diagram containing a ⟐ cannot have any ▽'s, which would necessarily be above the ⟐

[11] The 1-cell f is obtained from g by a (p, q)-shuffle, where g has p green's and q purple's. There is a certain representation of such shuffles by paths between corners of a $(p+1) \times (q+1)$ grid, from which the ◇ diagram can be obtained by placing a ◇ in every cell below the path.

attached to the domain and thus violate the rank condition. There also cannot be any $\bigcirc$'s, as if the green input to a $\bigcirc$ came from the domain, there would be no way to produce a green input for the $\oplus$. The input to a $\bigcirc$ cannot come from any other generator without violating the rank condition. Thus, the number of $\triangle$'s in a normal diagram containing a $\oplus$ is equal to the length of the codomain minus two, as $\triangle$ is the only generator that increases the number of strings (by one) and the generators $\bigcirc$ and $\triangledown$ which decrease the number of strings cannot appear. In particular, if f is equal to $\bullet\bullet$, then there can be no $\triangle$'s and the unique normal diagram in this case is a single $\oplus$. Otherwise, the green in the domain is connected to a $\triangle$ whose right output is the left input of the first $\oplus$. The normal diagram to f therefore has the form:



and is thus unique by the induction hypothesis. □

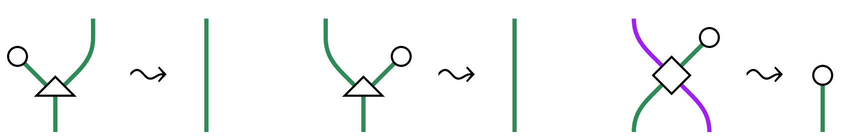
Proposition 4.5.6. *For any 1-cell f in $\mathscr{D}\tau$ there exists a unique normal string diagram from $\bullet\bullet$ to f.*

Proof. By Lemma 4.5.2 any such normal diagram can be bisected in a unique way into a normal diagram $B\colon \bullet\bullet \to f'$ with no $\Diamond$'s, followed by a diagram $A\colon f' \to f$ with only $\Diamond$'s. By Lemma 4.5.3 the 1-cell f' must be separated, so A is uniquely determined by f by Lemma 4.5.4 and B is uniquely determined from f' by Lemma 4.5.5. □

It remains to show that every string diagram from $\bullet\bullet$ is equivalent to a normal diagram. We can demonstrate this by showing that each generator can either be brought below every higher-rank generator or eliminated by a sequence of equivalence-preserving substitutions. We begin with the $\bigcirc$-type generators:

Lemma 4.5.7. *Every string diagram is equivalent to a string diagram where $\bigcirc$ does not occur above any other generator.*

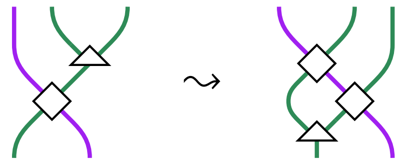
Proof. There are three ways this condition could fail for a general string diagram: the input could either be the left or right output of $\triangle$ or the right output of $\Diamond$. In any of these cases we can use one of the following relations to either remove the $\bigcirc$ or reduce the number of generators between it and the domain by one:



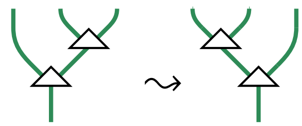
□

Lemma 4.5.8. *Every string diagram from a normal 1-cell is equivalent to a diagram where $\triangle$ never occurs above a generator of higher rank, or connected to the right output of another $\triangle$.*

Proof. Any $\triangle$ above a higher rank generator must have a $\Diamond$ along the green path from its input back to the domain, so any minimal violating $\triangle$ must directly follow a $\Diamond$. We can therefore pull the $\triangle$ through the $\Diamond$ as shown below, thus decreasing the number of ◇'s below the △:



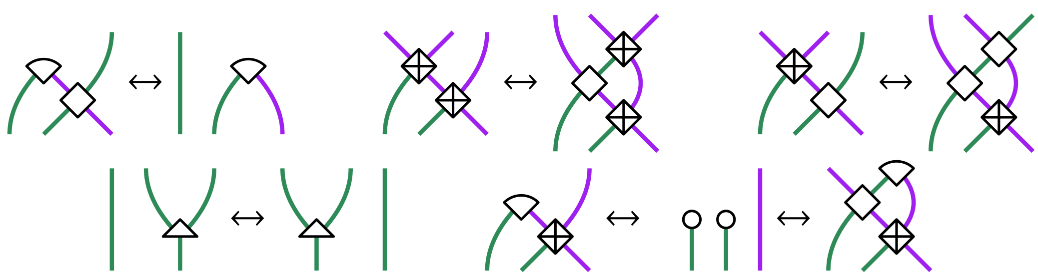
Repeating this for the minimal violating △ eventually results in the number of ◇'s below every △ reaching zero, at which point no △'s are above a higher rank generator. We then ensure that no △ is attached to the right output of another △ by repeated application of the following substitution:



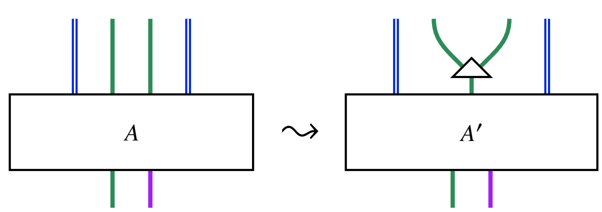
□

Note that we did not need to assume that the domain of the diagram is •• for Lemmas 4.5.7 and 4.5.8to hold. However, the requirement that the domain be normal will be necessary when considering the other generator types. Indeed, it is not the case that every string diagram from an arbitrary 2-cell is equivalent to a normal diagram. The value of the condition that the domain be a normal 1-cell is that it provides certain additional equivalence-preserving substitutions beyond those of the $\mathscr{D}\tau$ relations, as described in Lemmas 4.5.9 and 4.5.10.

Lemma 4.5.9. *For a string diagram from a normal 1-cell in $\mathscr{D}A$, the following substitutions preserve equivalence of diagrams:*

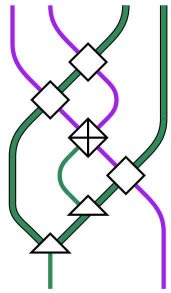


Proof. Notice that these substitutions have two adjacent green's in their domain, and that pre-pending a △ to the two green's in any of the above substitutions gives a relation that holds for general diagrams in $\mathscr{D}A$ (cf. 4.3.14). It suffices, therefore, to show that when the domain of the entire diagram is normal one can always arrange for a △ to be the input of the two green's by substitutions below the subdiagram in question. In other words, for every diagram A shown below, there exists a diagram A' such that the two diagrams below are equivalent:

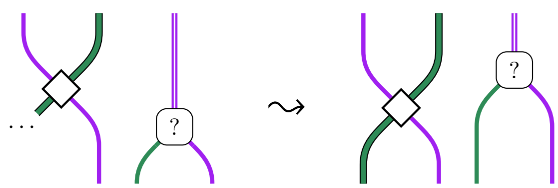


(4.27)

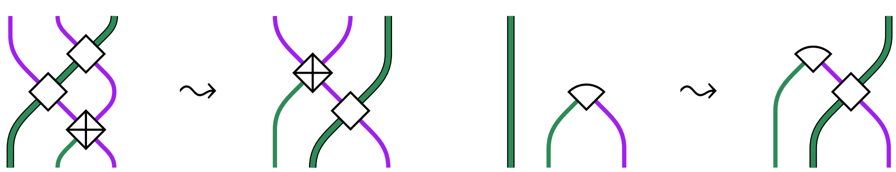
Since every generator has at most one green input, we can trace a path of green's from each of the adjacent green's in the output of A all the way back to the unique green in the domain. At some point the two paths merge, which must occur in a △, which we will refer to as the *source* of the two green's. We prove the existence of a diagram A' satisfying (4.27) by induction on the number of generators encountered on or between the green paths back to the source, noting that the case of zero generators is trivial (we don't count the source △). We will use *bounded* to mean "between or on" the green paths, and *interior* to mean bounded but not on the green paths. For example, there are five bounded generators for the two green paths in the following diagram (indicated in bold) of which one is interior:



First, we observe that we can replace A of (4.27) with an equivalent string diagram that has no interior generators. Any interior ○'s can easily be removed by Lemma 4.5.7. All remaining interior generators must occur on some interior green path between the bounding paths, as every generator has a green input. Interior green paths necessarily terminate at an interior generator as they cannot merge with or cross the bounding paths. Consider the terminus of the left-most interior green path, which is either a ◈ or a ⌓. Any purple to the left of this generator cannot be attached to the codomain of A, because the bounding green paths are adjacent at the codomain. It must therefore be attached to a generator whose green input is on the left-bounding green path (the purple is left of the left-most interior green path), and this generator is therefore a ◇. This ◇ can be pushed below the left-most terminal ◈ or ⌓ by string deformations (we use $\boxed{?}$ to represent both cases simultaneously):

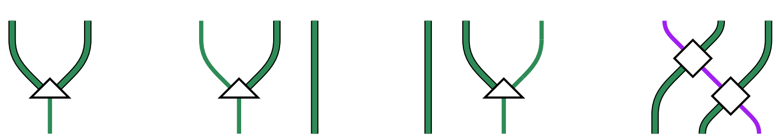


We may therefore assume that the left-most terminal interior generator is adjacent to the left bounding green path. The ◈ or ⌓ can then be removed from the interior using one of the substitutions from Lemma 4.5.9 by the induction hypothesis:



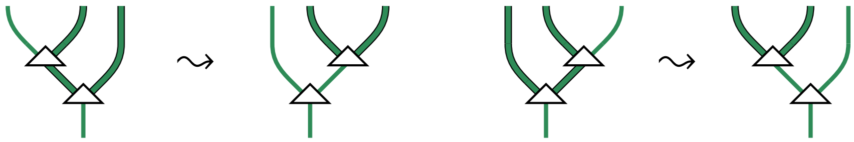
The induction hypothesis can be applied because there are strictly fewer generators bounded by the green paths from the two greens on the left of the domains than between the original bounding paths, thus we assume we can raise up a △ as their source to enable the substitution. Either substitution strictly decreases the number of interior terminating generators (i.e. ⌓'s and ◈'s). Repeating this process will therefore eventually remove all interior terminating generators, after which there can be no interior green paths and thus no interior generators.

Having removed all interior generators, there are three exhaustive cases for the last generators that occur along the green paths from their input to the subdiagram:

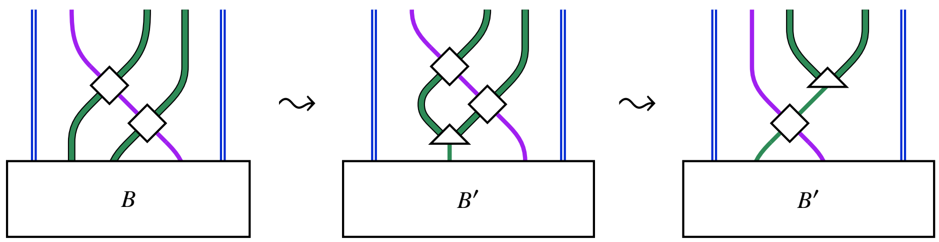


We rule out the possibility the left/right green path is the left/right output of a non-source △ by noting that this would produce an interior green path, which cannot exist in the absence of interior generators. We can also rule out the case where one of the green paths comes from a ◇ and the other is connected directly to the source △. This would result in an interior purple path which cannot terminate or originate in the interior, nor can it escape across the green path connected directly to the source △ which therefore contains no ◇'s. So if one path ends in a ◇, the other ends in either a non-source △ (cases 2 and 3) or another ◇ (case 4).

Case (1) is the desired configuration. For Cases (2) and (3), one can use the induction hypothesis to assume the source of the two green's is a △, in whose presence we can form the following substitutions:



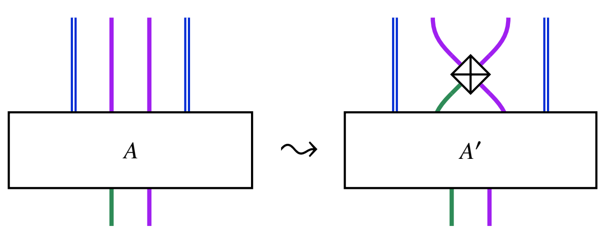
For case (4) we do the same, but this time we "pull through" the ◇ using another one of the $\mathscr{D}\tau$ relations:



□

Lemma 4.5.10. *The following substitutions preserve equivalence of diagrams from a normal 1-cell in $\mathscr{D}A$:*

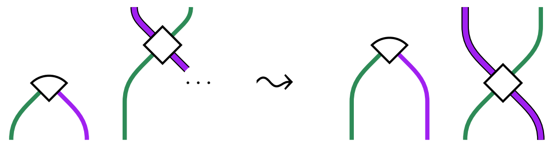
Proof. The domains of the diagrams in both substitutions are the same: a green followed by two purple's. If a ⟐ was attached as the source of the two purple's then we obtain on of the substitutions in Lemma 4.5.9. As with the proof for Lemma 4.5.9, we can show that if these subdiagrams occur in a larger diagram whose domain is a normal 1-cell then we can bring a ⟐ up to be the source of the two purple's by manipulations below the subdiagram. That is, for every diagram A as shown below on the left, there exists another diagram A' such that the following two diagrams are equivalent:



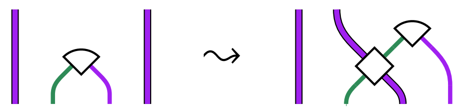
(4.28)

It follows that we can then perform the substitution in the presence of the ⟐, and then push the ⟐ back down the diagram. Because any generator with a purple output has exactly one purple input, we can trace a path of purple strings back from the subdiagrams to a source ⟐ at which they must meet. We demonstrate that an equivalence of the form (4.28) always hold by induction on the number of generators bounded by the two purple paths, observing that when this number is zero the result is trivial. First, we show that all interior generators can be removed. Generators of type ○ can be removed as already discussed in the proof for Lemma 4.5.7. We can also remove all interior △'s by Lemma 4.5.8, noting that an interior △ must occur on a green path above a ◇ where it crosses the left bounding purple path.

All remaining interior generators occur on an interior purple path that must terminate at a . Consider the which terminates the right-most interior purple path. Any green to its right must directly cross the right bounding purple path at a , which can be deformed to be below the :

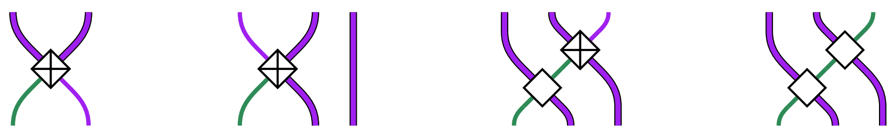


The can then be removed from the interior using the substitution on the right in Lemma 4.5.10 by the induction hypothesis.

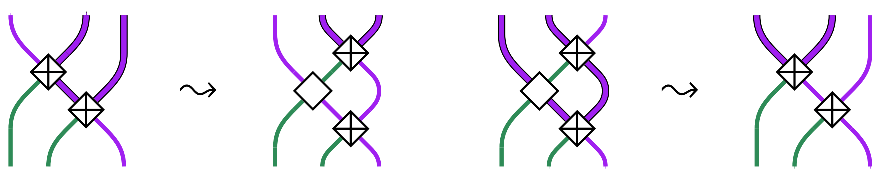


This substitution decreases the number of interior generators, so we may repeat this procedure until none remain.

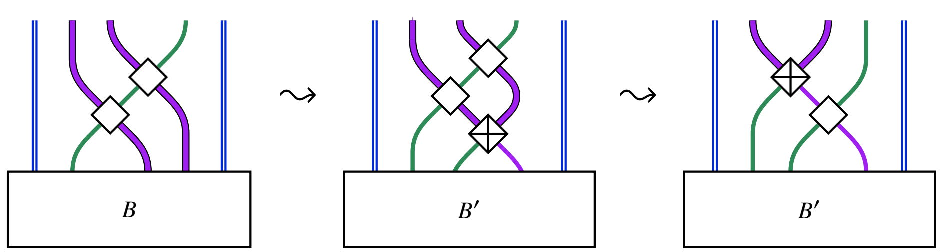
Once there are no generators strictly between the purple paths then there are four possible cases for the last generators to appear along the purple paths:



We rule out the possibility that the left/right purple is the left/right output of a non-source because this would result in an interior purple path, which cannot exist in the absence of interior generators. If the right purple path isn't directly connected to the source , then it generates an interior green path, which must cross the left purple path, making it impossible that the left purple path is directly connected to the source . Either it ends in a non-source (case (2)) or it ends in a where the interior green path crosses the left purple path (cases (3) and (4)). The four cases above are therefore exhaustive. Case (1) is the desired configuration. In cases (2) and (3) we use the induction hypothesis to raise a as the source of the two purple's, at which point we can apply one of the substitutions of Lemma 4.5.9:



For case (4) we use the induction hypothesis to raise a as the source of the two purple's, then "pull it through" using one of the substitutions of Lemma 4.5.9:



□

We now show that with these additional substitutions we can convert any string diagram from to a normal diagram, to which it is therefore equivalent.

Proposition 4.5.11. *Any string diagram for a 2-cell in $\mathscr{D}\tau$ from ●● is equivalent to a normal diagram.*

Proof. By Lemmas 4.5.7 and 4.5.8 we can ensure all $\bigcirc$'s and $\triangle$'s only occur below all higher-rank generators. Note that if the diagram contains any $\triangle$'s, then resolving the $\bigcirc$'s will eliminate all instances of $\bigcirc$, whose input can only be a green in the domain. A $\bigtriangledown$ will violate the rank condition if its purple input comes from another generator, or if its green path contains a $\Diamond$. Having moved all $\triangle$'s below $\Diamond$'s, this only happens if the green input of the $\bigtriangledown$ is directly connected to a $\Diamond$. So, for any violating $\bigtriangledown$ at least one of the following four cases holds:

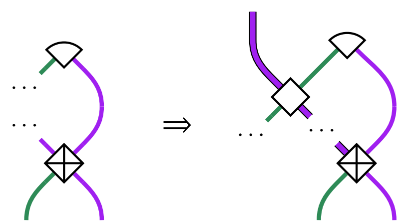
(a) The purple input of $\bigtriangledown$ comes from a $\Diamond$

(b) The green input of $\bigtriangledown$ comes from a $\Diamond$

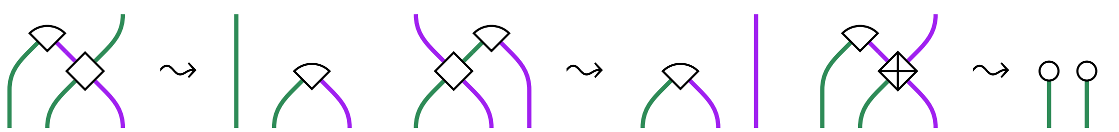
(c) The purple input of $\bigtriangledown$ is the left-hand output of a $\boxplus$

(d) The purple input of $\bigtriangledown$ is the right-hand output of a $\boxplus$

If we specifically consider the left-most violating $\bigtriangledown$[12] then if case (d) holds, so must case (a). This is because any purple path which comes from the left output of the $\boxplus$ cannot terminate in $\bigtriangledown$ (which would be rank-violating and left of the left-most such $\bigtriangledown$) and thus the green path from the green input to the left-most $\bigtriangledown$ must contain a $\Diamond$ where a purple path crosses it:

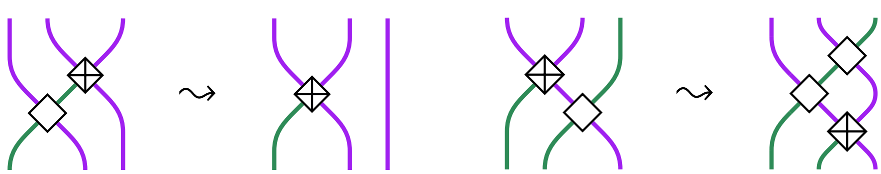


Thus, one of the first three cases must hold for the left-most violating $\bigtriangledown$, and we can either decrease the number of greater generators below this $\bigtriangledown$ or remove the $\bigtriangledown$ entirely by one of the substitutions from Lemmas 4.5.9 and 4.5.10:



Each of the substitutions above decreases the number of $\Diamond$'s or $\boxplus$'s in the diagram by at least one[13], and so this procedure can only be repeated a finite number of times, after which no violating $\bigtriangledown$'s will remain. This procedure can produce $\bigcirc$'s which will necessarily be rank-violating, so we need to re-apply substitutions to lower the $\bigcirc$'s according to Lemma 4.5.7 (which cannot disorder the other generator types).

Finally, we show that $\boxplus$'s can be brought below $\Diamond$'s. For a minimal violating $\boxplus$ it must be the case that either its green or purple input comes from a $\Diamond$. In either case we reduce the number of $\Diamond$'s beneath the violating $\boxplus$ by one of the following substitutions from Lemmas 4.5.9 and 4.5.10:



By this process the number of violating $\boxplus$'s eventually decreases, and thus eventually reaches zero. □

[12] i.e. left-most on the purple binary tree obtained by removing all the green's.

[13] lowering the $\bigcirc$'s after the substitution for case (c) could remove some additional $\Diamond$'s

Corollary 4.5.12. *The 1-cell* $\bullet\bullet$ *is the unique initial object in hom-category of the unique object of* $\mathscr{D}\tau$*.*

Proof. By Proposition 4.5.6 there exists a unique 2-cell admitting a normal diagram representation from $\bullet\bullet$ to any other 1-cell in $\mathscr{D}\tau$, and by Proposition 4.5.11 every 2-cell admits a normal diagram representation. $\square$

4.6 Related constructions

4.6.1 Dualisation

Having described the free completion of a 2-category $\mathcal{K}$ under oplax colimits, we can obtain descriptions of the free completion under all permutations of op/lax co/limits by appeal to duality. Let Λ denote the class of weights for lax limits of lax functors from 1-categories — i.e. lax morphism classifiers for $\Delta_{\mathbb{1}} : C^\ddagger \to \mathsf{Cat}$, where C is a 1-category and $C^\ddagger$ its lax functor classifier. We use the notation $\mathcal{K}^{\mathsf{cop}}$ to denote $(\mathcal{K}^{\mathsf{co}})^{\mathsf{op}} = (\mathcal{K}^{\mathsf{op}})^{\mathsf{co}}$.

Lemma 4.6.1. *For* $\mathcal{K}$, $\mathcal{L}$ *2-categories and* $F : \mathcal{K} \to \mathcal{L}$ *a 2-functor, the following are equivalent:*

(a) $\mathcal{K}$ *is* Ω*-cocomplete and* F *is* Ω*-cocontinuous.*

(b) $\mathcal{K}^{\mathsf{op}}$ *is* Ω*-complete and* F^{op} *is* Ω*-continuous.*

(c) $\mathcal{K}^{\mathsf{co}}$ *is* Λ*-cocomplete and* F^{co} *is* Λ*-cocontinuous.*

(d) K^{cop} *is* Λ*-complete and* F^{cop} *is* Λ*-continuous.*

Proof. We demonstrate $(a) \iff (d)$, with the other cases being analogous. A lax limit of a 2-functor $G : \mathcal{A} \to \mathcal{K}$ is the same as an oplax colimit of $G^{\mathsf{cop}} : \mathcal{A}^{\mathsf{cop}} \to \mathcal{K}^{\mathsf{cop}}$. In particular, the lax limit of a *lax* functor $G : \mathcal{A} \to \mathcal{K}$ is the same as the oplax colimit of the corresponding oplax functor $G^{\mathsf{cop}} : \mathcal{A}^{\mathsf{cop}} \to \mathcal{K}^{\mathsf{cop}}$, since $\left(\mathcal{A}^\ddagger\right)^{\mathsf{cop}} \cong (\mathcal{A}^{\mathsf{cop}})^\dagger$. It follows that $\mathcal{K}$ has and F preserves all lax limits of lax functors from 1-categories if and only if $\mathcal{K}^{\mathsf{cop}}$ and F^{cop} do so for oplax colimits from 1-categories. $\square$

Proposition 4.6.2. *For* $\mathcal{K}$ *a locally small 2-category:*

(a) $(\mathsf{Z}_{\mathcal{K}^{\mathsf{op}}})^{\mathsf{op}} : \mathcal{K} \to (\mathsf{F}_\Omega \mathcal{K}^{\mathsf{op}})^{\mathsf{op}}$ *is the free completion under* Ω*-limits.*

(b) $(\mathsf{Z}_{\mathcal{K}^{\mathsf{co}}})^{\mathsf{co}} : \mathcal{K} \to (\mathsf{F}_\Omega \mathcal{K}^{\mathsf{co}})^{\mathsf{co}}$ *is the free completion under* Λ*-colimits.*

(c) $(\mathsf{Z}_{\mathcal{K}^{\mathsf{cop}}})^{\mathsf{cop}} : \mathcal{K} \to (\mathsf{F}_\Omega \mathcal{K}^{\mathsf{cop}})^{\mathsf{cop}}$ *is the free completion under* Ω*-limits.*

Proof. For (c), because the 2-functor[14] $(-)^{\mathsf{cop}} : \mathsf{2CAT}^{\mathsf{co}} \to \mathsf{2CAT}$ is involutive, precomposition by $(\mathsf{Z}_{\mathcal{K}^{\mathsf{cop}}})^{\mathsf{cop}}$ gives an equivalence:

$$\begin{tikzcd}
\Lambda\text{−}\mathsf{Cts}((\mathsf{F}_\Omega \mathcal{K}^{\mathsf{cop}})^{\mathsf{cop}}, \mathcal{L}) \arrow[r, "((\mathsf{Z}_{\mathcal{K}^{\mathsf{cop}}})^{\mathsf{cop}})^*"] \arrow[d, leftrightarrow, "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathsf{2CAT}(\mathcal{K}, \mathcal{L}) \arrow[d, "\cong"] \\
(\Omega\text{−}\mathsf{Cocts}((\mathsf{F}_\Omega \mathcal{K}^{\mathsf{cop}}), \mathcal{L}^{\mathsf{cop}}))^{\mathsf{op}} \arrow[r, "(\mathsf{Z}^*_{\mathcal{K}^{\mathsf{cop}}})^{\mathsf{op}}"'] & (\mathsf{2CAT}(\mathcal{K}, L^{\mathsf{cop}}))^{\mathsf{op}}
\end{tikzcd}$$

whenever $\mathcal{L}$ is Λ-complete (thus $\mathcal{L}^{\mathsf{cop}}$ is Ω-cocomplete) because the map along the bottom is an equivalence by Corollary 4.3.22. Note that the pseudo-inverse is given by a *right* Kan extension in this case, as $(-)^{\mathsf{cop}}$ reverses the direction of 2-cells. A similar argument proves (a) and (b), with involutions $(-)^{\mathsf{op}}$ and $(-)^{\mathsf{co}}$ replacing $(-)^{\mathsf{cop}}$. $\square$

We can give more elementary definitions for these various free completions by "unwinding" some of the involutions. For example, an object of $(\mathsf{F}_\Omega \mathcal{K}^{\mathsf{co}})^{\mathsf{co}}$ is a 1-category C and an oplax functor $P : C \to \mathcal{K}^{\mathsf{co}}$, which can be identified with a lax functor $C \to \mathcal{K}$. We thereby obtain the following descriptions of these completions:

[14] we ignore the 3-cells, with respect to which $(-)^{\mathsf{cop}}$ is contravariant.

Definition 4.6.3 (The free completion under Ω-limits, $\mathsf{F}^{\Omega}\mathcal{K}$)**.** For locally-small $\mathcal{K}$, $\mathsf{F}^{\Omega}\mathcal{K} := (\mathsf{F}_{\Omega}\mathcal{K}^{\mathsf{op}})^{\mathsf{op}}$ admits the following description:

0-cells given by a 1-category C and an oplax functor $P\colon C \to \mathcal{K}$

1-cells from (C,P) to (D,Q) are given by a 1-functor $F\colon D \to C$ and an oplax transformation $\phi\colon PF \Rightarrow Q$

2-cells $(F,\phi) \Rightarrow (G,\psi)\colon (C,P) \to (D,Q)$ are given by a natural transformation $\gamma\colon G \Rightarrow F$ and a modulation $\Gamma\colon \phi\ P\gamma \to \psi$ ◊

Definition 4.6.4 (The free completion under Λ-colimits, $\mathsf{F}_{\Lambda}\mathcal{K}$)**.** For locally-small $\mathcal{K}$, $\mathsf{F}_{\Lambda}\mathcal{K} := (\mathsf{F}_{\Omega}\mathcal{K}^{\mathsf{co}})^{\mathsf{co}}$ admits the following description:

0-cells given by a 1-category C and a lax functor $P\colon C \to \mathcal{K}$

1-cells from (C,P) to (D,Q) are given by a 1-functor $F\colon C \to D$ and an oplax transformation $\phi\colon P \Rightarrow QF$

2-cells $(F,\phi) \Rightarrow (G,\psi)\colon (C,P) \to (D,Q)$ are given by a natural transformation $\gamma\colon G \Rightarrow F$ and a modulation $\Gamma\colon \phi \to Q\gamma\ \psi$ ◊

Definition 4.6.5 (The free completion under Λ-limits, $\mathsf{F}^{\Lambda}\mathcal{K}$)**.** For locally-small $\mathcal{K}$, $\mathsf{F}_{\Lambda}\mathcal{K} := (\mathsf{F}_{\Omega}\mathcal{K}^{\mathsf{cop}})^{\mathsf{cop}}$ admits the following description:

0-cells given by a 1-category C and a lax functor $P\colon C \to \mathcal{K}$

1-cells from (C,P) to (D,Q) are given by a 1-functor $F\colon D \to C$ and a lax transformation $\phi\colon PF \Rightarrow Q$

2-cells $(F,\phi) \Rightarrow (G,\psi)\colon (C,P) \to (D,Q)$ are given by a natural transformation $\gamma\colon F \Rightarrow G$ and a modulation $\Gamma\colon \phi \to \psi\ P\gamma$ ◊

The composition of 1-cells in these 2-categories is defined in the obvious way. The composition of 2-cells is given by appropriate horizontal and vertical reflections of the string diagrams in (4.9) and (4.10).

4.6.2 Restrictions

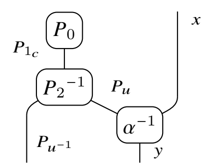
Other free completions related to $\mathsf{F}_{\Omega}\mathcal{K}$ are obtained by restricting $\mathsf{F}_{\Omega}\mathcal{K}$ to a full subcategory. In particular, if we choose a subclass of small categories $S \subseteq \mathsf{Cat}$, then we can consider the full sub-2-category of $\mathsf{F}_{\Omega}\mathcal{K}$ whose objects are oplax functors from a category in S (henceforth "S-oplax functors"). This 2-category, which we denote $\mathsf{F}_{\Omega[S]}\mathcal{K}$, is equivalent to the full subcategory of $[\mathcal{K}^{\mathsf{op}}, \mathsf{Cat}]$ whose objects are oplax colimits of S-oplax functors. If $\mathsf{F}_{\Omega[S]}\mathcal{K}$ is closed under S-oplax colimits in $\mathsf{F}_{\Omega}\mathcal{K}$, then it follows that $\mathsf{F}_{\Omega[S]}\mathcal{K}$ is the free cocompletion of $\mathcal{K}$ under oplax colimits of S-oplax functors and the class Ω_S of weights for such colimits is presaturated. Recalling how oplax colimits in F_{Ω} are computed via the extralax colimit, $\mathsf{F}_{\Omega[S]}\mathcal{K}$ will be closed under oplax colimits of S-oplax functors precisely when the Grothendieck construction of an S-oplax functor to the full sub-2-category $S \subseteq \mathsf{Cat}$ lands in S. One important example of a class S satisfying this property is $S = \{\mathbb{1}\}$, in which case $\mathsf{F}_{\Omega[S]}\mathcal{K}$ is simply the coKleisli completion of $\mathcal{K}$. Similarly, $\mathsf{F}_{\Omega[S]}\mathcal{K}$, $\mathsf{F}_{\Lambda[S]}\mathcal{K}$ and $\mathsf{F}_{\Lambda[S]}\mathcal{K}$ are respectively the Kleisli, co-Eilenberg-Moore and Eilenberg-Moore completions described in [LS02].

Remark 4.6.6. Letting $\mathsf{coKl}(\mathcal{K})$ denote the coKleisli completion of $\mathcal{K}$, objects of $\mathsf{coKl}^2(\mathcal{K})$ can be identified with certain "generalised" distributive laws of comonads which are dual to the *wreaths* of [LS02]. An element of $\mathsf{coKl}^2(\mathcal{K})$ can be identified with a lift of $\tau'\colon \mathbb{1}^{\dagger} \to \mathsf{Cat}$ along $\varpi_{\mathcal{K}}\colon \mathsf{F}_{\Omega}\mathcal{K} \to \mathsf{Cat}$, which under the adjunction of Proposition 4.3.8 corresponds to a 2-functor from $\mathscr{E}\tau' \cong \mathscr{D}\tau$. Thus, the 2-category $\mathscr{D}\tau$ is the classifier for such generalised distributive laws, whereas the classifier for "ordinary" distributive laws between comonads in $\mathcal{K}$ is given by the lax Gray tensor product, $\mathbb{1}^{\dagger} \boxminus \mathbb{1}^{\dagger}$. ◊

Other examples of classes S with the closure property describe above are finite categories, connected categories, codiscrete categories and groupoids. This last example will be relevant in Chapter 7, so we establish its closure property here:

Lemma 4.6.7. *The 2-category* Grpd *of groupoids is closed under oplax colimits indexed by oplax functors from groupoids.*

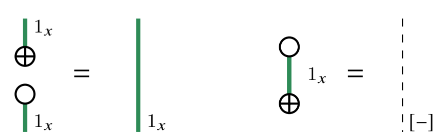
Proof. First note that all the 2-cells of Grpd are invertible, so any oplax functor $P: G \to \mathsf{Grpd}$ is automatically pseudo-functorial. We can show that the oplax colimit $\int$ of such a pseudofunctor P is again a groupoid by directly exhibiting an inverse for an arbitrary 1-cell. A 1-cell in $\int P$ is of the form (u, α) for $u: c \to d$ in G and $\alpha: P_u x \to y$ in P_d, and it has an inverse given by $\left(u^{-1}, P_0 x\, {P_2}^{-1} x\, P_{u^{-1}} \alpha^{-1}\right)$. We draw the second component of this inverse as a string diagram for clarification:



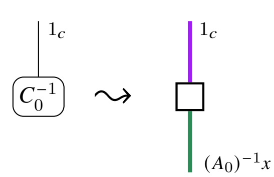
□

4.6.3 Cocompletion under normal oplax functors

If instead we take the full subcategory of $\mathsf{F}_\Omega\mathcal{K}$ whose objects are normal oplax functors — i.e. oplax functors $P: C \to \mathcal{K}$ where the P_0 cells are invertible — then the result $\mathsf{F}_{\Omega[\mathsf{N}]}\mathcal{K}$ is equivalent to the full subcategory of $[\mathcal{K}^{\mathsf{op}}, \mathsf{Cat}]$ whose objects are oplax colimits of normal oplax functors. Such colimits correspond to the class of weights of the form $\mathscr{Q}\mathbb{1}: \left(C^{\mathsf{N}}\right)^{\mathsf{op}} \to \mathsf{Cat}$ where C is a 1-category and $(-)^{\mathsf{N}}$ denotes the *normal* oplax functor classifier (obtained from the oplax functor classifier by requiring the C_0's be invertible). Given a normal oplax functor $B: C \to \mathsf{F}_{\Omega[\mathsf{N}]}\mathcal{K}$, we can show that the oplax colimit of B in $\mathsf{F}_\Omega\mathcal{K}$ is also a normal oplax functor, and thus $\mathsf{F}_{\Omega[\mathsf{N}]}$ is a free-cocompletion. By a slight modification of the definition of extralax cocone we obtain the appropriate notion of *normal-lax cocone* and *normal-lax colimit* such that the normal oplax functor $B: C \to \mathsf{F}_{\Omega[\mathsf{N}]}$ can be identified with a normal-lax cocone from $A = \varpi B$ to $\mathcal{K}$, and thence a 2-functor out of the normal-lax colimit $\mathscr{N}A$. A normal-lax cocone differs from an extralax cocone (Definition 4.3.1) only in that the component functors are required to be normal oplax, rather than oplax. Correspondingly, the definition of a normal-lax colimit requires an extra 2-cell generator, ⊕, and relations that declare it to be the inverse of ○:

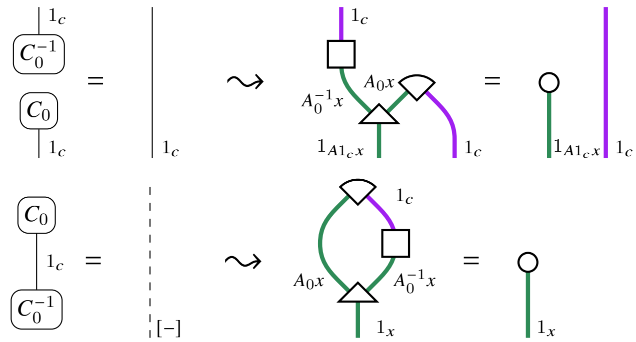


There is also the (θ) generator corresponding to the inverse of C_0 in C^{N}, denoted □:



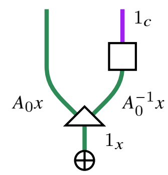
(4.29)

The relations for □ are obtained from those defining $({C_0}^{-1})$ as the inverse of (C_0):

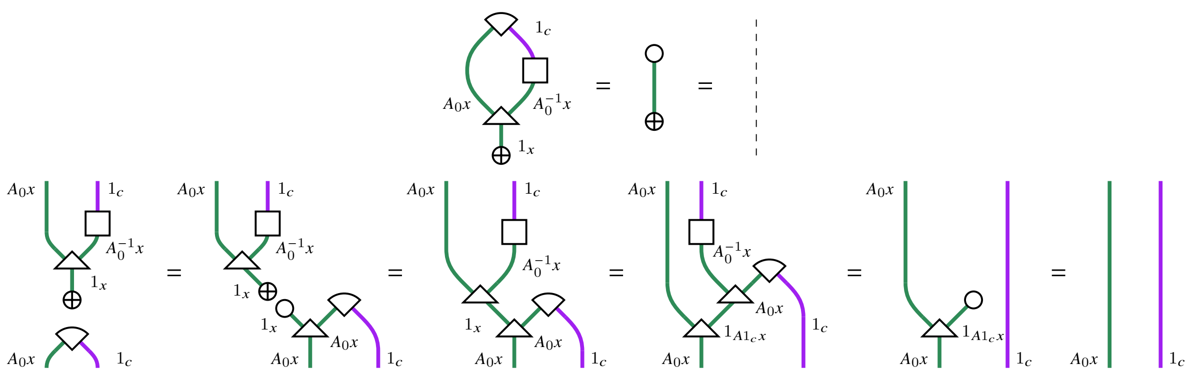


(4.30)

It follows from these relations that the ♢ generators are invertible, with inverse given by:

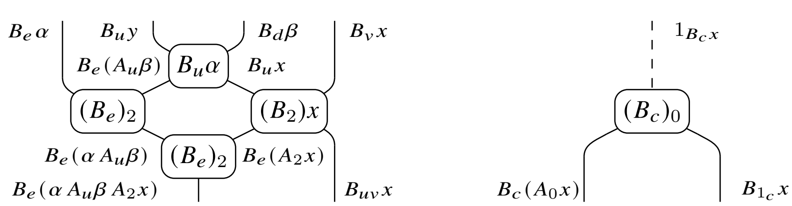


by the following string computation, using the relations of (4.30):



If $B\colon C \to \mathsf{F}_{\Omega[\mathsf{N}]}\mathcal{K}$ is a normal oplax functor with $A = \varpi B\colon C \to \mathsf{Cat}$, then the induced map $B^\flat\colon \mathscr{D}A \to \mathcal{K}$ factors through the canonical inclusion $\mathscr{D}A \to \mathscr{N}A$ and thus the images of ♢-type generators are invertible. The oplax colimit of B in $\mathsf{F}_\Omega\mathcal{K}$ as described in 4.4 is given by pre-composing $B^\flat$ with the oplax inclusion $\int A \to \mathscr{D}A$, whose oplax unit 2-cells are the images of ♢'s under $B^\flat$ and thus invertible. So the oplax colimit of B in $\mathsf{F}_\Omega\mathcal{K}$ is contained in the subcategory $\mathsf{F}_{\Omega[\mathsf{N}]}\mathcal{K}$, which is therefore the free cocompletion of $\mathcal{K}$ under oplax colimits of normal oplax functors.

Remark 4.6.8. It is then natural to ask whether the same is true with pseudofunctors substituted for normal oplax functors. That is, whether the oplax colimit in $\mathsf{F}_\Omega\mathcal{K}$ of a pseudofunctor from a 1-category $B\colon C \to \mathsf{F}_\Omega\mathcal{K}$ whose image lies in the full subcategory of pseudofunctors $\mathsf{F}_{\Omega[\mathsf{P}]}\mathcal{K}$ is again a pseudofunctor. The answer is "no". Consider the oplax-functoriality 2-cells for the colimit of $\overline{B}$ described in (4.24):



Because the image of B lands in $\mathsf{F}_{\Omega[\mathsf{P}]}\mathcal{K}$ the $(B_c)_*$ 2-cells are invertible. And, as it turns out, the 2-cell $(B_2)x$ is invertible (cf. Lemma 9.3.1). However, $B_u\alpha$ — the lax-naturality 2-cell for the lax transformation B_u — need not be invertible.

It follows that the class of oplax colimits of pseudofunctors isn't presaturated. The free cocompletion of $\mathcal{K}$ under this class is some full subcategory of $\mathsf{F}_\Omega\mathcal{K}$ which contains all the pseudofunctors but will in general contain some (normal) oplax functors as well. As a concrete example, assume $\mathcal{K}$ has all oplax colimits of pseudofunctors from 1-categories and $F\colon \mathbb{2} \to [\mathbb{2}, \mathcal{K}]_{\mathsf{lax}}$ is a strict 2-functor from the interval category to the category of 2-functors from $\mathbb{2}$ to $\mathcal{K}$ with lax transformations as morphisms. Such a 2-functor is equivalent to a functor $\mathcal{F}\colon \mathbb{2} \boxminus \mathbb{2} \to \mathcal{K}$ from the lax Gray tensor product, and there is a canonical isomorphism between the following oplax colimits:

$$\oint^{(x,y)\in\mathbb{2}\boxminus\mathbb{2}} \mathcal{F}(x,y) \cong \oint^{x\in\mathbb{2}} \oint^{y\in\mathbb{2}} F_x y$$

It follows that $\mathcal{K}$ must also have all oplax colimits of functors from $\mathbb{2}\boxminus\mathbb{2}$, which is a square filled with a non-invertible 2-cell. In general, such colimits cannot be expressed as oplax colimits of pseudo-functors from a 1-category which — by Remark 4.2.3 — are equivalent to oplax colimits of 2-functors from 2-categories with component-initial 1-cells that are preserved by composition. The oplax square isn't a 2-category with this property. This is an instance of an “iterated” oplax colimit of pseudofunctors from 1-categories (which happen to be strict) which cannot in general be expressed as a single oplax colimit of a pseudofunctor from a 1-category. Clearly the class of oplax colimits of strict functors from 1-categories similarly fails to be presaturated for the same reason. The problem is essentially the non-invertibility of the $\Diamond$ generators, which correspond to the lax naturality cells of the lax transformations of the 1-cells in $\mathsf{F}_\Omega\mathcal{K}$. If the image of B were contained in the non-full subcategory of pseudofunctors and *pseudo*-natural transformations, then the oplax colimit of B would again be a pseudofunctor. While the subcategory of pseudofunctors and pseudonatural transformations doesn't define a cocompletion, it is still a submonad of F_Ω and we will explore its properties in later chapters. ◊

4.7 Examples

We conclude this chapter with some examples of the free (co)completions under Ω-(co)limits of familiar 2-categories.

Example 4.7.1 (The free oplax cocompletion of a small discrete set)**.** Extending the observation that $\mathsf{F}_\Omega\mathbb{1} \simeq \mathsf{Cat}$, we note that for X a small discrete 2-category (i.e. a small set) we have $\mathsf{F}_\Omega X \simeq \mathsf{Cat}^{|X|}$. Under this equivalence the map $\varpi\colon \mathsf{F}_\Omega X \to \mathsf{Cat}$ corresponds to the coproduct map $\coprod\colon \mathsf{Cat}^{|X|} \to \mathsf{Cat}$. Given a function $f\colon X \to \mathsf{ob}(\mathcal{K})$ for $\mathcal{K}$ a Ω-cocomplete 2-category, the canonically induced Ω-cocontinuous map $\mathsf{Cat}^{|X|} \to \mathcal{K}$ is given by $(C_i)_{i\in X} \mapsto \coprod_{i\in X} C_i \odot f_i$. ◊

Example 4.7.2 ($\mathsf{F}_\Omega\mathsf{Cat}_\cong$)**.** Let $\mathsf{Cat}_\cong$ denote the locally groupoidal 2-category of small categories whose 2-cells are natural isomorphisms. Then objects of $\mathsf{F}_\Omega\mathsf{Cat}_\cong$ are pseudo-functors $C \to \mathsf{Cat}$ and the Grothendieck construction defines an equivalence $\mathsf{F}_\Omega\mathsf{Cat}_\cong \simeq \mathsf{OpFib}$, where OpFib is the 2-category of fibrations whose 1-cells are commuting squares with the “upstairs” functor opcartesian (F in Figure 4.5a), and 2-cells as shown in Figure 4.5b. The inclusion $\mathsf{Z}_{\mathsf{Cat}_\cong}\colon \mathsf{Cat}_\cong \to \mathsf{OpFib}$ sends a category to the identity on that category, and the projection $\varpi\colon \mathsf{F}_\Omega\mathsf{Cat}_\cong \to \mathsf{Cat}$ corresponds to the codomain projection $\mathsf{OpFib} \to \mathsf{Cat}$. By the universal property of $\mathsf{F}_\Omega\mathsf{Cat}_\cong$ any 2-functor $F\colon \mathsf{Cat}_\cong \to \mathcal{K}$ to a Ω-cocomplete 2-category induces a 2-functor $\widetilde{F}\colon \mathsf{OpFib} \to \mathcal{K}$ sending an opfibration $p\colon D \to A$ to oplax colimit of the induced pseudo presheaf postcomposed with F, $\oint^{a\in A} FD_a$. ◊

Example 4.7.3. Restricting the equivalence of Example 4.7.2 to various sub-2-categories of $\mathsf{Cat}_\cong$ gives corresponding subcategories of OpFib:

$$\begin{tikzcd} D \arrow[r, "F"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d, "q"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

(a) A 1-cell $(F, f) \colon p \to q$

$$\begin{tikzcd} D \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\overset{\Sigma}{\Downarrow}"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d, "q"] \\ A \arrow[r, bend right, "g"'] & B \end{tikzcd} = \begin{tikzcd} D \arrow[r, bend left, "F"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d, "q"] \\ A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\overset{\sigma}{\Downarrow}"] & B \end{tikzcd}$$

(b) A 2-cell $(\Sigma, \sigma) \colon (F, f) \Rightarrow (G, g)$

Figure 4.5: 1-cells and 2-cells in OpFib

(a) $\mathsf{F}_\Omega\mathsf{Grpd}$ is equivalent to the 2-category of opfibrations of groupoids. Note that every morphism in the total category of an opfibration of groupoids is opcartesian, so the opcartesianness condition on F in Figure 4.5a is vacuous.

(b) $\mathsf{F}_\Omega\mathsf{Cat}_1$, where Cat_1 denotes the underlying 1-category of Cat, is equivalent to the 2-category of *split* opfibrations where the F of Figure 4.5a is required to be split-opcartesian (i.e. preserve *chosen* opcartesian lifts).

(c) $\mathsf{F}_\Omega\mathsf{Set}$ is equivalent to the 2-category of discrete opfibrations. As with the groupoid case, the opcartesianness of F in the definition of the 1-cells becomes vacuous. ◊

Remark 4.7.4. The Grothendieck construction is also defined for oplax functors $C \to \mathsf{Cat}$ (as the oplax colimit), though the notion of fibration corresponding to the projection $\int P \to C$ is a weak one, as discussed in Section 2.2.5. In particular, one cannot recover the oplax presheaf P from this sort of fibration. To see that this is so, consider the case where $C = \mathbb{1}$, and thus $P \colon \mathbb{1} \to \mathsf{Cat}$ is a comonad in Cat and $\int P$ is its coKleisli category. It's not possible in general to recover the data for a comonad from the definition of its coKleisli category with the unique functor to $\mathbb{1}$. Similarly, one cannot recover a comonad morphism (i.e. 1-cell in $\mathsf{F}_\Omega\mathsf{Cat}$ between oplax functors from $\mathbb{1}$) from an arbitrary morphism between the coKleisli categories. ◊

Example 4.7.5 ($\mathsf{F}_\Omega\mathbb{2}$)**.** A functor to the arrow category, $\mathbb{2} := (0 \to 1)$, is sometimes called a *barrel*[15], with the fibre over $0 \in \mathbb{2}$ called the *bottom* of the barrel, and the fibre over 1 called the *top*. As described at [Joy22], a barrel $P \colon C \to \mathbb{2}$ can be identified with a profunctor $p \colon C_0{}^{\mathsf{op}} \times C_1 \to \mathsf{Set}$ from the bottom, C_0, to the top[16], C_1, defined by restricting the hom-functor of C to the inclusion $C_0{}^{\mathsf{op}} \times C_1 \to C^{\mathsf{op}} \times C$. This identification of barrels with profunctors underlies an equivalence $\mathsf{Cat}/\mathbb{2} \to \mathsf{Prof}$, where Prof is the category of profunctors between small categories where a 1-cell between $p \colon A \nrightarrow B$ and $q \colon C \nrightarrow D$ is a pair of functors $f \colon A \to C$, $g \colon B \to D$ and a natural transformation $\alpha \colon p \Rightarrow q(f-, g-)$. The pseudo-inverse $\mathsf{Col} \colon \mathsf{Prof} \to \mathsf{Cat}/\mathbb{2}$ to this equivalence is given by sending a profunctor $p \colon A \nrightarrow B$ to its *collage* (a.k.a. *co-graph*) $\pi_p \colon A \star_p B \to \mathbb{2}$. The objects of $\mathsf{Cat}/\mathbb{2}$ and $\mathsf{F}_\Omega\mathbb{2}$ are the same, but the 1-cells of $\mathsf{F}_\Omega\mathbb{2}$ are more permissive. There are also 2-cells, which we haven't yet accounted for in describing the equivalence $\mathsf{Cat}/\mathbb{2} \simeq \mathsf{Prof}$. We endow Prof with a 2-categorical structure by declaring a 2-cell $(f, g, \alpha) \Rightarrow (h, k, \beta) \colon p \to q$ to be a pair of natural transformations $\sigma \colon f \Rightarrow h$, $\tau \colon g \Rightarrow k$ making the following diagram commute:

$$\begin{tikzcd} p \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\circlearrowleft"] & q(f,g) \arrow[d, "q(f,\tau)"] \\ q(h,k) \arrow[r, "q(\sigma,k)"'] & q(f,k) \end{tikzcd}$$

With this definition of Prof as a 2-category, the equivalence $\mathsf{Cat}/\mathbb{2}$ extends to an equivalence of 2-categories. Recall that $\mathsf{F}_\Omega\mathcal{K}$ admits a description as a full subcategory of the Kleisli category for a monad on $\mathsf{2CAT}/\mathcal{K}$, which when

[15]This terminology appears to originate with [Joy22], where the equivalence between profunctors and barrels is described in detail.
[16]or vice versa, sometimes profunctors are defined with the $()^{\mathsf{op}}$ on the codomain.

$\mathcal{K} = \mathbb{2}$ restricts to a monad on $\mathsf{Cat}/\mathbb{2}$. The corresponding monad M on Prof sends a profunctor $p\colon A \nrightarrow B$ to the profunctor $p'\colon A \star_p B \nrightarrow B$ defined as [17]:

$$p'(x,y) = \begin{cases} p(x,y) & x \in A \\ B(x,y) & x \in B \end{cases}$$

The free cocompletion $\mathsf{F}_\Omega\mathbb{2}$ is therefore equivalent to the Kleisli category of M. For barrels $P\colon C \to \mathbb{2}$, $Q\colon D \to \mathbb{2}$ and a morphism between them in $\mathsf{F}_\Omega\mathbb{2}$, $F\colon C \to D$, $\phi\colon P \to QF$, we obtain a Kleisli arrow between the associated profunctors $p\colon C_0 \nrightarrow C_1$, $q\colon D_0 \nrightarrow D_1$ as follows:

(a) $f\colon C_0 \to D \cong D_0 \star_q D_1$ is the restriction of F to $C_0 \subseteq C$.

(b) $g\colon C_1 \to D_1$ is the restriction of F to C_1. The existence of $\phi\colon P \to QF$ implies that the image of C_1 is contained in D_1.[18]

(c) $\alpha\colon p \to q'(f-, q-)$ has component at $x \in C_0$, $y \in C_1$ given by:

$$p(x,y) = C(x,y) \xrightarrow{F} D(Fx,Fy) \cong q'(fx,gy)$$

The 2-functor $\mathsf{Kleisli}_\mathsf{M} \xrightarrow{\simeq} \mathsf{F}_\Omega\mathbb{2} \xrightarrow{\varpi} \mathsf{Cat}$ sends a profunctor to its collage. ◊

4.7.1 $\mathsf{Fam}(\mathcal{V})$-enriched categories with Fam-Mealy morphisms

If $\mathcal{V}$ is a small strict monoidal category, then the *suspension*[19], $\Sigma\mathcal{V}$ is a locally small 2-category. A lax functor from a codiscrete 1-category ∇X on a small set X to $\Sigma\mathcal{V}$ is then a $\mathcal{V}$-category with object set X and hom-object $X(x,y)$ given by the image in $\mathcal{V}$ of the unique morphism $x \to y$ in ∇X[20]. More generally, a lax functor from a 1-category C to $\Sigma\mathcal{V}$ assigns to each pair of objects $x, y \in C$ a family of objects in $\mathcal{V}$, $P_{x,y}\colon C(x,y) \to \mathsf{ob}(\mathcal{V})$. We can view this as an enrichment over $\mathsf{Fam}(\mathcal{V})$, rather than $\mathcal{V}$ itself. Note that $\mathsf{Fam}(\mathcal{V})$ inherits a monoidal structure from $\mathcal{V}$ by declaring the product of families $A\colon I \to \mathcal{V}$ and $B\colon J \to \mathcal{V}$ to be given by:

$$I \times J \xrightarrow{A\times B} \mathcal{V} \times \mathcal{V} \xrightarrow{\otimes} V$$

The product of 1-cells $\left(f, \phi_i\colon A_i \to B_{fi}\right)$ and $\left(g, \psi_j\colon C_j \to D_{gj}\right)$ is given by:

$$(f \times g, A_i \otimes C_j \xrightarrow{\phi_i\otimes\psi_j} B_{fi} \otimes D_{gj})$$

The unit for the monoidal product is the one-element family containing the unit for $\mathcal{V}$, which we will denote $\mathcal{I}$ to distinguish it from the unit I of $\mathcal{V}$. The associativity and unit isomorphisms are defined in the obvious way from those for $(\mathcal{V}, \otimes)$ and $(\mathsf{Cat}, \times)$. We will use $\boxtimes$ to denote this monoidal product on $\mathsf{Fam}(\mathcal{V})$.

Given a lax functor $P\colon C \to \Sigma\mathcal{V}$, we obtain a $\mathsf{Fam}(\mathcal{V})$-category with objects those of C by assigning to each pair of objects x, y in C the family $P_{x,y}\colon C(x,y) \to \mathcal{V}$. To describe the composition maps $P_{y,z} \boxtimes P_{x,y} \to P_{x,z}$ we must provide data of the form:

$$\begin{tikzcd} C(y,z) \times C(x,y) \arrow[d, "\mathsf{C}_{x,y,z}"'] \arrow[dr, "P_{y,z}\boxtimes P_{x,y}"] \arrow[dr, phantom, shift right=4, "\mathsf{D}_{x,y,z}\Downarrow"] & \\ C(x,z) \arrow[r, "P_{x,z}"'] & \mathcal{V} \end{tikzcd}$$

[17]The profunctor p' is the representable profunctor corresponding to the canonical inclusion $B \hookrightarrow A \star_p B$.
[18]If ϕ exists it is unique because $\mathbb{2}$ is a poset. Thus, a 1-cell of $\mathsf{F}_\Omega\mathbb{2}$ is a functor $F\colon C \to D$ satisfying $F(C_1) \subseteq D_1$.
[19]The one-object bicategory whose hom-category is given by $\mathcal{V}$, also called the *delooping* of $\mathcal{V}$.
[20]This is noted, for example, in [Bén67]

We obtain such maps by taking $\mathsf{C}_{x,y,z}\colon C(y,z)\times C(x,y)\to C(x,z)$ to be the composition in C, and then $\mathsf{D}_{f,g}\colon Pf\otimes Pg\to P(fg)$ to be the lax-functoriality 2-cell P_2. Similarly, the unit $\mathcal{I}\to P_{x,x}$ has index-set map given by the unit in C, $1_x : * \to C(x,x)$, and map on components given by $P_0\colon I\to P1_x$. It is straight-forward to verify that the axioms for these composition maps correspond precisely to those for composition in C and the axioms for the lax functoriality of P. Conversely, given any $\mathsf{Fam}(\mathcal{V})$-category E, we obtain an ordinary category E_1 by change-of-base with respect to the monoidal map $\mathsf{Fam}(!)\colon \mathsf{Fam}(V)\to\mathsf{Fam}(\mathbb{1})\cong\mathsf{Set}$. The composition data "D" and axioms for the $\mathsf{Fam}(\mathcal{V})$-category E then correspond precisely to the data and axioms for the lax-functoriality of the induced map $E_0\to\Sigma\mathcal{V}$. These two maps on objects between $\mathsf{Fam}(\mathcal{V})\text{-}\mathsf{Cat}$ and $\mathsf{F}_\Lambda(\Sigma\mathcal{V})$ are clearly inverse to each other, and this bijection on objects moreover commutes with the 2-functor $\varpi\colon\mathsf{F}_\Lambda(\Sigma\mathcal{V})\to\mathsf{Cat}$ and the change-of-base map $\mathsf{Fam}(\mathcal{V})\text{-}\mathsf{Cat}\to\mathsf{Fam}(\mathbb{1})\text{-}\mathsf{Cat}\simeq\mathsf{Cat}$.

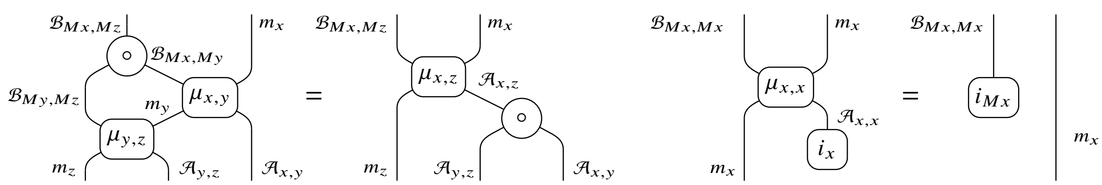
However, the 1-cells in $\mathsf{F}_\Lambda(\Sigma\mathcal{V})$ ("oplax triangles") do not correspond to the usual notion of $\mathsf{Fam}(\mathcal{V})$-enriched functor. Instead, they correspond to a certain class of *Mealy morphisms* as described in [Par12][21].

Definition 4.7.6 (Mealy morphism)**.** For monoidal $\mathcal{V}$ and two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, a Mealy morphism $M\colon\mathcal{A}\rightsquigarrow\mathcal{B}$ is given by

(a) a map on objects $M\colon\mathsf{ob}(\mathcal{A})\to\mathsf{ob}(\mathcal{B})$

(b) a map on objects $m\colon\mathsf{ob}(\mathcal{A})\to\mathsf{ob}(\mathcal{V})$

(c) for each $x,y\in\mathsf{ob}(\mathcal{A})$, a morphism $\mu_{x,y}\colon m_y\otimes\mathcal{A}(x,y)\to\mathcal{B}(Mx,My)\otimes m_x$ satisfying the following coherence conditions:



(4.31)

expressed in terms of string diagrams in $\Sigma\mathcal{V}$. ◊

Let $\mathcal{P}$ and $\mathcal{Q}$ denote the $\mathsf{Fam}(\mathcal{V})$-enriched categories corresponding to lax functors $P\colon C\to\Sigma\mathcal{V}$ and $Q\colon D\to\Sigma\mathcal{V}$. From a 1-cell $(F,\phi)\colon(C,P)\to(D,Q)$ we obtain a Mealy morphism from $M\colon\mathcal{P}\rightsquigarrow\mathcal{Q}$ defined as follows:

M is the underlying object map of F.

m maps $c\in\mathsf{ob}(C)=\mathsf{ob}(\mathcal{P})$ to the one-element $\mathcal{V}$-family containing the component of ϕ at c. We denote this family $\langle\phi_c\rangle$. Note that $\langle\phi_y\rangle\boxtimes\mathcal{P}_{x,y}\colon *\times C(x,y)\to\mathcal{V}$ is then isomorphic to the family $C(x,y)\to\mathcal{V}$ sending $u\colon x\to y$ to $\phi_y\otimes Pu$.

μ has component $\langle\phi_y\rangle\boxtimes\mathcal{P}_{x,y}\to\mathcal{Q}_{Fx,Fy}\boxtimes\langle\phi_x\rangle$ with map on indexing sets given by $F_{x,y}\colon C(x,y)\to D(Fx,Fy)$ and component at $u\colon x\to y$ given by $\phi_u\colon\phi_y\otimes Pu\to QFu\otimes\phi_x$. Note that the coherence conditions for μ in Definition 4.7.6 correspond precisely to the functoriality of F and oplax functoriality of ϕ.

The 1-cells of $\mathsf{F}_\Lambda(\Sigma\mathcal{V})$ therefore correspond to $\mathsf{Fam}(\mathcal{V})$-Mealy morphisms with the property that the map $m\colon\mathsf{ob}(C)\to\mathsf{ob}(\mathsf{Fam}(\mathcal{V}))$ factors through the inclusion $\mathcal{V}\to\mathsf{Fam}(\mathcal{V})$ which sends an object in $\mathcal{V}$ to the corresponding $*$-indexed family. We will call these special $\mathsf{Fam}(\mathcal{V})$-Mealy morphisms, "$\mathsf{Fam}$-Mealy morphisms". From a Fam-Mealy morphism (M,m,μ) between $\mathsf{Fam}(\mathcal{V})$-categories viewed as objects (C,P), (D,Q) of $\mathsf{F}_\Lambda(\Sigma\mathcal{V})$ one recovers the data of a functor $F\colon C\to D$ and oplax transformation $\phi\colon P\Rightarrow QF$ as follows:

(a) $M\colon\mathsf{ob}(C)\to\mathsf{ob}(D)$ provides the object map of F.

[21] What we call will call a Mealy morphism $\mathcal{A}\rightsquigarrow\mathcal{B}$ would actually be a Mealy morphism $\mathcal{A}^{\mathsf{op}}\rightsquigarrow\mathcal{B}^{\mathsf{op}}$ according to the definition chosen in [Par12]. Note that $\mathcal{A}^{\mathsf{op}}$ is in general a $\mathcal{V}^{\mathsf{rev}}$-category, though $\mathcal{V}^{\mathsf{rev}}\cong\mathcal{V}$ if $\mathcal{V}$ is symmetric.

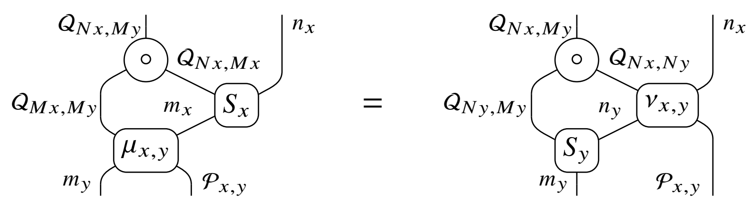
(b) $m\colon \mathsf{ob}(C) \to \mathsf{ob}(\mathcal{V})$ provides the components of ϕ.

(c) The maps $\mu\colon \langle \phi_y \rangle \boxtimes P_{x,y} \to Q_{Fx,Fy} \boxtimes \langle \phi_x \rangle$ consists of a map on indexing sets $C(x,y) \to D(Fx,Fy)$ — which provide the hom-functions for F — along with a family of maps $\phi_u\colon \phi_y \otimes Pu \to QFu \otimes \phi_x$ indexed by $u \in C(x,y)$ which provide the oplax-naturality data for ϕ. The lax-functoriality and oplax-naturality conditions for F and ϕ respectively both follow from the axioms given in (4.31). One visual indication of this fact, is that removing the "m" strings (whose indexing sets are the unit in $(\mathsf{Set}, \times)$) for those diagrams gives diagrams for the lax functoriality axioms, and that with the m strings present they are precisely the oplax-naturality diagrams (cf. Section 2.2.2).

With the obvious definition of composition of Fam-Mealy morphisms, this identification extends to an equivalence between the underlying 1-category of $\mathsf{F}_\Lambda(\Sigma\mathcal{V})$ and the category of $\mathsf{Fam}(\mathcal{V})$-categories with Fam-Mealy morphisms, which we denote $\mathsf{FamMealy}_{\mathcal{V}}$.

In [Par12], Paré also gives a description of 2-cells between Mealy morphisms called *Mealy cells*[22] which provide the necessary 2-dimensional structure on $\mathsf{FamMealy}_{\mathcal{V}}$ to obtain an equivalence $\mathsf{F}_\Lambda(\Sigma V) \simeq \mathsf{FamMealy}_{\mathcal{V}}$ of 2-categories.

Definition 4.7.7 (Mealy cell)**.** For Mealy morphisms $(M,m,\mu), (N,n,\nu)\colon \mathcal{P} \rightsquigarrow \mathcal{Q}$ between $\mathcal{V}$-categories, a *Mealy cell S* from (M,m,μ) to (N,n,ν) is given by a $\mathsf{ob}(\mathcal{P})$-indexed family of $\mathcal{V}$-morphisms $S_x\colon m_x \to \mathcal{Q}(Nx,Mx) \otimes n_x$ satisfying:



◊

Given a 2-cell $(\gamma,\Gamma)\colon (F,\phi) \Rightarrow (G,\psi)\colon (C,P) \to (D,Q)$, we obtain a Mealy cell $S\colon (M,m,\mu) \Rightarrow (N,n,\nu)\colon \mathcal{P} \rightsquigarrow \mathcal{Q}$ between the corresponding Fam-Mealy morphisms in $\mathsf{FamMealy}_{\mathcal{V}}$ as follows. Recall that $\mathcal{Q}_{Nx,Mx} \boxtimes n_x$ is given by the family $D(Nx,Mx) \to \mathcal{V}$ sending $u\colon Nx \to Mx$ to $Qu \otimes \psi_x$. A map in $\mathsf{Fam}(\mathcal{V})$ from m_x to $\mathcal{Q}_{Nx,Mx} \boxtimes n_x$ is thus given by an object $* \to D(Nx,Mx)$, which we take to be γ_x, and a morphism $\phi_x \to Q\gamma_x \otimes \psi_x$, which we take to be Γ_x. The coherence conditions for this Mealy 2-cell correspond exactly to the naturality of γ and coherence for the modulation Γ (compare the string diagrams of Definition 4.7.7 with those of Equation (4.8)). The composition of Mealy 2-cells is defined such that it coincides with the composition of the 2-cells in $\mathsf{F}_\Lambda(\Sigma\mathcal{V})$, (c.f (4.9) and (4.10), but upside-down). We therefore have an equivalence of 2-categories[23] $\mathsf{F}_\Lambda(\Sigma\mathcal{V}) \simeq \mathsf{FamMealy}_{\mathcal{V}}$.

Remark 4.7.8. The 2-categories $\mathsf{F}_\Omega(\Sigma\mathcal{V})$, $\mathsf{F}^\Omega(\Sigma\mathcal{V})$ and $\mathsf{F}^\Lambda(\Sigma\mathcal{V})$ all admit descriptions as less familiar variants of $\mathsf{FamMealy}_{\mathcal{V}}$ up to equivalence. For example, $\mathsf{F}_\Omega(\Sigma\mathcal{V}) \simeq \left(\mathsf{FamMealy}_{\mathcal{V}^{\mathsf{op}}}\right)^{\mathsf{co}}$ and $\mathsf{F}^\Lambda(\Sigma\mathcal{V}) \simeq \left(\mathsf{FamMealy}_{\mathcal{V}^{\mathsf{rev}}}\right)^{\mathsf{op}}$. ◊

Remark 4.7.9. A Mealy morphism $M\colon \mathcal{A} \rightsquigarrow \mathcal{B}$ with $m_a = I$ for all $a \in \mathsf{ob}(\mathcal{A})$ can be identified with a $\mathcal{V}$-functor $M'\colon \mathcal{A} \to \mathcal{B}$, and a Mealy cell between Mealy morphisms of this type can be identified with a $\mathcal{V}$-natural transformation $M' \Rightarrow N'$. Every $\mathsf{Fam}(\mathcal{V})$-Mealy morphism with $m_a = I$ is also a Fam-Mealy morphism, so there is a bijective locally fully-faithful inclusion $\mathsf{Fam}(\mathcal{V})\text{-}\mathsf{Cat} \hookrightarrow \mathsf{FamMealy}_{\mathcal{V}}$. Restricting the equivalence $\mathsf{FamMealy}_{\mathcal{V}} \simeq \mathsf{F}_\Lambda(\Sigma\mathcal{V})$ to this sub-2-category gives an equivalence between $\mathsf{Fam}(\mathcal{V})\text{-}\mathsf{Cat}$ and the locally full sub-2-category of $\mathsf{F}_\Lambda(\Sigma\mathcal{V})$ whose 1-cells involve *icons*, rather than general oplax transformations. ◊

[22] Our definition for Mealy cells is modified slightly to account for duality between Paré's Mealy cells and those defined here.
[23] For non-strict $\mathcal{V}$, $\mathsf{FamMealy}_{\mathcal{V}}$ will merely be a bicategory.

Example 4.7.10 (Categories with norms)**.** Taking $(\mathcal{V}, \otimes) = (\mathsf{R}_{\geq 0}, +)$ where the categorical structure on $\mathsf{R}_{\geq 0}$ is given by the $\geq$-ordering, $\mathsf{Fam}(\mathcal{V})$-categories are then the *normed categories*[24] described in [BG75]. These are categories where each morphism $f\colon x \to y$ has a norm $|f| \in \mathsf{R}_{\geq 0}$ satisfying $|1_x| = 0$ and $|fg| \leq |f| + |g|$. The Fam-Mealy morphisms between normed categories C and D are then given by a functor $F\colon C \to D$ on the underlying categories and an assignment of "weights" to objects, $|\cdot|\colon \mathsf{ob}(C) \to \mathsf{R}_{\geq 0}$ satisfying that for each $f\colon x \to y$ in C, $|y|+|f| \geq |g|+|x|$. This is probably more surprising than the notion of morphism given by $\mathsf{Fam}(\mathcal{V})$-functors, which has no weights on the objects of C and so resembles the definition of *short maps* between metric spaces. A Mealy cell between Fam-Mealy morphisms $F, G\colon C \rightsquigarrow D$ is a natural transformation $\gamma\colon G \Rightarrow F$ between the underlying functors satisfying $|x|_F \geq |\gamma_x| + |x|_G$ for all $x \in \mathsf{ob}(C)$. ◊

Example 4.7.11 (Categories with ideals)**.** Taking $\mathcal{V}$ to be the monoidal category of truth values, $\mathbb{2} = \{\bot \to \top\}$ with disjunction as the monoidal product, a $\mathsf{Fam}(\mathcal{V})$-category is a category C with a truth value $[f] \in \mathbb{T}$ for each morphism f satisfying $[1_c] = \bot$ and $[f] \vee [g] \implies [fg]$. If J denotes the subset of those morphisms with truth value $\top$, then these two conditions say that J is a *ideal* of the category which is proper on every endo-monoid. That is, J is closed under composition with arbitrary morphisms in C and $J \cap C(x,x)$ is a proper ideal of the monoid $C(x,x)$ for each $x \in C$. We will call an ideal in a category with this property a *proper ideal.* The Fam-Mealy morphisms are again somewhat less natural than the $\mathsf{Fam}(\mathcal{V})$-functors, which are just functors $F\colon C \to D$ that restrict to functions between the proper ideals. A Fam-Mealy morphism $(C, J) \rightsquigarrow (D, K)$ is instead a predicate $[-]\colon \mathsf{ob}(C) \to \mathbb{T}$ and an ordinary functor $F\colon C \to D$ satisfying that for each morphism $f\colon c \to d$, $[d] \vee [f] \implies [Ff] \vee [c]$. ◊

4.7.2 Enrichment in 2-Categories

A category enriched in a bicategory, $\mathcal{B}$, is a lax functor from a codiscrete category to $\mathcal{B}$. With this definition, a $\mathcal{V}$-category is equivalently a category enriched in the bicategory $\Sigma\mathcal{V}$. It stands to reason that a lax functor from a 1-category to a 2-category, $\mathcal{K}$, can be viewed as a sort of local-$\mathsf{Fam}(\mathcal{K})$-enriched category.

Definition 4.7.12 ($\mathfrak{F}\mathcal{K}$)**.** From a 2-category $\mathcal{K}$ we obtain a bicategory $\mathfrak{F}\mathcal{K}$ with the same objects and whose hom-categories are given by $\mathfrak{F}\mathcal{K}(x,y) = \mathsf{Fam}(\mathcal{K}(x,y))$. The composite of families $A\colon I \to \mathcal{K}(x,y)$ and $B\colon J \to \mathcal{K}(y,z)$ is given by:

$$J \times I \xrightarrow{B\times A} \mathcal{K}(y,z) \times \mathcal{K}(x,y) \xrightarrow{\mathsf{comp}} \mathcal{K}(x,z)$$

This composition fails to be strictly associative because the products of sets does. Identities and horizontal composition of 2-cells are as defined for the monoidal category $\mathsf{Fam}(\mathcal{V})$. ◊

Lax functors $C \to \mathcal{K}$ then correspond to lax functors from the codiscrete category on the object set of C into $\mathfrak{F}\mathcal{K}$ which when postcomposed by $\mathfrak{F}!\colon \mathfrak{F}\mathcal{K} \to \mathfrak{F}\mathbb{1} \cong \Sigma\mathsf{Set}$ give the Set-category C. One can then produce a notion of $\mathfrak{F}$-Mealy morphism between $\mathfrak{F}\mathcal{K}$-categories that will correspond to the morphisms of $\mathsf{F}_\Lambda\mathcal{K}$.

It's not clear that this is a simplifying perspective on $\mathsf{F}_\Lambda\mathcal{K}$, or that there are many naturally occurring examples of $\mathfrak{F}$-Mealy morphisms to apply it to. One justification for mentioning this enriched category perspective is that it fits thematically with an already established connection between enrichment and free cocompletions described in [GS16]. There, it is shown that the free cocompletion[25] in the bicategory of locally cocomplete bicategories under lax colimits from codiscrete categories sends $\mathcal{K}$ to the bicategory $\mathsf{Mod}(\mathcal{K})$ of $\mathcal{K}$-categories and *modules* between them. Moreover, by considering a suitable free cocompletion of a locally cocomplete $\mathcal{F}$-bicategory, $\mathcal{K}$, one obtains the $\mathcal{F}$-bicategory $\mathbb{M}\mathsf{od}(\mathcal{K})$ which, in the special case where $\mathcal{K}$ is inchordate, is just the inclusion $\mathsf{Cat}(\mathcal{K}) \to \mathsf{Mod}(\mathcal{K})$.

[24] Another slightly different notion of "categories with norms" with an extra condition is given in [Kub17]

[25] The meaning of free-cocompletion and lax colimit here is with respect to a certain monoidal bicategory Colim_∞ as the base of enrichment, rather than the monoidal 1-category Cat. With respect to this base, cocompletion under collages and arbitrary lax colimits of lax functors coincide.

Chapter 5

Oplax-Familial 2-Functors

We now turn our attention from the oplax fam construction to *oplax familial functors* — functors $K\colon \mathcal{A} \to \mathcal{B}$ for which the induced presheaves $\mathcal{B}(K-, b)\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ are in the saturation of the class Ω. The corresponding notion for ordinary Fam is essentially the parametric left adjoint, though there are a number of equivalent descriptions which we list below. Our two notions of oplax familial functor — one corresponding to F_Θ and the other corresponding to F_Ω — admit many of the same descriptions, and can be shown to share some properties with parametric left adjoints. In particular, they induce certain "generic factorisations" of morphisms of the form $Ka \to b$. One might expect these 2-categorical notions of familial functors to correspond to existing definitions, for example the familial 2-functors of Weber [Web07] and lax familial functors of Walker [Wal20]. Indeed, familial 2-functors in Weber's sense are special and dualised cases of our oplax familial functors, but the resemblance is stronger to the $\mathfrak{S}$-functors and $\mathfrak{P}$-functors introduced in Chapter 6, so we defer our discussion of these related notions until then. The main result of this chapter is that pointwise extensions along oplax-familial functors have a lifting property with respect to *lax* transformations, as well as 2-natural transformations. This property is applied to extend the free oplax-colimit completion from a 3-functor to a $\mathsf{Gray}_{\mathscr{L}}$-functor, i.e. one whose action is defined on the lax transformations in $2\mathsf{Cat}$ as well as the strict ones.

5.1 F_Φ-admissible Functors

Given a locally small category A, let us refer to a presheaf $X\colon A^{\mathsf{op}} \to \mathsf{Set}$ which is a small coproduct of representables as a *fam-presheaf.* These presheaves form the image of the fully-faithful functor $\mathsf{Fam}(A) \to [A^{\mathsf{op}}, \mathsf{Set}]$ which sends a family $M\colon I \to A$ to its *image presheaf* (i.e. the colimit of $\text{よ}M\colon I \to [A^{\mathsf{op}}, \mathsf{Set}]$). We will say a 1-functor $F\colon A \to B$ is *familial*[1] if the induced presheaf $B(F-, b)\colon A^{\mathsf{op}} \to \mathsf{Set}$ is a fam-presheaf for every $b \in B$. This condition is equivalent to any one of the following:

(a) F is a J-*left adjoint*, relative to the unit $J\colon A \to \mathsf{Fam}(A)$ to the Fam pseudomonad. That is, there is an isomorphism $B(F, 1) \cong A(J, R)$ for some $R\colon \mathsf{Fam}(B) \to A$ called a right-J adjoint. Right-J adjoints for this choice of J are also referred to as a *right multi-adjoints*, from [Die77].

(b) $\mathsf{Fam}(F)$ admits a right adjoint — I.e. F is Fam-admissible in the sense of Bunge and Funk, [BF99].

(c) Functors $G\colon A \to C$ into small-coproduct-complete categories admit left extensions along F which are preserved by coproduct-preserving functors.

(d) $J\colon A \to \mathsf{Fam}(A)$ admits a left extension along F preserved by coproduct-preserving functors — i.e. F is Fam-admissible, in the sense of Walker, [Wal18].

[1] Such functors are referred to as *familially representable* for the case where $B = \mathsf{Set}$ in [CJ95].

(e) F admits a factorisation as a left adjoint followed by a discrete opfibration

(f) Every morphism $u\colon Fa \to b$ can be expressed as a composite $u = v\,Fw$, where v is *generic* — i.e. locally terminal in $F \downarrow b$.

If the category A has an initial object, 0, then two further equivalent conditions are that the induced functors $F_a\colon a/A \to Fa/B$ all have right adjoints, or that $F_0\colon A \to F0/B$ in particular does. Such an F is referred to as a *parametric left adjoint.*

If we replace "*small coproduct* of representables" in the definition of fam-presheaf with "Φ-colimit of representables" for Φ some pre-saturated class[2] of small Cat-weights we obtain the following generalisation:

Definition 5.1.1 (F_Φ-functor)**.** For a pre-saturated class of small weights Φ, a 2-functor $K\colon \mathcal{A} \to \mathcal{B}$ is a F_Φ-functor if the induced Cat-presheaf $\mathcal{B}(K-, b)$ is a Φ-colimit of representables for all $b \in \mathcal{B}$. ◊

For example, if we let $\amalg$ denote the class of weights for 2-coproducts, then $\mathsf{F}_\amalg$-functors are what we might call *strict familial 2-functors*[3] The primary purpose of this chapter is to investigate F_Θ-functors and F_Ω-functors, the latter of which we might call *oplax-familial functors.*

As with familial functors, F_Φ-functors admit a number of equivalent definitions, some of which we describe and prove below. In particular, $F\colon \mathcal{A} \to \mathcal{B}$ is a F_Φ-functor precisely if it is F_Φ-*admissible.* The notion of an admissible 1-cell in a 2-category $\mathcal{A}$ with respect to a KZ-doctrine (M, μ, δ) was first given by Bunge and Funk in [BF99] as a 1-cell $f\colon a \to b$ whose image under M admits a right adjoint. In that work, it is shown that an equivalent property is that maps to M-cocomplete objects admit left extensions along f which are preserved by M-homomorphisms. In [Wal18] it is shown that it is sufficient to have this property with respect to the unit $a \to Ma$, and this is taken as the definition of M-admissible. We include both definitions of admissibility in our list of equivalent definitions below, and appeal to [BF99; Wal18] to justify their equivalence.

Proposition 5.1.2. *For Φ a pre-saturated class of small weights, the following conditions on a 2-functor $K\colon \mathcal{A} \to \mathcal{B}$ are equivalent:*

(a) K is a F_Φ-functor.

(b) K is a $\mathsf{Z}_\mathcal{A}$-left 2-adjoint, where $\mathsf{Z}_\mathcal{A} : \mathcal{A} \to \mathsf{F}_\Phi\mathcal{A}$ is the free Φ-cocompletion.

(c) $\mathsf{Z}_\mathcal{A}\colon \mathcal{A} \to \mathsf{F}_\Phi\mathcal{A}$ admits a left extension along K which is preserved by Φ-cocontinuous 2-functors.

(d) $\mathsf{F}_\Phi K$ admits a right adjoint.

Proof. Assume $K\colon \mathcal{A} \to \mathcal{B}$ is a F_Φ-functor, so that the image of $\mathcal{B}(K, 1)\colon \mathcal{B} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$, lands among Φ-colimits of representables (henceforth called Φ-presheaves). The inclusion $\mathsf{W}_\mathcal{A}\colon \mathsf{F}_\Phi\mathcal{A} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ is equivalent to the inclusion of the full-subcategory of Φ-presheaves because Φ is pre-saturated, so $\mathcal{B}(K, 1)$ factors through $\mathsf{W}_\mathcal{A}$ via some functor $R\colon \mathcal{B} \to \mathsf{F}_\Phi\mathcal{A}$. This 2-functor R defines the $\mathsf{Z}_\mathcal{A}$-relative right adjoint to K:

$$\mathcal{B}(Ka, b) \cong [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]\Big(よ_\mathcal{A} a, \mathcal{B}(K-, b)\Big) \cong [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}](\mathsf{W}_\mathcal{A}\mathsf{Z}_\mathcal{A} a, \mathsf{W}_\mathcal{A} Rb) \cong \mathsf{F}_\Phi\mathcal{A}(\mathsf{Z}_\mathcal{A} a, Rb) \tag{5.1}$$

using the fact that post-composition by $\mathsf{W}_\mathcal{A}$ is fully-faithful.

So we have (a) $\Rightarrow$ (b). Reversing the last two natural isomorphisms in (5.1) shows that if K is a $\mathsf{Z}_\mathcal{A}$-relative left adjoint with relative right adjoint R, then $\mathcal{B}(K-, b)$ is isomorphic to $\mathsf{W}_\mathcal{A} Rb$ so (b) $\Rightarrow$ (a).

[2]While the below definition makes sense for any class of weights, the condition that it be pre-saturated is necessary for it to coincide with the definition of F_Φ admissible, cf. the proof for Proposition 5.1.2.

[3]The name *familial 2-functor* is already used by Weber in [Web07] to describe a property that is both weaker and dual to being a $\mathsf{F}_\amalg$-functor.

Still assuming (a), there is a canonical 2-natural transformation $\eta_K \colon \mathsf{Z}_{\mathcal{A}} \Rightarrow R\,K$ from the fact that $\mathcal{B}(K, 1)$ is a left extension of $\text{よ}_{\mathcal{A}}$ along K (which is true for an arbitrary 2-functor, K) so that $\mathsf{W}_{\mathcal{A}}\,R \cong \mathcal{B}(K, 1)$ is a left extension of $\mathsf{W}_{\mathcal{A}}\,\mathsf{Z}_{\mathcal{A}} \cong \text{よ}_{\mathcal{A}}$ along K:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\text{よ}_{\mathcal{A}}"] \arrow[d, "K"'] \arrow[dr, "\mathsf{Z}_{\mathcal{A}}" description] \arrow[dr, phantom, "\cong", shift left=3] & {[\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}]} \\
\mathcal{B} \arrow[r, "R"'] \arrow[ur, phantom, "\Swarrow \eta_K", near start, shift right=3] & \mathsf{F}_\Phi\mathcal{A} \arrow[u, "\mathsf{W}_{\mathcal{A}}"']
\end{tikzcd}$$

The fully-faithful 2-functor $\mathsf{W}_{\mathcal{A}}$ reflects extensions, so η_K exhibits R as the left extension of $\mathsf{Z}_{\mathcal{A}}$ along K.

For (a) $\Rightarrow$ (c), it remains to show that the left extension $R \colon \mathcal{B} \to \mathsf{F}_\Phi\mathcal{A}$ is preserved by Φ-cocontinuous 2-functors. Assume for this purpose that $S \colon \mathsf{F}_\Phi\mathcal{A} \to C$ is Φ-cocontinuous. While the weights $\mathcal{B}(K-, b)$ are not necessarily in the saturation of Φ — they are not small whenever $\mathcal{A}$ isn't small — their colimits are nevertheless preserved by S because they are *extensions* of Φ-weights. Assuming $\mathcal{B}(K-, b)$ can be expressed as a Φ-colimit of representables as $\phi * \text{よ}T$ for $T \colon X \to \mathcal{A}$ some 2-functor from a small 2-category, we can show this property explicitly:

$$\begin{aligned}
S(\mathcal{B}(K-, b) * L) &\cong S\Big(\big(\phi * \text{よ}T\big) * L\Big) \\
&\cong S\Big(\phi * \big(\text{よ}T * L\big)\Big) \\
&\cong S(\phi * LT) \\
&\cong \phi * SLT \\
&\cong \big(\phi * \text{よ}T\big) * SL \\
&\cong \mathcal{B}(K-, b) * SL
\end{aligned}$$

We now observe that S preserves the extension R by an end calculus argument which essentially expresses that this is due to R being computed pointwise in terms of Θ-colimits. We make use of the fact that $\mathsf{W}_{\mathcal{A}}$ creates Θ-colimits to observe $R_b \cong \mathcal{B}(K-, b) * \mathsf{Z}_{\mathcal{A}}$.

$$\begin{aligned}
[\mathcal{B}, C](SR, H) &\cong \int_{b \in \mathcal{B}} C(SR_b, Hb) && \\
&\cong \int_{b \in \mathcal{B}} C(S(\mathcal{B}(K-, b) * \mathsf{Z}_{\mathcal{A}}), Hb) && (R_b \cong \mathcal{B}(K-, b) * \mathsf{Z}_{\mathcal{A}} \text{ since } \mathsf{W}_{\mathcal{A}}) \\
&\cong \int_{b \in \mathcal{B}} C((\mathcal{B}(K-, b) * S\mathsf{Z}_{\mathcal{A}}), Hb) && (S \text{ preserves } \Phi\text{-colimits}) \\
&\cong \int_{b \in \mathcal{B}} \int_{a \in \mathcal{A}} [\mathcal{B}(Ka, b), C(S\mathsf{Z}_{\mathcal{A}}a, Hb)] && (\text{definition of colimit}) \\
&\cong \int_{a \in \mathcal{A}} \int_{b \in \mathcal{B}} [\mathcal{B}(Ka, b), C(S\mathsf{Z}_{\mathcal{A}}a, Hb)] && (\text{Fubini theorem}) \\
&\cong \int_{a \in \mathcal{A}} C(S\mathsf{Z}_{\mathcal{A}}a, HKa) && (\text{Yoneda lemma}) \\
&\cong C^{\mathcal{A}}(S\mathsf{Z}_{\mathcal{A}}, HK) &&
\end{aligned}$$

So we have (a) $\Rightarrow$ (c). Proposition 15 from [Wal18] shows that (c) $\Leftrightarrow$ (d). To show (d) $\Rightarrow$ (b), assume that $\mathsf{F}_\Phi K$ has a right adjoint, $\mathcal{R} \colon \mathsf{F}_\Phi\mathcal{B} \to \mathsf{F}_\Phi\mathcal{A}$ and observe that K is a $\mathsf{Z}_{\mathcal{A}}$-left adjoint to $\mathcal{R}\,\mathsf{Z}_{\mathcal{B}}$:

$$\begin{aligned}
\mathcal{B}(Ka, b) &\cong \mathsf{F}_\Phi\mathcal{B}(\mathsf{Z}_{\mathcal{B}}Ka, \mathsf{Z}_{\mathcal{B}}b) \\
&\cong \mathsf{F}_\Phi\mathcal{B}(\mathsf{F}_\Phi K(\mathsf{Z}_{\mathcal{A}}a), \mathsf{Z}_{\mathcal{B}}b) \\
&\cong \mathsf{F}_\Phi\mathcal{A}(\mathsf{Z}_{\mathcal{A}}a, \mathcal{R}\mathsf{Z}_{\mathcal{B}}b)
\end{aligned}$$

□

Corollary 5.1.3. *The composition of F_Φ-functors is a F_Φ-functor.*

Proof. If $K\colon \mathcal{A} \to \mathcal{B}$ and $H\colon \mathcal{B} \to \mathcal{C}$ are F_Φ-functors, then $\mathsf{F}_\Phi(HK) = \mathsf{F}_\Phi H\,\mathsf{F}_\Phi K$ admits a right adjoint. □

5.2 Oplax-generic Factorisations

Considering the classes Θ — weights for oplax colimits — and Ω — weights for oplax colimits of oplax functors from 1-categories — in particular, F_Θ-functors and F_Ω-functors $K\colon \mathcal{A} \to \mathcal{B}$ are additionally characterised by the property that morphisms $Ka \to b$ admit a notion of "generic factorisation" that resembles generic factorisations for familial functor defined in [Web04] and developed in [Web07].

Recall from Chapter 3 the following result which classifies the locally discrete split 2-fibrations which are categories of elements of a $\mathcal{Q}$-coalgebra, and the related definition of *oplax-generic* object:

Proposition 3.4.20. *A locally discrete split 2-fibration $p\colon \mathcal{X} \to \mathcal{A}$ admits a $\mathfrak{Q}$-coalgebra structure if and only if every object of $\mathcal{X}$ admits a chosen cartesian morphism to a oplax generic object.*

Definition 3.4.18 (Oplax-generic object)**.** An object x in the domain of a locally discrete split 2-fibration $p\colon \mathcal{X} \to \mathcal{A}$ is called *oplax-generic* if it satisfies:

(a) For any morphism $f\colon y \to x$ and a chosen-cartesian $g\colon y \to z$ the category $(y \Downarrow \mathcal{X})(g, f)$ has a single connected component.

(b) If $f\colon y \to x$ and $g\colon y \to z$ are both chosen-cartesian, then there exists a unique chosen-cartesian $H\colon z \to x$. ◊

A 2-functor $K\colon \mathcal{A} \to \mathcal{B}$ is a F_Θ-functor if for all $b \in \mathcal{B}$, $\mathcal{B}(K-, b)$ is a $\mathcal{Q}$-coalgebra, which is equivalent to the locally discrete split 2-fibration $K \Downarrow b$ being a $\mathfrak{Q}$-coalgebra. Once we figure out what *chosen-cartesian* and *oplax generic* mean for $K \Downarrow b \to \mathcal{A}$, we will obtain the notion of generic factorisations appropriate to F_Θ-functors. We should expect this characterisation to resemble the generic-factorisation property for familial functors, which arises from the observation that categories of elements of fam-presheaves are those in which every object admits a morphism to a locally terminal object.

A morphism in $K \Downarrow b$ from $u\colon Kx \to b$ to $v\colon Ky \to b$ is a 1-cell $f\colon x \to y$ in $\mathcal{A}$ and a 2-cell $\sigma\colon u \Rightarrow v\,(Kf)$ in $\mathcal{B}$:

$$\begin{tikzcd} Kx \arrow[rr, "Kf"] \arrow[dr, "u"'] & \arrow[d, phantom, "\xRightarrow{\sigma}"] & Ky \arrow[dl, "v"] \\ & b & \end{tikzcd} \tag{5.2}$$

Such a morphism is *chosen-cartesian* with respect to the map $K \Downarrow b \to \mathcal{A}$ if σ is an identity. To say that a given object $u\colon Kx \to b$ admits a chosen-cartesian morphism to an oplax-generic object is therefore to say that u factors as $v\,(Kf)$ for some $f\colon x \to y$ and oplax-generic $v\colon Ky \to b$. We will refer to such a factorisation as an *oplax-generic factorisation.*

Now we consider which morphisms $u\colon Kx \to b$ are oplax-generic objects with respect to $K \Downarrow b \to \mathcal{A}$. The hypothesis of condition (a) from Definition 3.4.18 for a putative oplax-generic $u\colon Kx \to b$ involves a pair of triangles of the form shown in (5.2), where one triangle strictly commutes. Such a configuration can also be viewed as a square of the form shown on the right of (5.3):

$$
\begin{tikzcd}
Ky \arrow[r, "Kg"] \arrow[d, "Kf"'] \arrow[dr, "v" description] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] \arrow[dr, phantom, shift right=3, "\Leftarrow\sigma"] & Kz \arrow[d, "w"] \\
Kx \arrow[r, "u"'] & b
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
Ky \arrow[r, "Kg"] \arrow[d, "Kf"'] \arrow[dr, phantom, "\swarrow\sigma"] & Kz \arrow[d, "w"] \\
Kx \arrow[r, "u"'] & b
\end{tikzcd}
\tag{5.3}
$$

The conclusion of the first condition is that given such a square, the category $(v \Downarrow (K \Downarrow b)[(g, 1), (f, \sigma)])$ has a single connected component. The objects of this category are given by a morphism $h\colon z \to x$, a 2-cell $\tau\colon w \Rightarrow u\,(Kh)$, and a 2-cell $\alpha\colon h\,g \Rightarrow f$ such that $\sigma = u\,(K\alpha) \circ \tau\,(Kg)$:

$$
\begin{tikzcd}
& Kz \arrow[dr, "w"] \arrow[dd, "Kh" description] \arrow[dr, phantom, shift right=4, "\swarrow\tau"] & \\
Ky \arrow[ur, "Kg"] \arrow[dr, "Kf"'] \arrow[r, phantom, "K\alpha\swarrow"] & {} & b \\
& Kx \arrow[ur, "u"'] &
\end{tikzcd}
=
\begin{tikzcd}
& Kz \arrow[dr, "w"] \arrow[dd, phantom, "\sigma\Downarrow"] & \\
Ky \arrow[ur, "Kg"] \arrow[dr, "Kf"'] & & b \\
& Kx \arrow[ur, "u"'] &
\end{tikzcd}
\tag{5.4}
$$

For u to satisfy the first condition, such a diagonal filler must exist and be connected to any other diagonal filler. The morphisms between diagonal fillers are 2-cells between the underlying morphisms which commute with the 2-cells filling the triangles. For example, the 2-cell $\delta\colon k \Rightarrow h$ describes a morphism between the two diagonal fillers $(\alpha, K\delta\,\tau)$ and $(\alpha\,\delta, \tau)$ below:

$$
\begin{tikzcd}
& Kz \arrow[dr, "w"] \arrow[dd, bend right=40, "Kh" description] \arrow[dd, dotted, bend left=40, "Kk" description] & \\
Ky \arrow[ur, "Kg"] \arrow[dr, "Kf"'] \arrow[r, phantom, "K\alpha \Leftarrow \quad \overset{K\delta}{\Longleftarrow} \quad \overset{\tau}{\Leftarrow}"] & {} & b \\
& Kx \arrow[ur, "u"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& Kz \arrow[dr, "w"] \arrow[dd, dotted, bend right=40, "Kh" description] \arrow[dd, bend left=40, "Kk" description] & \\
Ky \arrow[ur, "Kg"] \arrow[dr, "Kf"'] \arrow[r, phantom, "K\alpha \Leftarrow \quad \overset{K\delta}{\Longleftarrow} \quad \overset{\tau}{\Leftarrow}"] & {} & b \\
& Kx \arrow[ur, "u"'] &
\end{tikzcd}
\tag{5.5}
$$

The second condition says that in the case where σ is an identity, there exists a unique $h\colon z \to x$ such that $w = u\,(Kh)$. A consequence of this condition is that h will additionally satisfy $h\,g = f$, because any $f'\colon y \to x$ satisfying $u\,(Kf') = u\,(Kf)$ (which includes $f' = h\,g$) must be equal to f by applying the second condition to the diagram below:

$$
\begin{tikzcd}
& Ky \arrow[dr, "u\,Kf"] \arrow[dd, phantom, "\circlearrowleft"] & \\
Ky \arrow[ur, equal] \arrow[dr, "Kf"'] & & b \\
& Kx \arrow[ur, "u"'] &
\end{tikzcd}
$$

In particular, if some $v\colon Ky \to b$ factors through an oplax-generic morphism $u\colon Kx \to b$ then it does so in a unique way. The morphism v may also admit a factorisation through some other oplax-generic morphism, but any two such factorisations will be isomorphic. For if v factorises through oplax-generic $u\colon Kx \to b$ and $u'\colon Kx' \to b$, there must exist unique $h\colon x \to x'$ and $k\colon x' \to x$ satisfying $u'\,(Kh) = u$ and $u\,(Kh') = u'$. It then follows that h is an isomorphism from x to x' and is moreover an isomorphism between the two oplax-generic factorisations of v.

We can summarise the properties of oplax generic morphisms as follows:

Definition 5.2.1 (Oplax-generic morphism, oplax-generic factorisation)**.** For $K\colon \mathcal{A} \to \mathcal{B}$ an object $u\colon Ka \to b$ is an *oplax generic morphism* if it has unique diagonal fillers for strictly commuting squares, and a unique connected component of diagonal fillers for weakly commuting squares as shown in (5.4). An oplax-generic factorisation for

$f\colon Ka \to b$ in $K \Downarrow b$ is a pair of a morphism $f_1\colon a \to \hat{f}$ in $\mathcal{A}$ and an oplax-generic morphism $f_2\colon K\hat{f} \to b$ such that $f_2\,(Kf_1) = f$. ◊

Note that oplax-generic factorisation is done on objects of $K \downarrow \mathcal{B}$, rather than morphisms of $\mathcal{B}$, so when we refer to an oplax-generic factorisation of $f\colon Ka \to b$ we really mean an oplax-generic factorisation of $(a, f : Ka \to b)$. In particular, oplax-generic factorisations of $f : Ka \to b$ and $f\colon Ka' \to b$ clearly won't coincide if $a \neq a'$.

From our earlier observation we also have:

Lemma 5.2.2. *If $f\colon Ka \to b$ admits two oplax generic factorisations, $Ka \xrightarrow{Kf_1} K\hat{f} \xrightarrow{f_2} b$ and $Ka \xrightarrow{Ks} Kx \xrightarrow{t} b$, then there is a unique isomorphism $d\colon x \to \hat{f}$ satisfying $d\,s = f_1$ and $f_2\,(Kd) = t$.*

With this interpretation of oplax-generic objects in $K \Downarrow b$ for general $b \in \mathcal{B}$ as oplax-generic morphisms, and of chosen-cartesian morphisms to these objects as factorisations, we obtain the following characterisation of F_Θ-functors:

Proposition 5.2.3. *A 2-functor $K\colon \mathcal{A} \to \mathcal{B}$ is a F_Θ-functor if and only if each object $f\colon Ka \to b$ in $K \Downarrow \mathcal{B}$ admits an oplax-generic factorisation.*

We can similarly characterise F_Ω-functors by adapting the recognition result for $\mathfrak{Q}$-coalgebras — Proposition 3.4.20 — to obtain a recognition result for categories of elements of Ω-presheaves. Given any Θ-presheaf, F, one can restrict the locally-discrete split 2-fibration from its category of elements, $P_F\colon \int F \to \mathcal{A}$ to its "generic core" to obtain a local discrete opfibration whose oplax-image presheaf is isomorphic to F. By Corollary 4.2.2, this oplax-image presheaf will be a Ω-presheaf if and only if the generic core has component-initial 1-cells, so F is a Ω-presheaf if and only if its category of elements is a $\mathfrak{Q}$-coalgebra *and* hom-categories between the generic objects have component-initial objects.

In fact, a slightly stronger condition will hold on the category of elements of a Ω-presheaf due to the following lemma:

Lemma 5.2.4. *For $p\colon X \to \mathcal{A}$ a $\mathfrak{Q}$-coalgebra, if all hom-categories between oplax-generics have component-initial 1-cells then so do all hom-categories into an oplax generic object.*

Proof. Assume $a, x \in X$ with x oplax-generic. By Proposition 3.4.20 there exists some chosen-cartesian $u\colon a \to y$ where y is oplax-generic. We will show that if $f\colon y \to x$ is component-initial in $X(y,x)$, then $f\,u$ is component-initial in $X(a,x)$.

First observe that every $v\colon a \to x$ admits a unique 2-cell σ from a 1-cell of the form $f\,u$ for component-initial $f\colon y \to x$ because $(a \Downarrow X)(u,x)$ has a unique connected component. For some other $w\colon a \to x$ with a 2-cell $\alpha\colon w \Rightarrow u$ there exists some other component-initial $g\colon y \to x$ and unique $\beta\colon g\,u \Rightarrow w$ and so $\alpha\,\beta\colon g\,u \Rightarrow v$ describes a morphism from u to v in $a \Downarrow X$. This morphism must be in the same component as $\sigma\colon f\,u \Rightarrow v$, so there exists some $\tau\colon f \Rightarrow g$ and thus a 2-cell $f\,u \Rightarrow w$ which is necessarily unique. It follows that $f\,u$ is initial in its connected component. □

It follows that oplax-generics for categories of elements of F_Ω-presheaves satisfy the following stronger property:

Definition 5.2.5 (Ω-generic object)**.** For $p\colon X \to \mathcal{A}$ a locally discrete split 2-fibration $p\colon X \to A$, an object $x \in X$ is Ω-generic if:

(a) For any $f\colon y \to x$ and chosen-cartesian $g\colon y \to z$, the category $(y \Downarrow X)(g, f)$ has an initial object.

(b) For any chosen-cartesian $f\colon y \to x$ and $g\colon y \to z$ there exists a unique chosen-cartesian $h\colon z \to x$. ◊

Lemma 5.2.6. *A presheaf $P\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ is a F_Ω-presheaf if and only if every object in its category of elements $p\colon X \to \mathcal{A}$ admits a chosen-cartesian morphism to a Ω-generic object.*

Proof. If objects of X admit chosen-cartesian morphisms to Ω-generics — which are in particular oplax-generic — then P is a F_Θ-presheaf by Proposition 3.4.20. Moreover, every oplax-generic will be Ω-generic because condition 2 of Definitions 3.4.18 and 5.2.5 ensure that any morphism from an oplax-generic to a Ω-generic will be an isomorphism. It follows that hom-categories between oplax-generics are hom-categories between Ω-generics, which by taking g in condition 1 of Definition 5.2.5 to be the identity on an oplax-generic must have component-initial 1-cells. It follows that the F_Θ-presheaf is moreover a F_Ω-presheaf.

Conversely, if P is a F_Ω-presheaf then all objects of X admit chosen-cartesian morphisms to oplax-generics and by Lemma 5.2.4 all hom-categories into oplax-generics have component-initial objects. It follows that for any oplax-generic $x \in A$ and morphisms $f\colon y \to x$, $g\colon y \to z$ where g is chosen-cartesian, the category $(y \Downarrow X)(g, f)$ has component-initial objects. Such component-initial objects exist and are initial because $(y \Downarrow X)(g, f)$ has a single connected component, so we conclude that all oplax-generics of X are Ω-generics. □

For $K\colon \mathcal{A} \to \mathcal{B}$ a 2-functor we will say an oplax-generic factorisation of $u\colon Ka \to b$ is Ω-*generic* whenever the oplax-generic component is Ω-generic with respect to K. The Ω-generic morphisms admit the following slightly stronger definition:

Definition 5.2.7 (Ω-generic morphism)**.** An object $f\colon Ka \to b$ of $K \Downarrow \mathcal{B}$ is Ω-generic if it has strict diagonal fillers for strictly commuting squares and an *initial* diagonal filler for weakly commuting squares of the form shown in (5.4). ◊

Corollary 5.2.8. *A 2-functor $K\colon \mathcal{A} \to \mathcal{B}$ is a F_Ω-functor if and only if all objects $u\colon Ka \to b$ in $K \Downarrow \mathcal{B}$ admit Ω-generic factorisations.*

5.2.1 $\mathcal{Q}$-coalgebras from generic factorisations

For $K\colon \mathcal{A} \to \mathcal{B}$ a F_Θ-functor, we can choose a representative for each isomorphism class of oplax generics in each $K \Downarrow b$ so that morphisms $Ka \to b$ admit unique oplax-generic factorisations through *chosen* oplax-generic morphisms. Assuming we have made such a choice, we will use the notation $Ka \xrightarrow{Ku_1} K\hat{u} \xrightarrow{u_2} b$ to denote the chosen oplax-generic factorisation of $u\colon Ka \to b$ (which will automatically be Ω-generic if K is also a F_Ω-functor). We can then describe a $\mathcal{Q}$-coalgebra structure on each $\mathcal{B}(K-, b)$ in terms of these factorisations. The component at $a \in \mathcal{A}$ is given on objects by:

$$Ka \xrightarrow{u} b \quad \mapsto \quad \left(a \xrightarrow{u_1} \hat{u}, F\hat{u} \xrightarrow{u_2} b\right) \in \oint^{x \in \mathcal{A}} \mathcal{A}(a, x) \times \mathcal{B}(Fx, b) = \mathcal{Q}\mathcal{B}(K-, b)_a$$

The action on morphisms is given by the “diagonal filler” property of oplax-generics:

$$\begin{tikzcd} Ka \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & K\hat{u} \arrow[dr, "u_2"] \arrow[dd, "K\hat{\alpha}"] & \\ Ka \arrow[ur, "Ku_1"] \arrow[dr, "Kv_1"'] \arrow[r, phantom, "\Leftarrow K\alpha_1"] & {} \arrow[r, phantom, "\Leftarrow \alpha_2"] & b \\ & K\hat{v} \arrow[ur, "v_2"'] & \end{tikzcd} \tag{5.6}$$

The oplax-genericity of v_2 merely provides the existence of a diagonal filler $\hat{\alpha}$, and the fact that any other such diagonal filler is connected to it in the sense of (5.5). But the 1-cells of $\mathcal{Q}\mathcal{B}(K-, b)$ are the connected components of such diagonal fillers (cf. (3.16)) so this mapping on 1-cells is well-defined in any case. Though when K is a F_Ω-functor, we can represent connected components of diagonal fillers by their component-initial element so that there exists a unique diagonal filler $(\hat{\alpha}, \alpha_2)$ which is component-initial in $(K \Downarrow b)(u_2, v_2)$.

5.2.2 Examples

Below we give some examples of F_Θ-functors, F_Ω-functors and $\mathsf{F}_\amalg$-functors. Each successive class of functors is a restriction of the previous: $\amalg \subset \Omega \subset \Theta$. Most naturally occurring examples of F_Θ-functors seem also to be $\mathsf{F}_\amalg$-functors, though we do exhibit (and in some cases "manufacture") separating examples.

Example 5.2.9 (Left adjoints)**.** Left adjoints are F_Φ-functors for Φ the saturation of the empty class of colimits, so they are in particular F_Φ-functors for $\Phi \in \{\amalg, \Omega, \Theta\}$. For a left adjoint $F\colon \mathcal{A} \to \mathcal{B}$, the oplax-generics to $b \in \mathcal{B}$ are universal arrows $F \to b$ and a collection of chosen oplax-generics amounts to a choice of right adjoint $G\colon \mathcal{B} \to \mathcal{A}$ and counit $\epsilon : FG \to 1_\mathcal{B}$. The chosen oplax-generic factorisation of $u\colon Fa \to b$ is then $Fa \xrightarrow{u'} FGb \xrightarrow{\epsilon_b} b$ where $u'\colon a \to Gb$ is the transpose of u. ◊

Example 5.2.10 (Functors $\langle a \rangle : \mathbb{1} \to \mathcal{A}$)**.** Any 2-functor with domain the terminal 2-category is a F_Ω-functor. The relative right $\mathsf{Z}_\mathbb{1}$-adjoint of $\langle a \rangle : \mathbb{1} \to \mathcal{A}$, is simply the representable $\mathcal{A}(a, -)\colon \mathcal{A} \to \mathsf{Cat}$ (recall that Cat is equivalent to $\mathsf{F}_\Omega\mathbb{1}$). Observe that for any $x \in \mathcal{A}$, $\mathcal{A}(a, x)$ is isomorphic to $\oint^{\mathcal{A}(a,x)} \mathbb{1}(*, *) \cong \mathcal{A}(a, x) \odot \mathbb{1}$. Every object $a \to b$ in $\langle a \rangle \Downarrow \mathcal{A} \cong a \Downarrow \mathcal{A}$ in $\mathcal{A}$ is Ω-generic, so Ω-generic factorisations exist trivially. The 2-functor $\langle a \rangle$ will only be a $\mathsf{F}_\amalg$-functor if each hom-category $\mathcal{A}(a, x)$ for $x \in \mathcal{A}$ is discrete, and will only be a left adjoint if each $\mathcal{A}(a, x)$ is isomorphic to $\mathbb{1}$ (i.e. a is initial). ◊

Example 5.2.11 (F_Θ-admissible 2-categories)**.** The notion of M-admissible 1-cell is extended in [BF99, Defn. 2.4] to define M-admissible objects as those whose unique 1-cell to the terminal object is M-admissible. With this terminology, we can consider what properties are required of a 2-category $\mathcal{A}$ for it to be F_Θ-admissible. We first observe that this property is equivalent to the identity 2-functor on $\mathcal{A}$ being a $\mathfrak{Q}$-coalgebra, since the identity on $\mathcal{A}$ is isomorphic to the projection $! \Downarrow * \to \mathcal{A}$ and $!\colon \mathcal{A} \to *$ is a F_Θ-functor if and only if this projection is a $\mathfrak{Q}$-coalgebra. We can therefore re-frame the property of F_Θ-admissibility as the property that every object in $\mathcal{A}$ admits a morphism to an object which is oplax-generic with respect to the locally split discrete 2-fibration $\mathcal{A} = \mathcal{A}$.

From Definition 3.4.18 and the fact that all 1-cells are chosen-cartesian with respect to the identity, an object x in $\mathcal{A}$ is oplax-generic if for any morphisms $f\colon y \to x$ and $g\colon y \to z$ there is a unique morphism $h\colon z \to x$ and the category $(y \Downarrow \mathcal{A})(g, f)$ has a unique connected component. The first condition clearly says that an oplax-generic object must be component-terminal in the underlying 1-category, and if an object is moreover component-terminal in the 2-categorical sense then the second condition will be satisfied as well. However, the property of being an oplax-generic object is, in general, weaker than being component-2-terminal. Let's assume $x \in \mathcal{A}$ is merely component-1-terminal (component terminal on the underlying 1-category), and consider what it means for $(y \Downarrow \mathcal{A})(g, f)$ to have a unique connected component for all $g\colon y \to z$ and $f\colon y \to x$.

The 1-cells in $y \Downarrow \mathcal{A}$ from f to g are given by a pair of a 1-cell $h\colon z \to x$ and a 2-cell $\alpha\colon hg \Rightarrow f$. Since x is component-1-terminal, such a 1-cell h is unique and f must be equal to hg, so objects of $(y \Downarrow \mathcal{A})(g, f)$ are given by 2-cells $\sigma\colon f \Rightarrow f$. A morphism between σ and $\tau\colon f \Rightarrow f$ in $(y \Downarrow \mathcal{A})(g, f)$ is given by a 2-cell $\rho\colon h \Rightarrow h$ such that $\sigma = \tau \circ \rho h$. It follows that $(y \Downarrow \mathcal{A})(g, f)$ will have a unique connected component for all $g\colon y \to z$ if $(y \Downarrow \mathcal{A})(f, f)$ does, since there is a functor $(y \Downarrow \mathcal{A})(f, f) \to (y \Downarrow \mathcal{A})(g, f)$ given by the "identity" on the $\sigma\colon f \Rightarrow f$ objects, and on 1-cells by sending $\rho\colon 1_x \Rightarrow 1_x$ to $\rho h\colon h \Rightarrow h$.

So considering now the case $(y \Downarrow \mathcal{A})(f,f)$, clearly one object in this category is $1_f \colon f \Rightarrow f$, so we can phrase the unique-connected-component condition as the property that all 2-cells $\alpha \colon f \Rightarrow f$ are in the same connected component as the identity. We will say (just within the scope of this example) that a 2-cell $\alpha : f \Rightarrow f$ is *near* if there exist 2-cells $\sigma, \tau \colon 1_x \Rightarrow 1_x$ such that $\alpha \circ \sigma f = \tau f$. On one hand, if α is near, then it is clearly connected via $\sigma \colon 1_x \Rightarrow 1_x$ to a 2-cell of the form τf, which is connected to the identity via τ. Thus, near 2-cells are in the connected component of the identity. We now show by induction that the converse is true, all 2-cells connected to the identity in $(y \Downarrow \mathcal{A})(f,f)$ are near.

First, observe that 1_f is clearly near, taking $\sigma = \tau = 1_{1_x}$. Now if ρ is a morphism from β to α, and α is near, then we have $\beta = \alpha \circ \rho f$. For σ, τ exhibiting α as near, we observe that β must also be near, witnessed by σ and $\tau \circ \rho$, using the fact that vertical (and horizontal) composition in $\mathcal{A}(x,x)(1_x,1_x)$ is commutative (by the Eckmann-Hilton argument):

$$\beta \circ \sigma f = \alpha \circ (\rho \circ \sigma) f = \alpha \circ (\sigma \circ \rho) f = (\tau \circ \rho) f$$

If we assumed instead that β were near, then α must be near as well, witnessed by $\rho \circ \sigma$ and τ.

$$\alpha \circ (\rho \circ \sigma) f = \beta \circ \sigma f = \tau f$$

So $\mathcal{A}$ is F_Θ-admissible precisely when the underlying 1-category $\mathcal{A}_0$ has component-terminal objects and every endo-2-cell on a unique 1-cell to a component-1-terminal object is *near*. This description is not particularly clarifying, but we can at least note that it is weaker than the existence of component-terminal objects: the suspension of a 1-object monoidal category (i.e. commutative monoid) is F_Θ-admissible since all 2-cells between identity 1-cells are near, but such 2-categories don't in general have component-terminal objects.

The property of being F_Ω-admissible is simpler. If a 2-category has enough component-1-terminal objects *and* the hom-categories into the 1-locally-terminal objects have component-initial objects — which is the second condition for Ω-genericity — then the component-1-terminal objects are, in fact, component-terminal. Choosing component-terminal objects $\{x_i\}_{i \in I}$ for each connected component of $\mathcal{A}$, we observe that $\coprod_{i \in I} \mathcal{A}(-,x_i) \cong \Delta_{\mathbb{1}} \cong \mathbb{1}(!-,*)$, so F_Ω-admissibility is actually equivalent to $\mathsf{F}_\amalg$-admissibility. ◊

Remark 5.2.12. More generally, any F_Ω-functor with codomain a locally-discrete 2-category will also be strict-familial. This is because for $K \colon \mathcal{A} \to \mathcal{B}$ with $\mathcal{B}$ locally-discrete, every 1-cell in $K \Downarrow b$ for any $b \in \mathcal{B}$ will be chosen-cartesian. As a result, hom-categories into an oplax-generic will have a unique object. Adding the condition that hom-categories into oplax-generics have component-initial objects forces these hom-categories to be isomorphic to the terminal category, which means the oplax-generics are also locally-terminal. The condition that every object in $K \Downarrow b$ admits a morphism to a locally-terminal object for every $b \in \mathcal{B}$ is equivalent to K being strict-familial. ◊

Example 5.2.13. For any small 2-category, $\mathcal{A}$, one can construct its *lax cocone classifier*, $i \colon \mathcal{A} \hookrightarrow \mathcal{A}_\Delta$, so-named because 2-functors $F \colon \mathcal{A}_\Delta \to \mathcal{K}$ are equivalent to lax cocones from Fi to some object in $\mathcal{K}$. The 2-category $\mathcal{A}_\Delta$ can be constructed by adding a single object, V, to $\mathcal{A}$ with hom-categories given by:

$$\mathcal{A}_\Delta(a,b) = \begin{cases} \mathcal{A}(a,b) & a,b \in \mathcal{A} \\ \oint^{x \in \mathcal{A}} \mathcal{A}(a,x) & a \in \mathcal{A},\ b = V \\ \varnothing & a = V\ \ b \in \mathcal{A} \\ \mathbb{1} & a = b = V \end{cases} \tag{5.7}$$

The non-trivial composition maps $\oint^{x \in \mathcal{A}} \mathcal{A}(b,x) \times \mathcal{A}(a,b) \to \oint^{x \in \mathcal{A}} \mathcal{A}(a,x)$ are those induced by the extra-natural family of composition maps $\gamma_x \colon \mathcal{A}(b,x) \times \mathcal{A}(a,b) \to \mathcal{A}(a,x)$. The inclusion $i \colon \mathcal{A} \to \mathcal{A}_\Delta$ is then a F_Θ-functor.

To prove this we need to consider the presheaf $\mathcal{A}_\Delta(i-,a)$ for the cases $a \in \mathcal{A}$ and $a = V$. For $a \in \mathcal{A}$, we have $\mathcal{A}_\Delta(i-,a) \cong \mathcal{A}(-,a)$, which is representable and thus a F_Θ-functor. For $a = V$ we have $\mathcal{A}_\Delta(i-,V) \cong \oint^{x\in\mathcal{A}} \mathcal{A}(-,x)$, which is patently the oplax image presheaf of $\mathcal{A} = \mathcal{A}$.

The oplax generic factorisation of a 1-cell $if\colon ia \to ib$ is $ia \xrightarrow{if} ib = ib$. A 1-cell $f\colon ia \to V$ is equivalently an object in $\oint^{x\in\mathcal{A}} \mathcal{A}(a,x)$ which is determined by a choice of $b \in \mathcal{A}$ and a 1-cell $u\colon a \to b$. The 1-cell f then factorises as:

$$ia \xrightarrow{iu} ib \xrightarrow{\lambda_b} V$$

where λ_b is one of the edges of the canonical lax cocone $i \rhd V$ corresponding to $* \xrightarrow{1_b} \mathcal{A}(b,b) \xrightarrow{e_b} \oint^{x\in\mathcal{A}} \mathcal{A}(b,x)$.

The inclusion $i\colon \mathcal{A} \to \mathcal{A}_\Delta$ will be a F_Ω-functor if $\mathcal{A}$ has component-initial 1-cells, and will be strict-familial if $\mathcal{A}$ is a disjoint sum of single-object 2-categories where each $\mathcal{A}(a,a)$ has an initial object (i.e. if $\mathcal{A}$ admits a bijective-on-objects locally initial functor from a locally discrete 2-category). $\lozenge$

Remark 5.2.14. The above construction is a special case of the *collage* of a profunctor, as described in Example 4.7.5. Replacing the second case in (5.7) with Wa for some weight $W\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ produces the collage of W viewed as a profunctor $\mathbb{1} \nrightarrow \mathcal{A}$. The canonical arrow $i\colon \mathcal{A} \to \mathcal{A}_W$ has the property that 2-functors $F\colon \mathcal{A}_W \to \mathcal{K}$ classify 2-natural transformations of the form $W \Rightarrow \mathcal{K}(Fi-,k)$, and the left-extension of $G\colon \mathcal{A} \to \mathcal{K}$ along $i\colon \mathcal{A} \to \mathcal{A}_W$ will compute $W * G$ as the image of $V \in \mathcal{A}_W$. Whenever W is in class Φ, $i\colon \mathcal{A} \to \mathcal{A}_W$ will be a F_Φ functor. The lax cocone construction of Example 5.2.13 is then simply the collage of the weight $\mathcal{Q}\Delta_{\mathbb{1}}\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$, which is a (free) $\mathcal{Q}$-coalgebra and therefore a F_Θ-presheaf. $\lozenge$

Example 5.2.15. Similar to the lax cocone construction is the inclusion of the comonad classifier coMnd into the *adjunction classifier*, Adj, of [SS86]. This is a F_Ω-functor. If we denote by C and M the two objects of Adj — where the free *left* adjoint goes from M to C — then $i\colon \mathsf{coMnd} \to \mathsf{Adj}$ is simply the inclusion of the full sub-2-category containing the object C.

For this reason the presheaf $\mathsf{Adj}(i-,\mathsf{C})$ is isomorphic to $\mathsf{coMnd}(-,*)$, which is representable and therefore a F_Θ-presheaf. On the other hand, $\mathsf{Adj}(i-,\mathsf{M})$ is isomorphic to $\oint^{\mathsf{coMnd}} \mathsf{coMnd}(*,*)$, which is the coKleisli category of the canonical comonad on $\mathsf{C} \in \mathsf{Adj}$. This is a Ω-colimit, and so inclusion $i\colon \mathsf{coMnd} \to \mathsf{Adj}$ is a F_Ω-functor, and in fact a F_κ-functor where κ is the weight for coKleisli objects[4].

Note that the hom-categories $\mathsf{Adj}(\mathsf{M},-)$ were irrelevant to determining whether $i\colon \mathsf{coMnd} \to \mathsf{Adj}$ is a F_Ω-functor. If we replaced $\mathsf{Adj}(\mathsf{M},\mathsf{M})$ with $\mathbb{1}$ and $\mathsf{Adj}(\mathsf{M},\mathsf{C})$ with $\varnothing$, we would simply obtain the lax cocone 2-category coMnd_Δ of Example 5.2.13 $\lozenge$

Example 5.2.16 (The unit for the monad)**.** In general, the unit $\mathsf{Z}_{\mathcal{A}}\colon \mathcal{A} \to \mathsf{F}_\Phi\mathcal{A}$ of a free cocompletion monad will be a F_Φ-functor. For an object $S \in \mathsf{F}_\Phi\mathcal{A}$ there exists some weight $\phi\colon C^{\mathsf{op}} \to \mathsf{Cat}$ and 2-functor $H\colon C \to \mathcal{A}$ such that $\mathsf{W}_{\mathcal{A}}(S) \cong \phi * \mathcal{A}(1,H)$. Then, observe:

$$\mathsf{F}_\Phi\mathcal{A}(\mathsf{Z}_{\mathcal{A}}-,S) \cong [\mathcal{A}^{\mathsf{op}},\mathsf{Cat}]\Big(よ_{\mathcal{A}}-, \phi * \mathcal{A}(1,H)\Big) \cong \phi * \mathcal{A}(1,H)$$

and $\phi * \mathcal{A}(1,H)$ is by definition a Φ-presheaf.

For the case $\Phi = \Theta$, we can consider what the oplax-generic morphisms for the unit $\mathsf{Z}_{\mathcal{A}}$ look like. Recall that the image of $a \in \mathcal{A}$ under $\mathsf{Z}_{\mathcal{A}}\colon \mathcal{A} \to \mathsf{F}_\Theta\mathcal{A}$ is $\langle a\rangle : \mathbb{1} \to \mathcal{A}$. A morphism in $\mathsf{F}_\Theta\mathcal{A}$ from $\langle a\rangle$ to $P\colon C \to \mathcal{A}$ in $\mathsf{F}_\Theta\mathcal{A}$ is a

[4] The class of weights containing only κ is pre-saturated, [LS02].

morphism in $\mathsf{2Cat}/\mathcal{A}$ from $\langle a\rangle$ to $\mathfrak{q}_P$:

$$\begin{tikzcd} \mathbb{1} \arrow[rr, "f"] \arrow[dr, "\langle a\rangle"'] & \circlearrowleft & \mathfrak{Q}_P \arrow[dl, "\mathfrak{q}_P"] \\ & \mathcal{A} & \end{tikzcd}$$

Such a morphism is specified by an arbitrary choice of object $c \in C$ and a morphism $u\colon a \to Pc$. To detect whether such a morphism is oplax-generic, it suffices to demonstrate that it is component-terminal in the cartesian core (property (b) of Definition 3.4.18) since any object with this property must be isomorphic to any other object it admits a chosen-cartesian morphism to, which includes in particular an oplax-generic object by the characterisation of F_Θ-algebras in Proposition 5.2.3.

Given two objects in $\mathsf{Z}_{\mathcal{A}} \Downarrow P$, represented by morphisms $u\colon a \to Px$ and $v\colon b \to Py$, a chosen-cartesian 1-cell from u to v is a 1-cell $s\colon a \to b$ satisfying $x = y$ and $u = v\,s$. Now, assume we have another object $w\colon c \to Pz$ in $\mathsf{Z}_{\mathcal{A}} \Downarrow P$ and chosen-cartesian 1-cells $s\colon u \to v$, $t\colon u \to w$. It follows that $x = y = z$ and the following diagram commutes:

$$\begin{tikzcd} & c \arrow[dr, "w"] & \\ a \arrow[ur, "t"] \arrow[dr, "s"'] \arrow[rr, phantom, "\circlearrowleft"] & & Px \\ & b \arrow[ur, "v"'] & \end{tikzcd} \tag{5.8}$$

The morphism $v\colon b \to Px$ will be an oplax-generic object of $\mathsf{Z}_{\mathcal{A}} \Downarrow P$ if and only if for all such commuting squares there exists some diagonal filler $h\colon c \to b$ satisfying $w = vh$ and $s = ht$. If v is invertible then $h = v^{-1}w$ satisfies these conditions, so the invertibility of v is a sufficient condition for v to be oplax generic. Taking $w = 1_{Px}$, $t = v$ and $s = 1_b$ demonstrates that it is also a necessary condition.

In particular, morphisms $\langle Px\rangle \to P$ of the form 1_{Px} are oplax-generic and are in bijection with the isomorphism classes of oplax-generics; so these can be our "chosen" oplax-generics for the purpose of determining unique oplax-generic factorisations. If we let (x, u) denote the morphism $\langle a\rangle \to P$ represented by $u\colon a \to Px$, then each such morphism admits the following oplax-generic factorisation:

$$\begin{tikzcd} \langle a\rangle \arrow[rr, "{(x,u)}"] \arrow[dr, "{(*,u)}"'] & \circlearrowleft & P \\ & \langle Px\rangle \arrow[ur, "{(x,1_{Px})}"'] & \end{tikzcd}$$

◊

5.3 Morphisms of F_Θ-functors

With the understanding that a F_Θ-functor $K\colon \mathcal{A} \to \mathcal{B}$ is one such that 1-cells $u\colon Ka \to b$ admit oplax-generic factorisations, a sensible notion of *morphism* of F_Θ-functors is a commuting square which preserves oplax-generic factorisations. That is, a morphism between F_Θ-functors $K\colon \mathcal{A} \to \mathcal{B}$ and $L\colon C \to \mathcal{D}$ is a pair of 2-functors $F\colon \mathcal{A} \to C$, $G\colon \mathcal{B} \to \mathcal{D}$ which form a commuting square:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[d, "L"] \\ \mathcal{B} \arrow[r, "G"'] & \mathcal{D} \end{tikzcd}$$

such that G maps K-oplax-generic arrows $u\colon Ka \to b$ to L-oplax-generic arrows $Gu\colon LFa \to Gb$. When K and L are endowed with a collection of *chosen* oplax-generic arrows and G preserves chosen oplax-generics, we say it is a *strict* morphism of F_Θ-functors. Note that if K and L are additionally F_Ω-functors, a (strict) morphism of F_Θ-functors between them will preserve (chosen) Ω-generic 1-cells.

Example 5.3.1 (Morphisms between free Θ-cocompletions)**.** We saw in Example 5.2.16 that the free Θ-cocompletion, $\mathsf{Z}_{\mathcal{A}}\colon \mathcal{A} \to \mathsf{F}_\Theta\mathcal{A}$, is a F_Θ-functor. For a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$, the commuting square:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "\mathsf{Z}_{\mathcal{A}}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{B} \arrow[d, "\mathsf{Z}_{\mathcal{B}}"] \\ \mathsf{F}_\Theta\mathcal{A} \arrow[r, "\mathsf{F}_\Theta F"'] & \mathsf{F}_\Theta\mathcal{B} \end{tikzcd}$$

is a strict morphism of F_Θ-functors relative to the chosen oplax-generics described in Example 5.2.16. Indeed, the image of an oplax-generic factorisation of $(x,u)\colon \langle a\rangle \to P$ under $\mathsf{F}_\Theta F$ is the same as the oplax generic factorisation of $\mathsf{F}_\Theta F(x,u) = (x, Fu)$:

$$\begin{tikzcd}[column sep=small] \langle a\rangle \arrow[rr, "{(x,u)}"] \arrow[dr, "{(*,u)}"'] & \arrow[d, phantom, "\circlearrowleft"] & P & \mapsto & \langle a\rangle \arrow[rr, "{(x,Fu)}"] \arrow[dr, "{(*,Fu)}"'] & \arrow[d, phantom, "\circlearrowleft"] & FP \\ & \langle Px\rangle \arrow[ur, "{(x,1_{Px})}"'] & & & & \langle FPx\rangle \arrow[ur, "{(x,1_{FPx})}"'] & \end{tikzcd}$$

◊

5.4 F_Θ-functors as Coalgebras

We have defined F_Θ-functors as those $K\colon \mathcal{A} \to \mathcal{B}$ for which the induced presheaf $\mathcal{B}(K-,b)\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ is a $\mathcal{Q}$-coalgebra for all b. In this section we shall see that the structure maps for these coalgebras $\mathcal{B}(K-,b) \to \mathcal{Q}\mathcal{B}(K-,b)$ are lax-natural in b and assemble into a coalgebra map for the corepresentable profunctor $\mathcal{B}(K,1)\colon \mathcal{A}^{\mathsf{op}} \times \mathcal{B} \to \mathsf{Cat}$ with respect to an "oplax-morphism in $\mathcal{A}$" classifier comonad on $\mathsf{Prof}(\mathcal{B},\mathcal{A})$. This characterisation of F_Θ-functors has an application in providing an adjunction between the oplax-morphism-in-$\mathcal{A}$ classifier for $\mathcal{B}(K,1)$ with the lax-morphism-in-$\mathcal{B}$ classifier. This in turn allows us to demonstrate that pointwise left-extensions along F_Θ-functors have an additional lifting property with respect to lax transformations, which we describe in Section 5.5.

Recall that for a locally small 2-category, $\mathcal{A}$, the oplax-morphism classifier is an oplax-idempotent comonad on $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ which sends a presheaf $X\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ to $\oint^{x\in\mathcal{A}} \mathcal{A}(-,x) \times X_x$. Similarly, the *lax* morphism classifier can be defined as a *lax*-idempotent comonad on $[\mathcal{A}, \mathsf{Cat}]$ which sends $X\colon \mathcal{A} \to \mathsf{Cat}$ to the oplax[5] coend $\oint^{a\in\mathcal{A}} X_a \times \mathcal{A}(a,-)$, as witnessed by the following calculation:

$$\begin{aligned} [\mathcal{A}, \mathsf{Cat}]_{\mathsf{lax}}(X,Y) &\cong \oint_{a\in\mathcal{A}} [X_a, Y_a] \\ &\cong \oint_{a\in\mathcal{A}} \left[X_a, \int_{b\in\mathcal{A}} [\mathcal{A}(a,b), Y_b]\right] \\ &\cong \int_{b\in\mathcal{A}} \oint_{a\in\mathcal{A}} [X_a \times \mathcal{A}(a,b), Y_b] \\ &\cong \int_{b\in\mathcal{A}} \left[\oint^{a\in\mathcal{A}} X_a \times \mathcal{A}(a,b), Y_b\right] && \text{(cf. Lemma 2.2.8)} \\ &\cong [\mathcal{A}, \mathsf{Cat}]\left(\oint^{a\in\mathcal{A}} X_a \times \mathcal{A}(a,-), Y\right) \end{aligned}$$

[5]This *lax* morphism classifier is also given by an *oplax* coend because we have changed the variance of $\mathcal{A}$ as well as the direction of laxness, cf. Remark 5.4.1.

Remark 5.4.1. Of course, one can also define the lax-morphism classifier on $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ and oplax-morphism classifier on $[\mathcal{A}, \mathsf{Cat}]$ using *lax* coends as:

$$X \mapsto \sqint^{a \in \mathcal{A}} \mathcal{A}(-, a) \times X_a \qquad X \mapsto \sqint^{a \in \mathcal{A}} X_a \times \mathcal{A}(a, -)$$

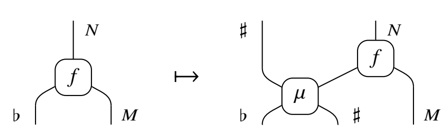
by the fact that $\oint^{\mathcal{A}^{\mathsf{op}}} \cong \sqint^{\mathcal{A}}$. These would be the relevant variants if we instead wanted to consider functors $K \colon \mathcal{A} \to \mathcal{B}$ such that $\mathcal{B}(b, K-)$ is an oplax colimit of representables. ◊

The 2-category of profunctors $\mathsf{Prof}(\mathcal{B}, \mathcal{A}) := [\mathcal{A}^{\mathsf{op}} \times \mathcal{B}, \mathsf{Cat}]$ admits lifts of both oplax-morphism classifier in $\mathcal{A}$ and the lax-morphism classifier in $\mathcal{B}$. The lifts are given respectively by the image of the oplax-morphism classifier on $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ under $[\mathcal{B}, -] \colon \mathsf{2Cat} \to \mathsf{2Cat}$ and the image of the lax-morphism classifier on $[\mathcal{B}, \mathsf{Cat}]$ under $[\mathcal{A}^{\mathsf{op}}, -]$. We denote these two comonads $(-)^\sharp$ and $(-)^\flat$ respectively, observe that they are oplax (resp. lax) idempotent, as these are preserved by 2-functors. Moreover, there is an invertible distributive law between these two comonads given by the Fubini rule for oplax coends:

$$\begin{aligned}
\left(M^\sharp\right)^\flat(a, b) &\cong \oint^{y \in \mathcal{B}} M^\sharp(a, y) \times \mathcal{B}(y, b) \\
&\cong \oint^{y \in \mathcal{B}} \left(\oint^{x \in \mathcal{A}} \mathcal{A}(a, x) \times M(x, y) \right) \times \mathcal{B}(y, b) \\
&\cong \oint^{y \in \mathcal{B}} \oint^{x \in \mathcal{A}} \mathcal{A}(a, x) \times M(x, y) \times \mathcal{B}(y, b) \\
&\cong \oint^{x \in \mathcal{A}} \oint^{y \in \mathcal{B}} \mathcal{A}(a, x) \times M(x, y) \times \mathcal{B}(y, b) \\
&\cong \oint^{x \in \mathcal{A}} \mathcal{A}(a, x) \times \oint^{y \in \mathcal{B}} M(x, y) \times \mathcal{B}(y, b) \\
&\cong \oint^{x \in \mathcal{A}} \mathcal{A}(a, x) \times M^\flat(x, b) \\
&\cong \left(M^\flat\right)^\sharp(x, b)
\end{aligned} \tag{5.9}$$

It is easy to see that this isomorphism preserves counits and the comultiplications for each comonad essentially by definition. For example, the counit for $\flat$ at $M^\sharp$ is defined by the oplax cowedge $M^\sharp(a, y) \times \mathcal{B}(y, b) \to M^\sharp(a, b)$ in y which in turn is defined by the oplax cowedge in x given by $\mathcal{A}(a, x) \times M(x, y) \times \mathcal{B}(y, b) \to \mathcal{A}(a, x) \times M(x, b) \to M^\sharp(a, b)$. The image under $\sharp$ of the counit for $\flat$ at M is induced by the oplax cowedge in x of $\mathcal{A}(a, x) \times M^\flat(x, b) \to \mathcal{A}(a, x) \times M(x, b) \to M^\sharp(a, b)$ whose components are induced by the oplax cowedge in y with components $\mathcal{A}(a, x) \times M(x, y) \times \mathcal{B}(y, b) \to \mathcal{A}(a, x) \times M(x, b) \to M^\sharp(a, b)$. That is to say, both maps (and indeed the two comultiplications) are induced by the universal property of both $M^{\sharp\flat}$ and $M^{\flat\sharp}$ as the oplax coend over $\mathcal{A}^{\mathsf{op}} \times \mathcal{B}$ of $\mathcal{A}(a, x) \times M(x, y) \times \mathcal{B}(y, b)$, and (5.9) is the unique comparison isomorphism between these two expressions of this oplax coend.

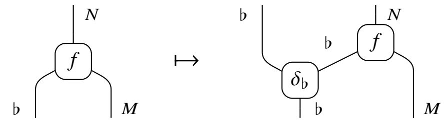
As a consequence of this distributive law, each of the comonads $\sharp$ and $\flat$ lifts to a comonad on the coKleisli category of the other. In particular, if we let $\mu \colon \flat\sharp \Rightarrow \sharp\flat$ denote the isomorphism shown in (5.9), then $\sharp$ lifts to a comonad $\overline{\sharp}$ on the coKleisli category for $\flat$ whose action on objects is $M \mapsto M^\sharp$ and on a coKleisli 1-cell $f \colon M^\flat \to N$ is given by:



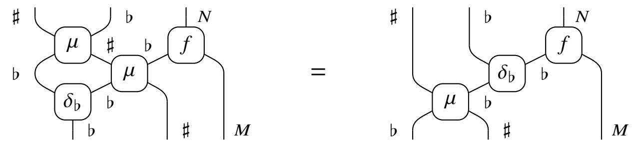
A 2-cell $\alpha \colon f \Rightarrow g \colon M^\flat \to N$ in $\mathsf{coKl}_\flat$ is sent to $\alpha^\sharp \mu_M$.

The distributive law μ then exhibits the canonical map $\flat' \colon \mathsf{coKl}_\flat \to \mathsf{Prof}(\mathcal{B}, \mathcal{A})$ as a morphism of comonads from $\overline{\sharp}$

to $\sharp$. The map $\flat'$ acts on objects by $M \mapsto M^\flat$. Its action on morphisms is defined in terms of the comultiplication $\delta_\flat$ for $\flat$ as:



Clearly each component $\mu_M\colon M^{\sharp^\flat} \to M^{\flat^\sharp}$ provides the data for the component of a transformation $\flat'\overline{\sharp} \Rightarrow \sharp\flat'$. The 1-naturality of this data corresponds to the equality of the following 1-cells in $\mathsf{Prof}(\mathcal{B},\mathcal{A})$ for any coKleisli 1-cell $f\colon M^\flat \to N$ in $\mathsf{coKl}_\flat$:



which follows from the fact that μ is a distributive law. The 2-naturality follows by replacing the 1-cell f with a 2-cell $\alpha\colon f \Rightarrow g$. That this 2-natural transformation is a morphism of comonads follows immediately from the fact that μ preserves the counit and comultiplication of $\sharp$.

Applying the same argument with the comonads $\sharp$ and $\flat$ exchanging roles shows that μ^{-1} endows the canonical map $\sharp'\colon \mathsf{coKl}_\sharp \to \mathsf{Prof}(\mathcal{B},\mathcal{A})$ with the structure of a comonad morphism from the lifted comonad $\overline{\flat}$ to $\flat$.

The coKleisli category $\mathsf{coKl}_\sharp$ can be viewed as the 2-category whose objects are profunctors $\mathcal{B} \nrightarrow \mathcal{A}$ and whose 1-cells are 2-natural transformations which are *oplax-natural* in $\mathcal{A}$ and *strictly 2-natural* in $\mathcal{B}$. More precisely: $\mathsf{coKl}_\sharp \cong [\mathcal{A}^{\mathsf{op}}, [\mathcal{B}, \mathsf{Cat}]]_{\mathsf{oplax}}$. We can demonstrate this by end calculus:

$$\begin{aligned}
\mathsf{coKl}_\sharp(M,N) &\cong \int_{(a,b)\in\mathcal{A}^{\mathsf{op}}\times\mathcal{B}} \Big[M^\sharp(a,b), N(a,b)\Big] \\
&\cong \int_{(a,b)\in\mathcal{A}^{\mathsf{op}}\times\mathcal{B}} \left[\oint^{x\in\mathcal{A}} \mathcal{A}(a,x)\times M(x,b), N(a,b)\right] \\
&\cong \oint_{x\in\mathcal{A}^{\mathsf{op}}} \int_{(a,b)\in\mathcal{A}^{\mathsf{op}}\times\mathcal{B}} [M(x,b), [\mathcal{A}(a,x), N(a,b)]] \\
&\cong \oint_{x\in\mathcal{A}^{\mathsf{op}}} \int_{b\in\mathcal{B}} \left[M(x,b), \left[\int_{a\in\mathcal{A}^{\mathsf{op}}} \mathcal{A}(a,x), N(a,b)\right]\right] \\
&\cong \oint_{x\in\mathcal{A}^{\mathsf{op}}} \int_{b\in\mathcal{B}} [M(x,b), N(x,b)]
\end{aligned} \tag{5.10}$$

By a similar argument we have $\mathsf{coKl}_\flat \cong [\mathcal{B}, [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]]_{\mathsf{oplax}}$. From this perspective, the lifted comonads $\overline{\sharp}$ and $\overline{\flat}$ can be described as the images of the oplax-morphism classifier on $\mathcal{A}^{\mathsf{op}}$ and lax-morphism classifier on $\mathcal{B}$ under the 2-functors $[\mathcal{B},-]_{\mathsf{lax}}$ and $[\mathcal{A}^{\mathsf{op}},-]_{\mathsf{oplax}}$ respectively.

This discussion of profunctors is relevant to F_Θ-functors by the following lemma:

Lemma 5.4.2. *A 2-functor $K\colon \mathcal{A} \to \mathcal{B}$ is a F_Θ-functor precisely if $\mathcal{B}(K,1)\colon \mathcal{A}^{\mathsf{op}}\times\mathcal{B} \to \mathsf{Cat}$ is a $\overline{\sharp}$-coalgebra.*

Proof. Of course, by "is a $\overline{\sharp}$-coalgebra" we mean "admits a $\overline{\sharp}$-coalgebra structure", though such structure will be unique up to isomorphism if it exists as $\overline{\sharp}$ is an oplax-idempotent comonad.

The simpler condition of a profunctor $M \in \mathsf{Prof}(\mathcal{B},\mathcal{A})$ admitting a $\sharp$-coalgebra is equivalent to the existence of a lift of $M\colon \mathcal{B} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ along the forgetful 2-functor $\mathsf{U}_{\mathcal{Q}}\colon \mathcal{Q}\text{-}\mathsf{coalg} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ by the universal property of $\mathcal{Q}\text{-}\mathsf{coalg}$ as an Eilenberg-Moore object (in $2\mathsf{Cat}^{\mathsf{co}}$). Stated explicitly, there is an isomorphism of morphisms over

$[\mathcal{B}, [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]]$:

$$\begin{tikzcd}[\mathcal{B}, \mathcal{Q}\text{-coalg}] \arrow[rr, "\cong"] \arrow[dr, "(\mathsf{U}_{\mathcal{Q}})_*"'] & \circlearrowleft & {[\mathcal{B}, \mathcal{Q}]\text{-coalg}} \arrow[dl, "\mathsf{U}_{[\mathcal{B},\mathcal{Q}]}"] \\ & {[\mathcal{B}, [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]]} & \end{tikzcd} \tag{5.11}$$

which is moreover an isomorphism between post-composition $[\mathcal{B}, \mathcal{Q}\text{-coalg}] \to [\mathcal{B}, [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]]$ and the forgetful map $[\mathcal{B}], \mathcal{Q}\text{-coalg} \to [\mathcal{B}, [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]]$.

The 2-category $\mathcal{Q}\text{-coalg}_{\mathsf{oplax}}$ has a similar — if less familiar — universal property as described [6]in [LS12, §4.3]. The universal property can be expressed as the following isomorphism over $[\mathcal{B}, [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]]$:

$$\begin{tikzcd}{\left[\mathcal{B}, \mathcal{Q}\text{-coalg}_{\mathsf{oplax}}\right]_{\mathsf{lax}}} \arrow[rr, "\cong"] \arrow[dr, "(\mathsf{U}_{\mathcal{Q}})_*"'] & \circlearrowleft & {[\mathcal{B}, \mathcal{Q}]_{\mathsf{lax}}\text{-coalg}_{\mathsf{oplax}}} \arrow[dl, "\mathsf{U}_{[\mathcal{B},\mathcal{Q}]_{\mathsf{lax}}}"] \\ & {[\mathcal{B}, [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]]_{\mathsf{lax}}} & \end{tikzcd} \tag{5.12}$$

We conclude from this that $M\colon \mathcal{B} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ admits a $\bar{\sharp}$-coalgebra precisely if it admits a lift along the forgetful functor $\mathsf{U}_{\mathcal{Q}}\colon \mathcal{Q}\text{-coalg}_{\mathsf{oplax}} \to [\mathcal{B}, [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]]$. Because $\mathcal{Q}$ is oplax-idempotent the forgetful functor $\mathsf{U}_{\mathcal{Q}}$ is fully-faithful, so a lift of M along $\mathsf{U}_{\mathcal{Q}}$ exists if and only if $M(-,b)$ is a Θ-presheaf for each $b \in \mathcal{B}$. A corepresentable profunctor $\mathcal{B}(K,1)$ satisfies this property precisely if $K\colon \mathcal{A} \to \mathcal{B}$ is a F_Θ-functor. □

A similar argument gives the following result:

Lemma 5.4.3. *Corepresentable profunctors $\mathcal{B}(K,1)$ are $\bar{\flat}$-coalgebras.*

Proof. Let $\mathsf{W}\colon [\mathcal{B}, \mathsf{Cat}] \to [\mathcal{B}, \mathsf{Cat}]$ denote the lax-morphism classifier. The 2-category $\mathsf{W}\text{-coalg}_{\mathsf{lax}}$ has the following universal property:

$$\begin{tikzcd}{[\mathcal{A}^{\mathsf{op}}, \mathsf{W}\text{-coalg}_{\mathsf{lax}}]_{\mathsf{oplax}}} \arrow[rr, "\cong"] \arrow[dr, "(\mathsf{U}_{\mathsf{W}})_*"'] & \circlearrowleft & {[\mathcal{A}^{\mathsf{op}}, \mathsf{W}]_{\mathsf{oplax}}\text{-coalg}_{\mathsf{lax}}} \arrow[dl, "\mathsf{U}_{[\mathcal{A}^{\mathsf{op}},\mathsf{W}]_{\mathsf{oplax}}}"] \\ & {[\mathcal{A}^{\mathsf{op}}, [\mathcal{B}, \mathsf{Cat}]]_{\mathsf{lax}}} & \end{tikzcd} \tag{5.13}$$

So, $M\colon \mathcal{A}^{\mathsf{op}} \to [\mathcal{B}, \mathsf{Cat}]$ admits a $\bar{\flat}$-coalgebra structure if and only if it admits a lift through the fully faithful[7] $\mathsf{U}_{\mathsf{W}}\colon \mathsf{W}\text{-coalg}_{\mathsf{lax}} \to [\mathcal{A}^{\mathsf{op}}, [\mathcal{B}, \mathsf{Cat}]]$ which is equivalent to $M(a,-)$ being a lax colimit of representables. This is true if M is corepresentable, so that $M(a,-) = \mathcal{B}(Ka,-)$ which is trivially a lax colimit of representables. □

Because $\sharp'\colon \mathsf{coKl}_\sharp \to \mathsf{Prof}(\mathcal{B},\mathcal{A})$ and $\flat'\colon \mathsf{coKl}_\flat \to \mathsf{Prof}(\mathcal{B},\mathcal{A})$ are morphisms of comonads, they map coalgebras to coalgebras. In particular, if $M\colon \mathcal{B} \nrightarrow \mathcal{A}$ admits a $\bar{\flat}$-coalgebra structure, then $M^\flat$ admits a $\flat$-coalgebra structure, and dually. So by Lemmas 5.4.2 and 5.4.3 we observe that for a F_Θ-functor $K\colon \mathcal{A} \to \mathcal{B}$:

(a) $\mathcal{B}(K,1)^\sharp$ admits a $\flat$-coalgebra structure: $h\colon \mathcal{B}(K,1)^\sharp \to \mathcal{B}(K,1)^{\sharp^\flat}$

(b) $\mathcal{B}(K,1)^\flat$ admits a $\sharp$-coalgebra structure: $k\colon \mathcal{B}(K,1)^\flat \to \mathcal{B}(K,1)^{\flat^\sharp}$

The comonad $\sharp$ is oplax-idempotent and $\flat$ is lax-idempotent, so h must be a left-adjoint section to the counit $\epsilon_\flat$ of the comonad $\flat$ and k must be a right-adjoint section to $\epsilon_\sharp$. Stringing these adjunctions together, we get an

[6]The description given in [LS12] is for $\mathscr{F}$-monads on $\mathscr{F}$-categories, which specialises to our application by some duality arguments and viewing 2-categories to "chordate" $\mathscr{F}$-categories.

[7]because W is lax-idempotent

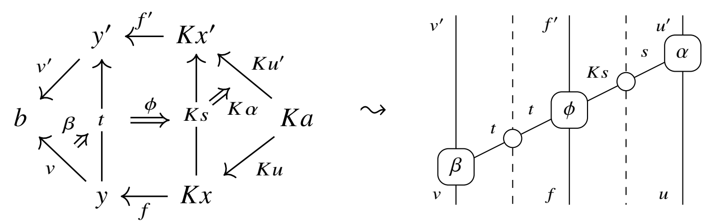
adjunction between $\mathcal{B}(K,1)^\sharp$ and $\mathcal{B}(K,1)^\flat$:

$$\begin{tikzcd} \mathcal{B}(K,1)^\flat \arrow[r, bend left, "k"] \arrow[r, phantom, "\top"] & \mathcal{B}(K,1)^{\flat\sharp} \arrow[l, bend left, "\epsilon_\sharp"] \arrow[r, bend left, "\mu^{-1}_{\mathcal{B}(K,1)}"] \arrow[r, phantom, "\circlearrowleft"] & \mathcal{B}(K,1)^{\sharp\flat} \arrow[l, bend left, "\mu_{\mathcal{B}(K,1)}"] \arrow[r, bend left, "\epsilon_\flat"] \arrow[r, phantom, "\top"] & \mathcal{B}(K,1)^\sharp \arrow[l, bend left, "h"] \end{tikzcd} \tag{5.14}$$

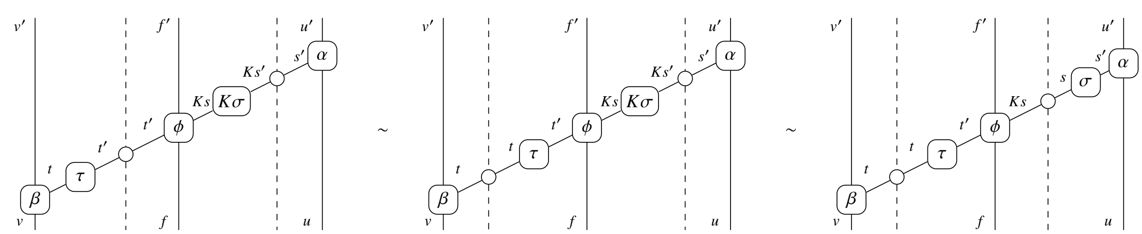
To better understand this adjunction we will try to give a more tangible description of the profunctor (up to isomorphism) given by $\mathcal{B}(K,1)^{\sharp\flat} \cong \mathcal{B}(K,1)^{\sharp\flat}$. From (5.9) we can express this profunctor as an oplax coend:

$$\mathcal{B}(K,1)^{\sharp\flat}(a,b) \cong \oint^{(x,y)\in\mathcal{A}\times\mathcal{B}} \mathcal{A}(a,x)\times\mathcal{B}(Kx,y)\times\mathcal{B}(y,b)$$

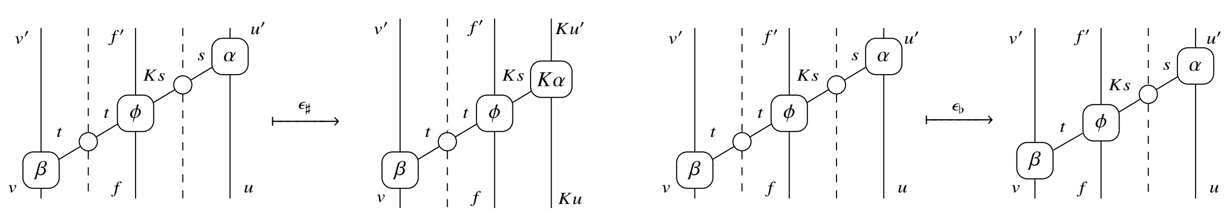
The objects of the category $\mathcal{B}(K,1)^{\sharp\flat}(a,b)$ can thus be identified with triples of the form $(a \xrightarrow{u} x\,,\ Kx \xrightarrow{f} y\,,\ y \xrightarrow{v} b)$. The morphisms $(u,f,v) \to (u',f',v')$ are equivalence classes on data of the following form, which we give once in both string-diagram and globular expressions:



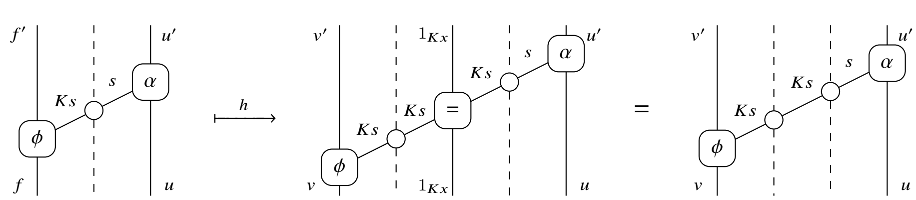
The equivalence classes are generated by relations shown below for 2-cells $\sigma\colon s \Rightarrow s'$, $\tau\colon t \Rightarrow t'$ (cf. 3.16):



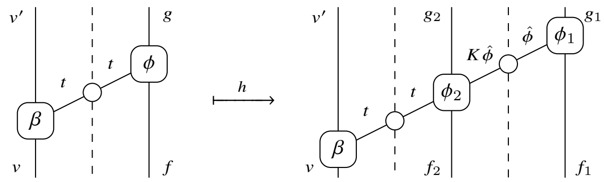
The counit maps for the comonads $\sharp$ and $\flat$ then act on objects and 1-cells as follows:



The $\flat$-coalgebra map $h\colon \mathcal{B}(K,1)^\sharp \Rightarrow \mathcal{B}(K,1)^{\sharp\flat}$ is given by the canonical lax-morphism classifier coalgebra structures on the representables $\mathcal{B}(Ka,-)$:



On the other hand, the $\sharp$-coalgebra map $k\colon \mathcal{B}(K,1)^\flat \Rightarrow \mathcal{B}(K,1)^{\flat\sharp}$ is given by the oplax-generic factorisation of 1-cells and diagonal fillers of (5.6):



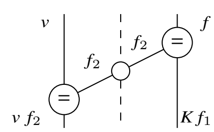
where ϕ_1, $\hat{\phi}$ and ϕ_2 are given by the diagonal filler for the following square:

$$
\begin{tikzcd}
 & K\hat{f} \arrow[dr, "tf_2"] \arrow[dd, phantom, "\phi\Downarrow"] & \\
Ka \arrow[ur, "Kf_1"] \arrow[dr, "Kg_1"'] & & x \\
 & K\hat{g} \arrow[ur, "g_2"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & K\hat{f} \arrow[dr, "tf_2"] \arrow[dd, "K\hat{\phi}"] & \\
Ka \arrow[ur, "Kf_1"] \arrow[dr, "Kg_1"'] \arrow[r, phantom, "\overset{\Leftarrow}{K\phi_1}"] & {} & x \arrow[l, phantom, "\overset{\Leftarrow}{\phi_2}"] \\
 & K\hat{g} \arrow[ur, "g_2"'] &
\end{tikzcd}
$$

Now consider the image of an object $(f,v) \in \mathcal{B}(K,1)^\flat$ moving from left to right and back in diagram (5.14):

$$
\left(Ka \xrightarrow{f} y \xrightarrow{v} b\right) \xrightarrow{k} \left(a \xrightarrow{f_1} \hat{f}, K\hat{f} \xrightarrow{f_2} y \xrightarrow{v} b\right) \xrightarrow{\epsilon_\flat} \left(a \xrightarrow{f_1} \hat{f}, K\hat{f} \xrightarrow{v\,f_2} b\right)
$$
$$
\left(Ka \xrightarrow{Kf_1} K\hat{f} \xrightarrow{v\,f_2} b\right) \xleftarrow{\epsilon_\sharp} \left(a \xrightarrow{f_1} \hat{f}, K\hat{f} \xrightarrow{=} K\hat{f} \xrightarrow{v\,f_2} b\right) \xleftarrow{h} \left(a \xrightarrow{f_1} \hat{f}, K\hat{f} \xrightarrow{v\,f_2} b\right)
$$

The counit ρ of the adjunction $\epsilon_\sharp\, h \dashv \epsilon_\flat\, k$ is given in terms of the counit $\rho_\flat$ of the adjunction $h \dashv \epsilon_\flat$ as $\epsilon_\sharp\, \rho_\flat\, k$, because the counit of $\epsilon_\sharp \dashv k$ is the identity. The component $\rho_{(f,v)} : (Kf_1, v\,f_2) \longrightarrow (f,v)$ is thus given by:



(5.15)

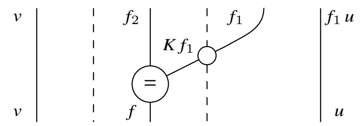
Similarly, the unit λ of the adjunction $\epsilon_\sharp\, h \dashv \epsilon_\flat\, k$ is given in terms of the unit $\lambda_\sharp$ of the adjunction $\epsilon_\sharp \dashv k$ as $\epsilon_\flat\, \lambda_\sharp\, h$. Before describing its components, first consider the action of $\epsilon_\flat\, k\, \epsilon_\sharp\, h$ on an object $(u,f) \in \mathcal{B}(Ka,b)^\sharp$:

$$
\left(Ka \xrightarrow{Ku} Kx \xrightarrow{f} b\right) \xleftarrow{\epsilon_\sharp} \left(a \xrightarrow{u} x, Kx \xrightarrow{=} Kx \xrightarrow{f} b\right) \xleftarrow{h} \left(a \xrightarrow{u} x, Kx \xrightarrow{f} b\right)
$$
$$
\left(Ka \xrightarrow{Ku} Kx \xrightarrow{f} b\right) \xrightarrow{k} \left(a \xrightarrow{(Ku)_1} \widehat{Ku}, K\widehat{Ku} \xrightarrow{(Ku)_2} Kx \xrightarrow{f} b\right) \xrightarrow{\epsilon_\flat} \left(a \xrightarrow{(Ku)_1} \widehat{Ku}, K\widehat{Ku} \xrightarrow{f\,(Ku)_2} b\right) \tag{5.16}
$$

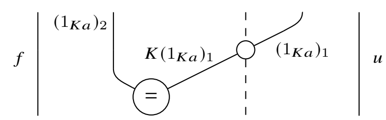
By the uniqueness of chosen oplax-generic factorisations we must have $(Ku)_1 = (1_{Kx})_1\, u$, $\widehat{Ku} = \widehat{1_{Kx}}$ and $(Ku)_2 = (1_{Kx})_2$, so that the image under $\epsilon_\flat\, k\, \epsilon_\sharp\, h$ of an arbitrary object (u,f) determined by the images of objects of the form $(x = x, Kx = Kx)$. This corresponds to the 2-naturality of $\epsilon_\flat\, k\, \epsilon_\sharp\, h$, as $(u,f) = \mathcal{B}(K,1)^\sharp_{(u,f)}(1_x, 1_{Kx})$.

The unit $\lambda_\sharp$ for the adjunction $\epsilon_\sharp \dashv k$ is given by the oplax-generic factorisation of the "middle" morphism $f\colon Kx \longrightarrow y$

— the component of $\lambda_\sharp$ at (u, f, v) is:

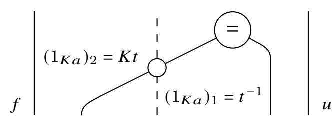


The unit λ for the adjunction $\epsilon_\sharp\, h \dashv \epsilon_\flat\, k$ is therefore given by oplax-generic factorisations of the identity morphisms 1_{Ka} — the component of λ at (u, f) in $\mathcal{B}(K,1)^\sharp$ is:

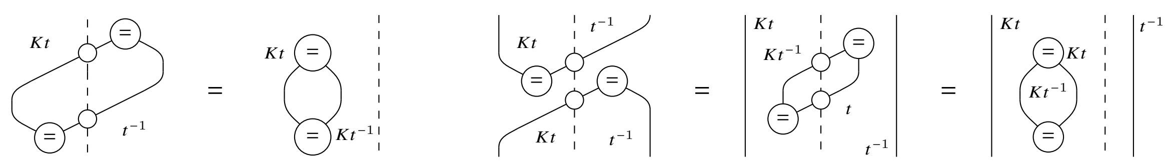


(5.17)

Note that $(1_{Ka})_1$ is not necessarily equal to 1_a and $(1_{Ka})_2$ is not necessarily 1_{Ka}; nor is 1_{Ka} necessarily oplax-generic. In the special case where $K \dashv R$ is a left adjoint, for example, the oplax generic factorisation of $Ka = Ka$ is $Ka \xrightarrow{K\eta_a} KRKa \xrightarrow{\epsilon_{Ka}} Ka$, which is not in general isomorphic to the trivial factorisation $Ka = Ka = Ka$ unless $K \dashv R$ forms an idempotent adjunction. However, if $(a, 1_{Ka})$ *is* oplax-generic, it follows that there must exist an isomorphism $t \colon \widehat{1_{Ka}} \to a$ such that $Kt = (1_{Ka})_2$ and $t\,(1_{Ka})_1 = 1_a$ by the uniqueness up-to-isomorphism of oplax-generic factorisations. The unit λ is therefore invertible whenever $(a, 1_{Ka})$ is oplax-generic for each $a \in \mathcal{A}$ with inverse given by:



which we demonstrate as follows, with components f and u suppressed:



We will say that a F_Θ-functor $K \colon \mathcal{A} \to \mathcal{B}$ is *normal* if $(a, 1_{Ka})$ is oplax-generic for each $a \in \mathcal{A}$. If K is moreover injective on objects, then we say it is *strictly-normal*, as each 1_{Ka} can be a *chosen* oplax-generic for each a, so that $(1_{Ka})_1 = 1_a$ and $(1_{Ka})_2 = 1_{Ka}$.

Lemma 5.4.4. *2-fully-faithful* F_Θ*-functors are normal.*

Proof. Assuming $K \colon \mathcal{A} \to \mathcal{B}$ is fully faithful, the morphism $(a, 1_{Ka})$ is an oplax generic object of $K \Downarrow Ka$ precisely if $(a, 1_a)$ is oplax-generic in $\mathcal{A} \Downarrow a$ by the isomorphism $K \Downarrow Ka \cong \mathcal{A} \Downarrow a$. The canonical $\mathcal{Q}$-coalgebra map for the representable $\mathcal{A}(-,a)$ sends $u \colon x \to a$ to $(u, 1_a) \in \mathcal{Q}\mathcal{A}(-,a)_x$, demonstrating that 1_a *is* oplax-generic in $\mathcal{A} \Downarrow a$ (the category of elements of $\mathcal{A}(-,a)$). □

We can summarise the observations made about the canonical coalgebra maps h and k in the following lemma:

Lemma 5.4.5. *If $K \colon \mathcal{A} \to \mathcal{B}$ is a F_Θ-functor then there exists an adjunction $\epsilon_\sharp\, h \dashv \epsilon_\flat\, k \colon \mathcal{B}(K,1)^\sharp \to \mathcal{B}(K,1)^\flat$ given in terms of $\sharp/\flat$-coalgebra structures k/h on $\mathcal{B}(K,1)^\flat/\mathcal{B}(K,1)^\sharp$ respectively. If K is strictly normal then k can be chosen such that $\epsilon_\flat\, k$ is a right-adjoint retraction to $\epsilon_\sharp\, h$.*

We can also describe morphisms of F_Θ-functors in terms of the corresponding $\sharp$-coalgebras. Recall from 5.3 that a strict morphism of F_Θ-functors from $K\colon \mathcal{A} \to \mathcal{B}$ to $L\colon C \to \mathcal{D}$ is a commuting square:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[d, "L"] \\ \mathcal{B} \arrow[r, "G"'] & \mathcal{D} \end{tikzcd}$$

where G sends chosen K-oplax-generic 1-cells to chosen L-oplax-generic 1-cells, and thus preserves oplax-generic factorisations. A choice of oplax-generic factorisations essentially describes the coalgebra maps $k\colon \mathcal{B}(K,1)^\flat \to \mathcal{B}(K,1)^{\flat\sharp}$, $l\colon \mathcal{D}(L,1)^\flat \to \mathcal{D}(L,1)^{\flat\sharp}$, so the fact that G preserves these factorisations should correspond to a notion of preservation of the coalgebra structures under some action by G and F.

Precomposition by $F^{\mathsf{op}}{\times}G\colon \mathcal{A}^{\mathsf{op}}{\times}\mathcal{B} \to C^{\mathsf{op}}{\times}\mathcal{D}$ defines a map $\mathsf{Prof}(\mathcal{D},C) \to \mathsf{Prof}(\mathcal{B},\mathcal{A})$ sending $p\colon C^{\mathsf{op}}{\times}\mathcal{D} \to \mathsf{Cat}$ to $p(F,G)$. This 2-functor moreover underlies a morphism between the $\sharp$-comonads on $\mathsf{Prof}(\mathcal{B},\mathcal{A})$ and $\mathsf{Prof}(\mathcal{D},C)$ and also between the $\flat$-comonads. For the $\sharp$ case, the map $\sigma_p\colon p(F,G)^\sharp \Rightarrow p^\sharp(F,G)$ has component at $(a,b) \in \mathcal{A}^{\mathsf{op}} \times \mathcal{B}$ induced by the (oplax) cowedge with component at $x \in \mathcal{A}$ defined as follows:

$$\mathcal{A}(a,x) \times p(Fx,Gb) \xrightarrow{F_{a,x} \times P(Fx,Gb)} C(Fa,Fx) \times p(Fx,Gb) \xrightarrow{\lambda_{Fx}} \oint^{y\in C} C(Fa,y) \times p(y,Gb)$$

The $\flat$ case is similar, the map $\tau_p\colon p(F,G)^\flat \Rightarrow p^\flat(F,G)$ being defined by the cowedge with component at $x \in \mathcal{B}$ given by:

$$p(Fa,Gx) \times \mathcal{B}(x,b) \xrightarrow{P(Fx,Gb) \times G_{x,b}} p(Fa,Gx) \times \mathcal{D}(Gx,Gb) \xrightarrow{\lambda_{Gx}} \oint^{y\in\mathcal{D}} p(Fa,y) \times \mathcal{D}(y,Gb)$$

For 2-functors F,G which form a commutative square between F_Θ-functors K and L, there is also in particular a morphism $G_{K,1}\colon \mathcal{B}(K,1) \to \mathcal{D}(GK,1) = \mathcal{D}(LF,1)$. The condition that F,G moreover form a strict morphism of F_Θ-functors is then equivalent to requiring that the following diagram in $\mathsf{Prof}(\mathcal{B},\mathcal{A})$ commutes:

$$\begin{tikzcd} \mathcal{B}(K,1)^\flat \arrow[r, "{G_{K,1}}^\flat"] \arrow[d, "k"'] & \mathcal{D}(LF,G)^\flat \arrow[rr, "\tau_{\mathcal{D}(L,1)}"] & & \mathcal{D}(L,1)^\flat(F,G) \arrow[d, "l_{F,G}"] \\ \mathcal{B}(K,1)^{\flat\sharp} \arrow[r, "{G_{K,1}}^{\flat\sharp}"'] & \mathcal{D}(LF,G)^{\flat\sharp} \arrow[r, "{\tau_{\mathcal{D}(L,1)}}^\sharp"'] & \mathcal{D}(L,1)^\flat(F,G)^\sharp \arrow[r, "\sigma_{\mathcal{D}(L,1)^\flat}"'] & \mathcal{D}(L,1)^{\flat\sharp}(F,G) \end{tikzcd}$$

If the diagram merely commutes up to isomorphism then (F,G) represents a non-strict morphism of F_Θ-algebras. In the case where K or L doesn't have chosen oplax-generics then there is no canonical choice of coalgebra structure maps k and l, so it is meaningless to ask for the above diagram to strictly commute.

5.5 Kan Extensions

A consequence of the adjunction between $\mathcal{B}(K,1)^\sharp$ and $\mathcal{B}(K,1)^\flat$ for $K\colon \mathcal{A} \to \mathcal{B}$ a F_Θ-functor, is that the universal property of pointwise left extensions along $\mathcal{K}$ extends in a weak sense to from strict 2-natural transformations between 2-functors to lax natural transformations. For context, recall that for any 2-functors $F\colon \mathcal{A} \to C$ and $G\colon \mathcal{B} \to C$, a left extension $\eta_F\colon F \Rightarrow \mathsf{lan}_K F\, K$ of F along K is characterised by the property that the map $[\mathcal{B},C](\mathsf{lan}_K F, G) \to [\mathcal{A},C](F,GK)$ given by whiskering on the right by K, then precomposing with the unit $\eta_F$is an isomorphism natural in G. This isomorphism extends in an obvious way to a (non-invertible) map between the categories of 2-functors and *lax* transformations $[\mathcal{B},C]_{\mathsf{lax}}(\mathsf{lan}_K F, G) \to [\mathcal{A},C]_{\mathsf{lax}}(F,GK)$. The weak sense in which the universal property of $\mathsf{lan}_K F$ applies to lax transformations for K a F_Θ-functor is that this map has a left adjoint if $\mathsf{lan}_K F$ is a pointwise extension. When K is a strictly-normal F_Θ-functor, this left adjoint is moreover a right inverse.

To see this, let F, G be functors as described above and observe that the profunctor $\mathcal{A} \nrightarrow \mathcal{A}$ given by $(x, y) \mapsto C(Fx, GKy)$ is isomorphic to $\int_{(a,b)\in\mathcal{A}\times\mathcal{B}}[\mathcal{A}(a,x)\times\mathcal{B}(Ky,b), C(Fa,Gb)]$ by the Yoneda lemma. The category of lax transformations (and modifications) from F to GK is given by the lax end of this profunctor:

$$\begin{aligned}
[\mathcal{B},C]_{\mathsf{lax}}(\mathsf{lan}_{\mathcal{K}}F, G) &\cong [\mathcal{A},C]_{\mathsf{lax}}(F, GK) \\
&\cong \int\!\!\!\!\!\square_{x\in\mathcal{A}} C(Fx, GKx) \\
&\cong \int\!\!\!\!\!\square_{x\in\mathcal{A}} \int_{(a,b)\in\mathcal{A}\times\mathcal{B}} [\mathcal{A}(a,x)\times\mathcal{B}(Ky,b), C(Fa,Gb)] \\
&\cong \int_{(a,b)\in\mathcal{A}\times\mathcal{B}} \left(\oint^{x\in\mathcal{A}} \mathcal{A}(a,x)\times\mathcal{B}(Kx,b), C(Fa,Gb)\right) \\
&\cong \mathsf{Prof}\Big(\mathcal{B}(K,1)^\sharp, C(F,G)\Big)
\end{aligned}$$

On the other hand, a pointwise extension $\mathsf{lan}_K F$ satisfies by definition the universal property that there exist isomorphisms $C(\mathsf{lan}_K F\, x, c) \cong \int_{a\in\mathcal{A}}[\mathcal{B}(Ka,x), C(Fa,c)]$ natural in $x \in \mathcal{A}$ and $c \in C$, from which we obtain:

$$C(\mathsf{lan}_K F\, x, Gy) \cong \int_{a\in\mathcal{A}} [\mathcal{B}(Ka,x), C(Fa,Gy)] \cong \int_{(a,b)\in\mathcal{A}\times\mathcal{B}} [\mathcal{B}(Ka,x)\times\mathcal{B}(y,b), C(Fa,Gb)]$$

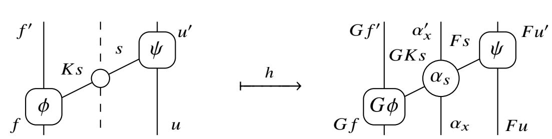
The category of lax transformations from $\mathsf{lan}_K F$ to G is given by the lax end of this profunctor:

$$\begin{aligned}
\int\!\!\!\!\!\square_{x\in\mathcal{B}} C(\mathsf{lan}_K F\, x, Gx) &\cong \int\!\!\!\!\!\square_{x\in\mathcal{B}} \int_{(a,b)\in\mathcal{A}\times\mathcal{B}} [\mathcal{B}(Ka,x)\times\mathcal{B}(x,b), C(Fa,Gb)] \\
&\cong \int_{(a,b)\in\mathcal{A}\times\mathcal{B}} \left[\oint^{x\in\mathcal{B}} \mathcal{B}(Ka,x)\times\mathcal{B}(x,b), C(Fa,Gb)\right] \\
&\cong \mathsf{Prof}\Big(\mathcal{B}(K,1)^\flat, C(F,G)\Big)
\end{aligned}$$

Pre-composing by the adjunction between $\mathcal{B}(K,1)^\sharp$ and $\mathcal{B}(K,1)^\flat$ therefore describes an adjunction between these two categories of lax transformations:

$$\begin{tikzcd}
{[\mathcal{A},C]_{\mathsf{lax}}(F,KG)} \arrow[r, shift left, "\mathsf{D}"] \arrow[r, phantom, "\perp"] & {[\mathcal{B},C]_{\mathsf{lax}}(\mathsf{lan}_K F, G)} \arrow[l, shift left, "\mathsf{E}"]
\end{tikzcd}$$

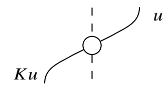
We can give an explicit description of the functors forming this adjunction. First, observe that for a lax transformation $\alpha\colon F \Rightarrow GK$ the corresponding map of profunctors $\mathcal{B}(K,1)^\sharp \to C(F,G)$ is given on objects by sending $a \xrightarrow{u} x, Kx \xrightarrow{f} b$ to $Gf\,\alpha_x\,Fu$, and its action on 1-cells is given by:



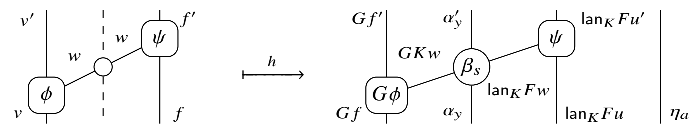
(5.18)

Conversely, from a profunctor map $S\colon \mathcal{B}(K,1)^\sharp \to C(F,G)$ one recovers the corresponding lax transformation $\sigma\colon F \Rightarrow GK$ by defining the component σ_a as the image of $(1_a, 1_{Ka}) \in \mathcal{B}(K,1)^\sharp_{a,Ka}$ under $S_{a,Ka}\colon \mathcal{B}(K,1)^\sharp_{a,Ka} \to [Fa, GKa]$, and by defining the lax-naturality 2-cell at $u\colon a \to x$ to be the image under $S_{a,Kx}\colon \mathcal{B}(K,1)^\sharp_{a,Kx} \to$

$[Fa, GKx]$ of the 1-cell:



On the other hand, from a lax transformation $\beta\colon \mathsf{lan}_K F \Rightarrow G$ one obtains the profunctor map $\mathcal{B}(K,1)^\flat \to C(F,G)$ which sends object $\left(Ka \xrightarrow{f} y, y \xrightarrow{v} b\right)$ to $Gv\,\beta_y\,\mathsf{lan}_K F f\,\eta_a$ and acts on 1-cells by:



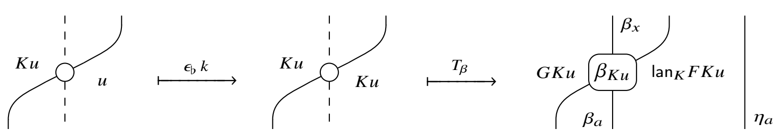
(5.19)

Recovering a lax transformation β from a profunctor map $T\colon \mathcal{B}(K,1)^\flat \to C(F,G)$ is a bit more subtle. Given such a map T, the corresponding β has component at $b \in \mathcal{B}$ given by the unique map $\mathsf{lan}_K Fb \to Gb$ induced by the cylinder $\mathcal{B}(K-,b) \to C(F-,Gb)$ sending $f\colon Ka \to b$ to $T(f,1_b)$ by the universal property of $\mathsf{lan}_K Fb$ as $\mathcal{B}(K-,b) * F$. That is, $\beta_b\colon \mathsf{lan}_K Fb \to Gb$ is the unique map such that for all $f\colon Ka \to b$ we have $\beta_b\,\mathsf{lan}_K Fb\,\eta_a = T(f,1_b)$ and the analogous condition for 2-cells $\sigma\colon f \to f'$. The lax-naturality component $\beta_v\colon Gv\,\beta_y \to \beta_b\,\mathsf{lan}_K Fv$ is the unique such 2-cell satisfying that for all $f\colon Ka \to y$, $\beta_v\,\mathsf{lan}_K Ff\,\eta_a$ is the morphism of $\mathcal{B}(K-,b)$-cylinders $T(f,v) \to T(vf,1_b)$ induced by the canonical map $(f,v) \to (vf,1)$ in $\mathcal{B}(K,1)^\flat_{a,b}$.

To understand how the map E acts on lax transformations, we must start with a lax transformation $\beta\colon \mathsf{lan}_K F \Rightarrow G$ and consider the images of objects of the form $(1_a, 1_{Ka})$ in $\mathcal{B}(K,1)^\flat_{a,Ka}$ and 1-cells of the form $(1_a, Kf) \to (f, 1_{Kx})$ under $\epsilon_\flat\,k$ composed with the profunctor map T_β induced by β. On objects, we obtain the component for $\mathsf{E}\beta$ at a as:

$$(a = a, Ka = Ka) \quad \overset{\epsilon_\flat k}{\longmapsto} \quad (Ka = Ka, Ka = Ka) \quad \overset{T_\beta}{\longmapsto} \quad \left(Fa \xrightarrow{\eta_a} \mathsf{lan}_K FKa \xrightarrow{\beta_{Ka}} GKa\right)$$

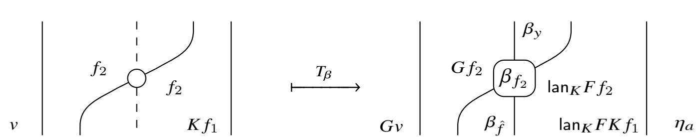
and the lax-naturality 2-cell at $u\colon a \to x$ as:



The map E therefore does, in fact, correspond to whiskering on the right by K then precomposing with $\eta\colon F \to \mathsf{lan}_K F\,K$.

We can also show that the maps D and E when restricted to the underlying categories of strict 2-natural transformations give the expected isomorphism which classifies the left extension. Let $\mathsf{D}_{\mathsf{strict}}$ and $\mathsf{E}_{\mathsf{strict}}$ denote these restrictions. The map $\mathsf{E}_{\mathsf{strict}}$ is clearly one half of the isomorphism, as demonstrated above. If we can show that the components of the counit of the adjunction $\mathsf{D} \dashv \mathsf{E}$ are identities at strict 2-natural transformations then it will follow that $\mathsf{D}_{\mathsf{strict}}$ defines a left-adjoint retraction-on-objects to the isomorphism $\mathsf{E}_{\mathsf{strict}}$, and therefore is the inverse to $\mathsf{E}_{\mathsf{strict}}$.

The component of the counit for the adjunction $\mathsf{D} \dashv \mathsf{E}$ at $\beta\colon \mathsf{lan}_K F \Rightarrow G$ is simply given by the image of the counit for the adjunction $\epsilon_\sharp\,h \dashv \epsilon_\flat\,k$ (given in (5.15)) under T_β:



(5.20)

which will be an identity whenever β is strictly 2-natural.

Summarising these observations and applying Lemma 5.4.5 we obtain the following result:

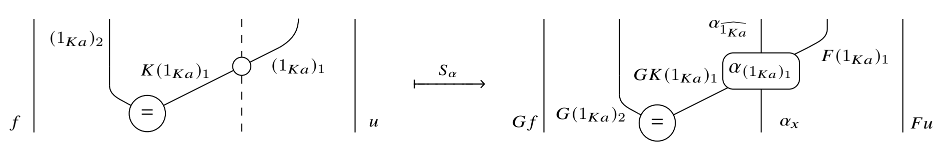
Lemma 5.5.1. *For F_Θ-functor $K\colon \mathcal{A} \to \mathcal{B}$ and 2-functors $F : \mathcal{A} \to C$ and $G\colon \mathcal{B} \to C$ 2-functors with $\mathsf{lan}_K F$ a pointwise left extension of F along K, the canonical map $[\mathcal{B}, C]_{\mathsf{lax}}(\mathsf{lan}_K F, G) \to [\mathcal{A}, C]_{\mathsf{lax}}(F, GK)$ admits a left adjoint which when restricted to strict 2-natural isomorphisms gives the isomorphism characterising $\mathsf{lan}_K F$. This left adjoint may be chosen to be a right inverse when K is moreover strictly-normal.*

We've seen that $\mathsf{D} \dashv \mathsf{E}$ restricts to the isomorphism characterising the left extension $\eta\colon F \Rightarrow \mathsf{lan}_K FK$, but in general this adjunction may contain a larger isomorphism. From any adjunction $L \dashv R\colon C \to D$ one obtains a (potentially empty) isomorphism obtained by restricting to the full subcategories $C' \subseteq C$ and $D' \subseteq D$ on objects at which the component of the unit and counit respectively are identities. By the description of the component of the $\mathsf{D} \dashv \mathsf{E}$ counit at $\beta\colon \mathsf{lan}_K F \Rightarrow G$ given in (5.20) we see that this counit component will be an identity precisely when β is strictly 2-natural at chosen oplax-generic morphisms. We will call lax transformations with this property *generic transformations*, or *K-generic transformations* when we wish to make the F_Θ-functor and associated oplax-generic factorisations explicit. The category of generic transformations $\mathsf{lan}_K F \Rightarrow G$ with arbitrary modifications as 1-cells will be denoted $[\mathsf{lan}_K F, G]_{\mathsf{gen}}$. It is convenient at this stage to note the following:

Lemma 5.5.2. *Generic transformations are closed under vertical composition and post-whiskering, and include identities.*

Proof. The lax-naturality 2-cell for a composite $F \overset{\alpha}{\Longrightarrow} G \overset{\beta}{\Longrightarrow} H$ at a 1-cell $f\colon x \to y$ in the domain of (e.g.) F, is given by $\beta_y\, \alpha_u \circ \beta_u\, \alpha_x$, so if α and β are both strictly natural at f, then so is $\beta \circ \alpha$. Similarly, if α is strictly natural at f, then $J\alpha$ will be, when J is some 2-functor from the codomain of F. The composition and post-whiskering results follow by assuming f is chosen oplax-generic for some F_Θ-functor. All strictly 2-natural transformations, including identities, are oplax generic. □

The component of the unit for the $\mathsf{D} \dashv \mathsf{E}$ adjunction at lax transformation $\alpha\colon F \Rightarrow GK$ is the image of the unit λ for the $\epsilon_\sharp\, h \dashv \epsilon_\flat\, k$ (shown in (5.17)) under the profunctor map $S_\alpha\colon \mathcal{B}(K,1)^\sharp \to C(F,G)$ induced by α:



The unit at α will therefore be an identity if α is strictly 2-natural at morphisms of the form $(1_{Ka})_1$. We will call lax transformations with this property *normal* transformations (or *K-normal* transformations) and denote the category of such transformations $[F, GK]_{\mathsf{norm}}$. Observe that if K is a strictly normal F_Θ-functor, then every lax transformation $F \Rightarrow GK$ is normal, as for such K, $(1_{Ka})_1 = 1_a$, and lax-naturality 2-cells at identities are automatically identities.

Lemma 5.5.3. *The adjunction of Lemma 5.5.1 restricts to an isomorphism $[F, GK]_{\mathsf{norm}} \cong [\mathsf{lan}_K F, G]_{\mathsf{gen}}$. If K is strictly natural then $[F, GK]_{\mathsf{norm}} = [F, GK]_{\mathsf{lax}}$.*

5.6 Application: F_Ω as a Lax-Gray Monad

The archetypal F_Θ-functor is the free cocompletion map $\mathsf{Z}_\mathcal{A}\colon \mathcal{A} \to \mathsf{F}_\Theta\mathcal{A}$. This 2-functor is injective on objects and the chosen oplax-generic factorisations as defined in Example 5.2.16 exhibit $\mathsf{Z}_\mathcal{A}$ as being strictly normal[8]. It follows from Lemma 5.5.1 that lax transformations of the form $F \Rightarrow G\mathsf{Z}_\mathcal{A}$ lift to (generic) lax transformations

[8] This also follows from the fact that $\mathsf{Z}_\mathcal{A}$ is fully faithful, by Lemma 5.4.4

$\mathsf{lan}_K F \Rightarrow G$ when $\mathsf{lan}_K F$ is a pointwise extension. In particular, consider any pair of 2-functors $F, G \colon \mathcal{A} \to \mathcal{B}$ and a lax transformation $\phi \colon F \Rightarrow G$. Post-whiskering by $\mathsf{Z}_\mathcal{B}$ gives a lax transformation $\mathsf{Z}_\mathcal{B}\phi \colon \mathsf{Z}_\mathcal{B}F \Rightarrow \mathsf{Z}_\mathcal{B}G$, which by the fact that $\mathsf{Z}_\mathcal{B}\, G = \mathsf{F}_\Theta G\, \mathsf{Z}_\mathcal{B}$ can be lifted to a generic transformation $\widetilde{\phi} \colon \mathsf{F}_\Theta F \Rightarrow \mathsf{F}_\Theta G$ given that $\mathsf{F}_\Theta F$ is a pointwise left extension of $\mathsf{Z}_\mathcal{B}\, F$ along $\mathsf{Z}_\mathcal{A}$. When ϕ is strictly 2-natural, the lift $\widetilde{\phi}$ is simply $\mathsf{F}_\Theta\phi$ because the lifting of lax transformations extends the lifting of strict transformations given the left extension property of $\mathsf{F}_\Theta F$ which was used to define $\mathsf{F}_\Theta F$ as a 2-functor.

In this section we will observe that this additional weak lax-transformation-lifting property of $\mathsf{F}_\Theta F$ can be used to define a $\mathsf{Gray}_\mathscr{L}$-functor structure on F_Θ, where $\mathsf{Gray}_\mathscr{L}$ is the category $\mathsf{2Cat}$ with monoidal product given by the lax Gray tensor product, $(- \boxminus -)$. One can then view $\mathsf{Gray}_\mathscr{L}$ as being enriched over itself in the usual way, as the monoidal structure is closed. The hom-$\mathsf{Gray}_\mathscr{L}$-objects in $\mathsf{Gray}_\mathscr{L}$ are the 2-categories of 2-functors, lax transformations and modifications. Because we wish to discuss 2-categories large enough to include Cat, we should also consider $\mathsf{GRAY}_\mathscr{L}$, the analogous construction starting with $\mathsf{2CAT}$, rather than $\mathsf{2Cat}$, and which therefore contains Cat as an object.

To show that the lifting

$$\mathsf{GRAY}_\mathscr{L}(\mathcal{A},\mathcal{B})(F,G) = [\mathcal{A},\mathcal{B}]_{\mathsf{lax}}(F,G) \to [\mathsf{F}_\Theta\mathcal{A}, \mathsf{F}_\Theta B]_{\mathsf{lax}}(\mathsf{F}_\Theta F, \mathsf{F}_\Theta G) = \mathsf{GRAY}_\mathscr{L}(\mathsf{F}_\Theta\mathcal{A}, \mathsf{F}_\Theta\mathcal{B})(\mathsf{F}_\Theta F, \mathsf{F}_\Theta G)$$

describes a genuine $\mathsf{Gray}_\mathscr{L}$-functor, we must show that the map preserves:

(a) vertical composition of lax transformations

(b) pre-whiskering and post-whiskering and by 2-functors

(c) interchangers

While doing so we will denote the action of F_Θ by $(-)'$ to stream-line the notation so that, for example, $\mathsf{F}_\Theta F \colon \mathsf{F}_\Theta\mathcal{A} \to \mathsf{F}_\Theta\mathcal{B}$ becomes $F' \colon \mathcal{A}' \to \mathcal{B}'$.

5.6.1 Preservation of vertical composition

Let's first consider why the 3-functor F_Θ preserves vertical composition of strict 2-natural transformations. *A priori* for each hom-category $[\mathcal{A},\mathcal{B}]$ we merely have:

(a) a mapping of objects $F \mapsto F'$ given by post-composing by $\mathsf{Z}_\mathcal{B}$ then taking the chosen left extension of $\mathsf{Z}_\mathcal{B}F$ along $\mathsf{Z}_\mathcal{A}$.

(b) an isomorphism on hom-categories $[\mathcal{A},\mathcal{B}](F,G) \to [\mathcal{A}',\mathcal{B}'](F',G')$ given by post-composition with the (fully-faithful) 2-functor Z_B followed by the canonical isomorphism defining the left extension F':

$$[\mathcal{A},\mathcal{B}'](\mathsf{Z}_\mathcal{B}F, G'\,\mathsf{Z}_\mathcal{A}) \cong [\mathcal{A}',\mathcal{B}'](F',G')$$

For now, we will refer to such data — a map on objects and collection of functors on hom-categories — as a *weak map* of 2-categories. We can justify that the weak map $\mathsf{F}_\Theta[\mathcal{A},\mathcal{B}]$ of 2-categories defined above is, in fact, a 2-functor as follows. First, we observe that $\mathsf{F}_\Theta[\mathcal{A},\mathcal{B}]$ factors as the 2-functor given by post-composition by $\mathsf{Z}_\mathcal{B}$, followed by a weak map L from the image of this post-composition, I, onto $[\mathcal{A}',\mathcal{B}']$. The image of L lands in the full sub-2-category of $[\mathcal{A}',\mathcal{B}']$ with objects of the form[9] F', which we denote $[\![\mathcal{A},\mathcal{B}]\!]$. Thus, the weak map $\mathsf{F}_\Theta[\mathcal{A},\mathcal{B}]$ is given by the following composition:

$$[\mathcal{A},\mathcal{B}] \xrightarrow{(\mathsf{Z}_\mathcal{B})_*} I \xrightarrow{L} [\![\mathcal{A},\mathcal{B}]\!] \hookrightarrow [\mathcal{A}',\mathcal{B}'] \tag{5.21}$$

[9]The 2-functors $\mathcal{A}' \to \mathcal{B}'$ which are of the form F' are equivalently those 2-functors which strictly preserve the chosen oplax colimits of 2-functors of the form $\mathsf{Z}_\mathcal{A}S$ for 2-functors $S \colon C \to \mathcal{A}$.

To show that $\mathsf{F}_\Theta[\mathcal{A},\mathcal{B}]$ is a 2-functor, it suffices to demonstrate that L is a 2-functor. The weak map L has an inverse weak map given by precomposition with $\mathsf{Z}_\mathcal{A}$, which is a 2-functor. The inverse of a 2-functor (as a weak map) must also be 2-functorial, so we conclude that L is a 2-functor and so is $\mathsf{F}_\Theta[\mathcal{A},\mathcal{B}]$.

We will now apply a similar argument to the weak mapping $\mathsf{G}_\Theta[\mathcal{A},\mathcal{B}]\colon [\mathcal{A},\mathcal{B}]_{\mathsf{lax}} \to [\mathcal{A}',\mathcal{B}']_{\mathsf{lax}}$ which has the same action on objects as $\mathsf{F}_\Theta[\mathcal{A},\mathcal{B}]$, but whose map on hom-categories is defined by post-composition with $\mathsf{Z}_\mathcal{B}$ followed by the canonical *adjunction* described in Lemma 5.5.1, $\mathsf{D}\colon [\mathcal{A},\mathcal{B}']_{\mathsf{lax}}(\mathsf{Z}_\mathcal{B}F, G'\,\mathsf{Z}_\mathcal{A}) \to [\mathcal{A}',\mathcal{B}'](F',G')$. We can factor the weak map $\mathsf{G}_\Theta[\mathcal{A},\mathcal{B}]$ as post-composition with $\mathsf{Z}_\mathcal{B}$ restricted to its image $\mathcal{I}$ followed by a weak map $\mathcal{W}_{\mathcal{A},\mathcal{B}}$ which lands in the full sub-2-category $[\![\mathcal{A},\mathcal{B}]\!]_{\mathsf{lax}}$ of $[\mathcal{A}',\mathcal{B}']_{\mathsf{lax}}$ whose objects are the 2-functors of the form F':

$$[\mathcal{A},\mathcal{B}]_{\mathsf{lax}} \xrightarrow{(\mathsf{Z}_\mathcal{B})_*} \mathcal{I} \xrightarrow{\mathcal{W}_{\mathcal{A},\mathcal{B}}} [\![\mathcal{A},\mathcal{B}]\!]_{\mathsf{lax}} \hookrightarrow [A',B']_{\mathsf{lax}}$$

The weak map $\mathcal{W}_{\mathcal{A},\mathcal{B}}$ has an inverse on objects but not on hom-categories, because its action on hom-categories is given by the left adjoint D, rather than an isomorphism. However, by Lemma 5.5.3 the functors D on hom-categories restrict to isomorphisms $[\mathcal{A},\mathcal{B}']_{\mathsf{lax}}(\mathsf{Z}_\mathcal{B}F, G'\,\mathsf{Z}_\mathcal{A}) \to [F',G']_{\mathsf{gen}}$. And by Lemma 5.5.2, generic transformations are closed under vertical composition and include identities, so the objects of $[\![\mathcal{A},\mathcal{B}]\!]_{\mathsf{lax}}$ along with the hom-categories $[F',G']_{\mathsf{gen}}$ do indeed form a locally full sub-2-category $[\![\mathcal{A},\mathcal{B}]\!]_{\mathsf{gen}}$ of $[\![\mathcal{A},\mathcal{B}]\!]_{\mathsf{lax}}$. We can therefore factorise $\mathsf{G}_\Theta[\mathcal{A},\mathcal{B}]$ further:

$$[\mathcal{A},\mathcal{B}]_{\mathsf{lax}} \xrightarrow{(\mathsf{Z}_\mathcal{B})_*} [\mathcal{A},\mathcal{B}']_{\mathsf{lax}} \xrightarrow{\mathcal{L}_{\mathcal{A},\mathcal{B}}} [\![\mathcal{A},\mathcal{B}]\!]_{\mathsf{gen}} \hookrightarrow [\![\mathcal{A},\mathcal{B}]\!]_{\mathsf{lax}} \hookrightarrow [A',B']_{\mathsf{lax}} \tag{5.22}$$

The weak map $\mathcal{L}_{\mathcal{A},\mathcal{B}}$ *does* have an inverse — pre-composition with $\mathsf{Z}_\mathcal{A}$ — which is 2-functorial, and therefore so is $\mathcal{L}_{\mathcal{A},\mathcal{B}}$, and so is $\mathsf{G}_\Theta[\mathcal{A},\mathcal{B}]$. We conclude that G_Θ preserves vertical composition of lax transformations, which is the horizontal composition in the 2-categories $\mathsf{G}_\Theta[\mathcal{A},\mathcal{B}]$.

5.6.2 Preservation of post-whiskering

We need to show that for 2-functors $F,G\colon \mathcal{A}\to\mathcal{B}$, $H\colon \mathcal{B}\to\mathcal{C}$, post-whiskering lax transformations $F\Rightarrow G$ by H is preserved by the map G_Θ. That is, the following diagram commutes:

$$\begin{tikzcd}
{[\mathcal{A},\mathcal{B}]_{\mathsf{lax}}} \arrow[r, "{\mathsf{G}_\Theta[\mathcal{A},\mathcal{B}]}"] \arrow[d, "H_*"'] & {[\mathcal{A}',\mathcal{B}']_{\mathsf{lax}}} \arrow[d, "H'_*"] \\
{[\mathcal{A},\mathcal{C}]_{\mathsf{lax}}} \arrow[r, "{\mathsf{G}_\Theta[\mathcal{A},\mathcal{C}]}"'] & {[\mathcal{A}',\mathcal{C}']_{\mathsf{lax}}}
\end{tikzcd} \tag{5.23}$$

Decomposing $\mathsf{G}_\Theta[\mathcal{A},X]$ as in (5.22), it suffices to show that each square of the following diagram commutes. Note that post-composition by H' does, in fact, restrict to a map $[\![\mathcal{A},\mathcal{B}]\!]_{\mathsf{gen}} \to [\![\mathcal{A},\mathcal{C}]\!]_{\mathsf{gen}}$ since $H'F' = HF'$ for $F' \in [\![\mathcal{A},\mathcal{B}]\!]_{\mathsf{gen}}$ and generic transformations are closed under post-whiskering.

$$\begin{tikzcd}
{[\mathcal{A},\mathcal{B}]_{\mathsf{lax}}} \arrow[r, "(\mathsf{Z}_\mathcal{B})_*"] \arrow[d, "H_*"'] \arrow[dr, phantom, "\mathbf{A}"] & {[\mathcal{A},\mathcal{B}']_{\mathsf{lax}}} \arrow[r, "\mathcal{L}_{\mathcal{A},\mathcal{B}}"] \arrow[d, "H'_*"] \arrow[dr, phantom, "\mathbf{B}"] & {[\![\mathcal{A},\mathcal{B}]\!]_{\mathsf{gen}}} \arrow[r, hook] \arrow[d, "H'_*"] \arrow[dr, phantom, "\mathbf{C}"] & {[\mathcal{A}',\mathcal{B}']_{\mathsf{lax}}} \arrow[d, "H'_*"] \\
{[\mathcal{A},\mathcal{C}]_{\mathsf{lax}}} \arrow[r, "(\mathsf{Z}_\mathcal{C})_*"'] & {[\mathcal{A},\mathcal{C}']_{\mathsf{lax}}} \arrow[r, "\mathcal{L}_{\mathcal{A},\mathcal{C}}"'] & {[\![\mathcal{A},\mathcal{C}]\!]_{\mathsf{gen}}} \arrow[r, hook] & {[\mathcal{A}',\mathcal{C}']_{\mathsf{lax}}}
\end{tikzcd} \tag{5.24}$$

Square A commutes by the fact that $\mathsf{Z}_\mathcal{C}\,H = H'\,\mathsf{Z}_\mathcal{B}$.

Square B commutes because precomposition by $\mathsf{Z}_\mathcal{A}$ is inverse to $\mathcal{L}_{\mathcal{A},X}$, and commutes with post-composition by H'. So $H'_*\,\mathcal{L}_{\mathcal{A},\mathcal{B}} = \mathcal{L}_{\mathcal{A},\mathcal{C}}\,\mathsf{Z}^*_\mathcal{A}\,H'_*\,\mathcal{L}_{\mathcal{A},\mathcal{B}} = \mathcal{L}_{\mathcal{A},\mathcal{C}}\,H'_*\,\mathsf{Z}^*_\mathcal{A}\,\mathcal{L}_{\mathcal{A},\mathcal{B}} = \mathcal{L}_{\mathcal{A},\mathcal{C}}\,H'_*$

Square C commutes trivially.

5.6.3 Preservation of pre-whiskering

We need to show that for 2-functors $F, G\colon \mathcal{A} \to \mathcal{B}$, $H\colon C \to \mathcal{A}$, pre-whiskering lax transformations $F \Rightarrow G$ by H is preserved by the map G_Θ. That is, the following diagram commutes:

$$\begin{tikzcd}
{[\mathcal{A},\mathcal{B}]_{\mathsf{lax}}} \arrow[r, "{\mathsf{G}_\Theta[\mathcal{A},\mathcal{B}]}"] \arrow[d, "H^*"'] & {[\mathcal{A}',\mathcal{B}']_{\mathsf{lax}}} \arrow[d, "(H')^*"] \\
{[C,\mathcal{B}]_{\mathsf{lax}}} \arrow[r, "{\mathsf{G}_\Theta[C,\mathcal{B}]}"'] & {[C',\mathcal{B}']_{\mathsf{lax}}}
\end{tikzcd} \tag{5.25}$$

The proof that G_Θ preserves post-whiskering is similar to the proof that pre-whiskering is preserved, though there is an important difference. While post-whiskering a generic transformation by a 2-functor always produces another generic transformation, the same isn't true for pre-whiskering. The notion of generic transformation preservation under pre-whiskering isn't necessarily even well-defined, given that being generic is a property reserved for lax transformations $\alpha\colon F \Rightarrow G\colon \mathcal{A} \to \mathcal{B}$ where $\mathcal{A}$ has chosen oplax-generic factorisation with respect to some F_Θ-functor. An appropriate setting for a notion of preservation of generic transformations under pre-whiskering is therefore a diagram of the following form:

$$\begin{tikzcd}
\mathcal{X}_1 \arrow[d, "M"'] & \mathcal{Y}_1 \arrow[d, "N"] \\
\mathcal{X}_2 \arrow[r, "S"'] & \mathcal{Y}_2
\end{tikzcd}$$

where M and N are both F_Θ-functors. One can then ask if N-generic transformations between functors out of $\mathcal{Y}_2$ are mapped under pre-whiskering by S to M-generic transformations. One class of 2-functors S for which this is true are those which together with some $T\colon \mathcal{X}_1 \to \mathcal{Y}_1$ form a strict morphism of F_Θ-functors from M to N. For if $\alpha\colon F \Rightarrow G\colon \mathcal{Y}_2 \to \mathcal{Z}$ is generic, then for any M-chosen-oplax-generic $g\colon Mx \to y$ in $\mathcal{X}$, Sg will be N-chosen-oplax-generic, and so $\alpha S_g = \alpha_{Sg}$ will be an identity.

The relevant examples here are 2-functors of the form $F'\colon \mathcal{A}' \to \mathcal{B}'$, which together with $F\colon \mathcal{A} \to \mathcal{B}$ are strict morphisms between the F_Θ-functors $\mathsf{Z}_\mathcal{A}$ and $\mathsf{Z}_\mathcal{B}$. Consequently, pre-whiskering by H' restricts from a map $[\mathcal{A}',\mathcal{B}']_{\mathsf{lax}} \to [C',\mathcal{B}']$ to a map $[\![\mathcal{A},\mathcal{B}]\!]_{\mathsf{gen}} \to [\![C,\mathcal{B}]\!]_{\mathsf{gen}}$. We can therefore decompose the diagram (5.25) into three squares as in the post-whiskering case:

$$\begin{tikzcd}
{[\mathcal{A},\mathcal{B}]_{\mathsf{lax}}} \arrow[r, "(\mathsf{Z}_\mathcal{B})_*"] \arrow[d, "H^*"'] \arrow[dr, phantom, "\mathbf{A}"] & {[\mathcal{A},\mathcal{B}']_{\mathsf{lax}}} \arrow[r, "\mathcal{L}_{\mathcal{A},\mathcal{B}}"] \arrow[d, "H^*"] \arrow[dr, phantom, "\mathbf{B}"] & {[\![\mathcal{A},\mathcal{B}]\!]_{\mathsf{gen}}} \arrow[r, hook] \arrow[d, "(H')^*"] \arrow[dr, phantom, "\mathbf{C}"] & {[\mathcal{A}',\mathcal{B}']_{\mathsf{lax}}} \arrow[d, "(H')^*"] \\
{[C,\mathcal{B}]_{\mathsf{lax}}} \arrow[r, "(\mathsf{Z}_\mathcal{B})_*"'] & {[C,\mathcal{B}']_{\mathsf{lax}}} \arrow[r, "\mathcal{L}_{C,\mathcal{B}}"'] & {[\![C,\mathcal{B}]\!]_{\mathsf{gen}}} \arrow[r, hook] & {[C',\mathcal{B}']_{\mathsf{lax}}}
\end{tikzcd} \tag{5.26}$$

Square A commutes because pre-composition commutes with post-composition.

Square B commutes by the fact that precomposition by Z_X is inverse to $\mathcal{L}_{X,\mathcal{B}}$, and $H'\,\mathsf{Z}_C = \mathsf{Z}_\mathcal{A}\,H$, so that:

$$(H')^*\,\mathcal{L}_{\mathcal{A},\mathcal{B}} = \mathcal{L}_{C,\mathcal{B}}\,\mathsf{Z}_C^*\,(H')^*\,\mathcal{L}_{\mathcal{A},\mathcal{B}} = \mathcal{L}_{C,\mathcal{B}}\,(H'\,\mathsf{Z}_C)^*\,\mathcal{L}_{\mathcal{A},\mathcal{B}} = \mathcal{L}_{C,\mathcal{B}}\,(\mathsf{Z}_\mathcal{A}\,H)^*\,\mathcal{L}_{\mathcal{A},\mathcal{B}} = \mathcal{L}_{C,\mathcal{B}}\,H^*\,\mathsf{Z}_\mathcal{A}^*\,\mathcal{L}_{\mathcal{A},\mathcal{B}} = \mathcal{L}_{C,\mathcal{B}}\,H^*$$

Square C obviously commutes.

5.6.4 Preservation of interchangers

For 2-functors $F, G\colon \mathcal{A} \to \mathcal{B}$, $S, T\colon C \to \mathcal{D}$ and lax transformations $\alpha\colon F \Rightarrow G$, $\beta\colon S \Rightarrow T$ there exists a modification $\alpha_\beta\colon T\alpha \circ \beta F \Rrightarrow \beta G \circ S\alpha$ which is the *interchanger* for the 2-cells α, β in the $\mathsf{Gray}_\mathscr{L}$-enrichment of $\mathsf{2CAT}$. The component of the modification α_β at $a \in \mathcal{A}$ is the lax-naturality 2-cell for α at β_a, i.e. $(\alpha_\beta)_a = \alpha_{\beta_a}$, whence the notation. For G_Θ to define a $\mathsf{Gray}_\mathscr{L}$ functor $\mathsf{GRAY}_\mathscr{L} \to \mathsf{GRAY}_\mathscr{L}$ it must send the interchanger for α, β to the interchanger for α', β'.

Because $[\![\mathcal{A}, C]\!]_{\mathsf{gen}}$ is a *locally* full sub-2-category of $[A', C']$, the interchanger $\alpha'_{\beta'}$ is necessarily a 2-cell in $[\![\mathcal{A}, C]\!]_{\mathsf{gen}}$. To show that the image of α_β under $\mathsf{G}_\Theta[\mathcal{A}, C] = \mathcal{L}_{\mathcal{A},C}\ (\mathsf{Z}_C)_*$ is equal to $\alpha'_{\beta'}$ it suffices to show that post-whiskering α_β by Z_C gives the same result as pre-whiskering $\alpha'_{\beta'}$ by $\mathsf{Z}_{\mathcal{A}}$ (which is inverse to $\mathcal{L}_{\mathcal{A},C}$). This is easily verified on components:

$$\left(\mathsf{Z}_C\,\alpha_\beta\right)_a = \mathsf{Z}_C\,\alpha_{\beta_a} = \alpha'_{\mathsf{Z}_{\mathcal{B}}\beta_a} = \alpha'_{\beta'\mathsf{Z}_{\mathcal{A}}a} = \left(\alpha'_{\beta'}\,\mathsf{Z}_{\mathcal{A}}\right)_a \tag{5.27}$$

5.6.5 The $\mathsf{Gray}_{\mathscr{L}}$-functors G_Θ and G_Ω

Having demonstrated that the lifting of lax transformations $\phi \mapsto \phi'$ preserves the $\mathsf{Gray}_{\mathscr{L}}$-structure of $\mathsf{GRAY}_{\mathscr{L}}$, we conclude that $\mathsf{G}_\Theta\colon \mathsf{GRAY}_{\mathscr{L}} \to \mathsf{GRAY}_{\mathscr{L}}$ is a well-defined $\mathsf{Gray}_{\mathscr{L}}$-functor which, in a sense, extends the free-cocompletion $\mathsf{F}_\Theta\colon \mathsf{2CAT} \to \mathsf{2CAT}$.

Because $\mathsf{F}_\Omega\colon \mathsf{2CAT} \to \mathsf{2CAT}$ is a full sub-2-functor (in fact, a sub-2-monad) of F_Θ, the $\mathsf{Gray}_{\mathscr{L}}$-functor structure on F_Θ restricts to define a $\mathsf{Gray}_{\mathscr{L}}$-functor G_Ω which extends F_Ω. For a lax transformation $\alpha\colon F \Rightarrow G\colon \mathcal{A} \to \mathcal{B}$, the image $\mathsf{G}_\Omega\alpha\colon \mathsf{G}_\Omega F \Rightarrow \mathsf{G}_\Omega G\colon \mathsf{G}_\Omega\mathcal{A} \to \mathsf{G}_\Omega\mathcal{B}$ actually admits a fairly elementary description. Recall that $\mathsf{G}_\Omega\alpha = \alpha'$ is defined in terms of the universal property of $\mathsf{G}_\Omega F$ as the pointwise left extension of $\mathsf{Z}_{\mathcal{B}}F$ along $\mathsf{Z}_{\mathcal{A}}$ (where $\mathsf{Z}_{\mathcal{A}}\colon \mathcal{A} \to \mathsf{F}_\Omega\mathcal{A}$ is now used to denote the free Ω-cocompletion). This universal property says that for each $(C, P) \in \mathsf{G}_\Omega\mathcal{A} = \mathsf{F}_\Omega\mathcal{A}$ (i.e. each oplax functor $P\colon C \to \mathcal{A}$) the object $(C, FP) = \mathsf{G}_\Omega F(C, P)$ is the $\mathsf{G}_\Omega\mathcal{A}(\mathsf{Z}_{\mathcal{A}}-, (C, P))$-weighted colimit of $\mathsf{Z}_{\mathcal{B}}F = \langle F-\rangle$. Thus, a map $J\colon (C, FP) \to X$ is determined by the pre-composition of $\mathsf{G}_\Omega\mathcal{B}(\langle F-\rangle, J)\colon \mathsf{G}_\Omega\mathcal{B}(\langle F-\rangle, (C, FP)) \to \mathsf{G}_\Omega\mathcal{B}(\langle F-\rangle, X)$ by the universal "cylinder" $\xi\colon \mathsf{G}_\Omega\mathcal{A}(\langle-\rangle, (C, P)) \Rightarrow \mathsf{G}_\Omega\mathcal{B}(\langle F-\rangle, (C, FP))$ whose component at $a \in \mathcal{A}$ acts on objects as follows:

$$\begin{tikzcd} \mathbb{1} \arrow[rr, "\langle c\rangle"] \arrow[dr, "\langle a\rangle"'] \arrow[rr, phantom, shift right=4ex, "\overset{f}{\Longrightarrow}"] & & C \arrow[dl, "P"] \\ & \mathcal{A} & \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \mathbb{1} \arrow[rr, "\langle c\rangle"] \arrow[dr, "\langle Fa\rangle"'] \arrow[rr, phantom, shift right=4ex, "\overset{Ff}{\Longrightarrow}"] & & C \arrow[dl, "FP"] \\ & \mathcal{B} & \end{tikzcd} \tag{5.28}$$

Note that the lax-transformation $f\colon \langle a\rangle \Rightarrow P\,\langle c\rangle$ above is merely a morphism $a \to Pc$.

In particular, the component of α' at $(C, P) \in \mathsf{G}_\Omega\mathcal{A}$ is defined such that the composition $\mathsf{G}_\Omega(\langle F-\rangle, \alpha') \circ \xi$ gives cylinder $\mathsf{G}_\Omega\mathcal{A}(\langle-\rangle, (C, P)) \Rightarrow \mathsf{G}_\Omega\mathcal{B}(\langle F-\rangle, (C, GP))$ whose component at $a \in \mathcal{A}$ acts on objects as follows:

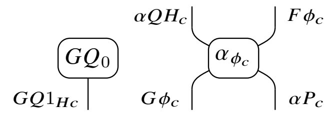
$$\begin{tikzcd} \mathbb{1} \arrow[rr, "\langle c\rangle"] \arrow[dr, "\langle a\rangle"'] \arrow[rr, phantom, shift right=4ex, "\overset{f}{\Longrightarrow}"] & & C \arrow[dl, "P"] \\ & \mathcal{A} & \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \mathbb{1} \arrow[r, equal] \arrow[ddrr, "\langle Fa\rangle"'] & \mathbb{1} \arrow[r, equal] \arrow[ddr, "\langle FPc\rangle"] \arrow[dl, phantom, "\overset{Ff}{\Rightarrow}"] & \mathbb{1} \arrow[r, "\langle c\rangle"] \arrow[dd, "\langle GPc\rangle"'] \arrow[dl, phantom, "\overset{\alpha_{Pc}}{\Rightarrow}"] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[ddl, "GP"] \\ & & & \\ & & \mathcal{B} & \end{tikzcd} \tag{5.29}$$

So the map $\alpha'_{(C,P)}$, which is composed of a functor $A_{(C,P)}\colon C \to C$ and a lax transformation $\widetilde{\alpha}_{(C,P)}$ must satisfy:

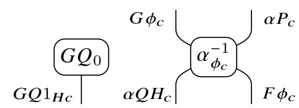
$$\begin{tikzcd} \mathbb{1} \arrow[r, equal] \arrow[dr, "\langle FPc\rangle"'] & \mathbb{1} \arrow[r, "\langle c\rangle"] \arrow[d, "\langle GPc\rangle"] \arrow[dl, phantom, "\overset{\alpha_{Pc}}{\Rightarrow}"] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[dl, "GP"] \\ & \mathcal{B} & \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbb{1} \arrow[r, "\langle c\rangle"] \arrow[dr, "\langle FPc\rangle"'] & C \arrow[r, "A_{(C,P)}"] \arrow[d, "FP"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\overset{\widetilde{\alpha}_{(C,P)}}{\Rightarrow}"] & C \arrow[dl, "GP"] \\ & \mathcal{B} & \end{tikzcd} \tag{5.30}$$

The functor $A_{(C,P)}$ must then be the identity on objects, and the component of $\widetilde{\alpha}_{(C,P)}$ at $c \in C$ is given by α_{Pc}. A similar argument which considers the action of the cylinders out of $\mathsf{G}_\Omega\mathcal{A}(\langle-\rangle, (C, P))$ on morphisms demonstrates that $A_{(C,P)}$ is indeed the identity on C and $\widetilde{\alpha}_{(C,P)}$ is simply the lax transformation $\alpha\,P$. To describe the lax-naturality 2-cell for $\mathsf{G}_\Omega\alpha$ at a 1-cell $(H, \phi)\colon (C, P) \to (D, Q)$ first we note that it has domain given by the composite $(H, G\phi)\,(1_C, \alpha P)$, which is $(H, G\phi\,\alpha P)$, and codomain given by the composite $(1_D, \alpha Q)\,(H, F\phi)$, which is $(H, \alpha\,QH\,F\phi)$. The natural transformation part of $\mathsf{G}_\Omega\alpha$ is 1_H, and the modification part has component at $c \in C$

given in terms of the the lax-naturality 2-cells for α:



Remark 5.6.1. Note that this 2-cell will be invertible whenever the lax-naturality 2-cells α_{ϕ_c} are, even though the modification component above may fail to be invertible as a 2-cell in $\mathscr{B}$ if GQ_0 isn't. The inverse also has 1_H as its natural transformation part, and modification part given by components:



It follows that $\mathsf{G}_\Omega \colon \mathsf{GRAY}_{\mathscr{L}} \to \mathsf{GRAY}_{\mathscr{L}}$ also restricts to a Gray-functor $\mathsf{GRAY} \to \mathsf{GRAY}$. ◊

We've now observed that $\mathsf{G}_\Omega\alpha$ is essentially given by post-whiskering with α. This is hardly surprising, as $\mathsf{F}_\Omega\alpha$ is defined in this way when α is strictly 2-natural; and because 1-cells of $\mathsf{F}_\Theta\mathscr{A}$ are lax triangles, post-composing a 0-cell by a lax transformation does indeed give a well-defined 1-cell. It's nevertheless interesting (to the author) that this natural definition of $\mathsf{G}_\Omega\alpha$ is obtainable by rather general arguments that arise from the universal properties of F_Θ and F_Ω, rather than our chosen constructions for them.

The extension of the free Θ-cocompletion and Ω-cocompletion from Cat-functors to $\mathsf{Gray}_{\mathscr{L}}$-functors can presumably be applied to some other free cocompletions. For example, there are those free cocompletions corresponding to pre-saturated sub-classes of Θ which also extend to $\mathsf{Gray}_{\mathscr{L}}$-functors as full sub-functors of G_Θ. Along with Ω, these include:

(a) $\Omega[\mathsf{N}]$, the class of weights for oplax colimits of *normal* oplax functors from 1-categories

(b) $\Omega[\mathbb{1}]$, the class of weights for coKleisli objects of comonads.

(c) $\Omega[\nabla]$, the class of weights for oplax colimits of oplax functors from *codiscrete* 1-categories[10].

There is also the prototypical Fam-construction (i.e. coproduct completion) on Set-categories. Free cocompletions on Cat (or CAT) naturally inherit the structure of a pseudo-monad, and Fam in particular is a pseudomonad whose underlying pseudofunctor is strictly 2-functorial. The class of weights for coproducts, $\amalg$, has an additional property that coproducts of representable presheaves on a 1-category A are coalgebras for the comonad W on $[A^{\mathsf{op}}, \mathsf{Set}]$ sending $K \colon A^{\mathsf{op}} \to \mathsf{Set}$ to $\coprod_{x \in \mathsf{ob}A} A(a,x) \times Kx$. We can call this comonad the *unnatural* transformation[11] classifier, because there is an unnatural transformation $\rho \colon K \to \mathsf{W}K$ for any presheaf $K \colon A^{\mathsf{op}} \to \mathsf{Set}$ given by sending $y \in Ka$ to $(1_a, y) \in \coprod_{x\in A} A(a,x) \times Kx$ such that precomposing by ρ_K induces a natural (in R) bijection:

$$[A^{\mathsf{op}}, \mathsf{Set}](\mathsf{W}K, R) \cong [A^{\mathsf{op}}, \mathsf{Set}]_{\mathsf{Unnat}}(K, R) \tag{5.31}$$

between natural transformations from $\mathsf{W}K$ and unnatural transformations from K. Though W is not idempotent, much of the analysis done for F_Θ-functors can be applied to $\mathsf{F}_\amalg$-functors by substituting $\mathscr{Q}$ with W, as we now briefly outline.

If for $P \colon B \nrightarrow A$ a profunctor we redefine $P^\sharp(a,b)$ as $\coprod_{x\in A} A(a,x) \times P(x,b)$ and $P^\flat(a,b)$ as $\coprod_{y\in B} P(a,y) \times B(y,b)$,

[10] such colimits (or their "lax" counterparts) are often called *collages*.

[11] an unnatural transformation $\alpha \colon F \Rightarrow G \colon C \to D$ is just a collection of morphisms $\alpha_c \colon Fc \to Gc$ indexed by $c \in \mathsf{ob}C$ without any naturality conditions.

then for any $\mathsf{F}_{\amalg}$-functor $K\colon A \to B$ there exist maps:

$$B(K,1)^{\sharp} \underset{r}{\overset{l}{\rightleftarrows}} B(K,1)^{\flat} \tag{5.32}$$

defined in terms of the W-coalgebra structures on $B(K-,b)$ for each $b \in B$, and on $B(Ka,-)$ for each $a \in A$. Equivalently, these maps can be defined in terms of the strictly generic factorisations of [Web04] for morphisms $u\colon Ka \to b$ as:

$$\left(a \xrightarrow{u} x, Kx \xrightarrow{f} b\right) \overset{l}{\longmapsto} \left(Ka \xrightarrow{Ku} Kx \xrightarrow{f} Kb\right) \qquad \left(Ka \xrightarrow{f} y \xrightarrow{v} b\right) \overset{r}{\longmapsto} \left(a \xrightarrow{f_1} \hat{a}, K\hat{a} \xrightarrow{v f_2} b\right)$$

These morphisms don't form an adjunction (they are maps in a 1-category) but when K is strictly normal — for the appropriate analogue of that condition in this setting — then rl will be the identity on $B(K,1)^{\sharp}$.

Using these morphisms one can define a lifting of unnatural transformations $F \Rightarrow GK\colon A \to C$ to unnatural transformations $\mathsf{lan}_K F \Rightarrow G$ where $\mathsf{lan}_K F$ is a pointwise extension. This lifting can be used to extend the 2-functor $\mathsf{Fam}\colon \mathsf{CAT} \to \mathsf{CAT}$ to a *sesqui-functor* from the sesqui-category SESQUI — which is CAT with the funny tensor product — to itself.

Despite the structural similarity between the endowment of a sesqui-functor structure on Fam and a $\mathsf{Gray}_{\mathscr{L}}$-functor structure on F_{Θ}, it's not clear to me whether or how this might generalise. It seems the necessary ingredients for these phenomena are:

(a) A system of comonads on $\mathcal{V}$-presheaf categories $[\mathcal{A}^{\mathsf{op}}, \mathcal{V}]$ which are lax-natural in $\mathcal{A}$.

(b) A saturated class of weights Φ which are coalgebras for these comonads.

The comonads $Q_{\mathcal{A}}$ on $[\mathcal{A}^{\mathsf{op}}, \mathcal{V}]$ define comonads on $\mathsf{Prof}(\mathcal{B}, \mathcal{A})$, for $\mathcal{A}$ and $\mathcal{B}$ $\mathcal{V}$-categories, by $QP(a,b) := (QP(-,b))a$. In terms of these profunctor comonads one can define a new closed structure for $\mathcal{V}$-Cat by associating to $\mathcal{A}, \mathcal{B} \in \mathcal{V}\mathsf{Cat}$ the coKleisli $\mathcal{V}$-category for Q on $\mathsf{Prof}(\mathcal{B}, \mathcal{A})$ restricted to those profunctors of the form $\mathcal{B}(F,1)$. In the cases of F_{Θ} and Fam it is to this new closed structure that the free-cocompletions could be extended.

Chapter 6

2-Fibrations

Our investigation into 2-categorical fam constructions will next consider properties analogous to Weber's condition of being a *familial 2-monad*. This condition involves certain maps being fibrations in a 2-category, and so establishing a "higher-dimensional" analogue requires that we consider 2-fibrations in 3-categories, and also in Gray-categories, as we shall see.

This chapter provides some concepts and results from the theory of 2-fibrations which will be necessary for this discussion of higher dimensional familial monads. In all other ways the chapter is self-contained, and in particular doesn't refer directly to the constructions F_Ω or F_Θ. Some knowledge of fibrations and 2-fibrations as described in [Buc14] is assumed, though we recall the definition of 2-fibration from Section 3.3 below. Note that this definition from [Buc14] differs slightly from an earlier definition of Hermida in [Her99] and the extension to bicategories given by Baković in [Bak].

6.1 Representability of 2-fibrations

Beginning with the definition of a Grothendieck fibration in Cat, one obtains a general notion of fibration in a 2-category by extending *representably*. That is, one says that a 1-cell $f : e \to b$ in a 2-category $\mathcal{A}$ is a fibration if $\mathcal{A}(-, f) : \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}^{\to}$ factors through Fib, the locally full sub-2-category of fibrations and cartesian morphisms. Equivalently, one can define a 2-cell $\alpha : a \Rightarrow b : x \to e$ in $\mathcal{A}$ to be generalised f-cartesian if its precomposition by arbitrary $g : y \to x$ is $\mathcal{A}(y, f)$-cartesian in $\mathcal{A}(y, e)$. A fibration in $\mathcal{A}$ is then a 1-cell $f : e \to b$ in $\mathcal{A}$ such that 1-cells in $\mathcal{A}(x, b)$ of the form $\beta : g \to fa$ have generalised-f-cartesian lifts in the usual way.

We will denote fibrations of the more general sort *representable* fibrations. The distinction is usually unnecessary, as the representable fibrations in Cat coincide with the Grothendieck fibrations. Though the terminology will prove useful when we adapt it to consider 2-fibrations in $\mathsf{2Cat}$, which we observe are not the same as representable 2-fibrations in the obvious sense.

Let's recall the definition of a 2-fibration, given earlier in Chapter 3:

Definition 3.3.2 (2-fibration)**.** A 2-functor $F : \mathcal{C} \to \mathcal{D}$ is a *2-fibration* if:

(a) 1-cells $u : d \to Fc$ have cartesian lifts to 1-cells $\bar{u} : u^*c \to c$

(b) 2-cells $\alpha : Ff \Rightarrow g$ have *opcartesian* lifts to 2-cells $\underline{\alpha} : f \Rightarrow \alpha_* g$

(c) The horizontal composition of opcartesian 2-cells is opcartesian

A 2-fibration with chosen (op)cartesian lifts of 1-cells and 2-cells (i.e. a cleavage) is *split* if these chosen lifts are preserved by vertical and horizontal composition in the obvious sense. ◊

For the following discussion, it will be convenient to rephrase this definition slightly using a stronger notion of opcartesianness for 2-cells. We will say a 2-cell is *strong* opcartesian if the result of whiskering it with arbitrary 1-cells is opcartesian in the usual sense (and therefore also strong-opcartesian). In a 2-category $\mathcal{A}$ where P-opcartesian 2-cells — for some $P\colon \mathcal{A} \to \mathcal{B}$ — are closed under horizontal composition, all opcartesian 2-cells are automatically strong-opcartesian. This is because whiskering a 2-cell θ with a 1-cell u is equivalent to horizontally compositing θ with the identity 2-cell at u, which is automatically opcartesian. Conversely, if all opcartesian 2-cells are strong then opcartesian 2-cells are closed under horizontal composition. For if $\theta\colon u \Rightarrow v$ and $\zeta\colon s \Rightarrow t$ are strong-opcartesian, then the horizontal composition $\zeta\,\theta\colon su \Rightarrow tv$ is the vertical composition $\zeta\, v \circ s\,\theta$ of opcartesian 2-cells, which is opcartesian. We therefore have the following equivalent definition of 2-fibration:

Lemma 6.1.1. *A 2-functor $F\colon C \to \mathcal{D}$ is a* 2-fibration *if*

(a) 1-cells $u\colon d \to Fc$ have cartesian lifts to 1-cells $\bar{u}\colon u^*c \to c$

(b) 2-cells $\alpha\colon Ff \Rightarrow g$ have strong *opcartesian lifts to 2-cells $\underline{\alpha}\colon f \Rightarrow \alpha_* g$*

We can extend this definition representably in two different ways, depending on whether we view a 2-functor as a 1-cell in $\mathsf{2Cat}$ or a 1-cell in Gray. Both notions will be relevant to Chapter 7, but we will find that it is the Gray-representable fibrations in Gray which correspond to the ordinary notion of 2-fibration, rather than the $\mathsf{2Cat}$-representable fibrations in $\mathsf{2Cat}$. We begin by describing the $\mathsf{2Cat}$-representable notion:

Definition 6.1.2 (Representable 2-fibration)**.** For a 1-cell $P\colon \mathcal{A} \to \mathcal{B}$ in a $\mathsf{2Cat}$-category $\mathbb{K}$ we say that a 2-cell $\sigma\colon F \Rightarrow G\colon C \to \mathcal{A}$ is *generalised P-cartesian* if for an arbitrary $H\colon X \to C$ in $\mathbb{K}$ the 2-cell σH is $\mathbb{K}(X, P)$-cartesian. A 3-cell $\Gamma\colon \sigma \Rrightarrow \tau\colon F \Rightarrow G\colon C \to \mathcal{A}$ is *strong generalised P-opcartesian* if for arbitrary $H\colon X \to C$ the 3-cell ΓH is a strong-$\mathbb{K}(X, P)$-opcartesian 2-cell in $\mathbb{K}(X, \mathcal{A})$. The 1-cell P is a *representable 2-fibration* if:

(a) for 1-cells $F\colon C \to \mathcal{A}$, $G\colon C \to \mathcal{B}$ and a 2-cell $\sigma\colon G \Rightarrow PF$ there exists $G'\colon C \to \mathcal{A}$ and a generalised cartesian $\sigma'\colon G' \Rightarrow F$ such that $P\sigma' = \sigma$.

(b) for 2-cells $\sigma : F \Rightarrow G\colon C \to \mathcal{A}$, $\tau\colon PF \Rightarrow PG$ and a 3-cell $\Gamma\colon P\sigma \Rrightarrow \tau$ there exists $\tau'\colon F \Rightarrow G$ and a strong generalised opcartesian $\Gamma'\colon \sigma \Rrightarrow \tau'$ such that $P\,\Gamma' = \Gamma$.

We say the representable 2-fibration P is *split* if the (op)cartesian liftings of 1-cells and 2-cells preserve horizontal and vertical composition. ◊

Remark 6.1.3. This definition is equivalent to the more concise and less wieldy condition that $\mathbb{K}(-, f)\colon \mathbb{K}^{\mathsf{op}} \to \mathsf{2Cat}^{\to}$ factor through $\mathsf{2Fib}$ — the 3-category of 2-fibrations, cartesian morphisms, vertical 2-natural transformations and vertical modifications. ◊

6.1.1 Representable 2-fibrations in 2Cat

We would now like to consider what representable 2-fibrations are in $\mathsf{2Cat}$, and in particular how they relate to ordinary 2-fibrations.

First, let's consider the generalised P-cartesian 1-cells of a 2-functor $P\colon \mathcal{A} \to \mathcal{B}$.

6.1.1.1 Generalised cartesian 1-cells

For the case $C = \mathbb{1}$, a 1-cell in $\mathsf{2Cat}(\mathbb{1}, \mathcal{A})$ is simply a 1-cell in $\mathcal{A}$, and it will be $\mathsf{2Cat}(\mathbb{1}, P)$-cartesian precisely if it is P-cartesian in $\mathcal{A}$. For general C, a generalised cartesian $\alpha\colon F \Rightarrow G\colon C \to \mathcal{A}$ should have its cartesianness preserved by pre-whiskering with $\langle c\rangle : \mathbb{1} \to C$ for arbitrary $c \in C$. Thus, if $\alpha\colon F \Rightarrow G$ is generalised-cartesian all of its components must be P-cartesian.

In fact, this condition is sufficient. Assume each component of a 2-natural transformation $\alpha\colon F \Rightarrow G\colon C \to \mathcal{A}$ is P-cartesian, and that there is some $\beta\colon H \Rightarrow G$ and $\gamma\colon PH \Rightarrow PF$ satisfying $P\alpha \circ \gamma = P\beta$. We now want to show that there is a unique lift of γ to $\gamma'\colon H \to F$ satisfying $\alpha \circ \gamma' = \beta$:

$$\begin{tikzcd} H \arrow[d, "\gamma'"'] \arrow[dr, "\beta"] \arrow[d, phantom, shift left=3ex, "\circlearrowleft"] & & & PH \arrow[d, "\gamma"'] \arrow[dr, "P\beta"] \arrow[d, phantom, shift left=3ex, "\circlearrowleft"] & \\ F \arrow[r, "\alpha"'] & G & \overset{P}{\longmapsto} & PF \arrow[r, "P\alpha"'] & PG \end{tikzcd}$$

We obtain the lift $\gamma'\colon H \to F$ by lifting each of its components using the cartesianness of the components of α. That is, $\gamma'_c\colon H_c \to F_c$ will be the unique 1-cell such that $P\gamma'_c = \gamma_c$ and $\alpha_c\gamma_c = \beta_c$. To see that γ' is 2-natural, observe that for $\sigma\colon u \Rightarrow v\colon c \to d$ some 2-cell in C we have $\gamma'_d\, H_\sigma = F_\sigma\, \gamma'_c$ because both 2-cells are equal when post-whiskered by α_c (by the naturality of β and α) and under the action of P (by the naturality of γ). This lift of γ to γ' is clearly the unique possible such lifting, because the lifting of each component was unique. For the 2-dimensional aspect of the cartesianness of α, assume we have 2-cells $\beta, \gamma\colon H \Rightarrow F$ and modifications $\Delta\colon \alpha\,\beta \Rrightarrow \alpha\,\gamma$, $\Gamma\colon P\beta \Rrightarrow P\gamma$ such that $P\Delta = P\alpha\,\Gamma$. We now want to demonstrate a unique lift of Γ to $\Gamma'\colon \beta \Rrightarrow \gamma$ satisfying $\alpha\,\Gamma' = \Delta$:

$$\begin{tikzcd} H \arrow[r, bend left, "\alpha\beta"] \arrow[r, bend right, "\alpha\gamma"'] \arrow[r, phantom, "\Downarrow\Delta"] & G & \overset{\alpha_*}{\longleftarrow} & H \arrow[r, bend left, "\beta"] \arrow[r, bend right, "\gamma"'] \arrow[r, phantom, "\Downarrow\Gamma'"] & F & \overset{P}{\longrightarrow} & PH \arrow[r, bend left, "P\beta"] \arrow[r, bend right, "P\gamma"'] \arrow[r, phantom, "\Downarrow\Gamma"] & PF \end{tikzcd}$$

This is achieved by lifting the components of Γ by the cartesianness of the components of α. We let $\Gamma'_c\colon \beta_c \Rightarrow \gamma_c$ be the such 2-cell satisfying $\alpha_c\,\Gamma'_c = \Delta_c$ and $P\Gamma'_c = \Gamma_c$. For this component-wise lifting of data to form a modification it must be the case that for each $u\colon c \to d$ in C, $Fu\,\Gamma_c = \Gamma_d\,H_u$, which follows from the fact that both of these 2-cells are equal under post-whiskering with α_d and under the action of P.

6.1.1.2 Strong generalised opcartesian 2-cells

Consider a modification $\Gamma\colon \sigma \Rrightarrow \tau\colon F \Rightarrow G\colon C \to \mathcal{A}$. If Γ is strong generalised opcartesian, it must have strong-opcartesian 2-cells as components in order for it to be stable under precomposition by 2-functors of the form $\langle c\rangle : \mathbb{1} \to C$. We now observe this condition is sufficient. Assume Γ has opcartesian 2-cell components, and that we are given modifications $\Delta\colon \sigma \Rrightarrow \rho$ and $\Xi\colon P\tau \Rrightarrow P\rho$ satisfying $\Xi \circ P\Gamma = P\Delta$. We want to exhibit a unique lift of Ξ to $\Xi'\colon \tau \Rrightarrow \rho$ satisfying $\Xi' \circ \Gamma = \Delta$:

$$\begin{tikzcd} \sigma \arrow[r, "\Gamma"] \arrow[dr, "\Delta"'] \arrow[dr, phantom, shift left=3ex, "\circlearrowleft"] & \tau \arrow[d, "\Xi'"] & & P\sigma \arrow[r, "P\Gamma"] \arrow[dr, "P\Delta"'] \arrow[dr, phantom, shift left=3ex, "\circlearrowleft"] & P\tau \arrow[d, "\Xi"] \\ & \rho & \overset{P}{\longmapsto} & & P\rho \end{tikzcd}$$

We obtain this lift by declaring the component $\Xi'_c\colon \tau_c \Rightarrow \rho_c$ to be the unique 2-cell satisfying $P\Xi'_c = \Xi_c$ and $\Xi'_c\,\Gamma_c = \Delta_c$. For $u\colon c \to d$ in C both $G_u\,\Xi'_c$ and $\Xi'_d\,F_u$ lie over the same 2-cell in $\mathcal{B}$ by the naturality of Ξ, and are equal under precomposition by the opcartesian 2-cell $G_u\,\Gamma_c = \Gamma_d\,F_u$ by the naturality of Δ, from which it follows $G_u\,\Xi'_c = \Xi'_d\,F_u$.

This shows that a modification with strong-opcartesian components is generalised opcartesian. It is moreover *strong* generalised opcartesian by the fact that whiskering on either side by an arbitrary 2-cell (i.e. 1-cell in $[C, \mathcal{A}]$) will produce a modification whose components are again strong-opcartesian. We conclude that a modification $\Gamma \colon \sigma \Rrightarrow \tau$ is a generalised strong-opcartesian 2-cell if and only if its components are strong-P-opcartesian 2-cells.

Remark 6.1.4. Our reason for considering *strong* opcartesian 2-cells, rather than the usual notion of opcartesian 2-cells, is that having opcartesian components is insufficient for a modification to be generalised opcartesian. Note that our justification for the naturality of the lifted modification Ξ' relied on the fact that $G_u \Gamma_c = \Gamma_d F_u$ was opcartesian, which doesn't necessarily follow from the components Γ_x each being opcartesian. ◊

If $P \colon \mathcal{A} \to \mathcal{B}$ is a representable 2-fibration then in particular $2\mathsf{Cat}(\mathbb{1}, P)$ — which is isomorphic to P — must be a 2-fibration. However, the converse is false: not every 2-fibration is a representable 2-fibration. We will observe why this is the case, and show instead that the *horizontally split* 2-fibrations are representable (horizontally split) 2-fibrations.

If P is a 2-fibration, then for 2-functors $F \colon C \to \mathcal{B}$, $G \colon C \to \mathcal{A}$ and a 2-natural transformation $\sigma \colon F \Rightarrow PG$, we can construct a new 2-functor $F' \colon C \to \mathcal{A}$ and 2-natural transformation $\sigma' \colon F' \to G$ whose component at $c \in C$ is the P-cartesian lift $\bar{\sigma_c} Gc \colon \sigma_c^* Gc \to Fc$.

$$
\begin{tikzcd}
 & \mathcal{A} \arrow[d, "P"] \\
C \arrow[ur, "G"] \arrow[r, "F"'] \arrow[r, phantom, shift left=3, "\Uparrow\sigma"] & \mathcal{B}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & \mathcal{A} \arrow[d, "P"] \\
C \arrow[ur, bend left=60, "G"] \arrow[ur, dashed, "F'"] \arrow[ur, phantom, bend left=30, "\sigma'"] \arrow[r, "F"'] \arrow[r, phantom, shift left=3, "\circlearrowright"] & \mathcal{B}
\end{tikzcd}
$$

The action of F' on 1-cells and 2-cells is given uniquely such that σ' becomes 2-natural. For example, $F'_u \colon F'_c \to F'_d$ is the unique 1-cell over F_u which satisfies $\bar{\sigma_d} G_d \; F_u = G_u \; \bar{\sigma_c} G_c$. With respect to this definition of F', σ' is 2-natural, and it is clearly cartesian and mapped to σ by P.

For 2-natural transformations $\sigma \colon F \Rightarrow G \colon C \to \mathcal{A}$, $\tau \colon PF \Rightarrow PG \colon C \to \mathcal{B}$ and a modification $\Gamma \colon P\sigma \Rrightarrow \tau$, we can attempt to lift Γ to a modification $\Gamma' \colon \sigma \Rrightarrow \tau'$ by constructing components $\Gamma'_c \colon \sigma_c \Rightarrow \tau_c$ as $\underline{\Gamma_c} \sigma_c \colon \sigma_c \Rightarrow (\Gamma_c)_* \tau_c$.

$$
\begin{tikzcd}
F \arrow[r, bend left=50, "\sigma"] \arrow[r, bend right=50, "\tau'"'] \arrow[r, phantom, "\Downarrow\Gamma'"] & G
\end{tikzcd}
\quad \overset{P}{\longmapsto} \quad
\begin{tikzcd}
PF \arrow[r, bend left=50, "P\sigma"] \arrow[r, bend right=50, "\tau"'] \arrow[r, phantom, "\Downarrow\Gamma"] & PG
\end{tikzcd}
$$

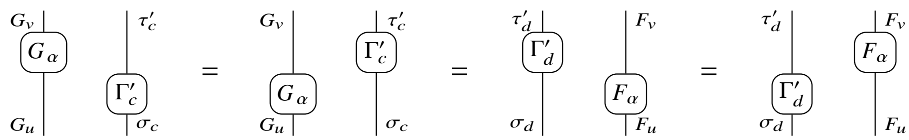
However, this time the naturality for the lifting of τ fails. For $u \colon c \to d$, both $G_u \Gamma'_c \colon G_u \sigma_c \Rightarrow G_u (\Gamma_c)_* \tau_c$ and $\Gamma'_d F_u \colon G_u \sigma_c \Rightarrow (\Gamma_d)_* \tau_d F_u$ are opcartesian 2-cells with the same domain which lie over the same 2-cell in $\mathcal{B}$, but they need not have the same codomain. There is, however, a canonical invertible 2-cell between $(\Gamma_d)_* \tau_d F_u$ and $G_u (\Gamma_c)_* \tau_c$. As we will show later, this 2-cell forms a pseudo-naturality 2-cell for a pseudo-natural lift of τ, rather than a 2-natural lift.

For there to exist a lifting of τ to a strictly 2-natural transformation, we need to be able to choose opcartesian lifts of 2-cells in $\mathcal{B}$ in a way that is stable under whiskering by arbitrary 1-cells. That is, whenever we have morphisms $u \colon x \to y$, $v \colon y \to z$ in $\mathcal{A}$, $w \colon Py \to Pz$ in $\mathcal{B}$ and a 2-cell $\alpha \colon Pv \Rightarrow Pw$, we require that the chosen lift $\underline{\alpha\, Pu}(w\, Pu)$ coincides with $u\, \underline{\alpha}(w)$:

$$
\begin{tikzcd}
x \arrow[r, bend left=40, "vu"] \arrow[r, bend right=40, "(\alpha\, Pu)_*(vu)"'] \arrow[r, phantom, "\Downarrow \underline{\alpha\, Pu}(vu)"] & z
\end{tikzcd}
\; = \;
\begin{tikzcd}
x \arrow[r, "u"] & y \arrow[r, bend left=50, "v"] \arrow[r, bend right=50, "\alpha_* v"'] \arrow[r, phantom, "\Downarrow \underline{\alpha} v"] & z
\end{tikzcd}
\quad \overset{P}{\longmapsto} \quad
\begin{tikzcd}
Px \arrow[r, "Pu"] & Py \arrow[r, bend left=50, "Pv"] \arrow[r, bend right=50, "w"'] \arrow[r, phantom, "\Downarrow\alpha"] & Pz
\end{tikzcd}
$$

along with the analogous property for post-whiskering. We will say that a cloven 2-fibration with this property is *horizontally split*[1].

Now with the additional assumption that P admits a horizontally split cleavage, our chosen lifts Γ'_c for the components of $\Gamma\colon P\sigma \Rrightarrow \tau$ will satisfy the property that $G_u\,\Gamma'_c = \Gamma'_d\,F_u$ which demonstrates simultaneously the 1-naturality of τ' and the naturality of $\Gamma'\colon \sigma \Rrightarrow \tau'$. The 2-naturality of τ' follows from the fact that for a 2-cell $\alpha\colon u \Rightarrow v$ in C, both $G_\alpha\,\tau'_c$ and $\tau'_d\,F_\alpha$ are equal under the image of P (by the 2-naturality of τ) and under vertical precomposition by $G_u\,\Gamma'_c = \Gamma'_d\,F_u$:



So horizontally split 2-fibrations are representable 2-fibrations. Moreover, they are horizontally split as representable 2-fibrations, since the stability of the lifts of 2-cells on components under whiskering by 1-cells translates to stability of the lifted modification under whiskering by a 2-natural transformation. We place this in a Lemma for later reference.

Lemma 6.1.5. *A 2-functor is a horizontally split 2-fibration if and only if it is a* generalised *horizontally split 2-fibration.*

It is easy to see that other splitting properties of a cleavage for a 2-fibration lift to and descend from corresponding properties for the representable 2-fibration, so we additionally have:

Corollary 6.1.6. *A 2-functor is a split 2-fibration if and only if it is a generalised split 2-fibration.*

Remark 6.1.7. Perhaps some explanation for the fact that general 2-fibrations aren't necessarily representable 2-fibrations, yet horizontally split 2-fibrations are, lies in the fact that horizontally split 2-fibrations are precisely those which arise from (certain) trihomomorphisms $\mathcal{A}^{\mathsf{op}} \to \mathsf{2Cat}$, whereas 2-fibrations more generally correspond to trihomomorphisms $\mathcal{A}^{\mathsf{op}} \to \mathsf{Gray}$ (cf. [Buc14, Remark 3.3.13]). ◊

6.1.2 Gray-representable fibrations

Remark 6.1.7 and the observation that general 2-fibrations may lift 2-natural transformation to pseudo-natural transformations suggests that 2-fibrations may be better described by Gray-representability. In this section we show that this is indeed the case.

Definition 6.1.8 (Gray-representable fibration)**.** For a 1-cell $P\colon \mathcal{A} \to \mathcal{B}$ in a Gray-category $\mathbb{K}$ we say that a 2-cell $\sigma\colon F \Rightarrow G\colon C \to \mathcal{A}$ is *Gray P-cartesian* if for an arbitrary $H\colon X \to C$ in $\mathbb{K}$ the 2-cell σH is $\mathbb{K}(X,P)$-cartesian. A 3-cell $\Gamma\colon \sigma \Rightarrow \tau\colon F \Rightarrow G\colon C \to \mathcal{A}$ is *strong Gray P-opcartesian* if for arbitrary $H\colon X \to C$ the 3-cell ΓH is a strong-$\mathbb{K}(X,P)$-opcartesian 2-cell in $\mathbb{K}(X,\mathcal{A})$. The 1-cell P is a *Gray-representable fibration* if:

(a) for 1-cells $F\colon C \to \mathcal{A}$, $G\colon C \to \mathcal{B}$ and a 2-cell $\sigma\colon G \Rightarrow PF$ there exists $G'\colon C \to \mathcal{A}$ and a Gray cartesian $\sigma'\colon G' \Rightarrow F$ such that $P\sigma' = \sigma$.

(b) for 2-cells $\sigma : F \Rightarrow G\colon C \to \mathcal{A}$, $\tau\colon PF \Rightarrow PG$ and a 3-cell $\Gamma\colon P\sigma \Rrightarrow \tau$ there exists $\tau'\colon F \Rightarrow G$ and a strong Gray opcartesian $\Gamma'\colon \sigma \Rrightarrow \tau'$ such that $P\Gamma' = \Gamma$. ◊

[1]Not to be confused with *globally split* 2-fibrations, where the chosen lifts of 1-cells strictly preserves composition

6.1.2.1 Gray-cartesian 1-cells and 2-cells

By observations made for the strict case, a pseudo-natural transformation $\sigma\colon F \Rightarrow G\colon \mathcal{C} \to \mathcal{A}$ must have P-cartesian components to be a Gray cartesian 1-cell. And as in the strict case, this is sufficient. Consider 2-functors $F, G, H\colon \mathcal{C} \to \mathcal{A}$, and pseudo-natural transformations $\sigma\colon F \Rightarrow H$, $\tau\colon G \Rightarrow H$ and $\rho\colon PF \Rightarrow PG$ where each 1-cell component of τ is P-cartesian and $P\tau\,\rho = P\sigma$. We must demonstrate that there is a unique pseudo-natural lift of ρ to $\rho'\colon F \Rightarrow G$ satisfying $\tau\,\rho' = \sigma$:

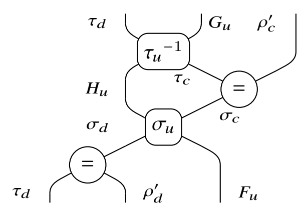
$$\begin{tikzcd} F \arrow[d, "\rho'"'] \arrow[dr, "\sigma"] \arrow[d, phantom, shift left=3ex, "\circlearrowleft"] & \\ G \arrow[r, "\tau"'] & H \end{tikzcd} \quad \overset{P}{\longmapsto} \quad \begin{tikzcd} PF \arrow[d, "\rho"'] \arrow[dr, "P\sigma"] \arrow[d, phantom, shift left=3ex, "\circlearrowleft"] & \\ PG \arrow[r, "P\tau"'] & PH \end{tikzcd}$$

We define the pseudonatural transformation $\rho'\colon F \Rightarrow G$ as having:

Component at $c \in \mathcal{C}$ given by the unique $\rho'_c\colon F_c \to G_c$ such that $P\rho'_c = \rho_c$ and $\tau_c\,\rho'_c = \sigma_c$.

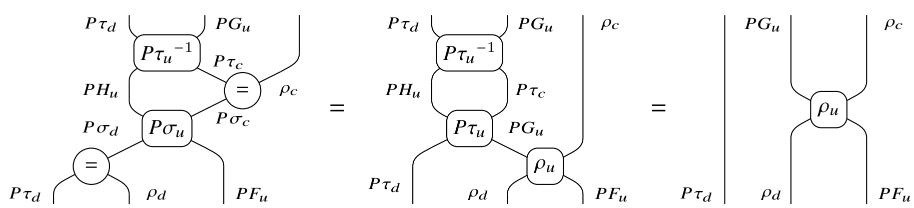
Naturality 2-cell at $u\colon c \to d$ being 2-cell $G_u\,\rho'_c \Rightarrow \rho'_c\,F_u$ constructed as follows:

(a) post-compose σ_u by the inverse of τ_u to obtain the following 2-cell in $\mathcal{A}$:



(6.1)

(b) Observe that the image of (6.1) under P is $P\tau_d\,\rho_u$:

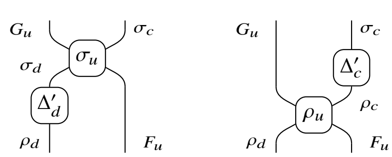


(6.2)

(c) Let ρ'_u be the unique lift of ρ_u such that $\tau_d\,\rho_u$ is equal to the 2-cell given in (6.1).

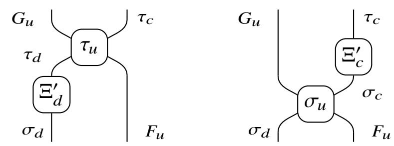
The uniqueness of such liftings transfers the coherence conditions from ρ, σ and τ to coherence conditions for the ρ'_u 2-cells. The invertibility of ρ'_u follows from uniqueness of 2-cell liftings and the fact that ρ_u is invertible. And we clearly have $\sigma = \tau \circ \rho'$ because $\tau_d\,\rho'_u$ is given by (6.1).

We conclude that ρ can be lifted to a pseudo-natural transformation ρ', and such a lifting is unique because its components are unique. A similar argument demonstrates the modification-lifting property for τ. Given $\rho, \sigma\colon F \Rightarrow G$ and modifications $\Gamma\colon \tau\,\rho \Rrightarrow \tau\,\sigma$, $\Delta\colon P\rho \Rrightarrow P\sigma$ such that $P\Gamma = P\tau\,\Delta$ lift the 2-cells $\Delta_c\colon P\rho_c \Rightarrow P\sigma_c$ to 2-cells $\Delta'_c\colon \rho_c \Rightarrow \sigma_c$ such that $P\Delta'_c = \Delta_c$ and $\tau_c\,\Delta'_c = \Gamma_c$ by the cartesianness of τ_c. Then observe that the following 2-cells in $\mathcal{A}$:



are equal under post-composition by τ_d (by the naturality of Γ) and under the image of P (by the naturality of Δ) and so must be equal by the cartesianness of τ_d.

The strong Gray opcartesian 2-cells are simply those modifications whose 2-cell components are strong-P-opcartesian, as we now demonstrate. Assume we have pseudonatural transformations $\rho, \sigma, \tau \colon F \Rightarrow G \colon \mathcal{C} \to \mathcal{A}$ and modifications $\Gamma \colon \rho \Rrightarrow \sigma$, $\Delta \colon \rho \Rrightarrow \tau$, $\Xi \colon P\sigma \Rrightarrow P\tau$ such that the components of Γ are P-opcartesian 2-cells and $P\Delta = \Xi \circ P\Gamma$; then we lift Ξ by defining components $\Xi'_c \colon \sigma_c \Rightarrow \tau_c$ to be the unique 2-cells satisfying $P\Xi'_c = \Xi_c$ and $\Xi'_c \circ \Gamma_c = \Delta_c$. The naturality of these components follows from the fact that the following 2-cells in $\mathcal{A}$:



are equal under the image of P by the naturality Ξ and are equal under vertical precomposition by $\Gamma_d\, F_u$ (which is an opcartesian 2-cell) by the naturality of Γ and Δ.

Lemma 6.1.9. *The Gray-representable fibrations in* Gray *are precisely the 2-fibrations.*

Proof. One direction is clear: if for some 2-functor $P \colon \mathcal{A} \to \mathcal{B}$ the Gray-functor $\mathsf{Gray}(-, P)$ factors through 2Fib then in particular $\mathsf{Gray}(\mathbb{1}, P)$ is a 2-fibration, and thus so is P.

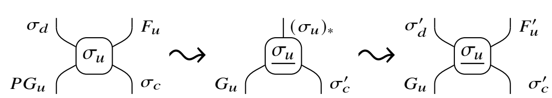
Conversely, if $P \colon \mathcal{A} \to \mathcal{B}$ is a 2-fibration we can demonstrate the requisite generalised lifting properties as follows.

For $F \colon \mathcal{C} \to \mathcal{A}$, $G \colon \mathcal{C} \to \mathcal{B}$ 2-functors and $\sigma \colon G \to PF$ a pseudo-natural transformation, we define components $\sigma'_c \colon F'_c \to G_c$ as $\bar{\sigma}_c G_c \colon \sigma^*_c G_c \to G_c$. We simultaneously define the action of F' on 1-cells and the pseudo-naturality 2-cells for σ' as follows:

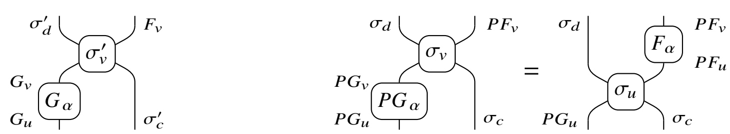
(a) Start with the pseudo-naturality 2-cell[2] $\sigma_u \colon PG_u\, \sigma_c \Rightarrow \sigma_d\, F_u$.

(b) Take the strong-opcartesian lift with domain $G_u\, \sigma'_c$

(c) Observe that the codomain of this lift lies over $\sigma_d\, F_u$, and so has a unique factorisation through σ'_d as $\sigma'_d\, F'_u$.

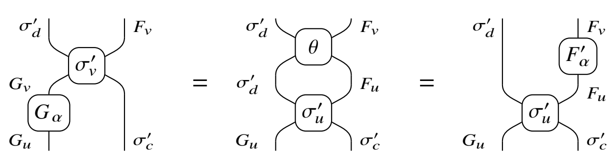


The uniqueness of lifts and pseudo-naturality of σ ensures F' is 1-functorial and σ' is pseudo-natural. For the action of F' on 2-cells, observe that for $\alpha \colon u \Rightarrow v$, the 2-cell on the left of (6.3) lies over the 2-cell on the right in $\mathcal{B}$:



(6.3)

So the 2-cell on the left factors through σ'_u, as $\theta \circ \sigma'_u$, and $\theta \colon \sigma'_d\, F'_u \Rightarrow \sigma'_d\, F'_v$ lies over $\sigma_d F_\alpha$, which means θ can be decomposed as $\sigma'_d\, F'_\alpha$ by the cartesianness of σ'_d:

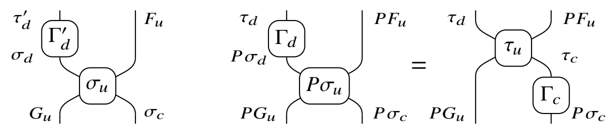


This simultaneously defines the action of F' on 2-cells and demonstrates the naturality of σ with respect to this action. Finally, note that the pseudo-naturality 2-cells σ'_u are invertible because they are (strong) opcartesian lifts of invertible 2-cells.

[2]Throughout this section we will refer to both these 2-cells and their inverses as "pseudo-naturality 2-cells", to be disambiguated by the given domain and codomain.

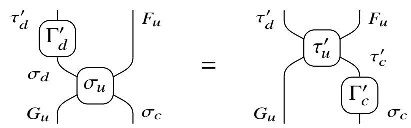
So $\sigma' \colon F' \Rightarrow G$ is a well-defined pseudonatural transformation, and is clearly Gray P-cartesian as its components are P-cartesian.

To demonstrate strong-opcartesian lifts of 2-cells, assume we have pseudo-natural transformations $\sigma \colon F \Rightarrow G \colon C \to \mathcal{A}$, $\tau \colon PF \Rightarrow PG$, and a modification $\Gamma \colon P\sigma \Rrightarrow \tau$. Define components $\Gamma'_c \colon \sigma_c \Rightarrow \tau'_c \colon F_c \to G_c$ to be the unique strong-opcartesian lifts of Γ_c with domain σ_c. To define the pseudo-naturality 2-cells for τ' and simultaneously observe the naturality of Γ'_c with respect to these 2-cells, first observe that the 2-cell in $\mathcal{A}$ on the left of (6.4) lies of the 2-cell in $\mathcal{B}$ on the right, for $u \colon c \to d$ in C:

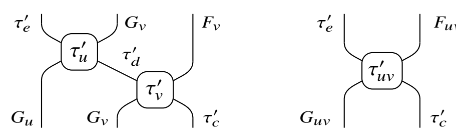


(6.4)

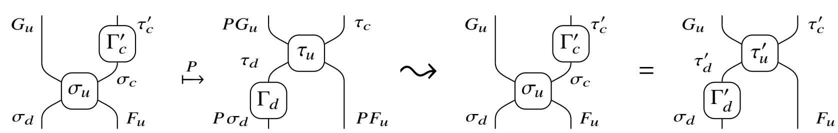
Because $G_u \Gamma'_c$ is opcartesian and lies over $PG_u \Gamma_c$, the 2-cell on the left of (6.4) factors through $G_u \Gamma'_c$, which provides the requisite 2-cell τ'_u:



To see that the 2-cells τ'_u are coherent, observe that the following 2-cells in $\mathcal{A}$ are equal under precomposition by $G_{uv} \Gamma_c$ by the coherence of σ, and are equal under P by the naturality of Γ and coherence of τ.



The 2-cells τ'_u are seen to be invertible by observing that there inverses can be lifted in much the same way:



These lifts of the inverse naturality 2-cells are inverses the lifts of the original naturality 2-cells by the fact that the following pairs 2-cells in $\mathcal{A}$ are each equal under precomposition by $G_u \Gamma_c$ and $F_u \Gamma_d$ respectively, and under the action of P:

We conclude that $\tau' \colon F \Rightarrow G$ is a well-defined pseudo-natural transformation, and that the 2-cells Γ'_c form a modification $\sigma \Rrightarrow \tau'$. This modification is Gray strong-P-opcartesian because its components are strong-P-opcartesian. □

6.2 Globally Split 2-Fibrations

It is shown in [Buc14] that there is a correspondence between presheaves valued in 2-categories and 2-fibrations via a 2-dimensional Grothendieck construction. We've appealed to this correspondence already in Chapter 3. More accurately, there is a family of such correspondences indexed by the various notions of "2-dimensional presheaf". For example, we saw in Chapter 3 that when we view Cat-presheaves $\mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ as locally-discrete 2-dimensional presheaves, the corresponding sort of 2-fibrations are the locally discrete split 2-fibrations. Buckley describes in [Buc14] notions of Grothendieck construction for the split-2-fibration–2Cat-presheaf correspondence, and for the fibred-bicategory–Bicat-presheaf correspondence[3]. In Chapter 7 we will encounter a flavour of 2-fibration lying between these two: *globally split* 2-fibrations. These are cloven 2-fibrations $P\colon \mathcal{A} \to \mathcal{B}$ satisfying the following two properties:

(a) the underlying 1-fibration of P is split;

(b) P is horizontally split, i.e. the chosen opcartesian lifts of 2-cells are strictly preserved by whiskering by arbitrary 1-cells.

These differ from split 2-fibrations in that opcartesian lifts of 2-cells need not preserve either horizontal or vertical composition[4]. In this section we will describe the corresponding notion of presheaf and Grothendieck construction for these fibrations, as well as the "free" globally split 2-fibration on a 2-functor.

6.2.1 Globally split 2-presheaves

The presheaves corresponding to globally split 2-fibrations over a 2-category $\mathcal{A}$ can be described by the following data:

(a) A map on objects $F\colon \mathsf{ob}\mathcal{A} \to \mathsf{2Cat}$

(b) Pseudo-functors on hom-categories $F\colon \mathcal{A}(b,a) \to \mathsf{Gray}(Fa, Fb)$ (note the contravariance) which satisfy:

   i) Whiskering is strictly preserved. For example, given $\begin{tikzcd} a \arrow[r, "g"] & b \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2, "\Downarrow\beta"] & c \end{tikzcd}$ in $\mathcal{A}$, we have $F_{\alpha g} = F_g F_\alpha$. Moreover, the pseudo-functoriality 2-cell $F_*\colon F_{\beta\circ\alpha} \Rightarrow F_\beta \circ F_\alpha$ post-whiskered by F_g gives the pseudo-functoriality 2-cell $F_{\alpha g \circ \beta g} \Rightarrow F_g F_\alpha \circ F_g F_\beta$. Similarly, $F_g F_*\colon F_g F_{1_f} \Rightarrow F_g 1_{F_f}$ is the pseudo-unitality 2-cell for fg, where $f\colon b \to c$. This describes naturality with respect to pre-whiskering in $\mathcal{A}$, but the same is true for post-whiskering.

   ii) for a diagram in $\mathcal{A}$ of the form $\begin{tikzcd} a \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\beta"] & c \end{tikzcd}$ the pseudo-functoriality 3-cell $F_*\colon F_\alpha F_t\ F_u F_\beta \Rightarrow F_v F_\beta\ F_\alpha F_s$ is the interchanger for F_β and F_α.

We will import terminology from the fibred-category perspective to refer to these maps as *globally split 2-presheaves.*

6.2.1.1 Globally split 2-presheaf classifiers

For ordinary fibrations in Cat, the corresponding presheaves are pseudofunctors from a 1-category, $A^{\mathsf{op}} \to \mathsf{Cat}$, but could equally well be described as Cat-functors from $\mathcal{A}^{\natural\,\mathsf{op}} \to \mathsf{Cat}$ from the pseudo-functor classifier[5] $\mathcal{A}^\natural$. We will show that there similarly exist classifiers for globally split 2-presheaves. Each of the pseudo-functors on hom-categories are classified by a strict 2-functor $F'_{a,b}\colon \mathcal{A}(b,a)^\natural \to \mathsf{Gray}(Fa, Fb)$, so it seems plausible (and is indeed true) that we can replace the 2-category $\mathcal{A}$ with some sort of 3-dimensional category $\widetilde{\mathcal{A}}$ whose hom-2-categories are the pseudo-functor classifiers of the hom-categories of $\mathcal{A}$ so that maps $\widetilde{\mathcal{A}}^{\mathsf{op}} \to \mathsf{Gray}$ classify presheaves of the sort described above. In this section we demonstrate the $\widetilde{\mathcal{A}}$ can be realised as a Gray-category, such that globally split 2-presheaves are classified by Gray-functors $\widetilde{\mathcal{A}}^{\mathsf{op}} \to \mathsf{Gray}$.

Our construction of $\widetilde{\mathcal{A}}$ will make use of the following maps:

Definition 6.2.1 ($\rho\colon A^\natural \otimes B^\natural \to (A \times B)^\natural$)**.** For 1-categories A and B there is a canonical 2-functor $\rho\colon A^\natural \otimes B^\natural \to (A \times B)^\natural$ which is the identity on objects and which for 1-cells $u\colon a \to a'$ in A and $v\colon b \to b'$ in B maps the generating 1-cells $[u]_b$ and $[v]_a$ to $[(u, 1_b)]$ and $[(1_a, v)]$ respectively. Both the domain and codomain of ρ are free

[3]In [Buc14, Remark 3.3.13] Buckley also comments on the notions of presheaf corresponding to general 2-fibrations.

[4]Note that if a globally split 2-fibration is vertically split, then it is automatically horizontally split as well.

[5]The usual notation for the pseudofunctor classifier of a 2-category $\mathcal{A}$ is $\mathcal{A}'$. We've used $\mathcal{A}^\natural$ to avoid conflict with the many other uses we've made of the $\mathcal{A}'$ notation.

on a graph, so there are no 1-cell relations we must ensure are preserved. The map on 2-cells is provided by the fact that both $A^\natural \otimes B^\natural$ and $(A \times B)^\natural$ admit fully-faithful projections to $A \times B$ which commute with the mapping on 1-cells just described. A 2-cell $\sigma\colon f \to g$ in $A^\natural \otimes B^\natural$ is thus mapped by ρ to the unique lift of its projection in $A \times B$ to a 2-cell between ρf and ρg. ◊

Remark 6.2.2. The 2-functor ρ is in fact a Gray-equivalence, in that there exists a 2-functor $\theta\colon (A \times B)^\natural \to A^\natural \otimes B^\natural$ such that $\theta\rho = 1_{A^\natural \otimes B^\natural}$ and there exists an invertible pseudonatural transformation (in fact, invertible *icon*) $\rho\theta \Rightarrow 1_{(A\times B)^\natural}$. Whereas $A^\natural \otimes B^\natural$ has invertible squares for each pair of morphisms $u\colon a \to a'$, $v\colon b \to b'$, $(A \times B)^\natural$ has a square filled with two invertible triangles:

$$\text{In } A^\natural \otimes B^\natural\colon \quad \begin{tikzcd} (a,b) \arrow[r, "{[u]_b}"] \arrow[d, "{{}_a[v]}"'] & (a',b) \arrow[d, "{{}_{a'}[v]}"] \\ (a,b') \arrow[r, "{[u]_{b'}}"'] \arrow[ur, phantom, "\Nearrow"] & (a',b') \end{tikzcd} \qquad \text{In } (A \times B)^\natural\colon \quad \begin{tikzcd} (a,b) \arrow[r, "{[(u,b)]}"] \arrow[d, "{[a,v]}"'] \arrow[dr, "{[u,v]}"] & (a',b) \arrow[d, "{[a',v]}"] \\ (a,b') \arrow[r, "{[u,b']}"'] & (a',b') \end{tikzcd}$$

Two canonical choices for the pseudo-inverse θ are given by deciding which of the two paths around the square to identity the diagonal with, i.e. whether $[u,v]$ will be mapped to $[u]_{b'}\ {}_a[v]$ or ${}_{a'}[v]\ [u]_b$. ◊

The maps ρ form part of the structure which exhibits $(-)^\natural\colon \mathsf{Cat} \to \mathsf{Gray}$ as a (well-understood) monoidal functor:

Lemma 6.2.3. *The canonical maps $\rho\colon A^\natural \otimes B^\natural \to (A \times B)^\natural$ along with the unique map $!\colon \mathbb{1}^\natural \to \mathbb{1}$ exhibit $(-)^\natural$ as a lax-monoidal 1-functor from $(\mathsf{Cat}, \times)$ to $(\mathsf{2Cat}, \otimes)$.*

Proof. We must demonstrate the maps ρ are 1-natural and preserve unitors and associators in the appropriate sense. For both, it suffices to show that the relevant diagrams commute for the underlying 1-categories. As observed in Definition 6.2.1, the extension of a 1-functor into a 2-category of the form $C^\natural$ is unique if it exists, so two 2-functors with codomain $C^\natural$ will be equal if their underlying 1-functors are equal. For naturality in A and B, assume we have 1-functors $F\colon A \to C$, $G\colon B \to D$ and consider the following square:

$$\begin{tikzcd} A^\natural \otimes B^\natural \arrow[r, "\rho_{A,B}"] \arrow[d, "F^\natural \otimes G^\natural"'] & (A \times B)^\natural \arrow[d, "(F\times G)^\natural"] \\ C^\natural \otimes D^\natural \arrow[r, "\rho_{C,D}"'] & (C \times D)^\natural \end{tikzcd}$$

The square clearly commutes on the generating 1-cells of $A^\natural \otimes B^\natural$ — the image of $[u]_b$ under both maps being $[(Fu, Gb)]$, for example — so we conclude that the above is a commuting square of 2-functors. For associativity and unitality diagrams of the following form commute (where the associators for $\times$ and $\otimes$ have been suppressed):

$$\begin{tikzcd} A^\natural \otimes B^\natural \otimes C^\natural \arrow[r, "\rho_{A,B} \otimes C^\natural"] \arrow[d, "A^\natural \otimes \rho_{B,C}"'] & (A \times B)^\natural \otimes C^\natural \arrow[d, "\rho_{A\times B, C}"] \\ A^\natural \otimes (B \times C)^\natural \arrow[r, "\rho_{A, B\times C}"'] & (A \times B \times C)^\natural \end{tikzcd} \qquad \begin{tikzcd} A^\natural \otimes \mathbb{1}^\natural \arrow[r, "A^\natural \otimes !"] \arrow[d, "\rho_{A,\mathbb{1}}"'] & A^\natural \otimes \mathbb{1} \arrow[d] \\ (A \times \mathbb{1})^\natural \arrow[r] & A^\natural \end{tikzcd}$$

That these diagrams commute is easily verified by considering the images of generating 1-cells. □

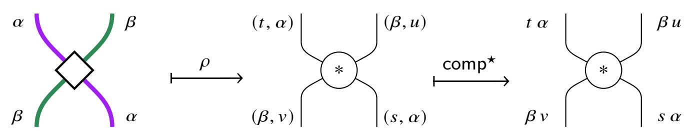
Definition 6.2.4 ($\widetilde{(-)}\colon \mathsf{2Cat} \to \mathsf{GrayCat}$)**.** We let $\widetilde{(-)}\colon \mathsf{2Cat} \to \mathsf{GrayCat}$ denote the canonical change-of-base 2-functor induced by the lax-monoidal functor $(-)^\natural\colon \mathsf{Cat} \to \mathsf{Gray}$. In particular, given a 2-category $\mathcal{A}$, $\widetilde{\mathcal{A}}$ is the

Gray-category with the same objects as $\mathcal{A}$, but with hom-2-category $\widetilde{\mathcal{A}}(a,b)$ given by the pseudo-functor classifier $\mathcal{A}(a,b)^\natural$. The composition and identities are given in terms of the canonical isomorphisms of Definition 6.2.1 as:

$$\mathbb{1} \xrightarrow{i_a{}^\natural} \mathcal{A}(a,a)^\natural \qquad \mathcal{A}(b,c)^\natural \otimes \mathcal{A}(a,b)^\natural \xrightarrow{\rho} (\mathcal{A}(b,c) \times \mathcal{A}(a,b))^\natural \xrightarrow{\mathsf{comp}^\natural} \mathcal{A}(a,c)^\natural \tag{6.5}$$

The coherence of this Gray-categorical structure follows from the general theory of change-of-base functors described in [EK66]. ◊

For a 2-category $\mathcal{A}$ there is a canonical trihomomorphism $\tau\colon \mathcal{A} \to \widetilde{\mathcal{A}}$ of the sort described in the definition of globally split 2-presheaves. The map τ is the identity on objects and on hom-categories is given by the canonical pseudofunctors $\mathcal{A}(a,b) \to \mathcal{A}(a,b)^\natural$. This map strictly preserves whiskering by the definition of horizontal composition in $\widetilde{\mathcal{A}}$: for a diagram $a \underset{v}{\overset{u}{\Downarrow\alpha}} b \xrightarrow{s} c$ in $\mathcal{A}$, the image of $[\alpha]_s \in \mathcal{A}(b,c) \otimes \mathcal{A}(a,b)$ under the composition map is $[\alpha]_s \overset{\rho}{\mapsto} [(s,\alpha)] \xmapsto{\mathsf{comp}^\natural} [s\,\alpha]$. Similarly, for a diagram of the form $a \underset{v}{\overset{u}{\Downarrow\alpha}} b \underset{t}{\overset{s}{\Downarrow\beta}} c$, we observe that the interchanger for $[\alpha]$ and $[\beta]$ in $\widetilde{\mathcal{A}}$ — the image of the corresponding ◇ in $\mathcal{A}(b,c)^\natural \otimes \mathcal{A}(a,b)^\natural$ — is given by the appropriate pseudo-functoriality 2-cell for $\mathcal{A}(a,c) \to \mathcal{A}(a,c)^\natural$ (denoted ⊛):



(6.6)

It follows that from any Gray-functor $\widetilde{\mathcal{A}}^{\mathsf{op}} \to \mathsf{Gray}$ one obtains a globally split 2-presheaf by precomposition with $\tau^{\mathsf{op}}\colon \mathcal{A}^{\mathsf{op}} \to \widetilde{\mathcal{A}}^{\mathsf{op}}$. Conversely, a globally split 2-presheaf determines a mapping on the generating 1-cells and 2-cells of $\widetilde{\mathcal{A}}^{\mathsf{op}}$ to Gray, and the preservation of whiskering and interchanger conditions correspond to the Gray-functoriality of the induced map. We conclude:

Lemma 6.2.5. *For $\mathcal{A}$ a 2-category, precomposition by $\tau^{\mathsf{op}}\colon \mathcal{A}^{\mathsf{op}} \to \widetilde{\mathcal{A}}^{\mathsf{op}}$ induces a bijection between $\left[\widetilde{\mathcal{A}}^{\mathsf{op}}, \mathsf{Gray}\right]$ and the set of globally split 2-presheaves on $\mathcal{A}$.*

6.2.2 The Grothendieck construction for globally split presheaves

We now describe in detail the Grothendieck construction for globally split 2-presheaves. The construction itself doesn't differ in any way from what is described in [Buc14] (aside from some dualisation) though it is important to observe that the structure on globally split 2-presheaves is sufficiently strict for the result to be a genuine 2-category. We will usually declare a globally split 2-fibration explicitly as a Gray-functor of the form $F\colon \widetilde{\mathcal{A}}^{\mathsf{op}} \to \mathsf{Gray}$, though for α a 2-cell in $\mathcal{A}$ we will use the notation F_α, rather than $F_{[\alpha]}$, to refer to the image under F of the corresponding generating 2-cell $[\alpha]$. We will typically refer to all the images under F of the coherence 3-cells in $\widetilde{\mathcal{A}}^{\mathsf{op}}$ as "F_*". Give that these 3-cells are uniquely determined by their domain and codomain, this notation should be unambiguous. So, for example, we might refer to a 3-cell $F_*\colon F_{\alpha\circ\beta} \Rrightarrow F_\alpha \circ F_\beta$ or $F_*\colon F_{1_u} \Rrightarrow 1_{F_u}$.

Construction 6.2.6 (Grothendieck construction for $F\colon \widetilde{\mathcal{A}}^{\mathsf{op}} \to \mathsf{Gray}$)**.** Let $\int F$ be the 2-category with:

0-cells pairs $(a \in \mathcal{A}, x \in Fb)$

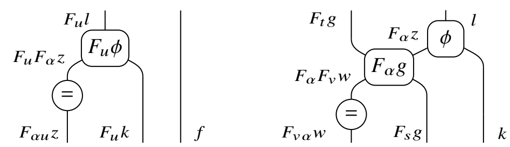
1-cells $(a,x) \to (b,y)$ given by pairs $(u\colon a \to b, f\colon x \to F_u y)$

2-cells $(u,f) \Rightarrow (v,g)$ given by pairs $\alpha\colon u \Rightarrow v$ in $\mathcal{A}$ and $\phi\colon F_\alpha y\, f \Rightarrow g$

Horizontal composition On 1-cells is given in the obvious way: $(v,g)\,(u,f) = (vu, F_u g\, f)$, $1_{(a,x)} = (1_a, 1_x)$. Now, given the following diagram in $\int F$:

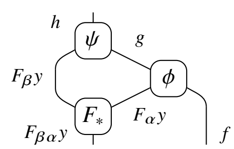
$$\begin{tikzcd}
(a,x) \arrow[r, "{(u,f)}"] & (b,y) \arrow[r, bend left, "{(s,k)}"{name=U}] \arrow[r, bend right, "{(t,l)}"'{name=D}] & (c,z) \arrow[r, "{(v,g)}"] & (d,w)
\arrow[from=U, to=D, phantom, "\Downarrow (\alpha,\phi)"]
\end{tikzcd} \tag{6.7}$$

the whiskering $(\alpha,\phi)\;(u,f)$ is given by $(\alpha\, u, X)$ where X is the 2-cell below on the left, and the whiskering $(v,g)\;(\alpha,\phi)$ is $(v\,\alpha, Y)$, with Y the 2-cell below on the right:



(6.8)

Vertical composition of 2-cells $(\beta,\psi)\circ(\alpha,\phi)$ is the pair formed by $\beta\,\alpha$ and the 2-cell:



(6.9)

◊

Lemma 6.2.7. *For $F\colon \widetilde{\mathscr{A}}^{\mathsf{op}} \to \mathsf{Gray}$, $\int F$ is a well-defined 2-category.*

Proof. We need to show that the whiskering and vertical composition satisfy the following conditions:

(a) Whiskering with a 1-cell preserves vertical composition of 2-cells

(b) Whiskering by a 1-cell twice is the same as whiskering by the composite

(c) pre-whiskering commutes with post-whiskering

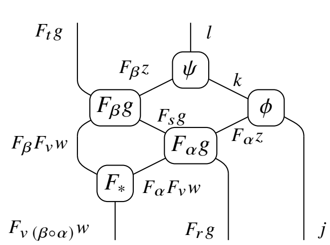
(d) Horizontal composition of 2-cells is uniquely defined in terms of whiskering and vertical composition.

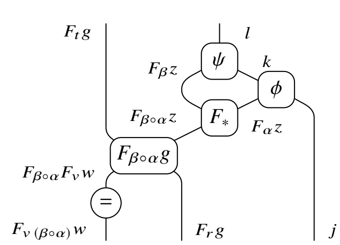
For (a), consider a diagram of the following form:

$$\begin{tikzcd}
(b,y) \arrow[r, bend left=50, "{(r,j)}"] \arrow[r, "{(s,k)}" description] \arrow[r, bend right=50, "{(t,l)}"'] & (c,z) \arrow[r, "{(v,g)}"] & (d,w)
\end{tikzcd} \tag{6.10}$$

with 2-cells $\Downarrow(\alpha,\phi)$ in the upper region and $\Downarrow(\beta,\psi)$ in the lower region.

Clearly the first component of both $((v,g)\,(\beta,\psi))\circ((v,g)\,(\alpha,\phi))$ and $(v,g)\,((\beta,\psi)\circ(\alpha,\phi))$ will be $v\,(\alpha\circ\beta)$. The second components will be, respectively:



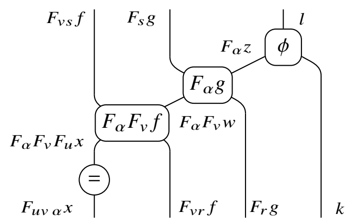


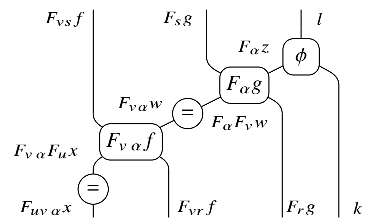
These are equal by the fact that $F_* \colon F_{\beta\circ\alpha} \Rrightarrow F_\beta F_\alpha$ defines a modification of pseudofunctors, and can thus be "pulled through" the $F_{\beta\circ\alpha} g$ naturality 2-cell. We omit the proof for the simpler case of pre-whiskering.

For (b) consider a diagram of the form:

$$\begin{tikzcd} (b,y) \arrow[r, bend left=50, "{(r,j)}"] \arrow[r, bend right=50, "{(s,k)}"'] \arrow[r, phantom, "\Downarrow(\alpha,\phi)"] & (c,z) \arrow[r, "{(v,g)}"] & (d,w) \arrow[r, "{(u,f)}"] & (a,x) \end{tikzcd} \tag{6.11}$$

The first components of $((u,f)(v,g))\,(\alpha,\phi)$ and $(u,f)\,((v,g)\,(\alpha,\phi))$ are both $uv\,\alpha$. The second components are respectively:



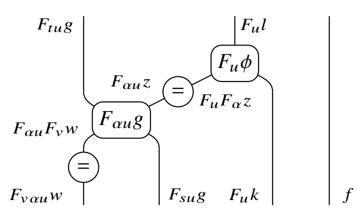


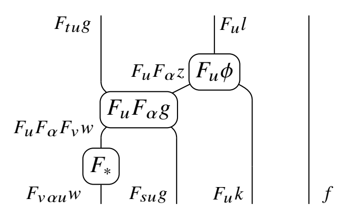
The equality of these 2-cells follows from the fact that $F_{v\alpha} = F_\alpha F_v$. Again, the case of pre-whiskering is much simpler, and therefore omitted.

For (c) the argument is similar. Consider a diagram of the form:

$$\begin{tikzcd} (a,x) \arrow[r, "{(u,f)}"] & (b,y) \arrow[r, bend left=50, "{(s,k)}"] \arrow[r, bend right=50, "{(t,l)}"'] \arrow[r, phantom, "\Downarrow(\alpha,\phi)"] & (c,z) \arrow[r, "{(v,g)}"] & (d,w) \end{tikzcd} \tag{6.12}$$

Both ways of whiskering this diagram clearly agree on the first component, which is $v\,\alpha\,u$. The second components are respectively:



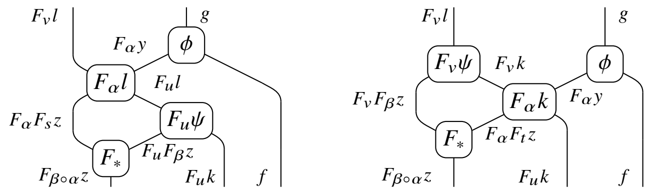


which are equal because $F_{\alpha u} = F_u F_\alpha$.

Finally, for (d) consider a diagram of the form:

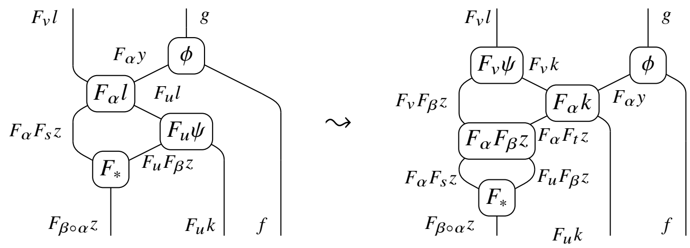
$$\begin{tikzcd} (a,x) \arrow[r, bend left=50, "{(u,f)}"] \arrow[r, bend right=50, "{(v,g)}"'] \arrow[r, phantom, "\Downarrow(\alpha,\phi)"] & (b,y) \arrow[r, bend left=50, "{(s,k)}"] \arrow[r, bend right=50, "{(t,l)}"'] \arrow[r, phantom, "\Downarrow(\beta,\psi)"] & (c,z) \end{tikzcd} \tag{6.13}$$

There are two ways to define the horizontal composition of 2-cells in terms of whiskering and vertical composition as $(t,l)\,(\alpha,\phi) \circ (\beta,\psi)\,(u,f)$ and $(\beta,\psi)\,(v,g) \circ (s,k)\,(\alpha,\phi)$. Both have first component $\alpha\,\beta$, and the second components are given respectively by:



(6.14)

To see that these diagrams represent the same 2-cell, first observe that by the pseudo-naturality of F_α, the 2-cell $F_u\psi$ can be pushed through $F_\alpha l$ as follows:



The 2-cell $F_\alpha F_\beta z$ is the component of the interchanger for F_α and F_β, which we observed earlier (e.g. in (6.6)) is a coherence 3-cell, so is of the form F_* and can be merged with the F_* 2-cell below it, giving the diagram on the right in (6.14). □

Construction 6.2.8 (The canonical 2-functor $|F|\colon \int F \to \mathcal{A}$)**.** Let $|F|\colon \int F \to \mathcal{A}$ be the 2-functor given by sending the 0-cells, 1-cells and 2-cells of $\int F$ to their first components. That is, the image of $(\alpha,\phi)\colon (u,f) \Rightarrow (v,g)\colon (a,x) \to (b,y)$ in $\int F$ under $|F|$ is $\alpha\colon u \Rightarrow v\colon a \to b$ in $\mathcal{A}$. We will occasionally refer to both $\int F$ and $|F|$ as "the Grothendieck construction" of F where this can be disambiguated by context. ◊

We now demonstrate that "globally split 2-presheaves" do, in fact, correspond to globally split 2-fibrations under this Grothendieck construction.

Lemma 6.2.9. *For $F\colon \widetilde{\mathcal{A}}^{\mathsf{op}} \to \mathsf{Gray}$ a Gray-functor, the map $|F|\colon \int F \to \mathcal{A}$ is a globally split 2-fibration.*

Proof. The cartesian liftings of 1-cells are given in the obvious way: for $u\colon a \to b$ and $(b,x) \in \int F$ we have the cartesian 1-cell $(u, 1_{F_u x})\colon (a, F_u x) \to (b,x)$. These liftings strictly preserve composition and identities, since $F_{uv} = F_v F_u$ and $F_{1_a} = 1_{F_a}$. For $\alpha\colon u \Rightarrow v\colon a \to b$ in $\mathcal{A}$ and $(u,f)\colon (a,x) \to (b,y)$ in $\int F$, we lift α to the opcartesian 2-cell $(\alpha, 1_{F_\alpha y\, f})\colon (u,f) \Rightarrow (v, F_\alpha y\, f)$. This lifting is merely pseudo-functorial, since F is pseudo-functorial on hom-categories. The preservation of whiskering under F corresponds to the property that the lifting of 2-cells is preserved under whiskering. That is, for $w\colon b \to c$ the lift of $w\,\alpha$, with domain $(w,g)\,(u,f)$ is the same as the lift of α with domain (u,f) post-whiskered by (w,g). It follows that the opcartesian lifts of 2-cells are, in fact, strong opcartesian, and so these liftings exhibit $|F|$ as a globally split 2-fibration. □

Construction 6.2.10 (The globally split 2-presheaf from a globally split 2-fibration)**.** Given any globally split 2-fibration $P\colon \mathcal{B} \to \mathcal{A}$, one obtains a globally split 2-presheaf $\mathcal{P}\colon \widetilde{\mathcal{A}}^{\mathsf{op}} \to \mathsf{Gray}$ in the expected way:

On 0-cells: $\mathcal{P}$ sends $a \in \mathcal{A}$ to the strict fibre $\mathcal{B}_a$.

On 1-cells for $u\colon a \to b$ in $\mathcal{A}$, the 2-functor $\mathcal{P}_u\colon \mathcal{B}_b \to \mathcal{B}_a$ acts by sending $x \in \mathcal{B}_b$ to $u^*x \in \mathcal{B}_a$. The action on 1-cells and 2-cells and 2-functoriality is determined by the cartesianness of $\bar{u}x\colon u^*x \to x$, for example $\mathcal{P}_u(f)$ for $f\colon x \to y$ is the unique map $f'\colon u^*x \to u^*y$ such that $\bar{u}y\, f' = f\, \bar{u}x$.

On 2-cells for $\alpha\colon u \Rightarrow v\colon a \to b$ in $\mathcal{A}$ the pseudonatural transformation $\mathcal{P}_\alpha\colon \mathcal{P}_u \Rightarrow \mathcal{P}_v$ has component at $x \in \mathcal{B}_b$ given by the unique 1-cell $\mathcal{P}_\alpha x\colon u^*x \to v^*x$ satisfying $\bar{v}x\, \mathcal{P}_\alpha x = \alpha_*(\bar{u}x)$. The pseudo-naturality 2-cell $\mathcal{P}_\alpha f$ for $f\colon x \to y$ is the unique 2-cell with the property that $\bar{v}y\, \mathcal{P}_\alpha f$ is equal to the comparison cell between the two opcartesian 2-cells $f\, \underline{\alpha}(\bar{u}x)$ and $\underline{\alpha}(\bar{u}y)\, \mathcal{P}_u f$ which both lie over $Pf\, \alpha$.

This map is clearly strictly 1-functorial by the fact that the fibration is globally split. The hom-functors are pseudo-functorial by the fact that P is locally an opfibration. The preservation of the cleavage by whiskering with 1-cells ensures that whiskering is strictly preserved by the associated trihomomorphism as required. It follows directly from the definitions of the pseudo-naturality 2-cells for $\mathcal{P}_\alpha$ that interchangers are given by the coherence 2-cells for the pseudo-functors on hom-categories (i.e. the opcartesianness of the lifting of 2-cells). ◊

We don't bother to define a Gray-categorical structure over globally split 2-fibrations, but we do note that Constructions 6.2.6 and 6.2.10 are inverse-up-to-isomorphism, in that:

(a) for $P\colon \mathcal{A} \to \mathcal{B}$ a globally split 2-fibration, the Grothendieck construction $\int \mathcal{P}$ will be isomorphic to P

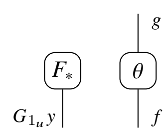
(b) for $F\colon \widetilde{\mathcal{A}}^{\mathsf{op}} \to \mathsf{Gray}$, the lifting by Construction 6.2.10 of $|F|\colon \int F \to \mathcal{A}$ will be isomorphic to F.

We will find it much more convenient to work directly with the Gray-category of globally split 2-presheaves, though the correspondence with globally split 2-fibrations will be of interest when we consider a notion of Gray-familial functor in Chapter 7.

6.2.3 Gray functoriality

For $\mathcal{A}$ a 2-category, both $\mathsf{GrayCat}\big(\widetilde{\mathcal{A}}^{\mathsf{op}}, \mathsf{Gray}\big)$ and $\mathsf{Gray}/\mathcal{A}$ are Gray-categories, and the map on objects $F \mapsto \int F$ can be extended to a Gray-functor. The details of this extension and proof of its Gray-functoriality are mostly unenlightening, though the existence of such an extension — and its left adjoint — will be relevant in Chapter 7. We begin with a definition and technical lemma to which we will frequently appeal in the proof of Gray-functoriality.

Definition 6.2.11 (Pure 1-cells and 2-cells)**.** Let $F\colon \widetilde{\mathcal{A}}^{\mathsf{op}} \to \mathsf{Gray}$. We say a 1-cell in $\int F$ of the form $(1_a, f)\colon (a, x) \to (a, y)$ is *pure*, and denote it $\langle f \rangle$. Similarly, a 2-cell $(u, f) \Rightarrow (u, g)$ whose first component is 1_u and whose second component is of the form:



is said to be pure, and is denoted $\langle \theta \rangle$. ◊

Lemma 6.2.12. *Pure 1-cells and 2-cells in $\int F$ satisfy the following composition rules (where 1-cells and 2-cells are indicated by Latin and Greek letters respectively):*

(a) $(u, f)\ \langle g \rangle = (u, fg)$

(b) $\langle g \rangle\ (u, f) = (u, F_u g\, f)$

(c) $\langle f \rangle\ \langle g \rangle = \langle fg \rangle$

(d) $(v, f)\ \langle \phi \rangle = \langle F_u f\, \phi \rangle$ *for* $\phi\colon g \Rightarrow h\colon x \to F_u y$

(e) $\langle \phi \rangle\ (u, f) = \langle F_u \phi\, f \rangle$ *for*

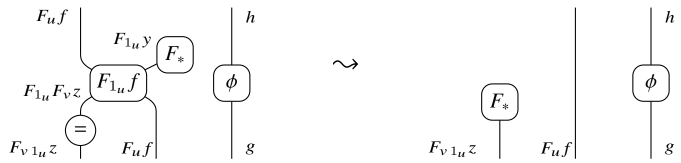
(f) $(\alpha, \phi) \circ \langle \psi \rangle = (\alpha, F_\alpha y \circ \psi)$

(g) $\langle \phi \rangle \circ (\alpha, \psi) = (\alpha, \phi \circ \psi)$

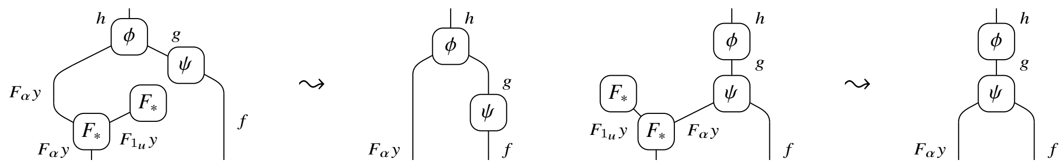
(h) $\langle\phi\rangle \circ \langle\psi\rangle = \langle\phi \circ \psi\rangle$

(i) $\langle\phi\rangle\ \langle\psi\rangle = \langle F_u\phi\,\psi\rangle$ *where* $\phi \colon g \Rightarrow g \colon x \to F_v y$

Proof. Equations (a), (b) and (c) follow immediately from the definition of composition in $\int F$. For (d) we appeal to the definition of post-whiskering given on the right in (6.8) and use the fact that $F_* \colon F_{1_u} \Rrightarrow 1_{Fu}$ is a modification:



Equation (e) follows immediately from the definition of pre-whiskering on the left in (6.8) and the fact that $F_u F_*$ is another coherence 3-cell. (f) and (g) follow from the definition of vertical composition in (6.9:



(h) is merely a special case of (f) or (g) and (i) follows from (d), (e) and (h). □

Construction 6.2.13 (The Grothendieck construction as a Gray functor)**.** We extend $F \mapsto |F|$ to a Gray functor $\int \colon \mathsf{GrayCat}\left(\widetilde{\mathcal{A}}^{\mathsf{op}}, \mathsf{Gray}\right) \to \mathsf{Gray}/\mathcal{A}$ whose action is as follows:

0-cells: $F \mapsto |F|$

1-cells: Given a Gray-transformation $\Gamma \colon F \to G$, we define $\int\Gamma \colon \int F \to \int G$ as the 2-functor with the following action:

0-cells $(a, x) \mapsto (a, \Gamma_a x)$

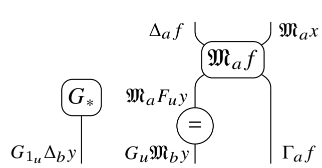
1-cells $(u, f) \colon (a, x) \to (b, y)$ is mapped to $(u, \Gamma_a f) \colon (a, \Gamma_a x) \to (b, \Gamma_b x)$. Note that the codomain of $\Gamma_a f$, $\Gamma_a F_u y$, is equal to $G_u \Gamma_b y$ by the Gray-naturality of Γ and that this mapping preserves composition and identities.

2-cells $(\alpha, \phi) \colon (u, f) \Rightarrow (v, g) \colon (a, x) \to (b, y)$ is mapped $(\alpha, \Gamma_a \phi)$. Note that $G_\alpha \Gamma_b y = \Gamma_a F_\alpha y$, so $\Gamma_a \phi$ has the appropriate domain. The 2-functoriality of each Γ_a and the Gray-functoriality of these components readily demonstrates that the whiskering (6.8) and vertical composition (6.9) of 2-cells is preserved by this mapping.

2-cells: Given a modification $\mathfrak{M} \colon \Gamma \Rrightarrow \Delta$ between Gray-transformations, we define a pseudo-natural transformation $\int\mathfrak{M} \colon \int\Gamma \to \int\Delta$ whose components are as follows:

0-cells: For $(a, x) \in \int F$, let $\int\mathfrak{M}_{(a,x)} \colon (a, \Gamma_a x) \to (a, \Delta_a x)$ be given by the pure 1-cell $\langle\mathfrak{M}_a x\rangle$.

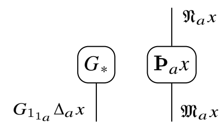
1-cells: For $(u, f) \colon (a, x) \to (b, y)$ in $\int F$, the pseudo-naturality 2-cell $\int\mathfrak{M}_{(u,f)}$ has domain[6] $\langle\mathfrak{M}_b y\rangle\ (u, \Gamma_a f) = (u, G_u \mathfrak{M}_b y\ \Gamma_a f)$ and codomain $(u, \Delta_a f)\ \langle\mathfrak{M}_a x\rangle = (u, \Delta_a f\ \mathfrak{M}_a x)$. Its first component is 1_u, and its second component is given by the following string diagram:



[6] One could just as well define the pseudo-naturality 2-cell as going in the other direction, since it is required to be invertible

That is, $\int \mathfrak{M}_{u,f} = \langle \mathfrak{M}_a f \rangle$. Note that $\mathfrak{M}_a \colon \Gamma_a \to \Delta_a$ is a pseudo-natural transformation, so $\mathfrak{M}_a f$ is invertible, from which it follows that the pseudo-naturality 2-cell $\langle \mathfrak{M}_a f \rangle$ is as well. We defer the proof that $\int \mathfrak{M}$ is a well-defined pseudo-natural transformation to Lemma 6.2.14

3-cells: Given a perturbation $\text{Þ} \colon \mathfrak{M} \to \mathfrak{N} \colon \Gamma \Rrightarrow \Delta \colon F \Rightarrow G$, whose data are modifications $\text{Þ}_a \colon \mathfrak{M}_a \Rrightarrow \mathfrak{N}_a$ for each $a \in \mathcal{A}$, we define a modification $\int \text{Þ} \colon \int \mathfrak{M} \Rrightarrow \int \mathfrak{N}$ by declaring its component at $(a,x) \in \int F$ to be the 2-cell $\langle \text{Þ}_a x \rangle : \langle \mathfrak{M}_a x \rangle \Rightarrow \langle \mathfrak{N}_a x \rangle : (a, \Gamma_a x) \to (a, \Delta_a x)$ which has as its first component 1_{1_a} and as its second component the 2-cell:



The proof that this defines a modification is deferred to Lemma 6.2.15. ◊

Lemma 6.2.14. *$\int \mathfrak{M}$ is a well-defined pseudo-natural transformation.*

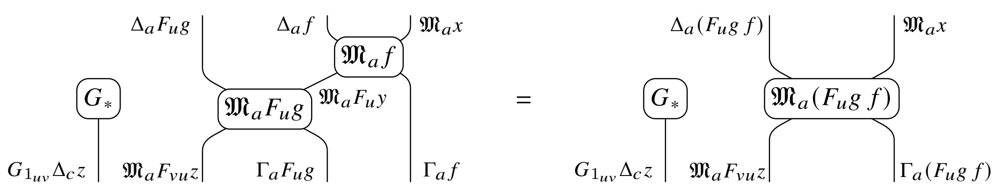
Proof. We need to demonstrate both functoriality of the pseudo-naturality squares and 2-naturality with respect to 2-cells in $\int F$. First, consider a composable pair of 1-cells in $\int F$:

$$(a,x) \xrightarrow{(u,f)} (b,y) \xrightarrow{(v,g)} (c,z)$$

If we paste the pseudo-naturality 2-cells for $\int \mathfrak{M}$ according to the following diagram:

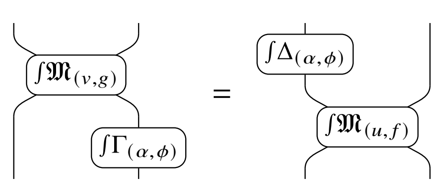
$$\begin{tikzcd}
(a, \Gamma_a x) \arrow[r, "{(u, \Gamma_a f)}"] \arrow[d, "\langle \mathfrak{M}_a x \rangle"'] & (b, \Gamma_b y) \arrow[r, "{(v, \Gamma_b g)}"] \arrow[d, dashed, "\langle \mathfrak{M}_b y \rangle"] \arrow[dl, phantom, "\overset{\langle \mathfrak{M}_a f \rangle}{\Longleftarrow}"] & (c, \Gamma_c z) \arrow[d, "\langle \mathfrak{M}_c z \rangle"] \arrow[dl, phantom, "\overset{\langle \mathfrak{M}_b g \rangle}{\Longleftarrow}"] \\
(a, \Delta_a x) \arrow[r, "{(u, \Delta_a f)}"'] & (b, \Delta_b y) \arrow[r, "{(v, \Delta_b g)}"'] & (c, \Delta_c z)
\end{tikzcd}$$

By repeated use of Lemma 6.2.12 we find that the pasting of these pure 2-cells is given by the pure 2-cell represented below on the left, which is equal to $\int \mathfrak{M}_{(v, \Gamma_b g)\,(u, \Gamma_a f)} = \langle \mathfrak{M}_a (F_u g) \rangle$ shown on the right by the pseudo-naturality of $\mathfrak{M}_a \colon \Gamma_a \Rightarrow \Delta_a$.

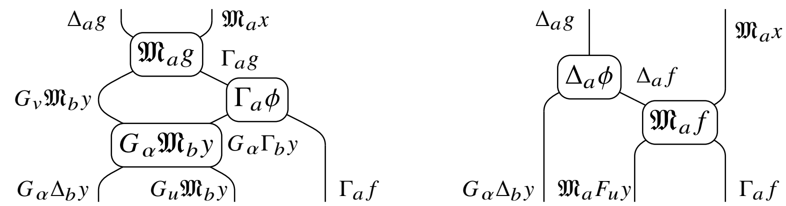


The case for preservation of identities proceeds similarly and is omitted.

For the naturality of $\int \mathfrak{M}$ with respect to 2-cells, consider a 2-cell $(\alpha, \phi) \colon (u,f) \Rightarrow (v,g) \colon (a,x) \to (b,y)$ in $\int F$. We need to show that the following 2-cells are equal in $\int G$:

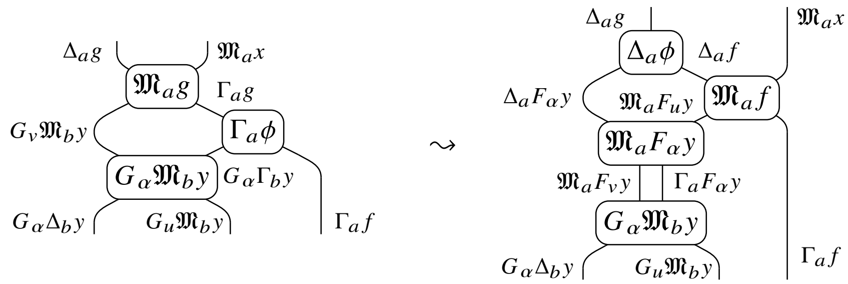


Observing that $\int \mathfrak{M}_{(v,g)} = \langle \mathfrak{M}_a g \rangle$ and $\int \mathfrak{M}_{(u,f)} = \langle \mathfrak{M}_a f \rangle$ we can use Lemma 6.2.12 to conclude that the first component of both 2-cells is α and the second components are respectively:



(6.15)

Showing that these are equal requires a bit more thought than the previous examples. Starting with the diagram on the left of (6.15), we can pull the $\Gamma_a\phi$ 2-cell through $\mathfrak{M}_a g$ by the fact that $\mathfrak{M}_a$ is a pseudo-natural transformation from Γ_a to Δ_a:

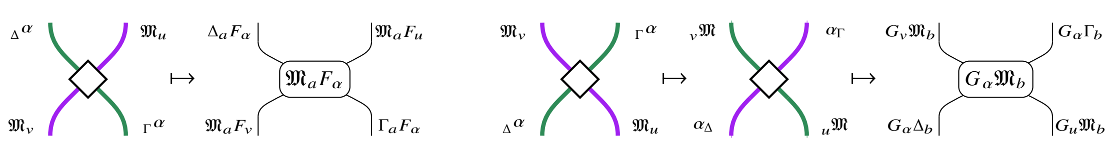


(6.16)

The diagram on the right of (6.16) is then equivalent to the diagram on the right of (6.15) because $\mathfrak{M}_a F_\alpha$ is inverse to $G_\alpha \mathfrak{M}_b$ by the naturality of $\mathfrak{M}$. This naturality property of Gray-modifications is perhaps not obvious, so we pause to consider why it is true. The 2-category $\mathsf{Gray}^{\widetilde{\mathcal{A}}^{\mathsf{op}}}(F,G)$ of Gray-natural transformations, Gray-modifications and Gray-perturbations is the end in Gray of $[F-,G-]\colon \mathcal{A}^{\mathsf{op}} \otimes \mathcal{A} \to \mathsf{Gray}$, with projections $\mathsf{E}_a\colon \mathsf{Gray}^{\widetilde{\mathcal{A}}^{\mathsf{op}}}(F,G) \to \mathsf{Gray}(Fa,Ga)$ which send a transformation, modification or perturbation to its component at a. The projections must form a wedge, meaning that for a and b in $\mathcal{A}$, the following diagram commutes:

$$\begin{tikzcd}
\mathsf{Gray}^{\widetilde{\mathcal{A}}^{\mathsf{op}}}(F,G) \otimes \mathcal{A}(a,b) \arrow[rr, "\mathsf{E}_a \otimes F_{a,b}"] \arrow[d, "\tau"'] & & \mathsf{Gray}(Fa,Ga) \otimes \mathsf{Gray}(Fb,Fa) \arrow[d, "\mathsf{comp}"] \\
\mathcal{A}(a,b) \otimes \mathsf{Gray}^{\widetilde{\mathcal{A}}^{\mathsf{op}}}(F,G) \arrow[r, "G_{a,b}\otimes \mathsf{E}_b"'] & \mathsf{Gray}(Gb,Ga) \otimes \mathsf{Gray}(Fb,Gb) \arrow[r, "\mathsf{comp}"'] & \mathsf{Gray}(Fb,Ga)
\end{tikzcd} \tag{6.17}$$

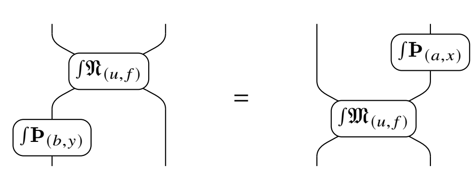
For $\mathfrak{M}\colon \Gamma \Rrightarrow \Delta$ a Gray-modification and $\alpha\colon u \Rightarrow v$ a 2-cell in $\mathcal{A}$, we have an 2-cell $\diamond$ in $\mathsf{Gray}^{\widetilde{\mathcal{A}}^{\mathsf{op}}}(F,G) \otimes \mathcal{A}(a,b)$ which "permutes" the corresponding 1-cells (cf. Section 2.2.7). Our result follows by considering the image of this $\diamond$ and its inverse under the upper and lower maps in the above square respectively:



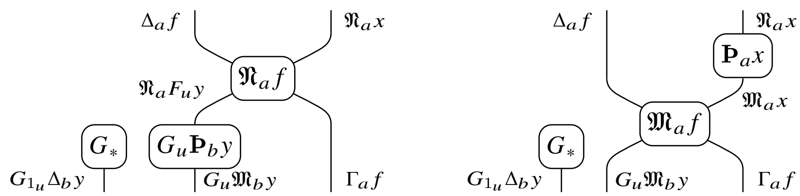
By the commutativity of (6.17) these 2-cells must be inverses. ◻

Lemma 6.2.15. $\int\mathbf{Þ}\colon \int\mathfrak{M} \Rrightarrow \int\mathfrak{N}$ *is a well-defined modification.*

Proof. We need to show that the following equality of 2-cells in $\int G$ holds for each $(u,f)\colon (a,x) \to (b,y)$.



All the 2-cells in these diagrams are pure, so we can apply Lemma 6.2.12 to express these 2-cells respectively as:



(6.18)

We use the naturality of $\mathbf{Þ}$ to replace $G_u \mathbf{Þ}_b y$ with $\mathbf{Þ}_a F_u y$ then observe the two 2-cells are equal by the naturality of the modification $\mathbf{Þ}_a : \mathfrak{M}_a \Rrightarrow \mathfrak{N}_a$. □

Lemma 6.2.16. *The map* $\int : \mathsf{GrayCat}\left(\widetilde{\mathcal{A}}^{\mathsf{op}}, \mathsf{Gray}\right) \to \mathsf{Gray}/\mathcal{A}$ *is* Gray*-functorial.*

Proof. This follows from many repeated elementary applications of Lemma 6.2.12, of which we describe only a few in detail. First, consider for 2-functors $F, G : \widetilde{\mathcal{A}}^{\mathsf{op}} \to \mathsf{Gray}$ the map on hom-categories $[F, G] \to [\int F, \int G]$. We will use $\diamond$ to denote the vertical composition of 2-cells in these 2-categories, and $\circ$ to denote horizontal composition. The 2-functoriality of these maps follows from arguments of the form:

2-cell Vertical composition: For $\mathbf{Ð} : \mathfrak{M} \Rrightarrow \mathfrak{N}$, $\mathbf{Þ} : \mathfrak{N} \Rrightarrow \mathfrak{O}$

$$\int(\mathbf{Þ} \diamond \mathbf{Ð})_{(a,x)} = \langle (\mathbf{Þ} \diamond \mathbf{Ð})_a x \rangle = \langle \mathbf{Þ}_a x \circ \mathbf{Ð}_a x \rangle = \langle \mathbf{Þ}_a x \rangle \circ \langle \mathbf{Ð}_a x \rangle = \int \mathbf{Þ}_{(a,x)} \circ \int \mathbf{Ð}_{(a,x)} = (\int \mathbf{Þ} \diamond \int \mathbf{Ð})_{(a,x)}$$

2-cell Horizontal composition: For $\mathbf{Þ} : \mathfrak{M} \Rrightarrow \mathfrak{N} : \Gamma \Rightarrow \Delta$, $\mathbf{Ð} : \mathfrak{O} \Rrightarrow \mathfrak{I} : \Delta \Rightarrow \Xi$:

$$\int(\mathbf{Ð} \circ \mathbf{Þ})_{a,x} = \langle \mathbf{Ð}_a x \, \mathbf{Ð}_a x \rangle = \left\langle F_{1_a} \mathbf{Ð}_a x \right\rangle \, \langle \mathbf{Þ}_a x \rangle = \langle \mathbf{Ð}_a x \rangle \, \langle \mathbf{Þ}_a x \rangle = (\int \mathbf{Ð} \circ \int \mathbf{Þ})_{a,x}$$

1-cell composition For $\mathfrak{N} : \Gamma \Rightarrow \Delta$, $\mathfrak{M} : \Delta \Rightarrow \Xi$ and $(u, f) : (a, x) \to (b, y)$ in $\int F$ we have:

$$\int(\mathfrak{M} \circ \mathfrak{N})_{(a,x)} = \langle (\mathfrak{M} \circ \mathfrak{N})_a x \rangle = \langle \mathfrak{M}_a x \, \mathfrak{N}_a x \rangle = \langle \mathfrak{M}_a x \rangle \, \langle \mathfrak{N}_a x \rangle = (\int \mathfrak{M} \circ \int \mathfrak{N})_{(a,x)}$$
$$\int(\mathfrak{M} \circ \mathfrak{N})_{(u,f)} = \langle (\mathfrak{M} \circ \mathfrak{N})_a f \rangle = \langle \mathfrak{M}_a f \, \mathfrak{N}_a x \circ \mathfrak{M}_a x \, \mathfrak{N}_a f \rangle = \langle \mathfrak{M}_a f \rangle \, \langle \mathfrak{N}_a x \rangle \circ \langle \mathfrak{M}_a x \rangle \, \langle \mathfrak{N}_a f \rangle = (\int \mathfrak{M} \circ \int \mathfrak{N})_{(u,f)}$$

To see that whiskering by Gray-natural transformations in $\mathsf{GrayCat}\widetilde{\mathcal{A}}^{\mathsf{op}}, \mathsf{Gray}$ is preserved, consider a perturbation $\mathbf{Þ} : \mathfrak{M} \to \mathfrak{N} : \Gamma \Rrightarrow \Delta : F \Rightarrow G$, a Gray-natural transformation $\Xi : G \Rightarrow H$ and a 2-cell $(\alpha, \phi) : (u, f) \Rightarrow (v, g) : (a, x) \to (b, y)$ in $\int F$. We observe the following:

$$\int(\Xi \, \Gamma)_{(\alpha,\phi)} = (\alpha, (\Xi \, \Gamma)_a \phi) = (\alpha, \Xi_a \Gamma_a \phi) = \int \Xi_{(\alpha, \Gamma_a \phi)} = (\int \Xi \, \int \Gamma)_{(\alpha,\phi)}$$
$$\int(\Xi \, \mathfrak{M})_{(u,f)} = \langle (\Xi \, \mathfrak{M})_a f \rangle = \langle \Xi_a \, \mathfrak{M}_a f \rangle = \int \Xi(\langle \mathfrak{M}_a f \rangle) = (\int \Xi \, \int \mathfrak{M})_{(u,f)}$$
$$\int(\Xi \, \mathbf{Þ})_{(a,x)} = \langle (\Xi \, \mathbf{Þ})_a x \rangle = \langle \Xi_a (\mathbf{Þ}_a x) \rangle = \int \Xi \, \langle \mathbf{Þ}_a x \rangle = (\int \Xi \, \int \mathbf{Þ})_{(a,x)}$$

Finally, we observe that interchangers are preserved. For $\mathfrak{M} : \Gamma \Rrightarrow \Delta : F \Rightarrow G$ and $\mathfrak{N} : \Xi \Rrightarrow \Lambda : G \Rightarrow H$ we have:

$$\left(\int \mathfrak{N}_{\int \mathfrak{M}}\right)_{(a,x)} = (\int \mathfrak{N})_{\langle \mathfrak{M}_a x \rangle} = \langle \mathfrak{N}_a (\mathfrak{M}_a x) \rangle = \left\langle (\mathfrak{N}_a)_{\mathfrak{M}_a} x \right\rangle = \langle (\mathfrak{N}_{\mathfrak{M}})_a x \rangle = (\int \mathfrak{N}_{\mathfrak{M}})_{(a,x)}$$ □

Lemma 6.2.17. *For* $\Gamma : F \Rightarrow G : \widetilde{\mathcal{A}}^{\mathsf{op}} \to \mathsf{Gray}$ *a* Gray*-natural transformation, the map* $\int \Gamma : \int F \to \int G$ *of globally-split 2-fibrations preserves chosen-cartesian 1-cells and chosen opcartesian 2-cells.*

Proof. Recall from Lemma 6.2.9 that the chosen-cartesian 1-cells of $\int F$ are those of the form $\left(u, 1_{F_u x}\right): (a, F_u x) \to (b, x)$ for $u: a \to b$ in $\mathcal{A}$ and $x \in F_b$. The image of such a 1-cell under $\int \Gamma$ is:

$$\left(u, \Gamma_a 1_{F_u x}\right) = \left(u, 1_{\Gamma_a F_u x}\right): (a, \Gamma_a F_u x) \to (b, \Gamma_b x)$$

which is clearly chosen-cartesian in $\int G$. Similarly, the chosen-opcartesian 2-cells are of the form:

$$\left(\alpha, 1_{F_\alpha y f}\right): (u, f) \Rightarrow (v, F_\alpha y f)$$

and the image under $\int \Gamma$ of such a 2-cell is the chosen-opcartesian 2-cell $\left(a, 1_{\Gamma_a F_\alpha y f}\right): (u, \Gamma_a f) \Rightarrow (v, \Gamma_a F_\alpha y f)$. $\square$

6.2.4 Free globally split 2-presheaves

We now describe a left adjoint to the Grothendieck construction $F \mapsto \int F$. This Construction is, in fact, essentially the same as the "free split 2-fibration" construction described in Section 3.3.2. The action on an object $P: \mathcal{B} \to \mathcal{A}$ is given by taking the free split 2-fibration for a 2-functor $P: \mathcal{B} \to \mathcal{A}$, identifying the resulting split 2-fibration with a strict presheaf $\mathcal{A}^{\mathsf{op}} \to \mathsf{2Cat}$ and then taking the induced Gray-functor $\widetilde{\mathcal{A}}^{\mathsf{op}} \to \mathsf{Gray}$, though it is more simple to describe the resulting presheaf directly:

Construction 6.2.18 ($\widehat{P}: \widetilde{\mathcal{A}}^{\mathsf{op}} \to \mathsf{Gray}$)**.** For $P: \mathcal{B} \to \mathcal{A}$, let P' denote the strict 2-functor $\mathcal{A}^{\mathsf{op}} \to \mathsf{Gray}$ which sends $a \in \mathcal{A}$ to the lax co-slice $a \Downarrow P$ and maps 1-cells and 2-cells to precomposition and pre-whiskering in the obvious way. Such a 2-functor in particular satisfies the conditions of a globally split 2-presheaf (with the action on hom-categories given by the "pseudo"-functors $\mathcal{A}(b, a) \xrightarrow{P'_{a,b}} \mathsf{2Cat}(a \Downarrow P, b \Downarrow P) \hookrightarrow \mathsf{Gray}(a \Downarrow P, b \Downarrow P)$) and so induces a Gray-functor $\widehat{P}: \widetilde{\mathcal{A}}^{\mathsf{op}} \to \mathsf{Gray}$. $\lozenge$

Rather than extending the map $P \mapsto \widehat{P}$ to a Gray-functor, we will demonstrate that $\widehat{P}$ co-represents the Gray-functor $\mathsf{Gray}/\mathcal{A}(P, |-|): [\widetilde{\mathcal{A}}^{\mathsf{op}}, \mathsf{Gray}] \to \mathsf{Gray}$, and that this isomorphism is Gray-natural. The mapping $P \mapsto \widehat{P}$ will then inherit Gray-functorial structure from these representations in a way that makes $P \mapsto \widehat{P}$ left Gray-adjoint to the Grothendieck construction.

Construction 6.2.19 ($\Phi: \left[\widetilde{\mathcal{A}}^{\mathsf{op}}, \mathsf{Gray}\right]\left(\widehat{P}, F\right) \cong \mathsf{Gray}/\mathcal{A}(P, |F|)$)**.** Let $P: \mathcal{B} \to \mathcal{A}$ and $\widehat{P}: \widetilde{\mathcal{A}}^{\mathsf{op}} \to \mathsf{Gray}$ be as in Construction 6.2.18. We establish some notation:

(a) For $x \in \mathcal{B}$, let $[\![x]\!] = (x, 1_{Px}) \in Px \Downarrow P$

(b) For $s: x \to y \in \mathcal{B}$, let $[\![s]\!] = (s, 1_{Ps}): [\![x]\!] \to (y, Ps)$ in $Px \Downarrow P$ (shown on the left below)

(c) For $\alpha: s \Rightarrow t: x \to y$, let $[\![\alpha]\!] = \alpha: (s, P\alpha) \Rightarrow [\![t]\!]: [\![x]\!] \to (y, Pt)$ in $Px \Downarrow P$ (shown on the right below)

$$\begin{tikzcd}
 & Px \arrow[d, "Ps"] \\
Px \arrow[ur, equal] \arrow[r, phantom, "\circlearrowleft"] \arrow[r, "Ps"'] & Py
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Px \arrow[d, "Ps"] \arrow[d, bend left=60, "Pt"] \arrow[d, phantom, shift left=4, "\overset{P\alpha}{\Rightarrow}"] \\
Px \arrow[ur, equal] \arrow[r, "Pt"'] \arrow[r, phantom, shift left=3, "\overset{P\alpha}{\Leftarrow}"] & Py
\end{tikzcd}$$

These data form the "generic" 0-cells, 1-cells and 2-cells of $\widehat{P}$ in the sense that:

(a) $(x, u) = \widehat{P}_u[\![x]\!] \in a \Downarrow P$

(b) $(s, \alpha) = \widehat{P}_\alpha[\![y]\!]\ \widehat{P}_u[\![s]\!]: (x, u) \to (y, v)$.

(c) $\sigma = \widehat{P}_\beta[\![y]\!]\ \widehat{P}_u[\![\sigma]\!]: (s, \alpha) \Rightarrow (t, \beta): (x, u) \to (y, v)$

Now consider a Gray-natural transformation $\Gamma\colon \widehat{P} \Rightarrow F$, whose data are a collection of 2-functors $\Gamma_a\colon a \Downarrow P \to F_a$. The Gray-naturality means that these 2-functors are determined by their action on the "generic" data. For example:

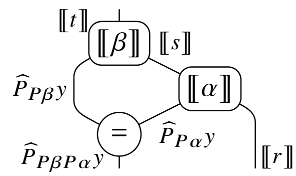
$$\Gamma_a(x,u) = F_u\Gamma_{Px}[\![x]\!] \qquad \Gamma_a(s,\alpha) = \Gamma_a\Big(\widehat{P}_\alpha[\![y]\!]\ \widehat{P}_u[\![s]\!]\Big) = F_\alpha\Gamma_{Py}[\![y]\!]\ F_u\Gamma_{Px}[\![s]\!] \tag{6.19}$$

We will use the shorthand γ_X for $\Gamma_{Px}[\![X]\!]$ for X a 0-cell, 1-cell or 2-cell in $\mathcal{B}$, and x the 0-cell domain of X.

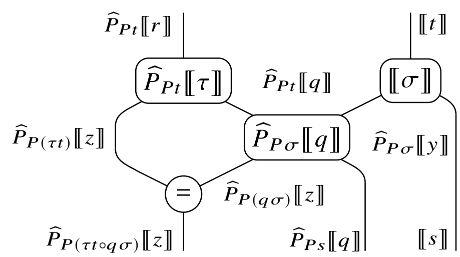
We then obtain a map $\overline{\Gamma}\colon P \to |F|$ sending n-cell X in $\mathcal{B}$ to $(PX, \gamma_X) \in \int F$. This is a well-defined 2-functor because generic data satisfy the following composition laws which mirror those of $\int F$:

(a) $[\![ts]\!] = \widehat{P}_{Ps}[\![t]\!]\ [\![s]\!]$

(b) For $\alpha\colon r \Rightarrow s$, $\beta\colon s \Rightarrow t$ in $\mathcal{B}$, vertical composition $[\![\beta \circ \alpha]\!]$ is given by:

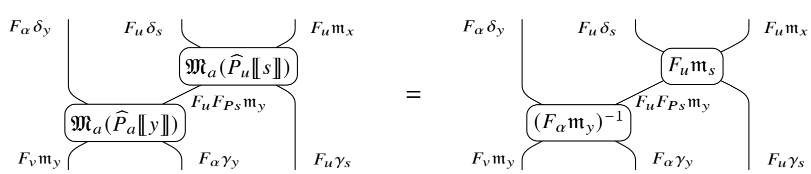


(c) for $\alpha : s \Rightarrow t$, $\beta\colon q \Rightarrow r$, the horizontal composition $[\![\beta\,\alpha]\!]$ is given by:



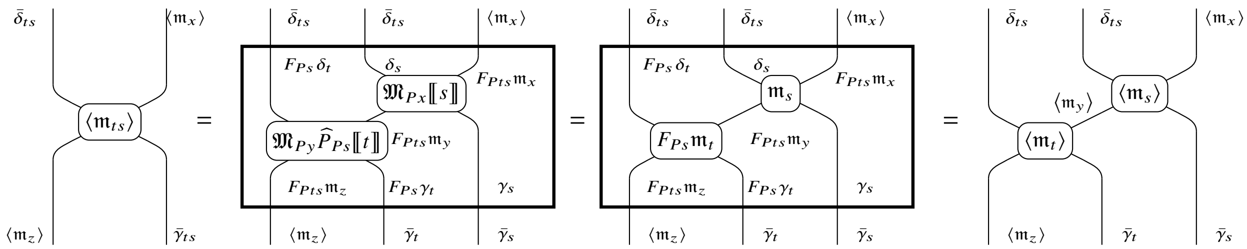
Conversely, any functor $H\colon \mathcal{B} \to \int F$ which is a map in the slice Gray$/\mathcal{A}$ from P to $|F|$ must be of the form $X \mapsto (PX, h_X)$ for X an n-cell in $\mathcal{B}$. One can recover a Gray-natural transformation $\mathfrak{h}\colon \widehat{P} \Rightarrow F$ by declaring $\mathfrak{h}_{Px}[\![X]\!] = h_X$ for x the 0-cell domain of X, and extending the definition of $\mathfrak{h}$ to arbitrary n-cells in each $a \Downarrow P$ according to (6.19). This is well-defined by the symmetry between the definitions of composition for generic data and for n-cells in $\int F$.

For a modification $\mathfrak{M}\colon \Gamma \Rrightarrow \Delta$, let $\mathfrak{m}_x := \mathfrak{M}_{Px}[\![x]\!]$ and let $\mathfrak{m}_s := \mathfrak{M}_{Px}[\![s]\!]$ for $s\colon x \to y$ in $\mathcal{B}$. The modification $\mathfrak{M}$ is completely determined by these data: for $(s,\alpha)\colon (x,u) \to (y,v)$ we have $\mathfrak{M}_a(x,u) = \mathfrak{M}_a\widehat{P}_u[\![x]\!] = F_u\mathfrak{m}_x$ and $\mathfrak{M}_a(s,\alpha) = \mathfrak{M}_a\Big(\widehat{P}_\alpha[\![y]\!]\ \widehat{P}_u[\![s]\!]\Big)$ which is given by:



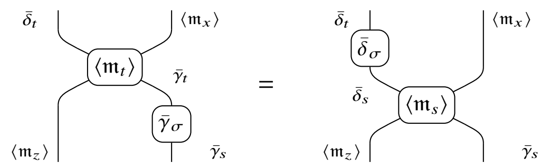
(6.20)

We obtain from $\mathfrak{M}$ a pseudonatural transformation $\overline{\mathfrak{M}}\colon \overline{\Gamma} \Rightarrow \overline{\Delta}$ whose component at object $x \in \mathcal{B}$ is $\langle\mathfrak{m}_x\rangle$ and whose pseudo-naturality 2-cell at $s\colon x \to y$ is given by $\langle\mathfrak{m}_s\rangle : \langle\mathfrak{m}_y\rangle\overline{\Gamma}_s \Rightarrow \overline{\Delta}_s\ \langle\mathfrak{m}_x\rangle$. To demonstrate the coherence condition for composition, observe that $\mathfrak{m}_{ts} = \mathfrak{M}_{Px}[\![ts]\!] = \mathfrak{M}_{Px}\left(\widehat{P}_{Ps}[\![t]\!]\,[\![s]\!]\right)$. Using the coherence of $\mathfrak{M}_{Px}$, we have:

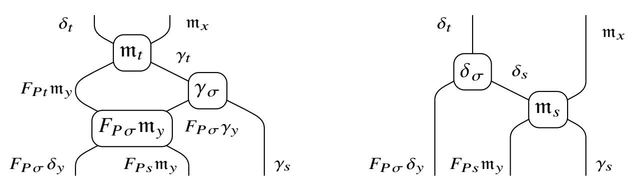


where a diagram contained in a black box represents the pure 2-cell in $\int F$ corresponding to the 2-cell in its interior.

For the 2-naturality, we need to demonstrate the following equalities of 2-cells in $\int F$:



Both 2-cells clearly have $P\sigma$ as their first component. Their second components are (after consolidation of F_* 2-cells):



(6.21)

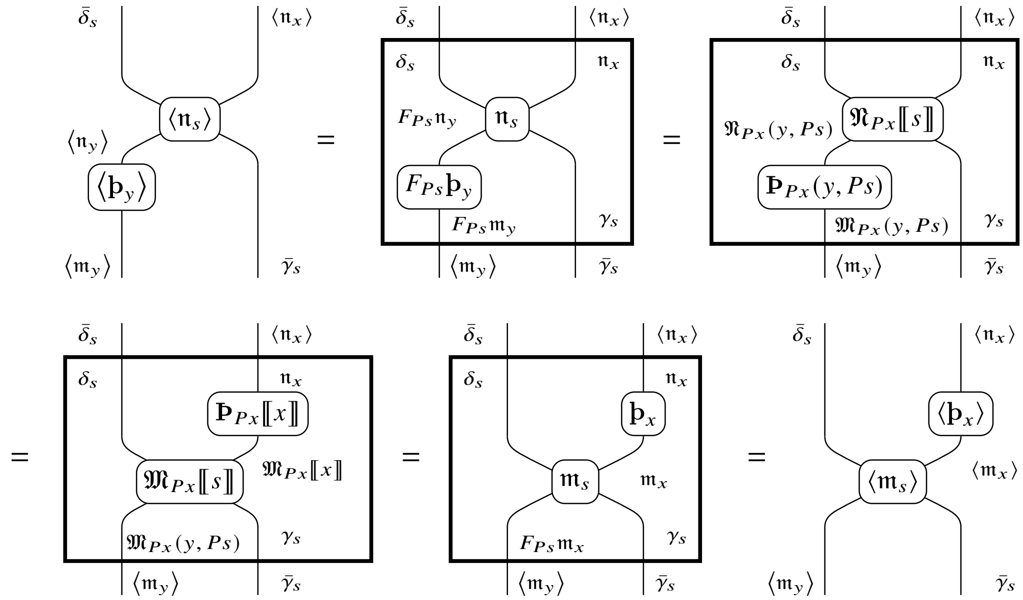
These 2-cells are equal by the same argument used to demonstrate the equality of the 2-cells in (6.15).

It's clear from the definition of $\mathfrak{m}_x$, that for $\mathfrak{M}\colon \Gamma \Rrightarrow \Delta$ and $\mathfrak{N}\colon \Delta \Rrightarrow \Xi$ we have $(\mathfrak{N} \circ \mathfrak{M})_{Px}[\![x]\!] = (\mathfrak{n} \circ \mathfrak{m})_x$. For the pseudo-naturality data at $s\colon x \to y$ in $\mathcal{B}$, we use Lemma 6.2.12 to observe that:

$$\overline{\mathfrak{N} \circ \mathfrak{M}}_s = \langle (\mathfrak{N} \circ \mathfrak{M})_{Px}[\![s]\!] \rangle = \langle \mathfrak{n}_x\, \mathfrak{m}_x \circ F_{Ps}\mathfrak{n}_x \mathfrak{m}_s \rangle = \langle \mathfrak{n}_s \rangle \langle \mathfrak{m}_x \rangle \circ \langle \mathfrak{n}_x \rangle \langle \mathfrak{m}_s \rangle = \left(\overline{\mathfrak{N}} \circ \overline{\mathfrak{M}}\right)_s$$

From a 1-cell $Z\colon \overline{\Gamma} \Rightarrow \overline{\Delta}$ in $\mathsf{Gray}/\mathcal{A}(P, \int F)$, which is necessarily a pseudonatural transformation whose component Z_x at $x \in \mathcal{B}$ is of the form $(1_{Px}, z_x)$ and whose pseudo-naturality 2-cells Z_s are of the form $(1_{Ps}, z_s)$ we can reconstruct the data of a modification $\mathfrak{Z}\colon \Gamma \Rrightarrow \Delta$ as described above. The result will be a well-defined modification by the pseudo-naturality of Z. Moreover, this reconstruction is clearly invertible, given that both the modification and the pseudonatural transformation are determined by the "z_X" data, which is preserved under both maps.

Finally, for a perturbation $\mathbf{Þ}\colon \mathfrak{M} \to \mathfrak{N}\colon \Gamma \Rrightarrow \Delta$, let $\mathfrak{þ}_x := \mathbf{Þ}_{Px}[\![x]\!]$, noting that $\mathbf{Þ}_a(x,u) = \mathbf{Þ}_a\widehat{P}_u[\![x]\!] = F_u\mathfrak{þ}_x$ by naturality. Then define a modification $\overline{\mathbf{Þ}}\colon \overline{\mathfrak{M}} \Rrightarrow \overline{\mathfrak{N}}$ whose component at $x \in \mathcal{B}$ is $\langle \mathfrak{þ}_x \rangle$. This is a well-defined modification due to the way pure 2-cells compose in $\int F$:



For vertical composition of perturbations $\mathbf{Þ}\colon \mathfrak{M} \to \mathfrak{N}$, $\mathbf{Ð}\colon \mathfrak{N} \to \mathfrak{O}$, we clearly have $(\mathbf{Ð} \circ \mathbf{Þ})_{Px}[\![x]\!] = \boldsymbol{\eth}_x \circ \mathfrak{p}_x$ (where $\boldsymbol{\eth}_x \coloneqq \mathbf{Ð}_{Px}[\![x]\!]$) and so $\overline{(\mathbf{Ð} \circ \mathbf{Þ})}_x = \langle \boldsymbol{\eth}_x \circ \mathfrak{p}_x \rangle = \langle \boldsymbol{\eth}_x \rangle \circ \langle \mathfrak{p}_x \rangle = \overline{\mathbf{Ð}} \circ \overline{\mathbf{Þ}}$. A similar argument shows that pre-whiskering and post-whiskering by modifications is preserved by the map $\mathbf{Þ} \mapsto \langle \mathfrak{p} \rangle$. Because $\mathbf{Þ}$ can be reconstructed from the $\mathfrak{p}_x$ data, which is equivalently the data of a vertical modification $\overline{\mathfrak{M}} \Rrightarrow \overline{\mathfrak{N}}$, the mapping $\mathbf{Þ} \mapsto \langle \mathfrak{p} \rangle$ is invertible. The naturality conditions for a vertical modification $\overline{\mathfrak{M}} \Rrightarrow \overline{\mathfrak{N}}$ coincide with the naturality conditions for the induced perturbation $\mathfrak{M} \to \mathfrak{N}$. ◊

Lemma 6.2.20. *For any $P\colon \mathcal{B} \to \mathcal{A}$ and $F\colon \widetilde{\mathcal{A}}^{\mathsf{op}} \to \mathsf{Gray}$, the isomorphism $\Phi\colon \left[\widetilde{\mathcal{A}}^{\mathsf{op}}, \mathsf{Gray}\right]\left(\widehat{P}, F\right) \cong \mathsf{Gray}/\mathcal{A}(P, |F|)$ is Gray-natural in F.*

Proof. For 1-naturality in F, assume $\mathbf{Þ}\colon \mathfrak{M} \to \mathfrak{N}\colon \Gamma \Rrightarrow\colon \Delta\colon \widehat{P} \Rightarrow F$ and $\Xi\colon F \Rightarrow G$. Then observe:

$$\int\Xi\, \overline{\Gamma}_X = \int\Xi_{(PX,\gamma_X)} = (PX, \Xi_{Px}\ \gamma_X) = (PX, \Xi_{Px}\ \Gamma_{Px}[\![X]\!]) = \overline{\Xi\, \Gamma}_X \qquad \text{(for } X \text{ any 0/1/2-cell with 0-domain } x\text{)}$$
$$\int\Xi\, \overline{\mathfrak{M}}_X = \int\Xi_{\langle \mathfrak{m}_X \rangle} = \langle \Xi_{Px}\ \mathfrak{m}_X \rangle = \langle \Xi_{Px}\ \mathfrak{M}_{Px}[\![X]\!] \rangle = \overline{\Xi\, \mathfrak{M}}_X \qquad \text{(for } X \text{ any 0/1-cell with 0-domain } x\text{)}$$
$$\int\Xi\, \overline{\mathbf{Þ}}_x = \int\Xi_{\langle \mathfrak{p}_x \rangle} = \langle \Xi_{Px}\ \mathfrak{p}_x \rangle = \langle \Xi_{Px}\ \mathbf{Þ}_{Px}[\![x]\!] \rangle = \overline{\Xi\, \mathbf{Þ}}_x \qquad \text{(for } x \text{ any 0-cell)}$$

Now assume there exists some other $\Lambda\colon F \Rightarrow G$ and $\mathfrak{J}\colon \Xi \Rrightarrow \Lambda$. We demonstrate naturality on 2-cells by observing that for $s\colon x \to y$ in $\mathcal{B}$ we have:

$$\int\mathfrak{J}\, \overline{\Gamma}_x = \int\mathfrak{J}_{(Px,\gamma_x)} = \langle \mathfrak{J}_{Px}\gamma_x \rangle = \langle \mathfrak{J}_{Px}\Gamma_{Px}[\![x]\!] \rangle = \overline{\mathfrak{J}\, \Gamma}_x$$
$$\int\mathfrak{J}_{\overline{\Gamma}_s} = \int\mathfrak{J}_{(Ps,\gamma_s)} = \langle \mathfrak{J}_{Px}\gamma_s \rangle = \langle \mathfrak{J}_{Px}\Gamma_{Px}[\![s]\!] \rangle = \overline{\mathfrak{J}\Gamma_s}$$
$$\int\mathfrak{J}_{\overline{\mathfrak{M}}_x} = \int\mathfrak{J}_{\langle \mathfrak{m}_x \rangle} = \langle \mathfrak{J}_{Px}\mathfrak{m}_x \rangle = \langle \mathfrak{J}_{Px}\mathfrak{M}_{Px}[\![x]\!] \rangle = \overline{\mathfrak{J}\mathfrak{M}_x}$$

Finally, assuming there also exists $\mathfrak{O}\colon \Xi \Rrightarrow \Lambda$ and $\mathbf{Ð}\colon \mathfrak{J} \to \mathfrak{O}$, we have:

$$\int\mathbf{Ð}\, \overline{\Gamma}_x = \int\mathbf{Ð}_{Px,\gamma_x} = \langle \mathbf{Ð}_{Px}\Gamma_{Px}[\![x]\!] \rangle = \langle (\mathbf{Ð}\ \Gamma)_{Px}[\![x]\!] \rangle = \overline{\mathbf{Ð}\, \Gamma}_x$$ □

Corollary 6.2.21. *The Grothendieck construction* $\int\colon \mathsf{GrayCat}\left(\widetilde{\mathcal{A}}, \mathsf{Gray}\right) \to \mathsf{Gray}/\mathcal{A}$ *has a left* Gray*-adjoint whose action on objects is given by* $P \mapsto \widehat{P}$*.*

Chapter 7

Familial Monads and Submonads

Fam's property of being a familial 2-monad. In this chapter we explore 3-dimensional versions of this notion and consider whether they hold for F_Ω. Observing that they do not, we introduce two sub-monads — $\mathfrak{S}$ and $\mathfrak{P}$ — obtained from F_Ω by considering only strict (resp. pseudo) functors and transformations. These sub-monads *do* satisfy natural analogues of the familial 2-monad property, though they are not free cocompletions.

7.1 Familial 2-monads

The notion of a familial 2-functors was introduced in [Web07] as a 2-dimensional analogue of the notion of parametric right-adjoint functor. First, let's recall the definition of *parametric right-adjoint*:

Definition 7.1.1 (Parametric right-adjoint)**.** When a category A has a terminal object, 1, a functor $T\colon A \to B$ is parametric right-adjoint if the induced functor $T_1\colon A \to B/T1$ is right-adjoint. ◊

Equivalently, T is parametric right-adjoint if T^{op} is familial, in the sense of the previous chapter.

Replacing A and B with 2-categories gives a reasonable definition of "parametric right-2-adjoint"; however, Weber's definition of Familial 2-functor includes an extra property:

Definition 7.1.2 (Familial 2-functor, familial 2-monad — from [Web07])**.** A *familial 2-functor* $K\colon \mathcal{A} \to \mathcal{B}$ is a parametric right-2-adjoint with the additional property that $K_1\colon \mathcal{A} \to \mathcal{B}/K1$ factors through the 2-category $\mathsf{Sp}(K1)$ of split fibrations over $K1$ and split morphisms (i.e. 2-functors which preserve *chosen*-cartesian 1-cells). A *familial 2-monad* is a 2-monad whose underlying 2-functor is a familial 2-functor and whose unit and multiplication are cartesian natural transformations. ◊

Remark 7.1.3. The term 2-monad usually means either Cat-monad or pseudomonad, but the most appropriate meaning for definition for our definition of familial 2-monad is something between to the two. Being in particular a familial 2-functor, the underlying endo-functor of a familial 2-monad must be strict, and the unit and multiplication must be strictly natural for the usual notion of cartesianness to apply[1]. But the definition is well-formed without the requirement that the associativity and identity laws hold strictly. Our use of the term "2-monad" will thus allow for these laws to merely hold up to isomorphism. After all, the eponymous familial 2-monad is $\mathsf{Fam}\colon \mathsf{Cat} \to \mathsf{Cat}$, which doesn't have strict associativity or identities. ◊

[1]Presumably one could define cartesianness of pseudonatural transformations in terms of bi-pullbacks, though we will not.

Remark 7.1.4. A familial 2-monad is automatically a cartesian monad because any parametric right adjoint preserves connected limits, in particular pullbacks. ◊

We can now consider whether $\mathsf{F}_\Omega\colon \mathsf{2Cat} \to \mathsf{2Cat}$ is an instance of some higher analogue of a familial 2-functor. First, let's consider the right adjoint property.

$$\begin{tikzcd} \mathsf{2CAT} \arrow[r, "\mathsf{F}_\Omega"] \arrow[dr, "(\mathsf{F}_\Omega)_1"'] \arrow[r, phantom, shift right=4ex, "\circlearrowright"] & \mathsf{2CAT} \\ & \mathsf{2CAT/Cat} \arrow[u, "\pi"'] \end{tikzcd} \tag{7.1}$$

A left adjoint L to $(\mathsf{F}_\Omega)_1$ would send a 2-functor $D\colon \mathcal{A} \to \mathsf{Cat}$ to some 2-category LD such that there is an isomorphism $\phi_{D,\mathcal{B}}\colon \mathsf{2CAT}(LD, \mathcal{B}) \cong \mathsf{2CAT/Cat}(D, \varpi_\mathcal{B})$ where $\varpi_\mathcal{B}\colon \mathsf{F}_\Omega\mathcal{B} \to \mathsf{Cat}$ is the canonical projection. It seems appropriate to ask that the isomorphism ϕ be a 2-natural isomorphism of 2-categories, so that L becomes a 3-adjunction. In fact, we have already shown that a left 1-adjoint $L\colon \mathsf{2CAT/Cat}$ exists on the underlying 1-categories, recall the following proposition from Chapter 4:

Proposition 4.3.8. *The map* $\mathsf{F}_\Omega/\mathbb{1}\colon \mathsf{2CAT} \to \mathsf{2CAT/Cat}$ *which sends* $\mathcal{K}$ *to* $\varpi_\mathcal{K}\colon \mathsf{F}_\Omega\mathcal{K} \to \mathsf{Cat}$ *has a left 1-adjoint which sends* $A\colon C \to \mathsf{Cat}$ *to* $\mathscr{E}A$.

However, the extralax colimit map cannot possibly be extended to a left 2-adjoint (or 3-adjoint) because the underlying 1-categories of $\mathsf{2CAT}(\mathscr{E}D, \mathcal{B})$ and $\mathsf{2CAT/Cat}(D, \varpi_\mathcal{B})$ aren't in general isomorphic. To see why, consider the case where $D\colon \mathcal{A} \to \mathsf{Cat}$ is the functor $\tau\colon \mathbb{1} \to \mathsf{Cat}$ which picks out the terminal category. The 2-category $\mathsf{2CAT/Cat}(\tau, \varpi_\mathcal{B})$ has:

objects given by oplax functors $\mathbb{1} \to \mathcal{B}$ (i.e. comonads in $\mathcal{B}$)

1-cells between $M, N\colon \mathbb{1} \to \mathcal{B}$ given by lax transformations from M to N (i.e. comonad morphisms)

2-cells given by modulations between the lax transformations, viewed as comodules.

On the other hand, $\mathscr{E}\tau$ is given by $\mathbb{1} \boxminus \mathbb{1}^\dagger \cong \mathbb{1}^\dagger$ (cf. the beginning of Section 4.3.2). The objects of $\mathsf{2CAT}(\mathbb{1}^\dagger, \mathcal{B})$ are certainly identifiable with oplax functors $\mathbb{1} \to \mathcal{B}$, but the 1-cells correspond to *strict* transformations, and the 2-cells are modifications between them. This demonstrates that $D \mapsto \mathscr{E}D$ fails to define a 2-left-adjoint to $(\mathsf{F}_\Omega)_1$, and also that no such 2-left-adjoint can exist. If such a 2-left-adjoint, L, were to exist, it must also be a 1-left-adjoint, and thus LD would be isomorphic to $\mathscr{E}D$ which we've just demonstrated does not satisfy the requisite 2-dimensional aspect of the universal property.

Given that F_Ω extends to a $\mathsf{Gray}_\mathscr{L}$-functor, G_Ω, we might ask instead that there is $\mathsf{Gray}_\mathscr{L}$-natural isomorphism:

$$\mathsf{GRAY}_\mathscr{L}/\mathsf{Cat}(D, \varpi_\mathcal{B}) \cong \mathsf{GRAY}_\mathscr{L}(\mathscr{E}D, \mathcal{B})$$

This also fails to hold, though in a more subtle way. Consider the constant-at-$\mathbb{1}$ functor $\rho = \Delta_\mathbb{1} : \mathbb{2} \to \mathsf{Cat}$ from the free-living arrow category. An arrow in $\mathsf{GRAY}_\mathscr{L}/\mathsf{Cat}(\rho, \varpi_\mathcal{B})$ is then equivalently a morphism from the co-arrow object $\Delta_\mathbb{1} : \mathbb{2} \boxminus \mathbb{2} \to \mathsf{Cat}$ to $\varpi_\mathcal{B}$, and thus to a 2-functor $\mathbb{2} \boxminus \mathbb{2} \to \mathsf{opLax}_{\mathsf{lax,modl}}(\mathbb{1}, \mathcal{B})$ into the 2-category of oplax functors, lax transformations and *modulations* (described in more detail in Section 9.2). On the other hand, $\mathscr{E}\rho \cong \mathbb{2} \boxminus \mathbb{1}^\dagger$, so the 2-category $\mathsf{GRAY}_\mathscr{L}(\mathscr{E}\rho, \mathcal{B})$ is isomorphic to $\left[\mathbb{2}, \mathsf{opLax}_{\mathsf{lax,modf}}(\mathbb{1}, \mathcal{B})\right]$ — functors into a 2-category of oplax functors, lax transformations and *modifications*. An arrow in this category is a 2-functor $\mathbb{2} \boxminus \mathbb{2} \to \mathsf{opLax}_{\mathsf{lax,modf}}(\mathbb{1}, \mathcal{B})$, which is not the same as a 2-functor into $\mathsf{opLax}_{\mathsf{lax,modl}}(\mathbb{1}, \mathcal{B})$ because $\mathbb{2} \boxminus \mathbb{2}$ has non-trivial 2-cells (it's the free living "lax square") and thus detects the difference between modulations and modifications.

Remark 7.1.5. The terminology of [LS02] would identify $\mathsf{opLax}_{\mathsf{lax,modf}}(\mathbb{1}, \mathscr{B})$ as the 2-category $\mathsf{coMnd}(\mathscr{B})$ of comonads in $\mathscr{B}$,, whereas $\mathsf{opLax}_{\mathsf{lax,modl}}(\mathbb{1}, \mathscr{B})$ is the cocompletion of $\mathscr{B}$ under coKleisli objects, $\mathsf{coKl}(\mathscr{B})$. These two 2-categories are known to be isomorphic on underlying 1-categories but differ on 2-cells, as we've observed above. ◊

There are various other ways one might seek to mould the theory of parametric right adjoints to fit F_Ω, though all such mouldings attempted by the author have run afoul of either a modules/lax-transformations mismatch on 1-cells or a modifications/modulations mismatch on 2-cells (or worse). One thing that seems interesting to note is that while strict 2-cells $D \to \varpi_{\mathscr{B}}$ in $\mathsf{2CAT}/\mathsf{Cat}$ don't lift to strict 2-cells $\mathscr{E}D \to \mathscr{B}$, they do seem to correspond to precisely those lax transformations which are strict on morphisms of the first type in $\mathscr{E}D$ (cf. Construction 4.3.3). Perhaps generalising the oplax-cocompletion to the context of $\mathscr{F}$-categories might provide a satisfying parametric right-adjoint perspective, though this hasn't been attempted.

In any case, the multiplication μ and unit η for the 2-monad F_Ω also happen not to be cartesian, so that it fails the second property of familial 2-functors. Consider the naturality square for μ at the unique morphism from a 2-category $\mathscr{A}$ to $\mathbb{1}$ (recall that $\mathsf{F}_\Omega\mathbb{1} \cong \mathsf{Cat}$):

$$\begin{tikzcd}
\mathsf{F}^2_\Omega\mathscr{A} \arrow[r, "\mu_{\mathscr{A}}"] \arrow[d, "\mathsf{F}_\Omega\varpi_{\mathscr{A}}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathsf{F}_\Omega\mathscr{A} \arrow[d, "\varpi_{\mathscr{A}}"] \\
\mathsf{F}_\Omega\mathsf{Cat} \arrow[r, "\mu_{\mathbb{1}}"'] & \mathsf{Cat}
\end{tikzcd}$$

The multiplication is defined in terms of the canonical oplax-colimits in $\mathsf{F}_\Omega\mathscr{A}$. In particular, the multiplication acts on object $B \colon C \to \mathsf{F}_\Omega\mathscr{A}$ in $\mathsf{F}^2_\Omega\mathscr{A}$ by sending it to its oplax colimit in $\mathsf{F}_\Omega\mathscr{A}$, which we recall from Section 4.4 is obtained as follows:

(a) Let $A = \varpi_{\mathscr{A}}\, B \colon C \to \mathsf{Cat}$

(b) Transpose the strict 2-functor $B' \colon C^\dagger \to \mathsf{F}_\Omega\mathscr{K}$ under the $\mathsf{2CAT}/\mathscr{K} \rightleftharpoons \mathsf{2CAT}/\mathsf{F}_\Omega\mathscr{K}$ adjunction of Corollary 4.3.9 to obtain a 2-functor $B^\flat \colon \mathscr{D}A \to \mathscr{A}$.

(c) The 1-category $\pi_1(\mathscr{D}A)$ is isomorphic to the Grothendieck construction $\int A$ and admits a canonical oplax functor $\lambda \colon \pi_1(\mathscr{D}A) \to \mathscr{D}A$

(d) The oplax colimit of B is $B^\flat\,\lambda \colon \int A \to \mathscr{A}$.

So in particular, under the identification $\mathsf{F}_\Omega\mathbb{1} \cong \mathsf{Cat}$ the multiplication $\mu_{\mathbb{1}}$ is simply given by the Grothendieck construction.

If we take the pullback of $\varpi_{\mathscr{A}}$ along $\mu_{\mathbb{1}} = \int$ we obtain a 2-functor $p \colon \mathsf{F}_\Omega\mathsf{Cat} \times_{\mathsf{Cat}} \mathsf{F}_\Omega\mathscr{A} \to \mathsf{F}_\Omega\mathsf{Cat}$ whose fibre over p at some object $A \colon C \to \mathsf{Cat}$ in $\mathsf{F}_\Omega\mathsf{Cat}$ is the 2-category $\mathsf{opLax}_{\mathsf{lax,modl}}(\int A, \mathscr{A})$. In particular, its fibre at $\tau \colon \mathbb{1} \to \mathsf{Cat}$ is given by $\mathsf{coKl}(\mathscr{A})$.

On the other hand, the fibre of $\mathsf{F}_\Omega\varpi_{\mathscr{A}}$ at τ is given by those objects $\mathbb{1} \to \mathsf{F}_\Omega\mathscr{A}$ which lift τ along $\varpi_{\mathscr{A}}$. An oplax functor $\mathbb{1} \to \mathsf{F}_\Omega\mathscr{A}$ which lifts τ is equivalently one which factors through $\mathsf{coKl}(\mathscr{A}) \hookrightarrow \mathsf{F}_\Omega\mathscr{A}$, thus the fibre of $\mathsf{F}_\Omega\varpi_{\mathscr{A}}$ over τ is isomorphic to $\mathsf{coKl}^2(\mathscr{A})$.

These two fibres — $\mathsf{coKl}(\mathscr{A})$ and $\mathsf{coKl}^2(\mathscr{A})$ — are in general not isomorphic (or equivalent), and so the above square is not in general a pullback. There is, of course, a canonical map from $\mathsf{F}^2_\Omega\mathscr{A}$ to the pullback. When restricted to the fibre over τ, this is just the multiplication map $\mathsf{coKl}^2(\mathscr{A}) \to \mathsf{coKl}(\mathscr{A})$ which being the multiplication for a colax idempotent 2-monad has both a left and right adjoint — given in terms of the unit η by $\mathsf{coKl}\,\eta$ and $\eta\,\mathsf{coKl}$ respectively. More generally, the objects of the fibre of $\mathsf{F}_\Omega\varpi_{\mathscr{A}}$ over an oplax functor $A \colon C \to \mathsf{Cat}$ can be identified with 2-functors $\mathscr{D}A \to \mathscr{A}$, and the map to the pullback sends such a 2-functor to its pre-composition with the oplax functor $\lambda\colon \int A \to \mathscr{D}A$. However, a 2-cell in the fibre of $\mathsf{F}_\Omega\varpi_{\mathcal{A}}$ at A is not as simple as a lax transformation between corresponding 2-functors from $\mathscr{D}A$ (because $\mathscr{D}$ doesn't preserve $\mathsf{Gray}_{\mathscr{L}}$ co-arrow objects).

Similarly, the unit is not a cartesian transformation. Observe that the pullback for the naturality square of the unit η at $!\colon \mathcal{A} \to \mathbb{1}$ is given by $\mathsf{coKl}(\mathcal{A})$, which is not in general isomorphic to $\mathcal{A}$:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "!"] \arrow[d, "\eta_{\mathcal{A}}"'] & \mathbb{1} \arrow[d, "\langle \mathbb{1} \rangle"] \\ \mathsf{F}_\Omega\mathcal{A} \arrow[r, "\varpi"'] & \mathsf{Cat} \end{tikzcd} \tag{7.2}$$

Though F_Ω fails to be a familial monad in a satisfying way, it does contain sub-2-monads which are in some sense familial monad properties and bear a structural similarity to Fam. One of these is $\mathsf{2Fam}$, the completion under coproducts for 2-categories, which is also the full sub-2-monad of F_Ω for which objects of $\mathsf{2Fam}(\mathcal{A})$ are strict functors from discrete categories into $\mathcal{A}$. Between $\mathsf{2Fam}$ and F_Ω are two other sub-monads — the $\mathfrak{S}$ and $\mathfrak{P}$ referred to in the introduction — which we will now describe.

7.2 Pseudo Fam and Strict Fam

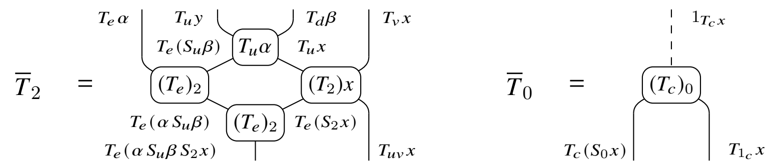
Consider first the full sub-2-category of $\mathsf{F}_\Omega\mathcal{A}$ (for $\mathcal{A}$ some 2-category) whose objects are pseudofunctors from a 1-category, i.e. those oplax functors $P\colon C \to \mathcal{A}$ whose P_0 and P_2 cells are invertible. Now, the free cocompletion $\mathsf{F}_\Omega\mathcal{A}$ is equivalent to the full sub-2-category of $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ whose objects are Ω-colimits of representables via the map sending an oplax functor $C \to \mathcal{A}$ to its oplax-image presheaf, so the full sub-2-category of $\mathsf{F}_\Omega\mathcal{A}$ at pseudofunctors is equivalent to the full sub-2-category of $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ whose objects are oplax-image presheaves of pseudofunctors from a 1-category. Let's call such presheaves $\mathfrak{p}$-presheaves, and let's call the full sub-2-category of $\mathsf{F}_\Omega\mathcal{A}$ at pseudofunctors $\mathfrak{p}\mathcal{A}$.

Now, for any 2-functor $F\colon \mathcal{A} \to \mathcal{B}$, $\mathsf{F}_\Omega F$ restricts to a 2-functor $\mathfrak{p}$ from $\mathfrak{p}\mathcal{A}$ to $\mathfrak{p}\mathcal{B}$, since post-composing a pseudofunctor, pseudo-natural transformation or modification by a 2-functor produces another such. The mapping $\mathcal{A} \mapsto \mathfrak{p}\mathcal{A}$ then automatically extends to a full sub-3-functor of F_Ω, and sub-$\mathsf{Gray}_{\mathscr{L}}$-functor of G_Ω because $\mathfrak{p}\mathcal{A} \subseteq \mathsf{F}_\Omega\mathcal{A}$ is a full sub-2-category.

However, it isn't a sub-2-monad of F_Ω. We observed in Remark 4.6.8 that a pseudofunctor $T\colon C \to \mathsf{F}_\Omega\mathcal{A}$ with image in $\mathfrak{p}\mathcal{A}$ may have an oplax colimit which lies outside $\mathfrak{p}\mathcal{A}$, i.e. the induced oplax functor $\overline{T}\colon \int S \to \mathcal{A}$, for $S = \varpi_{\mathcal{A}} T$, may fail to be a pseudofunctor. This demonstrates that $\mathfrak{p}$-presheaves are not a saturated class of weights, and that the multiplication of F_Ω does not restrict to a natural transformation $\mathfrak{p}^2 \to \mathfrak{p}$.

The obstruction to $\mathfrak{p}$-colimits in $\mathfrak{p}\mathcal{A}$ being pseudofunctors is that the 1-cells in $\mathfrak{p}\mathcal{A}$ are merely *lax* transformations. From the way $\overline{T}$ is defined — as the composite $\int S \xrightarrow{\lambda_S} \mathscr{D}S \xrightarrow{T^\flat} \mathcal{A}$ — the oplax functor $\overline{T}$ will in fact be a pseudofunctor whenever the $\diamondsuit$'s in $\mathscr{D}S$ are mapped to invertible 2-cells in $\mathcal{A}$ by $T^\flat$, which happens whenever T lands in the locally-full 2-category of pseudofunctors and *pseudo-natural* transformations, rather than arbitrary lax transformations. We prove this carefully in Section 9.3 by our construction of the *pseudolax* colimit, $\mathscr{Q}$, and in particular the observation that the pseudolax colimit $\mathscr{Q}S$ is obtained from the extralax colimit $\mathscr{D}S$ by freely inverting the 2-cells, and so its hom-categories are thus equivalence relations (Lemma 9.3.1). The proof given in that section involves carefully constructing and checking inverses for generating 2-cells, but the general idea here is to observe that if a 1-cell $u\colon c \to d$ of C is mapped to a 1-cell in $\mathfrak{p}\mathcal{A}$ with a pseudo-natural transformation $T_u\colon T_c \Rightarrow T_d S_u$ filling the triangle, then the 2-cell at the top of the diagram below for the oplax-functoriality composition 2-cell of $\overline{T}$ will be invertible.

All other 2-cells are invertible by the fact that T is a pseudofunctor and maps objects to pseudofunctors.



(7.3)

We will call this subcategory of pseudofunctors and pseudo-natural transformations $\mathfrak{P}\mathcal{A}$:

Definition 7.2.1 ($\mathfrak{P}\mathcal{A}$)**.** For a 2-category $\mathcal{A}$, $\mathfrak{P}\mathcal{A}$ is the 2-category with:

0-cells: a pair of a 1-category C and a pseudofunctor $P\colon C \to \mathcal{A}$

1-cells: from (C,P) to (D,Q) given by a pair of a functor $F\colon C \to D$ and a pseudonatural transformation $\phi\colon P \Rightarrow QF$.

2-cells: of type $(F,\phi) \Rightarrow (G,\psi)\colon (C,P) \to (D,Q)$ given by a pair of a natural transformation $\gamma\colon F \Rightarrow G$ and a modification $\Gamma\colon Q\gamma\,\phi \Rrightarrow \psi$.

◊

Remark 7.2.2. Note that the 2-cells of $\mathfrak{P}\mathcal{A}$ in Definition 7.2.1 are given by modifications, whereas those of $\mathsf{F}_\Omega\mathcal{A}$ are given by modulations. In fact, we could substitute "modulations" for "modifications" in our definition and it wouldn't change anything — when Q is a pseudofunctor, $Q\gamma$ will be pseudonatural (rather than a comodule) and a modulation between pseudo-natural transformations, viewed as comodules, is equivalent to a modification between the pseudo-natural transformations (cf. Remark 4.2.5 and Section 9.1). ◊

The map on objects $\mathcal{A} \to \mathfrak{P}\mathcal{A}$ also extends to a sub-3-functor of F_Ω, though given that $\mathfrak{P}\mathcal{A} \subset \mathsf{F}_\Omega\mathcal{A}$ isn't full we must also check that for $\alpha\colon F \Rightarrow G\colon \mathcal{A} \to \mathcal{B}$ a 2-natural transformation, the components of $\mathsf{F}_\Omega\alpha\colon \mathsf{F}_\Omega F \Rightarrow \mathsf{F}_\Omega G$ at pseudo-functors are 1-cells in $\mathfrak{P}\mathcal{B}$, which they are. The component at $P\colon C \to \mathcal{A}$ is:

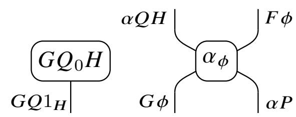
$$\begin{tikzcd} C \arrow[d, equal] \arrow[dr, "FP"] \arrow[dr, phantom, shift right=2ex, "\alpha P \Downarrow"] & \\ C \arrow[r, "GP"'] & \mathcal{B} \end{tikzcd} \tag{7.4}$$

and αP is 2-natural, so in particular pseudo-natural.

When α is lax-natural, however, the 1-cell (7.4) is *not* in $\mathfrak{P}\mathcal{A}$, so $\mathfrak{P}$ isn't a sub-$\mathsf{Gray}_{\mathscr{L}}$-functor of G_Ω. Though, as observed in Remark 5.6.1 G_Ω restricts from a $\mathsf{Gray}_{\mathscr{L}}$-functor to a Gray-functor, and when α is pseudo-natural the 1-cell (7.4) *does* lie in $\mathfrak{P}\mathcal{A}$. It follows that $\mathfrak{P}$ defines a sub-Gray-functor of G_Ω, as well as a sub-2-monad of F_Ω when restricted to $2\mathsf{Cat}$.

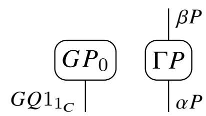
Definition 7.2.3 ($\mathfrak{P}\colon \mathsf{GRAY} \to \mathsf{GRAY}$)**.** For $F\colon \mathcal{A} \to \mathcal{B}$ a 2-functor $\mathfrak{P}F\colon \mathfrak{P}\mathcal{A} \to \mathfrak{P}\mathcal{B}$ is given by post-composition with F. If $\alpha\colon F \Rightarrow G$ is a pseudo-natural transformation, $\mathfrak{P}\alpha\colon \mathfrak{P}F \Rightarrow \mathfrak{P}G$ is the pseudo-natural transformation whose component at object (C,P) is given by post-composing with α. That is, $\mathfrak{P}\alpha_{(C,P)} = (1_c, \alpha P)$. The pseudo-naturality 2-cell of $\mathfrak{P}\alpha$ at a morphism $(H,\phi)\colon (C,P) \to (D,Q)$ is given in terms of the (invertible) interchanger $\alpha_\phi\colon G\phi\,\alpha P \Rrightarrow \alpha QH\,F\phi$ and the (invertible) modification $Q_0 : Q1_{1_D} \Rrightarrow 1_Q\colon Q \Rightarrow Q$ as $\big(1_H, \alpha_\phi \circ\ (GQ_0H * G\phi * \alpha P)\big)$ where $*$ indicates vertical composition of pseudo-natural transformations and $\circ$ indicates vertical composition of modifications. The string diagram for the modification component as a 2-cell in the 2-category $\mathsf{Pseudo}_{\mathsf{pseudo,modf}}(C,\mathcal{B})$

is:



(7.5)

Finally, for a modification $\Gamma\colon \alpha \Rrightarrow \beta\colon F \Rightarrow G\colon \mathcal{A} \to \mathcal{B}$, $\mathfrak{P}\Gamma$ is the modification whose component 2-cell at $P\colon C \to \mathcal{A}$ is represented by the following string diagram:



When $\alpha\colon F \Rightarrow G$ is a strictly 2-natural transformation, $\mathfrak{P}\alpha$ is also ((7.5) is an identity 2-cell in $\mathfrak{P}\mathcal{B}$). So we can also refer to the 3-functor $\mathfrak{P}\colon \mathsf{2CAT} \to \mathsf{2CAT}$ where appropriate. $\lozenge$

For a 2-category $\mathcal{A}$, we can also consider the subcategory $\mathfrak{S}\mathcal{A}$ of $\mathsf{F}_\Omega\mathcal{A}$ (and of $\mathfrak{P}\mathcal{A}$) whose objects are *strict* functors $P\colon C \to \mathcal{A}$ and whose 1-cells are given by strictly 2-natural transformations. As with the pseudo case, this restriction of the 1-cells to 2-natural transformations is necessary for the multiplication of F_Ω to restrict to a multiplication for $\mathfrak{S}$. To see that this is true, consider the oplax-functoriality 2-cells of (7.3) and note that if the image of $T\colon C \to \mathsf{F}_\Omega\mathcal{A}$ lands in $\mathfrak{S}\mathcal{A}$, then the oplax-functoriality 2-cells $(T_c)_*$ will be identities, and the lax-naturality 2-cell $T_u\alpha$ will be an identity. If T itself is strict, then T_2 is also an identity, so the above oplax-functoriality 2-cells for $\overline{T}$ are both identities as required.

With the 1-cells of $\mathfrak{S}$ constrained to strict 2-natural transformations, $\mathfrak{S}$ cannot be extended to a Gray-functor for the same reason $\mathfrak{P}$ couldn't be extended to a $\mathsf{Gray}_{\mathscr{L}}$-functor. However, it is a sub-3-functor of F_Ω and a sub-2-monad. In fact, $\mathfrak{S}$ admits a rather elementary definition as a 3-functor:

Definition 7.2.4 ($\mathfrak{S}\colon \mathsf{2CAT} \to \mathsf{2CAT}$)**.** Let $i\colon \mathsf{Cat} \to \mathsf{2CAT}$ denote the map which casts a 1-category as a locally discrete 2-category. For $\mathcal{A}$ a 2-category, $\mathfrak{S}\mathcal{A}$ is the lax slice $i \Downarrow \mathcal{A}$, which extends by general principles to a 3-functor $i \Downarrow (-)\colon \mathsf{2CAT} \to \mathsf{2CAT}$. $\lozenge$

Remark 7.2.5. The map $\mathfrak{S}$ is related to the 2-functor $\mathsf{Diag}\colon \mathsf{Cat} \to \mathsf{2Cat}$ described in [PT22]. In fact, the first definition for Diag in that paper is equal to $\mathfrak{S}i$; though later in the paper it is used to refer to the monad on Cat given by post-composing $\mathfrak{S}i$ by the "underlying category" map $\mathsf{2Cat} \to \mathsf{Cat}$. $\lozenge$

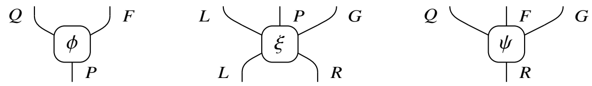
Neither $\mathfrak{P}$ nor $\mathfrak{S}$ are free cocompletions, for reasons already discussed. However, they do have properties not shared by F_Ω. The first we consider is their preservation of 2-fibrations.

7.3 Preservation of fibrations

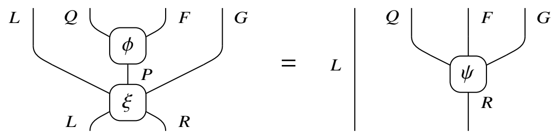
The Gray and Cat representability of 2-fibrations established in Chapter 6 allows us to give rather simple proofs for the preservation of 2-fibrations by $\mathfrak{S}$ and $\mathfrak{P}$. We begin by identifying for a 2-functor $L\colon \mathcal{A} \to \mathcal{B}$ the $\mathfrak{S}L$-cartesian and $\mathfrak{P}L$-cartesian 1-cells and 2-cells.

Lemma 7.3.1. *For a 2-functor $L\colon \mathcal{A} \to \mathcal{B}$, the $\mathfrak{S}L$-cartesian 1-cells of $\mathfrak{S}\mathcal{A}$ are those $(F,\phi)\colon (C,P) \to (D,Q)$ where $\phi\colon P \Rightarrow QF$ is generalised L-cartesian, and the strong-$\mathfrak{S}L$-opcartesian 2-cells of $\mathfrak{S}\mathcal{A}$ are those $(\gamma,\Gamma)\colon (F,\phi) \Rightarrow (G,\psi)$ such that Γ is generalised strong-L-opcartesian.*

Proof. Assume we have 1-cells $(F,\phi)\colon (C,P)\to(D,Q)$, $(G,\xi)\colon \mathfrak{S}_L(E,R)\to\mathfrak{S}_L(C,P)$, $(FG,\psi)\colon (E,R)\to(D,Q)$:

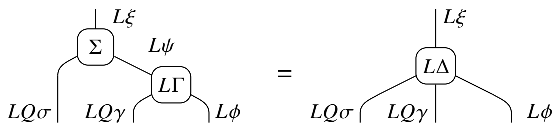


satisfying:



If ϕ is generalised L-cartesian, then so is ϕG, and so there exists a unique $\xi'\colon R\Rightarrow PG$ over ξ satisfying $\phi G\circ\xi'=\psi$. This amounts to a unique lift of the 1-cell $(G,\xi)\colon \mathfrak{S}_L(E,R)\Rightarrow\mathfrak{S}_L(C,P)$ which when post-composed with (F,ϕ) gives (FG,ψ). Taking G to be an identity shows that the condition that ϕ be generalised L-cartesian is also necessary.

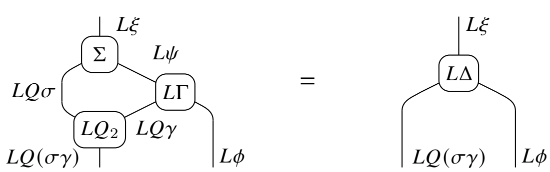
For the strong-$\mathfrak{S}L$-opcartesian 2-cells, assume we have 1-cells $(F,\phi),(G,\psi),(H,\xi)\colon (C,P)\to(D,Q)$ in $\mathfrak{S}\mathcal{A}$ and 2-cells $(\gamma,\Gamma)\colon (F,\phi)\Rightarrow(G,\psi)$, $(\sigma,\Sigma)\colon \mathfrak{S}_L(G,\psi)\Rightarrow\mathfrak{S}_L(H,\xi)$ $(\sigma\gamma,\Delta)\colon (F,\phi)\Rightarrow(H,\xi)$ satisfying:



If Γ is generalised strong-L-opcartesian then so is $(Q\sigma)\,\Gamma$, so Σ admits a unique lift to a modification $\Sigma'\colon Q\sigma\,\psi\Rrightarrow\xi$ such that $\Sigma'\circ Q\sigma\,\Gamma=\Delta$. This amounts to a unique lift of (σ,Σ) to a 2-cell $(\sigma,\Sigma')\colon (G,\psi)\Rightarrow(H,\xi)$ satisfying $(\sigma,\Sigma')\circ(\gamma,\Gamma)=(\sigma\gamma,\Delta)$ as required. Finally, note that whiskering (γ,Γ) with arbitrary 1-cells will produce a new 2-cell whose modification component is also generalised strong-L-opcartesian, so (γ,Γ) is strong-$\mathfrak{S}_L$-opcartesian. Taking σ to be an identity shows that the condition that Γ be generalised L-opcartesian is not just sufficient but also necessary. ◻

Lemma 7.3.2. *For a 2-functor $L\colon\mathcal{A}\to\mathcal{B}$, the $\mathfrak{P}L$-cartesian 1-cells are those $(F,\phi)\colon (C,P)\to(D,Q)$ where $\phi\colon P\Rightarrow QF$ is Gray L-cartesian, and the $\mathfrak{P}L$-opcartesian 2-cells are those $(\gamma,\Gamma)\colon (F,\phi)\Rightarrow(G,\psi)$ such that Γ is strong Gray-L-opcartesian 2-cell.*

Proof. The proof is essentially the same as that for Lemma 7.3.1. For 1-cells in particular the argument is exactly the same. For opcartesian 2-cells, the vertical composition of 2-cells is defined slightly differently, so the argument is slightly different. Assume we have 1-cells $(F,\phi),(G,\psi),(H,\xi)\colon (C,P)\to(D,Q)$ in $\mathfrak{P}\mathcal{A}$ and 2-cells $(\gamma,\Gamma)\colon (F,\phi)\Rightarrow(G,\psi)$, $(\sigma,\Sigma)\colon \mathfrak{P}_L(G,\psi)\Rightarrow\mathfrak{P}_L(H,\xi)$ and $(\sigma\gamma,\Delta)\colon (F,\phi)\Rightarrow(H,\xi)$. This time, the relevant hypothesis is the following equality of modifications:



(7.6)

The 2-cell Q_2 is an isomorphism, so automatically strong Gray-L-opcartesian. Which means the 2-cell below Σ on the left in (7.6) is the image under L of a Gray opcartesian 2-cell, so Σ can be lifted to a modification $\Sigma'\colon Q\sigma\,\psi\Rrightarrow\xi$ as required. ◻

Corollary 7.3.3. *The 3-functor $\mathfrak{S}\colon \mathsf{2CAT}\to\mathsf{2CAT}$ preserves split 2-fibrations.*

Proof. Assume $L\colon \mathcal{A} \to \mathcal{B}$ is a split 2-fibration. Given 0-cells $(C,P) \in \mathfrak{S}\mathcal{A}$, $(D,Q) \in \mathfrak{S}\mathcal{B}$ and a 1-cell $(F,\phi)\colon (D,Q) \to \mathfrak{S}_L(C,P) = (C,LP)$, we can lift $\phi\colon Q \Rightarrow LPF$ uniquely to a Gray L-cartesian $\phi'\colon Q' \Rightarrow PF$ which lies over ϕ. Then $(F,\phi')\colon (D,Q') \to (C,P)$ is a $\mathfrak{S}_L$-cartesian lift of (F,ϕ). Similarly, for 1-cells $(F,\phi)\colon (C,P) \to (D,Q)$, $(G,\psi)\colon (C,LP) \to (D,LQ)$ and a 2-cell $(\gamma,\Gamma)\colon (F,L\phi) \Rightarrow (G,\psi)$, we can lift $\Gamma : LQ\gamma\, L\phi \Rrightarrow \psi$ to a Gray L-opcartesian $\Gamma'\colon Q\gamma\,\phi \Rrightarrow \psi'$ so that $(\gamma,\Gamma')\colon (F,\phi) \Rightarrow (G,\psi')$ is an $\mathfrak{S}_L$-opcartesian lift of (γ,Γ). Because the 2-fibration L is representably split, so is the 2-fibration $\mathfrak{S}L$. □

Corollary 7.3.4. *The* Gray*-functor* $\mathfrak{P}\colon \mathsf{GRAY} \to \mathsf{GRAY}$ *preserves 2-fibrations.*

Proof. The proof proceeds in the same way as the proof for Corollary 7.3.3 with “Gray” replacing “generalised”. □

In the next two sections we use the fibration-preserving properties of $\mathfrak{S}$ and $\mathfrak{P}$ to demonstrate properties similar to those of Weber’s familial 2-monads.

7.4 $\mathfrak{S}$ is a 3-familial functor

Consider the factorisation of $\mathfrak{S}\colon \mathsf{2CAT} \to \mathsf{2CAT}$ through $\mathsf{2CAT}/\mathsf{Cat}$.

$$\begin{tikzcd} \mathsf{2CAT} \arrow[r, "\mathfrak{S}"] \arrow[dr, "\mathfrak{S}_1"'] & \mathsf{2CAT} \\ & \mathsf{2CAT}/\mathsf{Cat} \arrow[u, "\pi"'] \end{tikzcd}$$

For a Familial 2-functor $T\colon \mathcal{A} \to \mathcal{B}$ the 2-functor $T_1\colon \mathcal{A} \to \mathcal{B}/T1$ factors through split fibrations, so we might similarly ask whether $\mathfrak{S}_1$ factors through split 2-fibrations. That is, whether for a given 2-category, $\mathcal{A}$, the projection $\varpi\colon \mathfrak{S}\mathcal{A} \to \mathsf{Cat}$ is a split 2-fibration. Observe also that for a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$, the induced map $\mathfrak{S}F\colon \mathfrak{S}\mathcal{A} \to \mathfrak{S}\mathcal{B}$ preserves the chosen-cartesian 1-cells and chosen-opcartesian 2-cells of $\mathfrak{S}\mathcal{A}$, which we deduce from Lemma 7.3.1 to be respectively:

(a) 1-cells of the form $\big(F, 1_{QF}\big)\colon (C,QF) \to (D,Q)$ for $F\colon C \to D$, $Q\colon D \to \mathcal{A}$

(b) 2-cells of the form $\big(\gamma, 1_{(Q\gamma)\,\phi}\big)\colon (F,\phi) \Rightarrow (G,(Q\gamma)\,\phi)\colon (C,P) \to (D,Q)$ for $\phi\colon F \Rightarrow G\colon C \to D$.

We can also show this by demonstrating that $\mathfrak{S}_1$ factors through the 2-dimensional Grothendieck construction $\int\colon [\mathsf{Cat}^{\mathsf{op}}, \mathsf{2CAT}] \to \mathsf{2CAT}/\mathsf{Cat}$ which is equivalent to the inclusion of the 3-category of split fibrations and split cartesian morphisms $\mathsf{Sp}(\mathsf{Cat}) \hookrightarrow \mathsf{2CAT}/\mathsf{Cat}$. This observation will also provide a method for constructing the left 3-adjoint to $\mathfrak{S}_1$.

Let $\mathfrak{S}'\colon \mathsf{2CAT} \to [\mathsf{Cat}^{\mathsf{op}}, \mathsf{2CAT}]$ denote the 3-functor which sends a 2-category $\mathcal{A}$ to $\mathsf{2CAT}(i-,\mathcal{A})\colon \mathsf{Cat}^{\mathsf{op}} \to \mathsf{2CAT}$, where $i\colon \mathsf{Cat} \to \mathsf{2CAT}$ casts a 1-category as a locally-discrete 2-category and consider the composite:

$$\mathsf{2CAT} \xrightarrow{\mathfrak{S}'} [\mathsf{Cat}^{\mathsf{op}}, \mathsf{2CAT}] \xrightarrow{\int} \mathsf{2CAT}/\mathsf{Cat}$$

This 3-functor sends a 2-category $\mathcal{A}$ first to the presheaf $\mathsf{2CAT}(i-,\mathcal{A})$, then to $\int \mathsf{2CAT}(i-,\mathcal{A}) \cong i \Downarrow \mathcal{A}$ which is by definition $\mathfrak{S}\mathcal{A}$. Similar arguments considering the actions on 1-cells, 2-cells and 3-cells of $\mathsf{2CAT}$ show that $\int \mathfrak{S}'$ is isomorphic to $\mathfrak{S}_1$. For example, for $F\colon \mathcal{A} \to \mathcal{B}$ a 2-functor, $\int(i-,F) \cong i \Downarrow F$ is the same post-composition 2-functor as $\mathfrak{S}F$. Now, observe that:

(a) $\int$ has a left 3-adjoint given by the *free split 2-fibration* on a 2-functor $P\mathcal{A} \to \mathsf{Cat}$

(b) $\mathfrak{S}'$ has a left 3-adjoint given by the left extension of $i\colon \mathsf{Cat} \to \mathsf{2CAT}$ along the Yoneda embedding $\mathsf{Cat} \to [\mathsf{Cat}^{\mathsf{op}}, \mathsf{2CAT}]$ by the general theory of pointwise extensions. The action of this left adjoint on objects can be given in terms of coends as:

$$F \mapsto \int^{C \in \mathsf{Cat}} FC \times C$$

This means we can construct a left 3-adjoint for $\mathfrak{S}_1$ by composing the left 3-adjoints for $\int$ and $\mathfrak{S}'$:

$$\begin{tikzcd}
\mathsf{2CAT} \arrow[r, "\mathfrak{S}"] \arrow[d, "\mathfrak{S}'"'] \arrow[dr, "\mathfrak{S}_1"] & \mathsf{2CAT} \\
{[\mathsf{Cat}^{\mathsf{op}}, \mathsf{2CAT}]} \arrow[u, bend left, "\mathsf{lan}_{\jmath} i"] \arrow[r, "\int"] & \mathsf{2CAT/Cat} \arrow[u, "\pi"'] \arrow[l, bend left, "\mathsf{freeFib}"]
\end{tikzcd}$$

Composing these left adjoints gives a left 3-adjoint to $\mathfrak{S}_1$ which sends $P\colon \mathcal{A} \to \mathsf{Cat}$ to $\int^{C \in \mathsf{Cat}} (C \Downarrow P) \times C$, which is isomorphic to the category of elements of P:

$$\int^{c \in \mathsf{Cat}} (C \Downarrow P) \times C \cong \int^{c \in \mathsf{Cat}} \oint^{x \in \mathcal{A}} [C, Px] \times C \cong \oint^{x \in \mathcal{A}} \int^{c \in \mathsf{Cat}} [C, Px] \times C \cong \oint^{x \in \mathcal{A}} Px$$

This corresponds to the observation that a map $P \to \varpi_{\mathcal{B}}$ is an oplax cocone from iP to $\mathcal{B}$ in $\mathsf{2CAT}$, which is equivalent to a map from the oplax colimit $\int iP \cong i \int P$ to $\mathcal{B}$. So, $\mathfrak{S}\colon \mathsf{2CAT} \to \mathsf{2CAT}$ is a familial 3-functor in the following sense:

Lemma 7.4.1. *The 3-functor* $\mathfrak{S}_1\colon \mathsf{2CAT} \to \mathsf{2CAT/Cat}$ *has a left 3-adjoint and factors through* $\mathsf{Sp(Cat)}$.

7.5 $\mathfrak{S}$ is a 3-familial monad

We might call $\mathfrak{S}$ a familial 3-monad if its unit and multiplication are additionally cartesian, in the sense that their naturality squares are 3-pullbacks. Rather than considering the naturality squares for arbitrary morphisms in $\mathsf{2CAT}$, we need only consider the naturality squares at the unique morphisms from a 2-category to the terminal 2-category, $\mathbb{1}$. This is because for any 2-functor $F\colon \mathcal{A} \to \mathcal{B}$, the naturality square for F composes with the naturality square for $!\colon \mathcal{B} \to \mathbb{1}$ to give the naturality square for $!\colon \mathcal{A} \to \mathbb{1}$:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "\eta_{\mathcal{A}}"'] & \mathcal{B} \arrow[r, "!"] \arrow[d, dashed, "\eta_{\mathcal{B}}"] & \mathbb{1} \arrow[d, "\langle \mathbb{1} \rangle"] \\
\mathfrak{S}\mathcal{A} \arrow[r, "\mathfrak{S}F"'] & \mathfrak{S}\mathcal{B} \arrow[r, "\varpi"'] & \mathsf{Cat}
\end{tikzcd}$$

and the left-hand square will be a pullback if both the composite and the right-hand square are pullbacks. This is just as true with the multiplication, μ, substituted for the unit, η.

The naturality square for η at $!\colon \mathcal{A} \to \mathbb{1}$ is clearly a pullback square, as the fibre of $\varpi\colon \mathfrak{S}\mathcal{A} \to \mathsf{Cat}$ over $\mathbb{1} \in \mathsf{Cat}$ is $\mathsf{2CAT}(\mathbb{1}, \mathcal{A})$ which is isomorphic to $\mathcal{A}$.

Now, consider the naturality square for the multiplication at $!\colon \mathcal{A} \to \mathbb{1}$:

$$\begin{tikzcd}
\mathfrak{S}^2\mathcal{A} \arrow[r, "\mu_{\mathcal{A}}"] \arrow[d, "\mathfrak{S}\varpi_{\mathcal{A}}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathfrak{S}\mathcal{A} \arrow[d, "\varpi_{\mathcal{A}}"] \\
\mathfrak{S}\mathsf{Cat} \arrow[r, "\int"'] & \mathsf{Cat}
\end{tikzcd} \tag{7.7}$$

We know from Section 7.4 that $\varpi_{\mathcal{A}}$ is a split 2-fibration, and because $\mathfrak{S}$ preserves split 2-fibrations (Corollary 7.3.3) so is $\mathfrak{S}\varpi_{\mathcal{A}}$. Moreover, as we now explain, the map $(\mu_{\mathcal{A}}, \int)$ between these split fibrations preserves chosen cartesian maps.

As we've already observed, the chosen $\varpi_{\mathcal{A}}$-cartesian 1-cells are those of the form $(F, 1_{FQ})\colon (C, FQ) \to (D, Q)$. By Lemma 7.3.1 it follows that the chosen $\mathfrak{S}\varpi_{\mathcal{A}}$-cartesian 1-cells are those $(F, \phi)\colon (C, P) \to (D, Q)$ such that for each $c \in C$, $\phi_c\colon Pc \to QFc$ is $\varpi_{\mathcal{A}}$-cartesian in $\mathcal{A}$. Let $p\colon C \to \mathsf{Cat}$ and $q\colon D \to \mathsf{Cat}$ denote $\varpi_{\mathcal{A}}P$ and $\varpi_{\mathcal{A}}Q$ respectively so that (C, P) and (D, Q) may be identified with lax cocones $\lambda_P\colon p \rhd \mathcal{A}$, $\lambda_Q\colon q \rhd \mathcal{A}$, and let Φ denote $\varpi_{\mathcal{A}}\phi\colon p \Rightarrow q$. In terms of these lax cocones, the first component of $\mu_{\mathcal{A}}(F, \phi)$ is the canonical map from $\int p$ to $\int q$ induced by the transformation Φ; we call this map $\int\Phi$. The second component is the natural transformation ${\lambda_P}^\flat \Rightarrow {\lambda_Q}^\flat \int\Phi$ induced[2] by the maps ϕ_c between the legs of the cones λ_P and $\lambda_Q\,\Phi$. When each ϕ_c is an identity, the induced map ${\lambda_P}^\flat \Rightarrow \lambda_Q \int\Phi$ is as well, and so $\mu_{\mathcal{A}}(F, \phi)$ will be $\varpi_{\mathcal{A}}$-chosen cartesian. A similar argument shows that the image of a chosen $\mathfrak{S}\varpi_{\mathcal{A}}$-opcartesian 2-cell will be $\varpi_{\mathcal{A}}$-opcartesian.

To show that (7.7) is a pullback square, it therefore suffices to show that the induced map $\rho\colon \mathfrak{S}^2\mathcal{A} \to \mathfrak{S}\mathcal{A} \times_{\mathsf{Cat}} \mathfrak{S}\mathsf{Cat}$ is an isomorphism of fibrations over $\mathfrak{S}\mathsf{Cat}$. Because this map is certainly a map of fibrations, we need only show that it is an isomorphism on each fibre over $\mathfrak{S}\mathsf{Cat}$. To that end, consider an object $P\colon C \to \mathsf{Cat}$ in $\mathfrak{S}\mathsf{Cat}$. The fibre of $\mathfrak{S}\mathcal{A} \times_{\mathsf{Cat}} \mathfrak{S}\mathsf{Cat}$ above P is just $\mathsf{2CAT}(\int P, \mathcal{A})$. The fibre of $\mathfrak{S}\varpi_{\mathcal{A}}$ at P is isomorphic to $\mathsf{2CAT}/\mathsf{Cat}(P, \varpi_{\mathcal{A}})$, and the induced map ρ restricted to this fibre is the transposition map for the 3-adjunction $\int \dashv \mathfrak{S}_1$, and therefore an isomorphism. Thus, we have shown:

Proposition 7.5.1. *The 3-monad* $\mathfrak{S}\colon \mathsf{2CAT} \to \mathsf{2CAT}$ *is cartesian.*

7.6 $\mathfrak{P}$ is a Gray-familial functor

We now turn to the pseudo variant, $\mathfrak{P}$. Viewed as a 1-functor, $\mathfrak{P}_1\colon \mathsf{2CAT} \to \mathsf{2CAT}/\mathsf{Cat}$ does have a left adjoint given on objects by sending $F\colon \mathcal{A} \to \mathsf{Cat}$ to a *pseudo*-lax colimit $\mathscr{P}F$ (cf. Section 9.3). However, $\mathfrak{P}_1$ fails to have a left 2-adjoint as a 2-functor, for essentially the same reason $(\mathsf{F}_\Omega)_1$ does. For $\mathcal{A}$ a 2-category, the hom-2-category in $\mathsf{2CAT}/\mathsf{Cat}$ from $\tau\colon \mathbb{1} \to \mathsf{Cat}$ to $\varpi_{\mathcal{A}}\colon \mathfrak{P}\mathcal{A} \to \mathsf{Cat}$ has:

objects: pseudo functors $\mathbb{1} \to \mathcal{A}$

1-cells: pseudo-natural transformations $M \Rightarrow N\colon \mathbb{1} \to \mathcal{A}$

2-cells: modifications between pseudo-natural transformations

whereas the hom-2-category in $\mathsf{2CAT}$ from $\mathscr{P}\tau = \mathbb{1}^\natural$ to $\mathcal{A}$ is isomorphic to the 2-category with:

objects: pseudo functors $\mathbb{1} \to \mathcal{A}$

1-cells: strictly 2-natural transformations

2-cells: modifications

So $\mathsf{2CAT}/\mathsf{Cat}(\tau, \varpi_{\mathcal{A}})$ and $\mathsf{2CAT}(\mathscr{P}\tau, \mathcal{A})$ are not isomorphic, and thus $\mathfrak{P}_1$ cannot have a left 3-adjoint. However, it *does* have a left Gray-adjoint. Recall that $(\mathsf{G}_\Omega)_1\colon \mathsf{GRAY}_{\mathscr{L}} \to \mathsf{GRAY}_{\mathscr{L}}/\mathsf{Cat}$ was shown to not have a left $\mathsf{Gray}_{\mathscr{L}}$-adjoint because the 2-cells of $\mathsf{GRAY}_{\mathscr{L}}/\mathsf{Cat}(P, \varpi_{\mathcal{B}})$ were modulations, whereas those of $\mathsf{GRAY}_{\mathscr{L}}(\mathscr{E}P, \mathcal{B})$ were modifications. The fact that modulations and modifications between pseudo-natural transformations between pseudofunctors are equivalent should then suggest that $\mathfrak{P}$ does indeed define a left Gray-adjoint to $\mathfrak{P}_1$. We can show that this is the case by an argument analogous to that given for the $\mathfrak{S}$ case.

In [Buc14, Remark 2.3.3], Buckley observes that for a given 2-category $\mathcal{B}$, the map $\varpi_{\mathcal{B}}\colon \mathfrak{P}\mathcal{B} \to \mathsf{Cat}$ is the Grothendieck construction of the map $\mathcal{M}\colon \mathsf{Cat}^{\mathsf{op}} \to \mathsf{GRAY}$ sending $C \in \mathsf{Cat}$ to $\mathsf{Pseudo}(C, \mathcal{B})$, which is isomorphic to $\mathsf{Gray}\left(C^\natural, \mathcal{B}\right)$. The map $\mathcal{M}$ is not a 2-functor, essentially because $C \mapsto C^\natural$ fails to be a 2-functor. Indeed,

[2] Recall that ${\lambda_P}^\flat$ is used to denote the map out of the oplax colimit $\int p$ induced by the lax cocone λ_P.

for $\gamma\colon F \Rightarrow G\colon C \to D$ and $\delta\colon G \Rightarrow H$ natural transformations in Cat, the induced pseudo-natural transformations $\delta^\natural\gamma^\natural$ and $(\delta\gamma)^\natural\colon F^\natural \Rightarrow H^\natural\colon D^\natural \to C^\natural$ are merely isomorphic, with components at $c \in C^\natural$ given by $[\delta_c][\gamma_c]$ and $[\delta_c\gamma_c]$ respectively. Instead, the map $(-)^\natural\colon \mathsf{Cat} \to \mathsf{Gray}$ is a globally split (co)-2-presheaf, in the sense of Section 6.2.1. It clearly comprises the relevant data: a map $C \mapsto C^\natural$ on objects and *pseudo*-functors $(-)^\natural_{C,D}\colon \mathsf{Cat}(C,D) \to \mathsf{Gray}\big(C^\natural, D^\natural\big)$ whose pseudo-functoriality 2-cells are the unique modifications of the form $\delta^\natural\gamma^\natural \Rrightarrow \delta\gamma^\natural$ and $1_{F^\natural} \Rrightarrow 1_F{}^\natural$ with components $[\delta]_c[\gamma]_c \Rightarrow [\delta_c\gamma_c]$ and $1_{Fc} \Rightarrow [1_{Fc}]$ respectively. We then verify that this data satisfies the necessary conditions of a globally split co-2-presheaf:

(a) Whiskering is strictly preserved: for a functor $F\colon C \to D$, $F^\natural\colon C^\natural \to D^\natural$ is defined on generators by $F^\natural[f] = [Ff]$, so that for a natural transformation $\delta\colon G \Rightarrow H\colon B \to C$ and functor $K\colon A \to B$ we have:

$$\left(F^\natural\,\delta^\natural\right)_c = F^\natural\left(\delta^\natural_c\right) = F^\natural[\delta_c] = [F\delta_c] = (F\delta)^\natural_c \qquad \left(\delta^\natural\,F^\natural\right)_c = \delta^\natural_{F^\natural c} = [\delta_{Fc}] = (\delta F)^\natural_c$$

(b) for a diagram in Cat of the form $\begin{tikzcd} A \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & C \end{tikzcd}$ the pseudo-functoriality 2-cell $(\alpha K)^\natural\,(F\beta)^\natural \Rrightarrow (G\beta)^\natural\,(\alpha H)^\natural$ is the interchanger for $\beta^\natural$ and $\alpha^\natural$. This is true because $C^\natural$ is given locally by equivalence relations, so there is at most one modification between 2-cells with D as their 0-cell codomain.

This globally-split 2-presheaf is therefore corresponds to a Gray-functor $(-)^\natural\colon \widetilde{\mathsf{Cat}} \to \mathsf{Gray}$, and the map $\mathcal{M}$ corresponds to the Gray-functor $\mathsf{GRAY}\big((-)^\natural, \mathcal{B}\big)\colon \widetilde{\mathsf{Cat}}^{\mathsf{op}} \to \mathsf{GRAY}$. The Grothendieck construction for such a Gray functor, described in Construction 6.2.6 coincides with Buckley's Grothendieck construction defined for the underlying globally split 2-presheaf, so we conclude that the projection $\left|\mathsf{GRAY}\big((-)^\natural, \mathcal{B}\big)\right|\colon \int \mathsf{GRAY}\big((-)^\natural, \mathcal{B}\big) \to \mathsf{Cat}$ is isomorphic to $\varpi_{\mathcal{B}}\colon \mathfrak{P}\mathcal{B} \to \mathsf{Cat}$.

It's not too difficult to verify directly that this is true. The objects of this Grothendieck construction are pairs of a category C and a 2-functor $P'\colon C^\natural \to \mathcal{B}$, which is equally given by a pseudofunctor $P\colon C \to \mathcal{B}$. A 1-cell from (C,P) to (D,Q) is given by a pair of a functor $F\colon C \to D$ and a 1-cell in $\mathsf{GRAY}\big(C^\natural, \mathcal{B}\big)$ from P' to $\mathsf{GRAY}\big(F^\natural, \mathcal{B}\big)(Q') = Q'\,F^\natural$. But $Q'\,F^\natural = (QF)'$, so this is equivalently a pseudo-natural transformation $P \Rightarrow QF$, and so on.

We can observe moreover that the map on objects $\mathcal{B} \mapsto \left[(-)^\natural, \mathcal{B}\right]$ extends to a Gray-functor $\left[(-)^\natural, 1\right]\colon \mathsf{GRAY} \to \left[\widetilde{\mathsf{Cat}}^{\mathsf{op}}, \mathsf{GRAY}\right]$ — the Gray-profunctor co-represented by $(-)^\natural\colon \widetilde{\mathsf{Cat}} \to \mathsf{GRAY}$ — and $\mathfrak{P}_1\colon \mathsf{GRAY} \to \mathsf{GRAY}/\mathsf{Cat}$ factors through $\left[\widetilde{\mathsf{Cat}}^{\mathsf{op}}, \mathsf{Gray}\right]$ as follows:

$$\begin{tikzcd} \mathsf{GRAY} \arrow[rr, "\mathfrak{P}"] \arrow[d, "{\left[(-)^\natural,1\right]}"'] \arrow[drr, "\mathfrak{P}_1"] & \arrow[dr, phantom, "\circlearrowleft"] & \mathsf{GRAY} \\ \left[\widetilde{\mathsf{Cat}}^{\mathsf{op}}, \mathsf{GRAY}\right] \arrow[rr, "\int"'] \arrow[ur, phantom, "\circlearrowleft"] & & \mathsf{GRAY}/\mathsf{Cat} \arrow[u, "\pi"'] \end{tikzcd}$$

We have already seen that the lower triangle commutes on objects, and it is not difficult to verify that for a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$, the map $\int\left[(-)^\natural, F\right]\colon \int\left[(-)^\natural, \mathcal{A}\right] \to \int\left[(-)^\natural, \mathcal{B}\right]$ is given by post-composition of the second component with F, as required (cf. Construction 6.2.13).

From this factorisation we conclude:

Lemma 7.6.1. $\mathfrak{P}_1\colon \mathsf{GRAY} \to \mathsf{GRAY}$ *has a left* Gray*-adjoint, and factors through the inclusion of the locally-full sub-*Gray*-category of globally split 2-fibrations and cartesian maps which strictly preserve chosen-cartesian 1-cells and chosen-opcartesian 2-cells.*

Proof. We can construct a left Gray-adjoint to $\mathfrak{P}_1$ by composing the left Gray-adjoint to $\int$ described in Construction 6.2.18 (Corollary 6.2.21) with the left Gray-adjoint to $\left[(-)^\natural, 1\right]$, which is given by the left extension L of $(-)^\natural$ along the Yoneda embedding:

$$\begin{tikzcd}
\widetilde{\mathsf{Cat}} \arrow[r, "(-)^{\natural}"] \arrow[d, "よ"'] & \mathsf{GRAY} \\
\left[\widetilde{\mathsf{Cat}}^{\mathsf{op}}, \mathsf{GRAY}\right] \arrow[ur, "L"'] &
\end{tikzcd}$$

The action of L on objects can be expressed in terms of coends in Gray as:

$$F \mapsto \int^{C \in \mathsf{Cat}} FC \otimes C^{\natural}$$

Taking the composition of these two left Gray-adjoints gives a Gray-functor $\mathscr{P}\colon \mathsf{GRAY}/\mathsf{Cat} \to \mathsf{GRAY}$ whose action on objects is given by:

$$\left(\mathcal{A} \xrightarrow{P} \mathsf{Cat}\right) \mapsto \int^{C \in \widetilde{\mathsf{Cat}}} \widehat{P} \otimes C^{\natural} \cong \int^{C \in \widetilde{\mathsf{Cat}}} (C \Downarrow P) \otimes C^{\natural} \tag{7.8}$$

And finally, we note that $\mathfrak{P}_1$ factors through the inclusion of globally split 2-fibrations and split morphisms of such fibrations because the Gray-variant of the Grothendieck construction does. □

Remark 7.6.2. Just as the extralax colimit provides a left-1-adjoint for $(\mathsf{F}_\Omega)_1$, the *pseudolax* colimit provides a left 1-adjoint for $\mathfrak{P}_1$, which must therefore be isomorphic as a 1-functor to the left Gray-adjoint $\mathscr{P}$ just defined. In particular, for $P\colon C \to \mathsf{Cat}$ a pseudofunctor from a 1-category and $P'\colon C^{\natural} \to \mathsf{Cat}$ its classifying 2-functor, the image of P' under $\mathscr{P}$ must be isomorphic to the pseudolax colimit $\mathcal{Q}P$. This observation will be relevant to demonstrating a cartesianness property for the multiplication of $\mathfrak{P}$ in the next section. ◊

Remark 7.6.3. It's worth considering why this trick cannot be used to show that $(\mathsf{G}_\Omega)_1$ has a left $\mathsf{Gray}_{\mathscr{L}}$-adjoint. We might expect that there is a variant of the $\mathcal{A} \mapsto \widetilde{\mathcal{A}}$ construction which replaces $\mathcal{A}(x,y)$ with $\mathcal{A}(x,y)^{\dagger}$, but this cannot be defined in the same way as for $\widetilde{\mathcal{A}}$ because there is no equivalence $(C \times D)^{\dagger} \simeq C^{\dagger} \boxminus C^{\dagger}$ (the square in Remark 6.2.2 cannot be recovered from the triangles if the 2-cells aren't invertible). Correspondingly, a natural transformation $\alpha\colon F \Rightarrow G\colon C \to D$ between 1-functors induces a *comodule* between the corresponding 2-functors $F^{\dagger} \Rightarrow G^{\dagger}\colon C^{\dagger} \to D^{\dagger}$, rather than a lax transformation. ◊

7.7 $\mathfrak{P}$ is a Familial Gray-monad

Consider the naturality square for the unit η of the monad $\mathfrak{P}$ at $!\colon \mathcal{A} \to \mathbb{1}$ (recall that by arguments given for the $\mathfrak{S}$ case, this is the only naturality square we need consider):

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "!"] \arrow[d, "\eta_{\mathcal{A}}"'] & \mathbb{1} \arrow[d, "\langle \mathbb{1} \rangle"] \\
\mathfrak{P}\mathcal{A} \arrow[r, "\varpi"'] & \mathsf{Cat}
\end{tikzcd} \tag{7.9}$$

This square is *not* a pullback square. If we take the pullback of $\varpi\colon \mathfrak{P}\mathcal{A} \to \mathsf{Cat}$ along $\langle \mathbb{1} \rangle : \mathbb{1} \to \mathsf{Cat}$, we get the 2-category $\mathsf{Pseudo}(\mathbb{1}, \mathcal{A})$ of pseudofunctors, pseudonatural transformations and modification, which we can identify with $\mathsf{Gray}\left(\mathbb{1}^{\natural}, \mathcal{A}\right)$. This category is not isomorphic to $\mathcal{A}$. However, there is a Gray-equivalence between these 2-categories. Note that $\mathcal{A}$ can be identified with $\mathsf{Gray}(\mathbb{1}, \mathcal{A})$ and that the canonical map from $\mathcal{A}$ to the pullback corresponds to the map $\mathsf{Gray}(\mathbb{1}, \mathcal{A}) \to \mathsf{Gray}\left(\mathbb{1}^{\natural}, \mathcal{A}\right)$ given by precomposition by the unique 2-functor $!\colon \mathbb{1}^{\natural} \to \mathbb{1}$. This 2-functor is not fully-faithful, so it isn't a 2-equivalence, but it is a Gray-equivalence, whose inverse is given by the unique 2-functor[3] $\rho\colon \mathbb{1} \to \mathbb{1}^{\natural}$. There is an invertible pseudo-natural transformation — in fact, an *icon* — between

[3]Note that the 2-functor ρ is distinct from the canonical *pseudo*-functor $\mathbb{1} \to \mathbb{1}^{\natural}$ arising from the universal property of $\mathbb{1}^{\natural}$ as the pseudo-functor classifier.

the identity on $\mathbb{1}^\natural$ and $\rho\,!\colon \mathbb{1}^\natural \to \mathbb{1} \to \mathbb{1}^\natural$ (and clearly $!\,\rho = 1_{\mathbb{1}}$). The image of a Gray-equivalence under $\mathsf{Gray}(-, \mathcal{A})$ will be another Gray-equivalence, so we conclude that $\mathcal{A}$ is Gray-equivalent to the pullback, though not isomorphic. We might call $\mathcal{A}$ a Gray-bipullback, or – adopting the *n*Lab naming convention — simply a Gray-pullback. While this term might seem more appropriate to the case where the square may be allowed to commute merely up to invertible pseudo-natural isomorphism, we note that because one of the legs of the square in (7.9) is a 2-fibration — which by Lemma 6.1.9 is a Gray-representable notion — the Gray-pullback $\mathcal{A}$ also has a universal property with respect to such weakly-commuting squares as shown in [JS93]. Moreover, the fact that the square (7.9) strictly commutes was a result of a convenient choice of presentation for the free Ω-cocompletion $\mathsf{F}_\Omega\mathcal{A}$ and definition for the unit — in general, a free cocompletion will only have pseudo-natural unit and multiplication — so it seems appropriate that the pullback should be defined by a universal property with respect to weakly-commuting squares. It's important to note that this condition is still strict enough that considering only the naturality squares for the unique maps to $\mathbb{1}$ is sufficient. Considering the following diagram:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "\eta_{\mathcal{A}}"'] & \mathcal{B} \arrow[r, "!"] \arrow[d, "\eta_{\mathcal{B}}"] & \mathbb{1} \arrow[d, "\langle \mathbb{1} \rangle"] \\
\mathfrak{P}\mathcal{A} \arrow[r, "\mathfrak{P}F"'] & \mathfrak{P}\mathcal{B} \arrow[r, "\varpi"'] & \mathsf{Cat}
\end{tikzcd}$$

If $\mathcal{B}$ is Gray-equivalent to the pullback of the right square, then its pullback along $\mathfrak{P}F$ will be Gray-equivalent to the pullback of the combined square by the fact that taking pullbacks defines a Gray-functor and thus preserves Gray-equivalences. Then if $\mathcal{A}$ is Gray equivalent to the pullback of the combined square, it is also Gray-equivalent to the pullback of $\mathcal{B}$, by composition of Gray-equivalences.

Now consider the naturality square for the multiplication, μ at $!\colon \mathcal{A} \to \mathbb{1}$:

$$\begin{tikzcd}
\mathfrak{P}^2\mathcal{A} \arrow[r, "\mu_{\mathcal{A}}"] \arrow[d, "\mathfrak{P}\varpi_{\mathcal{A}}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{P}\mathcal{A} \arrow[d, "\varpi_{\mathcal{A}}"] \\
\mathfrak{P}\mathsf{Cat} \arrow[r, "\int"'] & \mathsf{Cat}
\end{tikzcd} \tag{7.10}$$

We will find that just as for the unit square this is not a 2-pullback, but merely a Gray-pullback. We can show this by reasoning similar to what was applied to the $\mathfrak{S}$-case.

First, observe that both $\varpi_{\mathcal{A}}$ and $\mathfrak{P}\varpi_{\mathcal{A}}$ are 2-fibrations: the former by the fact that $\mathfrak{P}_1$ factors through globally split 2-fibrations[4], and the latter by the fact that $\mathfrak{P}$ preserves 2-fibrations (Corollary 7.3.4).

Second, we observe that $(\mu_{\mathcal{A}}, \int)$ is a morphism of 2-fibrations. For the preservation of cartesian 1-cells, it suffices to show that $\mu_{\mathcal{A}}$ maps *chosen*-cartesian 1-cells in $\mathfrak{P}^2\mathcal{A}$ to cartesian 1-cells in $\mathfrak{P}\mathcal{A}$. A 1-cell of $\mathfrak{P}^2\mathcal{A}$ is given by a pseudo-commuting triangle as shown below:

$$\begin{tikzcd}
C \arrow[dr, "P"] \arrow[d, "F"'] \arrow[dr, phantom, shift right=3, "\phi\Downarrow"] & \\
D \arrow[r, "Q"'] & \mathfrak{P}\mathcal{A}
\end{tikzcd}$$

Let $\vec{P}_c\colon \hat{P}_c \to \mathcal{A}$ denote the image of c under P, let $\vec{P}_u\colon \vec{P} \Rightarrow \vec{P}_{c'}\,\hat{P}_u$ denote the image of $u\colon c \to c'$ under P and let $\vec{\phi}_c\colon \vec{P}_c \Rightarrow \vec{Q}_{Fc}\,\hat{\phi}_c$ denote the component of ϕ at $c \in C$ as a 1-cell in $\mathfrak{P}\mathcal{A}$. In terms of this data, the image of (F, ϕ) under $\mu_{\mathcal{A}}$ is the 1-cell in $\mathfrak{P}\mathcal{A}$ which we will denote:

$$\begin{tikzcd}
\int\hat{P} \arrow[dr, "\int\vec{P}"] \arrow[d, "\int\hat{\phi}"'] \arrow[dr, phantom, shift right=3, "\int\vec{\phi}\Downarrow"] & \\
\int\hat{Q} \arrow[r, "\int\vec{Q}"'] & \mathcal{A}
\end{tikzcd}$$

[4]Alternatively, note that $\varpi_{\mathcal{A}}$ is isomorphic to the image of the 2-fibration $!\colon \mathcal{A} \to \mathbb{1}$ under $\mathfrak{P}$.

The 2-functor $\int\hat{\phi}$ maps $\left(c \in C, x \in \hat{P}_c\right)$ to $\left(Fc, \hat{\phi}_c x\right)$ and the component of $\int\vec{\phi}$ at (c,x) is given by the morphism $\vec{\phi}_c x \colon \vec{P}_c x \to \vec{Q}_{Fc}\left(\hat{\phi}_c x\right)$ in $\mathcal{A}$. Now, ϕ will be chosen-cartesian if each of its components are $\varpi_{\mathcal{A}}$ chosen-cartesian, which means that their pseudo-natural transformation part is an identity. That is, each $\vec{\phi}_c \colon \vec{P}_c \Rightarrow \vec{Q}_{Fc}\hat{\phi}_c$ must be an identity transformation. If this is the case, then each component of the pseudonatural transformation $\int\vec{\phi}$ is an identity. This doesn't necessarily mean the transformation is an identity transformation, but it is nevertheless cartesian, since the cartesian 1-cells in $\mathfrak{P}\mathcal{A}$ with respect to the projection $\varpi_{\mathcal{A}} \colon \mathfrak{P}\mathcal{A} \to \mathsf{Cat}$ are precisely those 1-cells whose pseudo-natural transformation part has invertible components. So we conclude that $\mu_{\mathcal{A}}$ preserves cartesian 1-cells.

A similar argument shows that for a 2-cell $(\gamma, \Gamma) \colon (F, \phi) \Rightarrow (G, \psi)$ in $\mathfrak{P}^2\mathcal{A}$, the modification part of the image under $\mu_{\mathcal{A}}$ is the 2-cell in $\mathfrak{P}\mathcal{A}$ whose modification part $\int\vec{\Gamma}$, has component at $(c,x) \in \int\hat{P}$ given by $\vec{\Gamma}_c x$:

$$\begin{tikzcd}
 & \vec{Q}_{Fc}\hat{\phi}_c x \arrow[r, "\vec{Q}_{\gamma_c}\hat{\phi}_c x"] \arrow[d, phantom, "\Downarrow \vec{\Gamma}_c x"] & \vec{Q}_{Gc}\hat{Q}_{\gamma_c}\hat{\phi}_c \arrow[dr, "\vec{Q}_{Gc}\hat{\Gamma}_c x"] & \\
\vec{P}_c x \arrow[ur, "\vec{\phi}_c x"] \arrow[rrr, "\vec{\psi}_c x"'] & & & \vec{Q}_{Gc}\hat{\phi}_c x
\end{tikzcd} \tag{7.11}$$

Where $\hat{\Gamma}_c$ denotes the natural transformation part of the 2-cell Γ_c in $\mathfrak{P}\mathcal{A}$, and $\vec{\Gamma}_c$ denotes the modification part. When (γ, Γ) is a chosen-opcartesian 2-cell, each component Γ_c must be $\varpi_{\mathcal{A}}$-chosen opcartesian, meaning its modification part $\vec{\Gamma}_c$ must be an identity and so in particular all the 2-cells of the form shown in (7.11) must be identities, and thus $\int\vec{\Gamma}$ is (chosen) opcartesian.

Having shown that $(\int, \mu_{\mathcal{A}})$ forms a map of 2-fibrations $\mathfrak{P}\varpi_{\mathcal{A}}$ to $\varpi_{\mathcal{A}}$, the induced map $S \colon \mathfrak{P}^2\mathcal{A} \to \mathfrak{P}\mathcal{A} \times_{\mathsf{Cat}} \mathfrak{P}\mathsf{Cat}$ is also a map of 2-fibrations. To show that this map is a Gray-equivalence, it therefore suffices to show that it is a Gray-equivalence *on fibres* since by the general properties of 2-fibrations, fibre-wise pseudo-inverses can be extended to a pseudo-inverse for S.

The fibre of $\mathfrak{P}\varpi_{\mathcal{A}}$ at some object $P \colon C \to \mathsf{Cat}$ of $\mathfrak{P}\mathsf{Cat}$ is isomorphic to $\mathsf{GRAY/Cat}(P', \varpi_{\mathcal{A}})$, where $P' \colon C^{\natural} \to \mathsf{Cat}$ is the strict 2-functor classifying the pseudofunctor P. By the adjunction $\mathscr{P} \dashv \mathfrak{P}_1$, this in turn is isomorphic to $\mathsf{GRAY}(\mathcal{Q}P, \mathcal{A})$ — where $\mathcal{Q}P$ is the pseudo-lax colimit — by the fact that $\mathcal{Q}P \cong \mathscr{P}P'$ (Remark 7.6.2). On the other hand, the fibre of $\mathfrak{P}\mathcal{A} \times_{\mathsf{Cat}} \mathfrak{P}\mathsf{Cat}$ over P is isomorphic to $\mathsf{GRAY}\left((\int P)^{\natural}, \mathcal{A}\right)$. Now, the multiplication for $\mathfrak{P}$ is computed by sending an object of $\mathfrak{P}^2\mathcal{A}$ in the fibre over P to the corresponding 2-functor $\mathcal{Q}P \to \mathcal{A}$ and then precomposing by the canonical pseudofunctor $\lambda \colon \int P \to \mathcal{Q}P$. So the map S_P between fibres from $\mathsf{GRAY}(\mathcal{Q}P, \mathcal{A})$ to $\mathsf{GRAY}\left((\int P)^{\natural}, \mathcal{A}\right)$ is given by precomposition with the strict 2-functor $\overline{\lambda} \colon (\int P)^{\natural} \to \mathcal{Q}P$ which classifies the pseudofunctor $\lambda \colon \int P \to \mathcal{Q}P$.

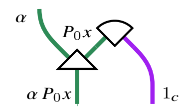
We can define a Gray-pseudo-inverse $\mathcal{T} \colon \mathcal{Q}P \to (\int P)^{\natural}$ by acting as the identity on objects, and on morphisms by the following mapping of generating 1-cells for $u \colon c \to d$ in C and $\alpha \colon x \to y$ in P_c:

$$[u]_x \mapsto \left[\left(u, 1_{P_u x}\right)\right] \qquad [\alpha] \mapsto [(1_c, \alpha\, P_0 x)]$$

Both $\mathcal{Q}P$ and $(\int P)^{\natural}$ are locally equivalence relations given by the fibres of the projections $\mathcal{Q}P \to \pi_1(\mathcal{Q}P) \cong \int P$ and $(\int P)^{\natural} \to \pi_1\left((\int P)^{\natural}\right) \cong \int P$, so the mapping $\mathcal{T}$ on generating 1-cells extends to a 2-functor precisely if it commutes with these two projections, which it does: note that the normal 1-cells in the connected components of $[u]_x$ and $[\alpha]$ are respectively $\left[1_{P_u x}\right][u]_x$ and $[\alpha\, P_0 x][1_c]_x$ (cf. Lemma 4.3.19) which under the isomorphism $\pi_1(\mathcal{Q}P) \cong \int P$ are identified with the 1-cells $\left(u, 1_{P_u x}\right)$ and $(1_c, \alpha\, P_0 x)$.

The map $\overline{\lambda} \colon (\int P)^{\natural} \to \mathcal{Q}P$ also commutes with the projections down to $\int P$. There therefore exists a pseudo-natural transformation $\sigma \colon 1_{\mathcal{Q}P} \Rightarrow \overline{\lambda}\mathcal{T}$ whose components are identities and whose naturality 2-cell at a 1-cell $f \in \mathcal{Q}P$ is given by the unique (invertible) 2-cell $\overline{\lambda}\mathcal{T}(f) \Rightarrow f$. The uniqueness of 2-cells in $\mathcal{Q}P$ ensures that this choice satisfies the axioms of an invertible icon. The same argument applies to constructing an invertible icon τ from $\mathcal{T}\overline{\lambda}$ to $1_{(\int P)^{\natural}}$.

The existence and definition of these invertible icons is perhaps clarified by an example of the pseudo-naturality 2-cells. For $\alpha\colon x \to y$ in P_c, the image of $[\alpha]$ under $\overline{\lambda}\mathcal{T}$ is $[\alpha P_0 x][1_c]_x$, and the 2-cell $\sigma_{[\alpha]}$ is the 2-cell relating $[\alpha]$ to its normal 1-cell:



The image of $[(u,\alpha)]\colon (c,x) \to (d,y)$ in $(\int P)^\natural$ under $\mathcal{T}\overline{\lambda}$ is $[(1_d, \alpha\,P_0P_u x)]\big[(u, 1_{P_u x})\big]$. The component of τ at the generating 1-cell $[(u,\alpha)]$ is the unique 2-cell in $(\int P)^\natural$ corresponding to the property that $(1_d, \alpha\,P_0P_u x)\big(u, 1_{P_u x}\big) = (u,\alpha)$ in $\int P$.

Remark 7.7.1. We can give a more abstract description of the map $\mathcal{T}\colon \mathcal{Q}P \to (\int P)^\natural$ by appealing to the comonadic structure of $(-)^\natural$ as an endo-2-functor on the 2-category of 2-categories, 2-functors and *invertible icons*. Coalgebras for this comonad are 2-categories whose underlying 1-category is free on a graph are coalgebras for this comonad [Lac02a], which includes $\mathcal{Q}P$. The map $\mathcal{T}$ is then given in terms of a coalgebra structure map $\rho\colon \mathcal{Q}P \to (\mathcal{Q}P)^\natural$ by the composite $\mathcal{Q}P \xrightarrow{\rho} (\mathcal{Q}P)^\natural \xrightarrow{\mathsf{p}_1{}^\natural} (\int P)^\natural$. ◊

We conclude that for any $P\colon C \to \mathsf{Cat}$, the 2-categories $(\int P)^\natural$ and $\mathcal{Q}P$ are Gray-equivalent, and that the image of this equivalence under the Gray-functor $\mathsf{Gray}(-,\mathcal{A})$ gives a Gray-equivalence between the fibres over P of the pullback $\mathfrak{P}\mathcal{A} \times_{\mathsf{Cat}} \mathfrak{P}\mathsf{Cat}$ and $\mathfrak{P}^2\mathcal{A}$ which exhibits the induced map $S\colon \mathfrak{P}^2\mathcal{A} \to \mathfrak{P}\mathcal{A} \times_{\mathsf{Cat}} \mathfrak{P}\mathsf{Cat}$ as a Gray-equivalence. Consequently, we have the following "familial Gray-monad" property of $\mathfrak{P}$:

Proposition 7.7.2. *The* Gray*-functor* $\mathfrak{P}\colon \mathsf{GRAY} \to \mathsf{GRAY}$ *satisfies the property that* $\mathfrak{P}_1\colon \mathsf{GRAY} \to \mathsf{GRAY}/\mathsf{Cat}$ *factors through globally-split 2-fibrations and has a left* Gray*-adjoint. Moreover, the naturality squares of the underlying 2-monad are* Gray*-pullbacks.*

Remark 7.7.3. We refrain from referring to $\mathfrak{P}$ as a Gray-monad — referring instead to the underlying 2-monad — because it is certainly not a Gray-monad in the sense of $\mathcal{V}$-monads for general monoidal $\mathcal{V}$. Though it fails to be a Gray monad in the same way that $\mathfrak{S}$ fails to be a $\mathsf{2Cat}$-monad and indeed Fam fails to be a Cat-monad: it's associativity and unit don't hold strictly. The unit and multiplication maps are Gray-natural — essentially by construction — so this appears to be its only obstacle to $\mathfrak{P}$ being a genuine Gray-monad. The appropriate notion, then, is probably some partially-strict *pseudo*-Gray-monad in the same way that the 2-monads of this section are partially-strict pseudomonads, rather than Cat-monads (cf. Remark 7.1.3) though we don't attempt to make this precise. ◊

7.7.1 Application: Composition of opfibrations from the indexed perspective

When a monad $T\colon C \to C$ on a finitely complete category is cartesian, one can define for any algebra $\alpha\colon TA \to A$ an internal category by the span $TA \xleftarrow{T\alpha} T^2A \xrightarrow{\mu_A} TA$. The identities and composition for the internal category are given in terms of the unit and multiplication for T:

$$
\begin{tikzcd}
 & TA \arrow[dl, equal] \arrow[dr, equal] \arrow[d, "T\eta"] & \\
TA \arrow[r, phantom, "\circlearrowleft"] & T^2A \arrow[l, "T\alpha"] \arrow[r, "\mu_A"'] & TA \arrow[l, phantom, "\circlearrowleft"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & & T^3A \arrow[dl, "T^2\alpha"'] \arrow[dr, "\mu T_A"] \arrow[ddd, bend left, "T\mu_A"] & & & \\
 & & T^2A \arrow[dll, "T\alpha"'] \arrow[dr, "\mu_A"'] & & T^2A \arrow[dl, "T\alpha"] \arrow[drr, "\mu_A"] & & \\
TA & & & TA & & & TA \\
 & & & T^2A \arrow[ulll, "T\alpha"] \arrow[urrr, "\mu_A"'] & & &
\end{tikzcd}
$$

The 2-functor $!\colon \mathsf{Cat} \to \mathbb{1}$ is an algebra for the monad $\mathfrak{P}\colon \mathsf{GRAY} \to \mathsf{GRAY}$, which is cartesian in a weak sense, so we can consider the correspondingly weak notion of category internal to $\mathsf{2CAT}$ given by the span $\mathsf{Cat} \xleftarrow{\varpi} \mathfrak{P}\mathsf{Cat} \xrightarrow{\int} \mathsf{Cat}$.

This describes something like a double category whose (let's say) horizontal morphisms are functors and whose vertical morphisms from D to C are given by pseudofunctors $P\colon C \to \mathsf{Cat}$ with $\int P = D$. The "identity" vertical morphisms are given by the unit:

$$\begin{tikzcd} & \mathsf{Cat} \arrow[dl, equal] \arrow[dr, equal] \arrow[dd, "\mathfrak{P}\eta"] & \\ \mathsf{Cat} \arrow[r, phantom, "\cup"] & {} \arrow[r, phantom, "\cong"] & \mathsf{Cat} \\ & \mathfrak{P}\mathsf{Cat} \arrow[ul, "\varpi"] \arrow[ur, "\int"'] & \end{tikzcd}$$

That is, the vertical identity at $C \in \mathsf{Cat}$ is the pseudo-functor $\Delta_{\mathbb{1}}\colon C \to \mathsf{Cat}$. The fact that the unit law for $\mathfrak{P}$ holds only up to isomorphism means that this "vertical identity" isn't literally a vertical morphism from C to C. Rather, it goes from $\int\Delta_{\mathbb{1}}$ to C, and $\int\Delta_{\mathbb{1}}$ is merely isomorphic to C.

For the composition, given pseudo-functors $P\colon C \to \mathsf{Cat}$ and $Q\colon \int P \to \mathsf{Cat}$ which represent vertical morphisms $\int P \to C$ and $\int Q \to \int P$ respectively, we find the composite by first lifting the pair (P, Q) to the Gray-pullback, $\mathfrak{P}^2\mathsf{Cat}$, and then projecting down to $\mathfrak{P}\mathsf{Cat}$ via $\mathfrak{P}(\int)$. The result is a pseudo-functor $Q \diamond P\colon C \to \mathsf{Cat}$ which sends c to the Grothendieck construction on $Pc \hookrightarrow \int P \xrightarrow{Q} \mathsf{Cat}$. Once again, the domain for the composition, $\int(Q \diamond P)$ is merely isomorphic to $\int Q$ because the associativity for $\mathfrak{P}$ holds only up to isomorphism.

Of course, a similar construction is possible for the monad $\mathfrak{S}$, whose morphisms can be thought of as indexed representations of *split* fibrations, rather than arbitrary fibrations.

7.8 $\mathfrak{P}$-functors and $\mathfrak{S}$-functors

In Chapter 5 we characterised F_Θ-functors and F_Ω-functors as those $K\colon \mathcal{A} \to \mathcal{B}$ such that morphisms of the form $f\colon Ka \to b$ admit certain generic factorisations. More generally, for Φ some pre-saturated class of colimits $K\colon \mathcal{A} \to \mathcal{B}$ is a F_Φ-functor if for each $b \in \mathcal{B}$ the presheaf $\mathcal{B}(K-, b)\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ is a Φ-colimit of representables. This is equivalent to the condition that the profunctor $\mathcal{B}(K, 1)\colon \mathcal{B} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ factors through $\mathsf{F}_\Phi\mathcal{A}$. There are two ways we might think to adapt this definition to define $\mathfrak{P}$-functors and $\mathfrak{S}$-functors:

(a) Require $\mathcal{B}(K-, b)$ to be an oplax-image presheaf of a pseudo (resp. strict) functor from a 1-category — henceforth $\mathfrak{P}$-presheaf (resp. $\mathfrak{S}$-presheaf)

(b) Require $\mathcal{B}(K, 1)\colon \mathcal{B} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ to factor through the (non-full) inclusion $\mathfrak{P}\mathcal{A} \hookrightarrow \mathsf{F}_\Omega\mathcal{A} \hookrightarrow [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ (resp. $\mathfrak{S}\mathcal{A} \hookrightarrow \mathsf{F}_\Omega\mathcal{A} \hookrightarrow [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$).

Condition (b) is clearly stronger, and it is this definition we shall choose. These sorts of functors lie on the spectrum between $\mathsf{F}_\amalg$-functors and F_Θ-functors due to the position of $\mathfrak{P}$ and $\mathfrak{S}$ in the following chain of monads:

$$\mathsf{F}_\amalg \subset \mathfrak{S} \subset \mathfrak{P} \subset \mathsf{F}_\Omega \subset \mathsf{F}_\Theta \tag{7.12}$$

So $\mathfrak{S}$-functors and $\mathfrak{P}$-functors are in particular F_Ω-functors, and thus satisfy the same Ω-generic factorisation properties.

In fact, we can characterise $\mathfrak{P}$-functors and $\mathfrak{S}$-functors by generic factorisation properties. We start with a recognition lemma for $\mathfrak{P}$-presheaves:

Lemma 7.8.1. *A presheaf $X\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ is an oplax-image presheaf for a pseudo-functor from a 1-category if and only if it is a Ω-presheaf and the composition of component-initial 1-cells between generic objects in its category of elements $P\colon \mathsf{el}X \to \mathcal{A}$ is component-initial.*

Proof. Given any oplax-image presheaf, $Y\colon \mathcal{A}^{\mathrm{op}} \to \mathsf{Cat}$, one can construct its category of elements $P_Y\colon E_Y \to \mathcal{A}$, and then take the *generic core*: the restriction of this projection to the full sub-2-category on its generic objects. Denote the resulting local discrete opfibration $Q_Y\colon D \to \mathcal{A}$. Recall from Chapter 3 that the oplax-image presheaf of Q_Y is again Y. To say that Y is a $\mathfrak{P}$-presheaf is equivalent to requiring that it is the oplax-image presheaf of a strict 2-functor $C^{\natural} \to \mathcal{A}$ for C a 1-category. If D has component-initial 1-cells in each connected component of its hom-categories, then it admits a component-initial bijective-on-objects functor into it from $\pi_1(D)^{\dagger}$. If, moreover, the (potentially nullary) composition of component-initial 1-cells in D is always component-initial, then it also admits a locally-initial b.o.o functor from $\pi_1(D)^{\natural}$. The oplax-image presheaf of a 2-functor is constant under precomposition by a locally-initial b.o.o functor, so if D has component-initial 1-cells which are closed under composition, then the oplax-image presheaf of $Q_Y\colon D \to \mathcal{A}$ is equally the oplax-image presheaf of a 2-functor from $\pi_1(D)^{\natural}$, and thus a $\mathfrak{P}$-presheaf.

Conversely, if Y is the oplax-image presheaf of the pseudo-functor $F\colon C \to \mathcal{A}$, then the induced $F'\colon C^{\natural} \to \mathcal{A}$ must factor through the local discrete opfibration $Q\colon D \to \mathcal{A}$ via a b.o.o locally-initial functor $H\colon C^{\natural} \to D$. Locally-initial 2-functors create component-initial objects, so we conclude that D has component-initial 1-cells in each connected component of its hom-categories, and that the composition of component-initial 1-cells is component-initial. □

Essentially the same proof gives the following result for $\mathfrak{S}$-presheaves:

Lemma 7.8.2. *A presheaf $X\colon \mathcal{A}^{\mathrm{op}} \to \mathsf{Cat}$ is a $\mathfrak{S}$-presheaf if and only if it is a Ω-presheaf and component-initial 1-cells can be chosen in each connected component of the hom-categories between generic objects in its category of elements $P\colon \mathsf{el}X \to \mathcal{A}$ in such a way that the composition of chosen component-initial 1-cells is chosen component-initial, and identities are chosen component-initial.*

Proof. The key observation is that a 2-category D having chosen component-initial 1-cells closed under composition is equivalent to the existence of a bijective-on-objects locally-initial functor from a 1-category. □

From these lemmas we can characterise those 2-functors $K\colon \mathcal{A} \to \mathcal{B}$, satisfying the weaker condition (a) above:

Lemma 7.8.3. *For a 2-functor $K\colon \mathcal{A} \to \mathcal{B}$, $\mathcal{B}(K-, b)$ will be a $\mathfrak{P}$-presheaf (resp. $\mathfrak{S}$-presheaf) if the (potentially nullary) composition of (chosen) component-initial 1-cells between oplax-generic morphisms in $K \Downarrow b$ is (chosen) component-initial.*

However, $\mathfrak{P}$ and $\mathfrak{S}$ functors must additionally satisfy the property that for $f\colon b \to b'$ the morphism $\mathcal{B}(K-, b) \to \mathcal{B}(K-, b')$ is the image of a 1-cell in $\mathfrak{P}\mathcal{A}$ (resp. $\mathfrak{S}\mathcal{A}$). To determine what this entails, we consider what $\mathcal{B}(K-, f)\colon \mathcal{B}(K-, b) \to \mathcal{B}(K-, b')$ looks like as a 1-cell in $\mathsf{F}_{\Omega}\mathcal{A}$.

For K a F_{Ω}-functor, $\mathcal{B}(K-, b)$ is the oplax-image presheaf of the oplax functor from the 1-category whose objects are the generic morphisms to b, and whose 1-cells are the component-initial 1-cells between generic morphisms (assume we have chosen one from each connected component). Composition of 1-cells is given by taking the composition in $K \Downarrow b$ then taking the chosen component-initial object in its connected component. Denote this 1-category D_b. The oplax functor $P_b\colon D_b \to \mathcal{A}$ is the composition of the oplax "inclusion" $D_b \hookrightarrow K \Downarrow b$ with the projection $K \Downarrow b$. So, for example, a generic morphism $u\colon Ka \to b$ is sent to $a \in \mathcal{A}$. For $f\colon b \to b'$ in $\mathcal{B}$, the induced 1-cell from $P_b\colon D_b \to \mathcal{A}$ to $P_{b'}\colon D_{b'} \to \mathcal{A}$ in $\mathsf{F}_{\Omega}\mathcal{A}$ is given by the functor $D_f\colon D_b \to D_{b'}$ sending object $u\colon Ka \to b$ to the generic component of the factorisation of $fu\colon Ka \to b'$, which we denote $(fu)_2\colon K\widehat{fu} \to b'$. The action on a 1-cell

$(s,\sigma)\colon (x,u) \to (y,v)$ is given by taking the component-initial diagonal filler of the following diagram:

$$\begin{tikzcd} Kx \arrow[r, "K(fu)_1"] \arrow[d, "Ks"'] \arrow[dr, phantom, "\Downarrow f\sigma"] & K\widehat{fu} \arrow[dr, "(fu)_2"] & \\ & & b \\ Ky \arrow[r, "K(fv)_1"'] & K\widehat{fv} \arrow[ur, "(fv)_2"'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} Kx \arrow[r, "K(fu)_1"] \arrow[dd, "Ks"'] & K\widehat{fu} \arrow[dr, "(fu)_2"] \arrow[d, dashed] & \\ & Ks' \arrow[l, Leftarrow, "K\sigma_1"'] \arrow[r, phantom, "\Swarrow \sigma_2"] \arrow[d] & b' \\ Ky \arrow[r, "K(fv)_1"'] & K\widehat{fv} \arrow[ur, "(fv)_2"'] & \end{tikzcd} \tag{7.13}$$

That is, the image of (s,σ) under D_f is (s',σ_2).

$$\begin{tikzcd} Kx \arrow[dd, "Ks"'] \arrow[dr, "u"] & \\ \arrow[r, phantom, "\sigma\Downarrow"] & b \\ Ky \arrow[ur, "v"'] & \end{tikzcd} \quad \overset{D_f}{\longmapsto} \quad \begin{tikzcd} Kx \arrow[d, dashed] \arrow[dd, bend right=60, "Ks"'] \arrow[dr, "u"] & & \\ Ks' \arrow[d] \arrow[r, phantom, "\sigma'\Downarrow"] & b \arrow[r, "f"] & b' \\ Ky \arrow[ur, "v"'] & & \end{tikzcd}$$

For example, the above diagram indicates that (s,σ) would be sent to the component-initial $(s', f\sigma')$, where $fv\,K\alpha \circ f\sigma' = f\sigma$.

The lax transformation $P_f\colon P_b \to P_{b'}\,D_f$ is essentially given by the square in the diagram on the right of (7.13). That is, its component at $u\colon Kx \to b$ is $(fu)_1\colon x \to \widehat{fu}$, and the lax-naturality 2-cell at (s,σ) is given by $\sigma_1\colon s'\,(fu)_1 \Rightarrow (fv)_1\,s$. The invertibility of the lax-naturality 2-cell is equivalent to the condition that $((fv)_1\,s, f\sigma)$ is component-initial in the hom-category from fu to $(fv)_2$ in $K \Downarrow b'$. This is because it is isomorphic to the precomposition of the component-initial (s',σ_2) by the chosen-cartesian 1-cell from fu to $(fu)_2$, and such 1-cells are also component-initial by (the proof of) Lemma 5.2.4. Thus, P_f will be pseudo-natural precisely if for all component-initial 1-cells (s,σ) between oplax generics, the component-initial is preserved in the sense that the 1-cell on the right below is component-initial:

$$\begin{tikzcd} Kx \arrow[dd, "Ks"'] \arrow[dr, "u"] & \\ \arrow[r, phantom, "\sigma\Downarrow"] & b \\ Ky \arrow[ur, "v"'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} Kx \arrow[dd, "s"'] \arrow[dr, "u"] & \\ \arrow[r, phantom, "\sigma\Downarrow"] & b \arrow[d, "f"] \\ Ky \arrow[ur, "v"] \arrow[d, "K(fv)_1"'] \arrow[r, phantom, "\circlearrowright"] & b' \\ K\widehat{fv} \arrow[ur, "(fv)_2"'] & \end{tikzcd}$$

Similarly, for the lax-naturality 2-cells to be identities, it must be the case that chosen component-initial 1-cells are sent to chosen component-initial 1-cells into oplax-generics, where we extend the meaning of chosen component-initial 1-cell to include the composition of chosen cartesian 1-cells into oplax-generics with a chosen component-initial 1-cell to another oplax-generic. We will refer to these component-initial 1-cell preservation properties as "(strict) preservation of component-initial 1-cells into oplax-generics under post-whiskering". We can therefore characterise $\mathfrak{P}\mathcal{A}$ and $\mathfrak{S}\mathcal{A}$ functors as follows:

Lemma 7.8.4. *A 2-functor $K\colon \mathcal{A} \to \mathcal{B}$, is a $\mathfrak{P}$-functor (resp. $\mathfrak{S}$-functor) precisely if (chosen) component-initial lax triangles into oplax-generic morphisms are closed under (potentially nullary) composition and (strictly) preserved by post-whiskering in $K \Downarrow \mathcal{B}$.*

We can relate the notions of $\mathfrak{P}$-functors and $\mathfrak{S}$ functors to existing notions of familial functors. Our $\mathfrak{S}$-functors are also, as we show below, Weber's co-op-familial[5] 2-functors [Web07] and very similar to the *strictly oplax op-familial* 2-functors of Walker [Wal20] (a special case of his much more general class of *lax familial* pseudofunctors between bicategories). There is a subtle distinction between the two, however. Whereas a familial 2-functor $T\colon \mathcal{A} \to \mathcal{B}$ must satisfy the property that $T/1\colon \mathcal{A} \to \mathcal{B}/T1$ factors through the 2-category split fibrations *and split morphisms*,

[5] The addition of "co's" and "op's" is an attempt to dualise Walker's and Weber's terminology beyond its usage in the respective papers. It is intended to expresses that K will be familial in their terminologies if $\mathcal{K}^{\mathrm{cop}}$ is a $\mathfrak{S}$-functor.

Walker's notion of strictly lax familial seems equivalent to requiring that each $Tt_a\colon Ta \to T1$ is a split fibration (cf. Definition 5.5 and Theorem 5.8 from [Wal20]). There's also the more trivial distinction that the domain of a familial 2-functor is required to have a terminal object, though this seems to merely be a matter of convenience. Below we essentially reprove Walker's Theorem 5.8, [Wal20], modulo some dualisation and for the case where $T/1$ factors through split fibration and split morphisms.

Lemma 7.8.5. *A 2-functor $K\colon \mathcal{A} \to \mathcal{B}$ from a 2-category with an initial object $I \in \mathcal{A}$ is a $\mathfrak{S}$-functor if and only if $K^{\mathsf{cop}}\colon \mathcal{A}^{\mathsf{cop}} \to \mathcal{B}^{\mathsf{cop}}$ is a familial 2-functor.*

Proof. First, assume we have a $\mathfrak{S}$-functor $K\colon \mathcal{A} \to \mathcal{B}$. To show that K^{cop} is a familial 2-functor (Definition 7.1.2), we must show that:

(a) for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ the map $\mathcal{B}(Ki_a, b)\colon \mathcal{B}(Ka, b) \to \mathcal{B}(KI, b)$ is a split *op*fibration where $i_a\colon I \to a$ is the unique morphism from the initial object. Moreover, precomposition in $\mathcal{A}$ and postcomposition in $\mathcal{B}$ induce split-cartesian maps of these split opfibrations.

(b) The map $I/K\colon \mathcal{A} \to KI/\mathcal{B}$ has a *right* adjoint

For (a), we use the property that $\mathcal{B}(K-, b) \cong \oint^{c\in C} \mathcal{A}(-, Pc)$ for some strict functor from a 1-category $P\colon C \to \mathcal{A}$. The map $\mathcal{B}(Ki_a, b)$ is then isomorphic to the projection $\oint^{c\in C} \mathcal{A}(a, Pc) \to \oint^{c\in C} \mathcal{A}(I, Pc) \cong \oint^{c\in C} \mathbb{1} \cong C$ which is the Grothendieck opfibration for the strict presheaf $\mathcal{A}(a, P-)\colon C \to \mathsf{Cat}$ and thus a split opfibration. In fact, the map $\mathcal{B} \to \mathsf{Cat}^{\to}$ sending b to $\mathcal{B}(Ki_a, b)$ is up to isomorphism given by the composite:

$$\mathcal{B} \xrightarrow{R} \mathfrak{S}\mathcal{A} \xrightarrow{\mathfrak{S}(\mathcal{A}(a,-))} \mathfrak{S}\mathsf{Cat} \xrightarrow{\simeq} \mathsf{SpOpFib} \hookrightarrow \mathsf{Cat}^{\to}$$

where R is the factorisation of $\mathcal{B}(K, 1)\colon \mathcal{B} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ through $\mathfrak{S}$, and $\mathsf{SpOpFib}$ denotes the 2-category of split opfibrations and split opfibration morphisms in Cat, which was observed to be equivalent to $\mathfrak{S}\mathsf{Cat} \cong \mathsf{F}_{\Omega}\mathsf{Cat}_1$ in Example 4.7.3. So in particular, for $u\colon b \to b'$ the map $\mathcal{B}(Ka, u)\colon \mathcal{B}(Ka, b) \to \mathcal{B}(Ka, b')$ is a map of split opfibrations. If instead we fix b and consider a morphism $v\colon a \to a'$, the induced map between the opfibrations $\oint^{c\in C} \mathcal{A}(a', Pc) \to \oint^{c\in C} \mathcal{A}(a, Pc)$ is the image of the component in $\mathfrak{S}\mathsf{Cat}$ of the 2-natural transformation $\mathfrak{S}v_*\colon \mathfrak{S}(\mathcal{A}(a', -)) \Rightarrow \mathfrak{S}(\mathcal{A}(a, -))$ at (C, P) in $\mathfrak{S}\mathcal{A}$, whose image in $\mathsf{Cat}^{\to}$ is therefore a map of split opfibrations.

$$\begin{tikzcd} \mathcal{B} \arrow[r, "R"] & \mathfrak{S}\mathcal{A} \arrow[r, bend left, "\mathfrak{S}(\mathcal{A}(a',-))"] \arrow[r, bend right, "\mathfrak{S}(\mathcal{A}(a,-))"'] \arrow[r, phantom, "\Downarrow\mathfrak{S}(v^*)"] & \mathfrak{S}\mathsf{Cat} \arrow[r, "\simeq"] & \mathsf{SpOpFib} \arrow[r, hook] & \mathsf{Cat}^{\to} \end{tikzcd}$$

For (b), note that by the isomorphisms $\mathcal{B}(K-, b) \cong \oint^{c\in C} \mathcal{A}(-, Pc)$, we have in particular $\mathcal{B}(KI, b) \cong \oint^{c\in C} \mathcal{A}(I, Pc) \cong C$ so we can view P as a functor $\mathcal{B}(KI, b) \to \mathcal{A}$ and $\oint^{s\in\mathcal{B}(KI,b)} \mathcal{A}(-, Ps)$ as being fibred over $\mathcal{B}(KI, b)$. The fibre of $\mathcal{B}(Ki_a, b)\colon \mathcal{B}(Ka, b) \to \mathcal{B}(KI, b)$ at $s\colon KI \to b$ is isomorphic to $KI/\mathcal{B}(Ki_a, s)$ which is therefore also isomorphic to the fibre of $\oint^{s\in\mathcal{B}(KI,b)} \mathcal{A}(a, Ps)$ at s, i.e. $\mathcal{A}(a, Ps)$. So, we have isomorphisms $KI/\mathcal{B}(Ki_a, s) \cong \mathcal{A}(a, Ps)$ for each $a \in \mathcal{A}$ and $s \in KI/\mathcal{B}$. These isomorphisms are moreover 2-natural in a, by the fact that $\mathcal{B}(K-, b) \cong \oint^{s\in\mathcal{B}(KI,b)} \mathcal{A}(-, Ps)$ is a 2-natural transformation in $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$, so we can extend the mapping $s \mapsto Ps$ to a 2-functor $KI/\mathcal{B} \to \mathcal{A}$ which is right adjoint to I/K. We note incidentally that this right adjoint will send $s\colon KI \to b$ to a $\hat{s} \in \mathcal{A}$ which fits in a generic factorisation s, $KI \xrightarrow{Ki_{\hat{s}}} K\hat{s} \xrightarrow{s_2} b$ for s; the map s_2 being the counit.

Conversely, we assume now that $K\colon \mathcal{A} \to \mathcal{B}$ has the property that I/K factors through split co-op-fibrations and has a right adjoint, R, and show it follows that K is a $\mathfrak{S}$-functor. The image of a under I/K being a split co-op-fibration means in particular that $\mathcal{B}(Ki_a, b)\colon \mathcal{B}(Ka, b) \to \mathcal{B}(KI, b)$ is a split opfibration, so there exists some corresponding strict 2-functor from a locally discrete 1-category $P_{a,b}\colon \mathcal{B}(KI, b) \to \mathsf{Cat}$ which on objects maps $s\colon KI \to b$ to $KI/\mathcal{B}(Ki_a, s)$, such that $\mathcal{B}(Ka, b) \cong \oint^{s\colon KI\to b} P_{a,b}s$. Each category $KI/\mathcal{B}(Ki_a, s)$ is isomorphic to $\mathcal{A}(a, Rs)$, so $P_{a,b}$

can up-to-isomorphism be chosen to have its mapping on objects given by $s \mapsto \mathcal{A}(a, Rs)$. Moreover, for $v\colon a \to a'$ the maps $\mathcal{B}(Kv, b)\colon \mathcal{B}(Ka', b) \to \mathcal{B}(Ka, b)$ are by hypothesis split maps of opfibrations, and thus the induced maps on fibres $KI/\mathcal{B}(Ki_{a'}, s) \to KI/\mathcal{B}(Ki_a, s)$ are strictly natural in s. These induced maps translate under the isomorphisms $KI/\mathcal{B}(Ki_a, s) \cong \mathcal{A}(a, Rs)$ to the maps $\mathcal{A}(v, Rs)\colon \mathcal{A}(a', Rs) \to \mathcal{A}(a, Rs)$ — by the 2-naturality in $\mathcal{A}$ of these isomorphisms — so the maps $\mathcal{A}(v, Rs)$ form a strictly natural transformation $P_{a',b} \Rightarrow P_{a,b}\colon \mathcal{B}(KI, b) \to \mathsf{Cat}$. Consequently, for any 2-cell $\alpha\colon s \Rightarrow t\colon KI \to b$, the below square commutes:

$$\begin{tikzcd}
\mathcal{A}(a, Rs) \arrow[r, "P_{a,b}\alpha"] \arrow[d, "\mathcal{A}(v,Rs)"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{A}(a, Rt) \arrow[d, "\mathcal{A}(v,Rt)"] \\
\mathcal{A}(a', Rs) \arrow[r, "P_{a',b}\alpha"'] & \mathcal{A}(a', Rt)
\end{tikzcd}$$

It follows that the maps $P_{-,b}\alpha\colon \mathcal{A}(-, Rs) \Rightarrow \mathcal{A}(-, Rt)$ are strictly 2-natural transformations between representables, and so are represented by morphisms $\mathcal{R}_b\alpha\colon Rs \to Rt$. The assignments $\alpha \mapsto \mathcal{R}_b\alpha$ assemble into the mapping on 1-cells of a strict functor $\mathcal{R}_b\colon \mathcal{B}(KI, b) \to \mathcal{A}$ by the functoriality of $P_{a,b}$, and we have $P_{-,b}\square \cong \mathcal{A}(-, \mathcal{R}\square)$. It then follows that $\mathcal{B}(K-, b) \cong \oint^{s\colon KI \to b} \mathcal{A}(-, \mathcal{R}_b s)$.

This shows that for each $b \in \mathcal{B}$, $\mathcal{B}(K-, b)$ is a $\mathfrak{S}$-presheaf. To additionally show that the maps between these presheaves induced by $u\colon b \to b'$ lie in the image of $\mathfrak{S}\mathcal{A} \hookrightarrow [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ we use the fact that the maps between split opfibrations $\mathcal{B}(Ki_a, u)\colon \mathcal{B}(Ki_a, b) \to \mathcal{B}(Ki_a, b')$ are split. Such split maps correspond to strictly 2-natural transformations $P_{a,b} \Rightarrow \left(P_{a,b'} \circ \mathcal{B}(KI, u)\right)\colon \mathcal{B}(KI, b) \to \mathsf{Cat}$, and are thus induced by a strictly 2-natural transformation $\rho_u\colon \mathcal{R}_b \Rightarrow \mathcal{R}_{b'}\ \mathcal{B}(KI, u)\colon \mathcal{B}(KI, b) \to \mathcal{A}$ which constitutes a 1-cell $(\mathcal{B}(KI, u), \rho_u)\colon (\mathcal{B}(KI, b), \mathcal{R}_b) \to (\mathcal{B}(KI, b'), \mathcal{R}_{b'})$ in $\mathfrak{S}\mathcal{A}$ between the appropriate objects.

To summarise, we have constructed an explicit factorisation of $\mathcal{B}(K, 1)\colon \mathcal{B} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ through $\mathfrak{S}\mathcal{A}$ by the mapping on objects $b \mapsto (\mathcal{B}(KI, b), \mathcal{R}_b)$ and on 1-cells $u \mapsto (\mathcal{B}(KI, u), \rho_u)$, which therefore exhibits K as a $\mathfrak{S}$-functor.

This completes both directions of the proof. □

A very similar proof gives the corresponding result for $\mathfrak{P}$-functors:

Lemma 7.8.6. *A 2-functor $K\colon \mathcal{A} \to \mathcal{B}$ from a 2-category with an initial object $I \in \mathcal{A}$ is a $\mathfrak{P}$-functor if and only if $I/K\colon \mathcal{A} \to KI/\mathcal{B}$ factors through the category of co-op-fibrations under KI and has a right adjoint.*

Proof. First, assume that K is a $\mathfrak{P}$-functor, and thus that $\mathcal{B}(K, 1)\colon \mathcal{B} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ factors through the inclusion $\mathfrak{P}\mathcal{A} \hookrightarrow [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ via some 2-functor $R\colon \mathcal{B} \to \mathfrak{P}\mathcal{A}$. Then $\mathcal{B}(Ki_a, -)\colon \mathcal{B} \to \mathsf{Cat}^{\to}$ is given up-to-isomorphism by the following composite:

$$\mathcal{B} \xrightarrow{R} \mathfrak{P}\mathcal{A} \xrightarrow{\mathfrak{P}(\mathcal{A}(a,-))} \mathfrak{P}\mathsf{Cat} \xrightarrow{\simeq} \mathsf{OpFib} \hookrightarrow \mathsf{Cat}^{\to}$$

where the equivalence $\mathfrak{P}\mathsf{Cat} \simeq \mathsf{OpFib}$ is the one established in 4.7.2 by the fact[6] that $\mathsf{F}_\Omega\mathsf{Cat}_{\cong} \cong \mathfrak{P}\mathsf{Cat}$. So the map I/K does indeed send objects to co-op-fibrations. Moreover, for each $v\colon a \to a'$ and for any $b \in \mathcal{B}$, the map between op-fibrations $\mathcal{B}(Ki_v, b)\colon \mathcal{B}(Ki_{a'}, b) \to \mathcal{B}(Ki_a, b)$ is given by the component of the following natural transformation at b, where v^* denotes the natural transformation $\mathcal{A}(v, -)\colon \mathcal{A}(a', -) \Rightarrow \mathcal{A}(a, -)$

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R"] & \mathfrak{P}\mathcal{A} \arrow[r, bend left, "\mathfrak{P}(\mathcal{A}(a',-))"] \arrow[r, bend right, "\mathfrak{P}(\mathcal{A}(a,-))"'] \arrow[r, phantom, "\Downarrow\mathfrak{P}(v^*)"] & \mathfrak{P}\mathsf{Cat} \arrow[r, "\simeq"] & \mathsf{OpFib} \arrow[r, hook] & \mathsf{Cat}^{\to}
\end{tikzcd}$$

[6]More generally: $\mathfrak{P}\mathcal{K} \cong \mathsf{F}_\Omega\mathcal{K}$ whenever $\mathcal{K}$ is locally groupoidal, i.e. a (2,1)-category. We explore this further in the next section.

and is therefore a map of opfibrations, as required. For the existence of a right adjoint, we note that there is an isomorphism of opfibrations $\rho_{a,b} : \mathcal{B}(Ki_a, b) \cong \oint^{s\colon KI\to b} \mathcal{A}(a, P_b s)$ where $P_b\colon \mathcal{B}(KI, b) \to \mathcal{A}$ is the image of b under $R\colon \mathcal{B} \to \mathfrak{P}\mathcal{A}$. This isomorphism of opfibrations induces isomorphisms on the fibres $KI/\mathcal{B}(Ki_a, s) \cong \mathcal{A}(a, Ps)$ for each $a \in \mathcal{A}$ and $s\colon KI \to b$. These isomorphisms are in fact 2-natural in $a \in \mathcal{A}$, because $\mathcal{B}(K-, b) \cong \oint^{s\colon KI\to b} \mathcal{A}(-, Ps)$ is strictly 2-natural by assumption, and so in particular the corresponding square of opfibration morphisms commutes strictly for each $v\colon a \to a'$:

$$\begin{tikzcd}
\mathcal{B}(Ki_{a'}, b) \arrow[r, "\rho_{a',b}"] \arrow[d, "\mathcal{B}(Ki_v,b)"'] \arrow[dr, phantom, "\circlearrowleft"] & \left(\oint^{s\colon KI\to b} \mathcal{A}(a', P_b s)\right) \arrow[d, "\oint^{s\colon KI\to b} \mathcal{A}(v, P_b s)"] \\
\mathcal{B}(Ki_a, b) \arrow[r, "\rho_{a,b}"'] & \left(\oint^{s\colon KI\to b} \mathcal{A}(a, P_b s)\right)
\end{tikzcd}$$

The map on objects $s \mapsto P_b s$ therefore extends to a 2-functor $\mathcal{P}\colon KI/\mathcal{B} \to \mathcal{A}$ which is right-adjoint to I/K.

Conversely, assume that $K\colon \mathcal{A} \to \mathcal{B}$ satisfies the properties that I/K factors through co-op-fibrations and has a right adjoint. Similarly to the $\mathfrak{S}$-case, we note that the first of these properties implies the existence of *pseudo*-functors $P_{a,b}\colon \mathcal{B}(KI, b) \to \mathsf{Cat}$ satisfying $\oint^{s\colon KI\to b} P_{a,b} \cong \mathcal{B}(Ka, b)$ and $P_{a,b}s = \mathcal{A}(a, Rs)$. For $v\colon a \to a'$ the maps $\mathcal{B}(Kv, b)\colon \mathcal{B}(Ka', b) \to \mathcal{B}(Ka, b)$ are cartesian maps of opfibrations, and thus the induced maps on fibres $KI/\mathcal{B}(Ki_{a'}, s) \to KI/\mathcal{B}(Ki_a, s)$ are *pseudo*-natural in s. These induced maps translate under the (natural in $\mathcal{A}$) isomorphisms $KI/\mathcal{B}(Ki_a, s) \cong \mathcal{A}(a, Rs)$ to the maps $\mathcal{A}(v, Rs)\colon \mathcal{A}(a', Rs) \to \mathcal{A}(a, Rs)$ which therefore are also pseudo-natural in s as transformations $P_{a',b} \Rightarrow P_{a,b}\colon \mathcal{B}(KI, b) \to \mathsf{Cat}$. So for any 2-cell $\alpha\colon s \Rightarrow t\colon KI \to b$, the below square commutes up-to-isomorphism:

$$\begin{tikzcd}
\mathcal{A}(a, Rs) \arrow[r, "P_{a,b}\alpha"] \arrow[d, "\mathcal{A}(v,Rs)"'] \arrow[dr, phantom, "\phi_{v,\alpha}\nearrow"] & \mathcal{A}(a, Rt) \arrow[d, "\mathcal{A}(v,Rt)"] \\
\mathcal{A}(a', Rs) \arrow[r, "P_{a',b}\alpha"'] & \mathcal{A}(a', Rt)
\end{tikzcd} \tag{7.14}$$

Pseudonatural transformations between representables aren't in bijection with maps between the representing objects, but they do induce such maps. For each $\alpha\colon s \Rightarrow t$ we obtain a map $\mathcal{R}_b\alpha\colon Rs \to Rt$ by the image of $1_{Rs} \in \mathcal{A}(Rs, Rs)$ under $P_{Rs,b}\alpha\colon \mathcal{A}(Rs, Rs) \to \mathcal{A}(Rs, Rt)$. The 2-natural transformation between presheaves $\mathcal{A}(-, \mathcal{R}_b\alpha)\colon \mathcal{A}(-, Rs) \Rightarrow \mathcal{A}(-, Rt)$ is not *equal* to the pseudo-natural transformation $P_{-,b}\alpha$, but it is isomorphic in that there exists an invertible modification between them in $\mathsf{Gray}(\mathcal{A}^{\mathsf{op}}, \mathsf{Cat})$ whose components are determined by components of the isomorphism ϕ_α filling the square in (7.14). Because $P_{a,b}\colon \mathcal{B}(KI, b) \to \mathsf{Cat}$ is merely pseudo-functorial, the mapping $\alpha \mapsto \mathcal{R}_b\alpha$ defines a pseudofunctor $R\colon \mathcal{B}(KI, b) \to \mathcal{A}$ with the property that $\mathcal{A}(a, R-)$ is isomorphic to $P_{a,b}$ in $\mathsf{Pseudo}(\mathcal{B}(KI, b), \mathsf{Cat})$, the 2-category of pseudofunctors, pseudo-natural transformations and modifications. In fact, there is an invertible icon σ between them: for $\alpha\colon s \Rightarrow t\colon KI \to b$, the component of σ at α is the natural isomorphism $P_{a,b}\alpha \Rightarrow \mathcal{A}(a, \mathcal{R}_b\alpha)\colon \mathcal{A}(a, Rs) \to \mathcal{A}(a, Rt)$ whose component at $f\colon a \to Rs$ is given by:

$$(P_{a,b}\alpha) f = \left(P_{a,b}\alpha \circ \mathcal{A}(f, Rs)\right) 1_{Rs} \xrightarrow{\phi_{f,\alpha} 1_{Rs}} \left(\mathcal{A}(f, Rt) \circ P_{Rs,b}\alpha\right) 1_{Rs} = \mathcal{A}(f, Rt)\, \mathcal{R}_b\alpha = \mathcal{A}(a, \mathcal{R}_b\alpha)\, f$$

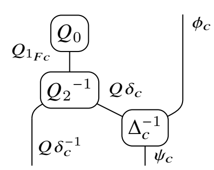
This invertible icon between $\mathcal{A}(a, \mathcal{R}_b-)$ and $P_{a,b}$ (and in general any invertible pseudonatural transformation of presheaves) induces an invertible cartesian map of opfibrations $\oint^{s\colon KI\to b} \mathcal{A}(a, \mathcal{R}_b s) \cong \oint^{s\colon KI\to b} P_{a,b}s \cong \mathcal{B}(Ka, b)$. Because these icons $\sigma_a\colon \mathcal{A}(a, \mathcal{R}_b-) \cong P_{a,b}$ are *strictly* 2-natural in $\mathcal{A}$, we more generally have $\mathcal{B}(K-, b) \cong \oint^{s\colon KI\to b} \mathcal{A}(-, \mathcal{R}_b s)$, which demonstrates that $\mathcal{B}(K-, b)\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ is a $\mathfrak{P}$-presheaf. The fact that $Ki_a\colon KI \to Ka$ is a co-op-fibration in $\mathcal{B}$ means that for $u\colon b \to b'$, the maps $\mathcal{B}(Ka, u)\colon \mathcal{B}(Ka, b) \to \mathcal{B}(Ka, b')$ are maps of opfibrations. It follows that the induced maps between presheaves $KI/\mathcal{B}(Ki_a, -) \Rightarrow KI/\mathcal{B}(Ki_a, u-)\colon \mathcal{B}(KI, b) \to \mathsf{Cat}$ are pseudo-natural, rather than merely lax-natural. Mapping across the $I/K \dashv R$ adjunction then shows the corresponding map $\mathcal{A}(a, \mathcal{R}_b-) \Rightarrow \mathcal{A}(a, \mathcal{R}_{b'}u_*-)$ is pseudo-natural as well, and so is the induced $\psi\colon \mathsf{R} \Rightarrow Q\ \mathcal{B}(KI, u)\colon \mathcal{B}(KI, b) \to \mathcal{A}$ which underlies the 1-cell in $\mathsf{F}_\Omega\mathcal{A}$ from $(\mathcal{B}(KI, b), \mathcal{R}_b)$ to $(\mathcal{B}(KI, b'), \mathcal{R}_{b'})$ over $\mathcal{B}(Ka, u)$. This 1-cell in $\mathsf{F}_\Omega\mathcal{A}$ therefore lies in $\mathfrak{P}\mathcal{A}$. It follows that $\mathcal{B}(K,1)\colon \mathcal{B} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ factors through $\mathfrak{P}\mathcal{A}$, and so K is a $\mathfrak{P}$-functor. □

So $\mathfrak{P}$-functors are slightly more general than familial 2-functors but less general than Weber's lax familial functors (dualised appropriately).

7.9 Groupoid families

Another way in which we can obtain new monads from F_Ω is by restricting its action to some subcategory of $\mathsf{2CAT}$. For example, we could restrict F_Ω to the full sub-3-category of (2,1)-categories, i.e. those 2-categories whose 2-cells are all invertible. We will call this sub-3-category $\mathsf{GrpdCat}$ (since (2,1) categories are just Grpd-categories). Because oplax functors to a (2,1)-category are pseudo-functors, and lax natural transformations between them are pseudo-natural, we observe that for $\mathcal{K} \in \mathsf{GrpdCat}$, $\mathsf{F}_\Omega\mathcal{K} \cong \mathfrak{P}\mathcal{K}$. So $\mathfrak{P}\mathcal{K}$ is the free Ω-cocompletion of $\mathcal{K}$, though even restricted to $\mathsf{GrpdCat}$, $\mathfrak{P}$ is not a free-cocompletion monad as $\mathfrak{P}\mathcal{K}$ will typically not be a (2,1)-category, even if $\mathcal{K}$ is. For example, $\mathbb{1}$ is a (2,1)-category and $\mathfrak{P}\mathbb{1} \cong \mathsf{Cat}$ is not. To achieve a free-cocompletion on Grpd we will also restrict the sorts of indexing categories, as described in Section 4.6.2.

Recall in particular from Lemma 4.6.7 of that section that the free cocompletion of a 2-category $\mathcal{A}$ under oplax colimits of oplax functors from *groupoids* is given by the full subcategory of $\mathsf{F}_\Omega\mathcal{A}$ whose objects are the oplax functors from groupoids. We will call this full submonad F_Γ and the class of weights Γ. In particular, we have $\mathsf{F}_\Gamma\mathbb{1} \cong \mathsf{Grpd}$, and more generally $\mathsf{F}_\Gamma\mathcal{K}$ will be a (2,1)-category whenever $\mathcal{K}$ is. We can prove this by explicitly giving the inverse for a 2-cell $(\delta,\Delta)\colon (F,\phi) \Rightarrow (G,\psi)\colon (C,P) \to (D,Q)$ in $\mathsf{F}_\Gamma\mathcal{K}$. The natural transformation part of the inverse 2-cell will be $\delta^{-1}\colon G \Rightarrow F$, and the modification part will have component at $c \in C$ given by:



Because $\mathsf{GrpdCat}$ is a full subcategory of $\mathsf{2Cat}$, it follows that $\mathsf{F}_\Gamma\colon \mathsf{2Cat} \to \mathsf{2Cat}$ restricts to a monad on $\mathsf{GrpdCat}$, which we denote $\mathfrak{G}\colon \mathsf{GrpdCat} \to \mathsf{GrpdCat}$. This monad is also a free-cocompletion with respect to a class of Grpd-weights for oplax colimits of oplax (equiv. pseudo) functors from groupoids. These are the weights of the form $@\Delta_{\mathbb{1}}\colon C^{\natural\,\mathsf{op}} \to \mathsf{Grpd}$ for C a groupoid. The images of these presheaves do genuinely lie in Grpd, since $@\Delta_{\mathbb{1}c} \cong \oint^{x\in C^\natural} C^\natural(c,x)$ which is isomorphic to the Grothendieck construction of pseudofunctor $C \xrightarrow{C^\natural(c,-)} \mathsf{Grpd}$, which is a groupoid (cf. Lemma 4.6.7).

To see that $\mathfrak{G}$ is the free cocompletion with respect to this class, which we shall call Γ', we can verify directly that the fully-faithful canonical inclusion $\mathsf{Z}_\mathcal{K}\colon \mathcal{K} \to \mathfrak{G}\mathcal{K}$ satisfies the required universal property. To that end, we assume that $\mathcal{L}$ is some other Γ'-cocomplete (2,1)-category to show that precomposition with $\mathsf{Z}_\mathcal{K}$ induces a natural isomorphism of categories:

$$\Gamma'\text{-}\mathsf{coCts}(\mathfrak{G}\mathcal{K},\mathcal{L}) \cong \mathsf{GrpdCat}(\mathcal{K},\mathcal{L})$$

That this is true follows from the property of $\mathsf{F}_\Gamma\mathcal{K} \cong \mathfrak{G}$ as the free Γ-cocompletion for the class of Cat-weights, Γ, by the following observations:

(a) a (2,1)-category is Γ'-cocomplete if and only if it is Γ-cocomplete as a 2-category

(b) $\Gamma'\text{-}\mathsf{coCts}(\mathfrak{G}\mathcal{K},\mathcal{L}) \cong \Gamma\text{-}\mathsf{coCts}(\mathfrak{G}\mathcal{K},\mathcal{L})$ for $\mathcal{K}$ and $\mathcal{L}$ (2-1)-categories

(c) $\mathsf{GrpdCat}(\mathcal{K},\mathcal{K}) \cong \mathsf{2Cat}(\mathcal{K},\mathcal{L})$ for $\mathcal{K}$ and $\mathcal{L}$ (2,1)-categories

Since $\mathfrak{G}\mathcal{K}$ is also isomorphic to $\mathfrak{P}\mathcal{K}$, it inherits the following properties when viewed as a $\mathsf{Gray}_{(2,1)}$-functor, where $\mathsf{Gray}_{(2,1)}$ is the category of groupoids enriched over itself with the Gray monoidal product.

Proposition 7.9.1. *The free-cocompletion monad on* $\mathsf{GrpdCat}$ *for* $\mathfrak{G}$*-colimits preserves 2-fibrations, is a parametric right* $\mathsf{Gray}_{(2,1)}$*-adjoint, and has naturality squares for its multiplication and unit given by* $\mathsf{Gray}_{(2,1)}$*-pullbacks.*

Proof. The 2-fibrations in $\mathsf{Gray}_{(2,1)}$ are just those 2-functors which are 2-fibrations in $\mathsf{2Cat}$, and thus preserved by $\mathfrak{G}$ by the fact that they are preserved by $\mathfrak{P}$. The parametric right $\mathsf{Gray}_{(2,1)}$-adjoint property follows by the same arguments as those given for $\mathfrak{P}$ adapted to $\mathsf{Gray}_{(2,1)}$:

(a) $\mathfrak{G}_1 \colon \mathsf{GRAY}_{(2,1)} \to \mathsf{GRAY}_{(2,1)}/\mathsf{Grpd}$ factors through the Grothendieck construction for globally split Grpd-presheaves as:

$$\mathsf{GRAY}_{(2,1)} \xrightarrow{\mathsf{GRAY}_{(2,1)}\left((-)^{\natural},1\right)} \left[\widetilde{\mathsf{Grpd}}, \mathsf{GRAY}_{(2,1)}\right] \xrightarrow{\int} \mathsf{GRAY}_{(2,1)}/\mathsf{Grpd}$$

(b) The left adjoint for the Grothendieck construction is genuinely a $\mathsf{GRAY}_{(2,1)}$-valued presheaf, as for $P \colon \mathscr{A} \to \mathsf{Grpd}$ from a (2,1)-category each $G \Downarrow P$ for $G \in \mathsf{Grpd}$ is a (2,1)-category.

(c) $\mathsf{GRAY}_{(2,1)}\left((-)^{\natural},1\right)$ has a left $\mathsf{Gray}_{(2,1)}$-adjoint computed as a left extension of $(-)^{\natural} \colon \mathsf{Grpd} \to \mathsf{GRAY}_{(2,1)}$ along the Yoneda embedding by the same general argument as was applied in the $\mathfrak{P}$ case.

composing these two left $\mathsf{Gray}_{(2,1)}$-adjoints gives a left $\mathsf{Gray}_{(2,1)}$-adjoint to $\mathfrak{G}_1$ which exhibits $\mathfrak{G}$ as a parametric right $\mathsf{Gray}_{(2,1)}$-adjoint.

For the property that the naturality squares are $\mathsf{Gray}_{(2,1)}$-pullbacks, we note that $\mathsf{GrpdCAT}$ is in particular a reflective sub-1-category of $\mathsf{2CAT}$ (whose reflection freely inverts all 2-cells in a 2-category) and thus its inclusion creates limits. In particular, pullbacks of cospans of (2,1)-categories are (2,1)-categories, and are also pullbacks in the full-subcategory of (2,1)-categories. Now, the naturality squares for $\mathfrak{P}$ are not strict pullbacks, but instead admit a Gray-equivalence from the top-left of the square to the strict pullback. It follows that the naturality squares for $\mathfrak{G}$ have this same property, since Gray-equivalences between (2,1)-categories are also $\mathsf{Gray}_{(2,1)}$-equivalences in $\mathsf{Gray}_{(2,1)}$. □

The completion of (2,1)-categories under "pseudo-families" of groupoids seems to bear the strongest resemblance to the Fam-construction for 1-categories, being both a free cocompletion and satisfying a good notion of "familiality".

Chapter 8

Closing Remarks

The ordering of the material in this thesis is more-or-less reverse to the order in which each of these topics were explored. In particular, this project began with an investigation into the $\mathfrak{G}$ construction and a related full submonad $\mathfrak{G}_{\mathsf{N}}$ whose objects are *normal* pseudofunctors, those with identity pseudo-unit 2-cells P_0. The purpose of this investigation was to find examples of the two-dimensional comprehension categories described in [Gar09] as models for Garner's two-dimensional type theory, ML_2.

Briefly, the two-dimensional model's of type theory described in [Gar09, Definition 4.6.2] are (2,1)-categorical analogues of Jacob's *comprehension categories* [Jac93], which involve a globally split 2-fibration $p\colon \mathfrak{T} \to \mathfrak{C}$ in $\mathsf{GrpdCat}$ of "types" over "contexts" which factors through the codomain projection $\mathfrak{C}^2 \xrightarrow{\mathsf{cod}} \mathfrak{C}$ via a map $E\colon \mathfrak{T} \to \mathfrak{C}$ called the *comprehension* map. The map E should be cartesian, and factor through the inclusion of normal isofibrations, and satisfy some other more subtle properties. Such structures have internal languages which satisfy the structural rules for ML_2, whereas the logical rules correspond to the existence of certain adjunctions to reindexing maps, among other things. The $\mathfrak{G}_{\mathsf{N}}$ construction on Grpd provides such a two-dimensional model corresponding to the *groupoid model* of Hofmann and Streicher [HS98]. In particular, the globally split 2-fibration $p\colon \mathfrak{G}_{\mathsf{N}}\mathsf{Grpd} \to \mathsf{Grpd}$ is given by ϖ_{Grpd}, factors through the equivalence between $\mathfrak{G}_{\mathsf{N}}(\mathsf{Grpd})$ and the category of normal isofibrations in Grpd given by the Grothendieck construction. For this model, the unit types and sum types are essentially given by the identity and composition for the weak internal category structure associated to the $\mathfrak{G}_{\mathsf{N}}$-algebra $\mathsf{Grpd} \cong \mathfrak{G}_{\mathsf{N}}\mathbb{1} \to \mathbb{1}$ as described in 7.7.1. The properties they are required to satisfy then follow from general properties of $\mathfrak{G}_{\mathsf{N}}$ as a free cocompletion, and in particular as a lax-idempotent monad. The effort to establish this free-cocompletion property then lead to an exploration of similar, more general free cocompletions, and then descriptions of their corresponding classes of weights, and then in a number of other directions without returning to the ever-receding original problem. We now feel, however, that the necessary background for this problem has been sufficiently considered, and the intention remains to explore such two dimensional models further. Some of the questions originally posed which remain relevant are:

(a) Do other algebras for $\mathfrak{G}_{\mathsf{N}}$, i.e. (2,1)-categories closed under oplax-colimits of normal pseudo-functors from groupoids, generate interesting two-dimensional models of type theory?

(b) For a (2,1)-category $\mathcal{K}$ with a terminal object t, there is a canonical map $\mathsf{Grpd} \to \mathfrak{G}_{\mathsf{N}}\mathcal{K}$ given by the image of $\langle t \rangle : \mathbb{1} \to \mathcal{K}$ under $\mathfrak{G}_{\mathsf{N}}$ which seems to provide a 2-dimensional analogue of the family model over sets [Jac93, Example 4.14], and thus, perhaps, a 2-comprehension category with unit. What properties of $\mathcal{K}$ correspond to this model having products, or sums, or other type constructions? In the family model over sets for a 1-category C, for example, products in the internal language correspond to the existence of infinite products in the category C.

(c) can product types in the groupoid model be explained in terms the oplax-idempotent completion under oplax *limits*, and its interaction with $\mathfrak{G}_{\mathsf{N}}$?

Aside from this application to type theory, there are a number of possible extensions of the results in this work which might be interesting. For example:

(a) Working with *bicategories*, rather than 2-categories. Results from Walker's work on lax familial functors as pseudofunctors between bicategories as well as Buckley's fibred bicategories already extend some of the ideas in this thesis to the bicategorical setting and suggest further extension is possible.

(b) Working with the $\mathscr{F}$-categories of [LS12], rather than 2-categories. In a few places we've specialised results or ideas developed in the more general setting of $\mathscr{F}$-categories, particularly from the paper [LS12] by Lack and Shulman — e.g. rigged limits, weak morphism classifiers, the universal property of T-$\mathbb{A}\mathsf{lg}_{\mathsf{w}}$. Aside from the possibility of providing more general results, working in this section might allow for a better treatment of oplax notions of functors. A frequent obstacle to establishing properties of F_Ω which are analogous to those of $\mathfrak{S}$ and $\mathfrak{P}$ has been that comodules and lax transformations don't coincide; the comodules are strictly more general. It is possible that considering instead lax-$\mathscr{F}$-functors and transformations provides a more satisfying structure. Or perhaps a virtual double category setting could provide a good extension, since comodules form horizontal morphisms in such a structure.

We also identify a few of the results presented for which the proofs we presented were slightly mysterious, or laborious, which suggests there might exist more effective arguments and perspectives:

(a) Lemma 4.3.12, the proof that $\mathfrak{i}_K \Downarrow p$ is isomorphic to $\mathsf{F}_\Omega \mathfrak{Q}_p$, for any 2-functor $p\colon \mathcal{X} \to \mathcal{K}$. It seems it may more generally be true that lax slice categories $\mathsf{F}_\Theta \Downarrow p$ are isomorphic to $\mathsf{F}_\Theta \mathfrak{Q}_p$, from which the result would follow. For a monad T on C, there is an induced monad on C/Tc whose Kleisli category is isomorphic to the slice category Kl_T/c. There may be a similar property for lax slices which explains the result above.

(b) Corollary 4.5.12 which essentially shows that 2-categories of the form $\mathscr{D}A$ have enough initial objects. There may be a simpler approach that doesn't demonstrate this explicitly by combinatorial arguments.

(c) Echoing the remarks made at the end of Chapter 5, the extensions of Fam from a Cat-functor to a Sesqui-functor and of F_Ω from a $2\mathsf{Cat}$-functor to a $\mathsf{Gray}_{\mathscr{L}}$-functor seem to be two instances of a more general phenomenon. These extension properties, then, might be better described by a more general theory.

Chapter 9

Appendices

9.1 Modules and Modulations

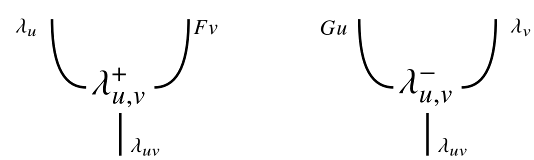
For a 2-category $\mathcal{K}$, the 2-cells of the category $\Omega\mathcal{K}$ can be described using the language of *comodules* and *modulations* as described in [Coc+03]. Comodules and modules provide a more general notion of 2-cells between lax/oplax functors. Both oplax and lax transformations can be cast as either comodules or modules, but there exist comodules and modules that do not arise in this way. In particular, comodules are general enough to describe the result of whiskering a transformation—lax, or oplax—on the right by an oplax functor. We will describe comodules and modulations between them in the context of functors from a 1-category to a 2-category.

9.1.1 Comodules

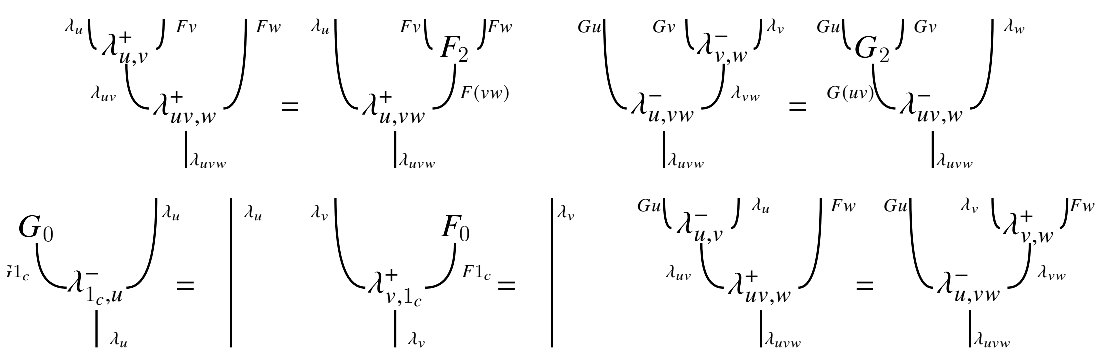
Given two oplax functors, $F, G\colon C \to \mathcal{K}$ from a 1-category to a 2-category, a *comodule* $\lambda\colon F \Rightarrow G$ consists of the following data:

(a) For each $u\colon c \to c'$ a morphism $\lambda_u\colon Fc \to Gc'$

(b) For each composable pair $c \xrightarrow{v} c' \xrightarrow{u} c''$ a pair of 2-cells in $\mathcal{K}$ of the form:



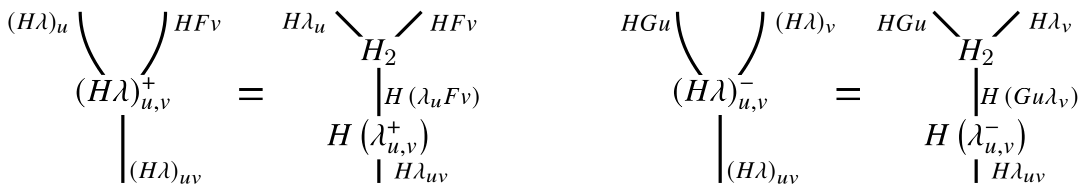
These data satisfy the following composition and unit conditions:



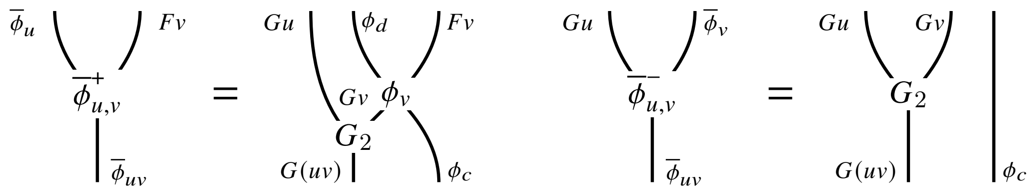
To give two oplax functors $F, G\colon C \to \mathcal{K}$ and a comodule between them is equivalent to giving an oplax functor

$\widehat{\alpha}\colon C \times 2 \to \mathcal{K}$ where 2 is the "free-living arrow". A lax functor of this form gives a module between lax functors.

Given a comodule $\lambda\colon F \Rightarrow G\colon C \to \mathcal{K}$ and an oplax functor $H\colon \mathcal{K} \to \mathcal{L}$, we obtain another comodule $H\lambda$ with 1-cell components given by $(H\lambda)_u = H(\lambda_u)$ and 2-cell components as shown below:



Given an lax transformation $\phi\colon F \Rightarrow G\colon C \to \mathcal{K}$, we can produce a corresponding comodule $\overline{\phi}\colon F \Rightarrow G$ whose 1-cell components are given by $\overline{\phi}_u = Gu \circ \phi_{\mathrm{dom}(u)}$. The 2-cell components are given by:



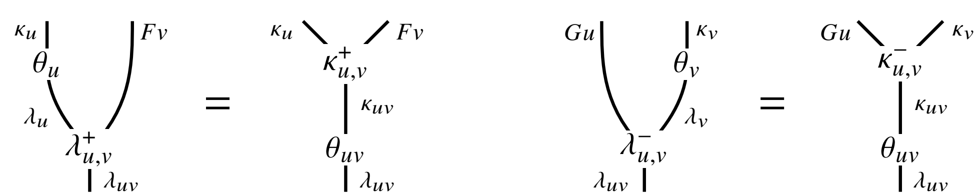
The name *module* comes (presumably) from the example of a module between functors from the terminal category $\mathbb{1}$ to the suspension of a monoidal category V. Because we are only considering modules between functors to 2-categories here, we should require that the monoidal category be strict. In that context, we can make the translations shown in Table 9.1.

Op/lax functor	$\rightsquigarrow$	Co/monoid in V
Icon between op/lax functors	$\rightsquigarrow$	Co/monoid morphism
Op/lax transformation of lax functors	$\rightsquigarrow$	Left/right-free bimodule of monoids
Op/lax transformation of oplax functors	$\rightsquigarrow$	Right/left-free bimodule of comonoids
Module between lax functors	$\rightsquigarrow$	Bimodule of monoids
Comodule between oplax functors	$\rightsquigarrow$	Bicomodule of comonoids
Modification between op/lax transformations	$\rightsquigarrow$	Left/right free co/module morphism
Modulation between co/modules	$\rightsquigarrow$	Morphism of bicomodules

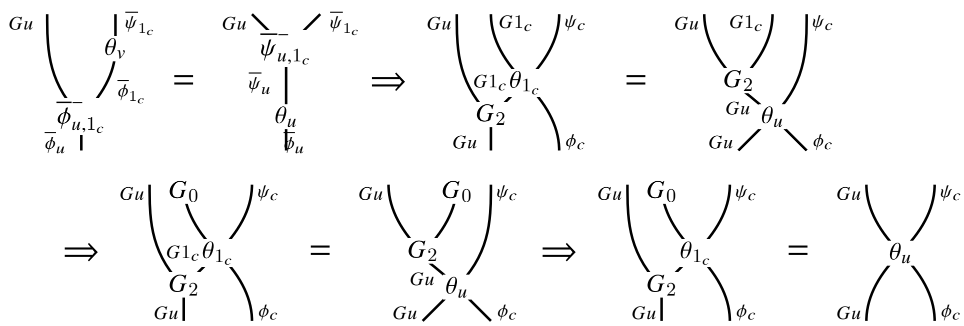
Table 9.1: Why the name *modules*?

9.1.2 Modulations

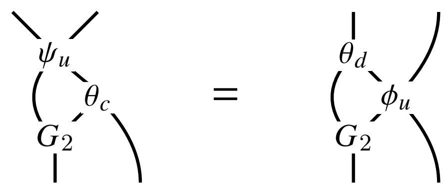
Modulations provide a notion of morphism between comodules. For two comodules $\lambda, \kappa\colon F \Rightarrow G\colon C \to \mathcal{K}$ a modulation $\theta\colon \lambda \to \kappa$ consists of 2-cells $\theta_u\colon \lambda_u \to \kappa_u$ for each 1-cell $u\colon c \to d$ required to commute with the + and − 2-cells of λ and κ in the following sense:



An interesting special case is a modulation between comodules arising from lax transformations. Here, we can consider the axiom for the $(-)$ 2-cells in the case $v = 1_c$ and observe the following:

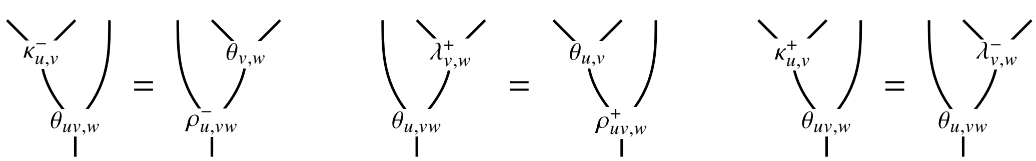


The data for such a modulation is therefore completely determined by the θ_{1_c} components. In fact, it is determined by the 2-cells $\theta_c \coloneqq G_0\psi_c \circ \theta_{1_c} \colon G1_c\phi_c \to \psi_c$. Re-expressing the coherence conditions for modulations in terms of these θ_c components, we arrive at a single non-tautological axiom:

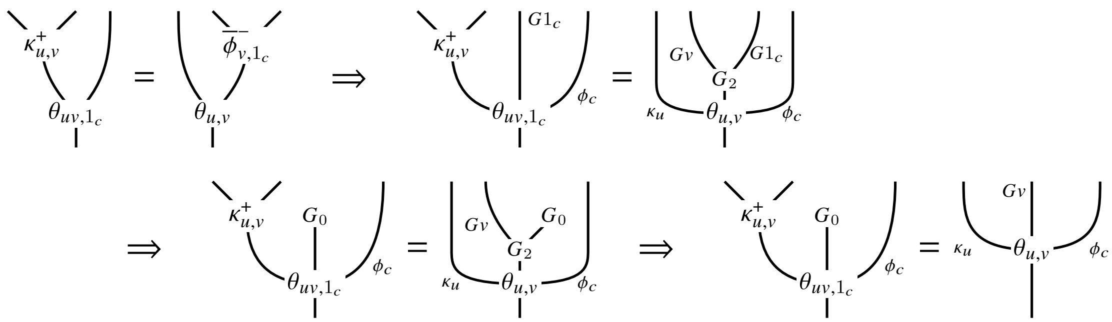


In fact, all that was required for this simplification was that the codomain was the comodule of a lax transformation. The reduction in the data for such modulations can be viewed as a generalisation of the fact that a module morphism to a free module is equivalent to a morphism to the generator of the free module.

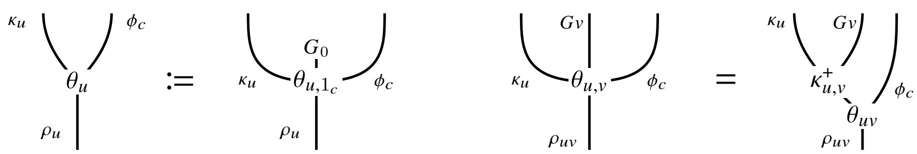
Modules cannot be vertically composed in general. Given modules $\lambda \colon F \Rightarrow G$ and $\kappa \colon G \Rightarrow H$, one can describe a universal property that the composite $\kappa\lambda$ should satisfy, but the existence of a module satisfying this property is not guaranteed when the codomain of F, G, H is not locally cocomplete. This is true for general bimodules of monoids in a monoidal category that is not necessarily cocomplete. One can never-the-less consider a notion of a modulation from κ after λ to some other module $\gamma \colon F \Rightarrow H$ called a *bimodulation*. More generally, one can speak of multi-modulations from a single comodule to a sequence of contiguous comodules. The composition $\kappa\lambda$ is then defined to be a representation of the functor $\mathsf{Multimod}(-; \kappa, \lambda) \colon \mathsf{Mod}_{[F,H]} \to \mathsf{Set}$. A bimodulation $\theta \colon \rho \to [\kappa, \lambda]$ is given by a collection of 2-cells $\theta_{u,v} \colon \rho_{uv} \to \kappa_u\lambda_v$ for each composable pair of 1-cells $c \xrightarrow{v} d \xrightarrow{u} e$ in C satisfying the following coherence conditions:



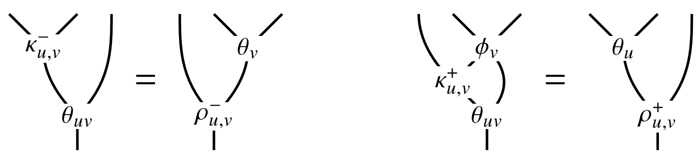
We can show that in the case where λ is a comodule induced by a lax transformation, any such bimodulation is equivalent to a modulation from ρ to a single comodule. This comodule is therefore the composition $\kappa\lambda$, and thus precomposition of comodules by lax transformations is always possible. Again, there is an analogy to the usual notion of modules, where it is always possible to tensor a left-free bimodule on the left. We first consider the third axiom for a bimodulation where $\lambda = \overline{\phi}$ and $w = 1_c$:



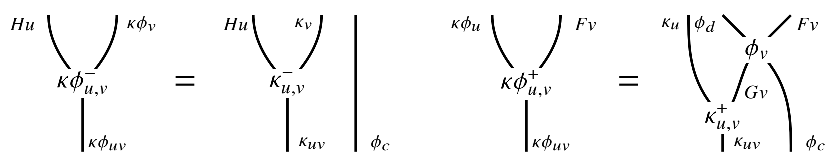
It follows that the data of such a bimodule can be reduced to 2-cells $\theta_u \colon \rho_u \longrightarrow \kappa_u\phi_c$:



The original axioms on the $\theta_{u,v}$ data reduces to a pair of axioms on the θ_u data:

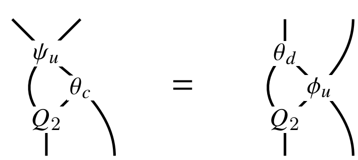


These data and axioms correspond to a modulation from ρ to a comodule $\kappa\phi \colon F \Rightarrow H$ whose component at $u \colon c \longrightarrow d$ in C is given by $\kappa_u\phi_c$ and whose (+) and (−) 2-cells are given by:



Thus the composition $\kappa\overline{\phi}$ exists and is given by the data above.

These are the required observations to make sense of a modulation $Qs \circ \phi \longrightarrow \psi \colon P \Rightarrow QG \colon C \longrightarrow \mathcal{K}$ where $P \colon C \longrightarrow \mathcal{K}$ and $Q \colon D \longrightarrow \mathcal{K}$ are oplax functors, $\phi \colon P \longrightarrow QF$ and $\psi \colon P \longrightarrow QG$ are lax transformations, and $s \colon F \Rightarrow G$ is a natural transformation. The data consists of 2-cells $\theta_c \colon Qs_c\phi_c \Rightarrow \psi_c$ in $\mathcal{K}$ for each $c \in C$ satisfying the single axiom below:



(9.1)

9.1.3 Alternative descriptions

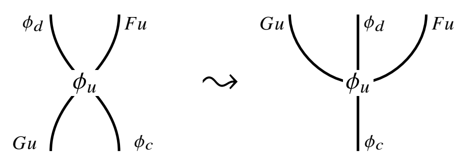
For a given 1-category, C, one can consider the *twisted arrow category* of C, $\mathsf{Tw}(C)$ which has as objects the morphisms of C. A morphism between $f \colon a \longrightarrow b$ and $g : c \longrightarrow d$ is a pair of morphisms $u : a \longrightarrow c$ and $v \colon d \longrightarrow b$ such that $f = vgu$. Such a morphism can be thought of as a "twisted" commuting square:

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[d, "f"'] & c \arrow[d, "g"] \\ b & d \arrow[l, "v"] \end{tikzcd}$$

Given a pair of oplax functors $F, G\colon C \to \mathcal{K}$ (where $\mathcal{K}$ is a 2-category) a comodule from F to G can equivalently be described as a lax transformation filling the diagram below:

$$\begin{tikzcd} & C \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow\phi"] & \\ \mathsf{Tw}(C) \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & {} & \mathcal{K} \\ & C^{\mathrm{op}} \arrow[ur, "G^{\mathrm{op}}"'] & \end{tikzcd}$$

The functors π_1 and π_2 are projection onto the domain and codomain respectively. For such a definition to make sense, we must extend the notion of lax transformation to a map from a covariant to contravariant functor. The obvious way to do this is to replace the data of a 2-cell filling the square corresonding to each morphism $u\colon c \to d$ in C with a 2-cell filling the corresponding twisted square. From a string diagram perspective, this looks like bending the "Gu" string so that it attaches at the top, rather than the bottom (where G is a general contravariant codomain):

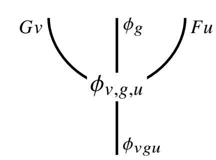


The lax transformation between the two functors from $\mathsf{Tw}(C)$ therefore has data given by:

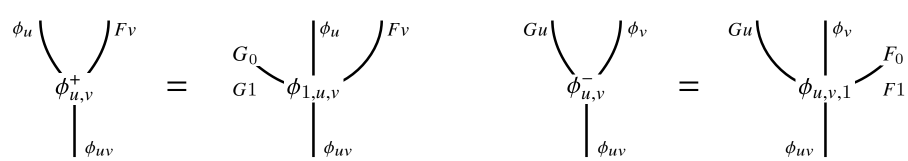
(a) For each morphism $f\colon a \to b$ in C a 1-cell $\phi_f\colon Fa \to Gb$ in $\mathcal{K}$

(b) For each composable triple (i.e. morphism in $\mathsf{Tw}(C)$) a 2-cell in $\mathcal{K}$:

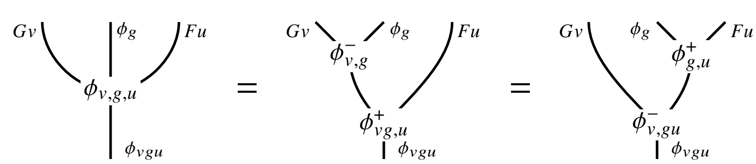
$$a \xrightarrow{u} c \xrightarrow{g} d \xrightarrow{v} b \quad \rightsquigarrow$$



The data of the 2-cells $\phi^+_{u,v}$ and $\phi^-_{u,v}$ from the definition at the start of this section can be recovered as:



And one can obtain the data of a lax transformation from the twisted category from the data in the original definition:



Moreover, the coherence data on the twisted category lax transformations maps precisely onto the coherence data for the original definition of comodules. One can equally show that modifications between such lax transformations from the twisted category are in correspondence with modulations between the associated comodules.

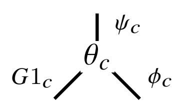
9.2 Relation between the extralax colimit and the Gray tensor product

There is a sense in which the extralax colimit of a 2-functor from a 2-category $A\colon C \to \mathsf{Cat}$ can be viewed as a "dependent" Gray tensor product (of the lax sort). As noted in the text, in the particular case of a constant functor $\Delta^C_D\colon C \to \mathsf{Cat}$ from a 1-category (i.e. locally discrete category) we have $\mathscr{E}(\Delta^C_D) \cong C \boxminus D^\dagger$. We could define another notion of the tensor of a 2-category, C with a 1-category D, written $C\widetilde{\oplus}D$ as being a representation of the functor:

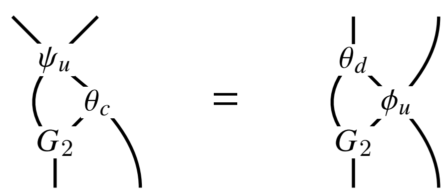
$$\mathsf{2Cat} \to \mathsf{2Cat}, \quad \mathcal{K} \mapsto \mathsf{2Cat}(C, \mathsf{opLax}_{\mathsf{lax}}(D,\mathcal{K}))$$

where $\mathsf{opLax}_{\mathsf{lax}}(D,\mathcal{K})$ denotes the 2-category of oplax functors, lax transformations and modifications. Then we would have $C\widetilde{\oplus}D \cong C \boxminus D^\dagger$

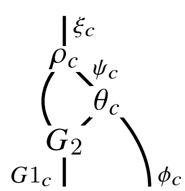
However, whenever the codomain of the constant functor, C, has some 2-dimensional structure it is no longer true that $\mathscr{E}(\Delta^C D) \cong C\widetilde{\boxminus}D$. This suggests that there is something wrong with the 2-cells of $\mathsf{opLax}_{\mathsf{lax}}(D,\mathcal{K})$. If instead we use modulations rather than modifications as the 2-cells of this category, then the issue is resolved. Recall from Section 9.1 that a modulation between lax transformations $\phi, \psi\colon F \to G\colon D \to \mathcal{K}$ (viewed as comodules) is given by assigning to each object $c \in D$ a 2-cell in $\mathcal{K}$ of the form:



such that for each 1-cell $u\colon c \to d$ in D the following condition is satisfied:



When expressed in this form, the vertical composition of modulations $\theta\colon \phi \Rightarrow \psi$ and $\rho\colon \psi \to \xi$ has component at $c \in D$ given by:



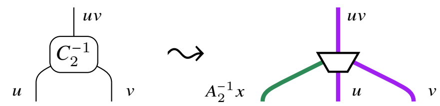
This describes a 2-category of modulations between lax transformations between oplax functors, $\mathsf{opLax}_{\mathsf{lax,modl}}(D,\mathcal{K})$. For any pair of a 2-category C and 1-category D there is a 2-functor:

$$\mathcal{K} \mapsto \mathsf{2Cat}\big(C, \mathsf{opLax}_{\mathsf{lax,modl}}(D,\mathcal{K})\big)$$

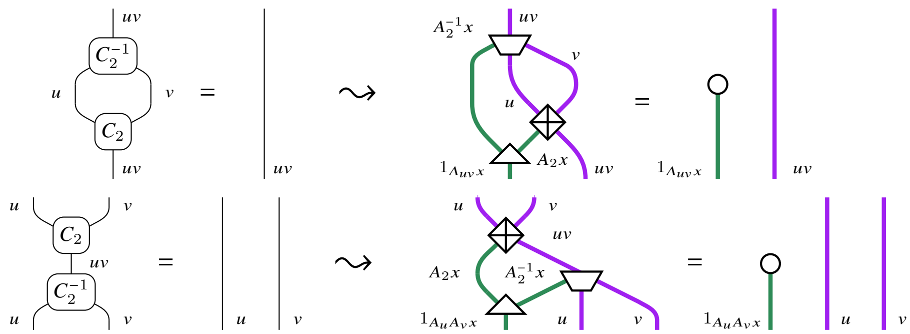
which is represented by a 2-category we can denote $C \boxtimes D$. This representing 2-category satisfies $\mathscr{E}(\Delta^C_D) \cong C \boxtimes D$ for any 2-category C, and for C a 1-category we have $C \boxtimes D \cong C \boxminus D^\dagger$.

9.3 The Pseudo-lax colimit

There is an appropriate notion of pseudo-lax cocone and pseudo-lax colimit which one obtains from the normal lax colimit by additionally requiring the $\triangle$'s be invertible — we will use $\bigtriangledown$ to indicate the inverses — that the $\diamondsuit$'s are invertible (corresponding to the pseudo-naturality of the faces of the pseudo-lax cocone) and adding generators corresponding to the inverses of C_2:

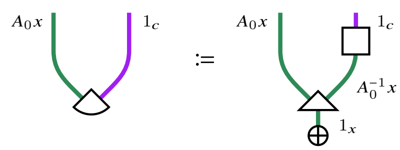


with corresponding relations:

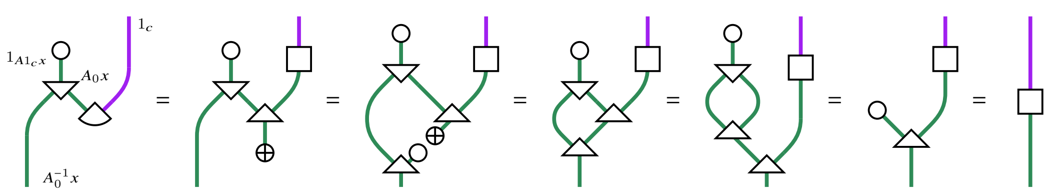


(9.2)

This gives one presentation for $@A$, though we will show that another presentation is given by taking the presentation for the extralax colimit $@A$ and requiring that all generator types are invertible. We've already observed in 4.6.3 that the ◇ generator is invertible in the presence of the □ generator, with inverse given by:

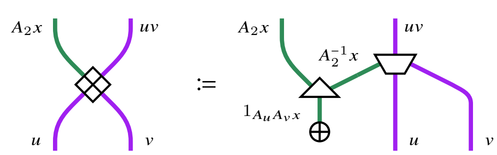


In the presence of the ▽ generator, the generator □ of (4.29) can be defined in terms of inverses to ◇ and △, denoted ◇ and ▽ respectively, and so can be omitted from the presentation:

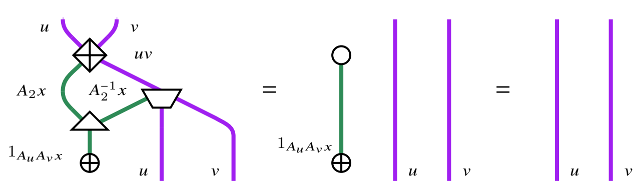


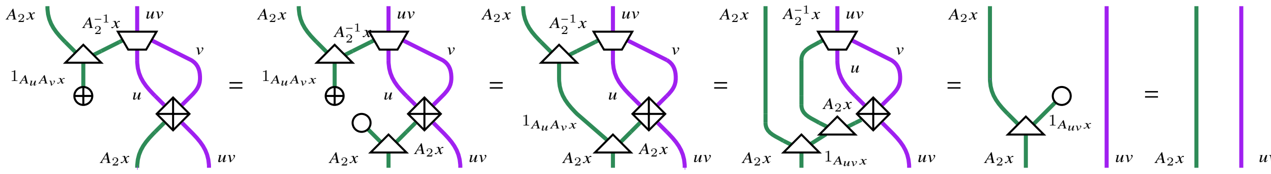
(9.3)

We can similarly define an inverse for the ⟡ generators in terms of ⏢ as:

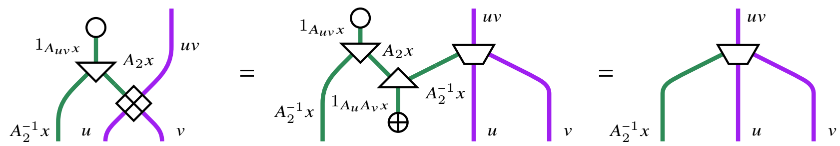


whose property as an inverse is easily verified from the relations in (9.2):





And the generator can be defined in terms of the inverse of using a string diagram argument similar to that given in (9.3):



Thus we can replace the generator with a formal inverse to . This leaves us with only the generators of $\mathscr{D}A$ and their formal inverses. It follows that every 2-cell in $\mathscr{Q}A$ is invertible. This property, along with the existence of component-initial 1-cells in connected components (the *normal* 1-cells), shows that $\mathscr{Q}A$ is locally an equivalence relation. That is, there is at most one 2-cell between any two 1-cells, and that 2-cell is invertible. Whether two 1-cells in $\mathscr{Q}A$ are connected by a 2-cell can be determined by reducing each 1-cell to its connected normal 1-cell using the algorithm described in Lemma 4.3.19 and verifying that the normal 1-cells are equal. We can relate this to the projection $\mathsf{p}_1 \colon \mathscr{Q}A \to \pi_1(\mathcal{A}) \cong \int A$:

Lemma 9.3.1. *For $A \colon C \to \mathsf{Cat}$ a pseudofunctor from a 1-category, the hom-categories of $\mathscr{Q}A$ are locally equivalence relations given by the fibres over 1-cells of $\mathsf{p}_1 \colon \mathscr{Q}A \to \int A$, and the canonical map $\int A \to \mathscr{Q}A$ is a pseudo-functor.*

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