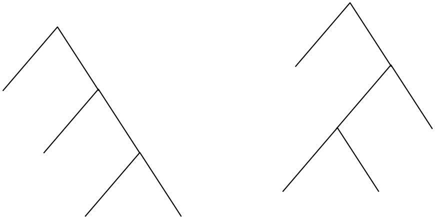
Thompson's group $F \subseteq T \subseteq V$



Definitions, why is F interesting?

An investigation of properties of F:

1. Finite?
2. Abelian?
3. Torsion-free?
4. Finitely generated?
5. Finitely presented?
6. Simple?
7. Solvable?
8. Solvable word problem?
9. Abelianisation?
10. Centre?
11. Cohomological dimension?
12. F_∞? FP_∞?
13. Amenable?
14. QFA?

$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Definitions

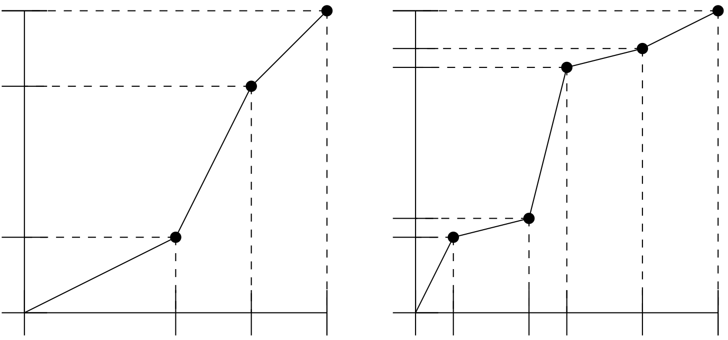
Definition (F)**:** Thompson's group F is the group of associative laws definable on a binary opertation.

Definition (F)**:** The group of "order-preserving" autormorphisms of the free Jónsson-Tarski algebra on a single generator.

Definition (F)**:** The automorphism group of (any of) the objects in the free monoidal category generated by a single object A and an isomorphism $A \otimes A \xrightarrow{\phi} A$

Definition (F)**:** The group of order-preserving automorphisms of $[0, 1]$ that :

1. Are Piecewise-linear
2. Have finitely many singularities ('breakpoints')
3. Each breakpoint is a *dyadic rational*, i.e. of the form $\frac{a}{2^n}$
4. The gradient of each linear region is a power of 2



Definition (F)**:** Pairs of rooted binary trees with equal numbers of nodes (with a non-obvious group operation).

Definition $((F, \phi_F, x_F))$**:** The initial object in the category with:

1. Objects: triples consisting of
   (a) A group G
   (b) A group endomorphism $G \xrightarrow{\phi_G} G$
   (c) A group element $x_G \in G$

   $c_{x_G}(g) = x_G^{-1} g x_G$

   such that $\phi_G^2 = c_{x_G} \circ \phi_G$
2. Morphisms from $(G, \phi_G, x_G) \to (H, \phi_H, x_h)$: group homomorphisms $f : G \to H$ such that $f \circ \phi_G = \phi_H \circ f$ and $f(x_G) = x_H$

$$\begin{tikzcd} x_H \arrow[d, "f"'] & \in G \arrow[d, "f"'] \arrow[r, "\phi_G"] & G \arrow[d, "f"] \\ x_H & \in H \arrow[r, "\phi_H"] & H \end{tikzcd}$$

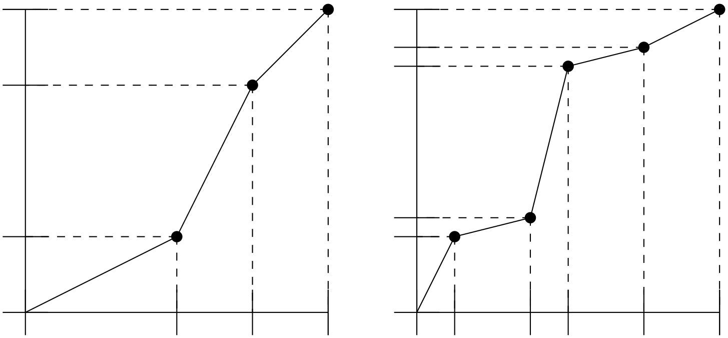
Definition (F)**:** The fundamental group of the initial object in the following category:

1. Objects: triples consisting of
   (a) A pointed topological space X
   (b) A basepoint preserving map $\psi_X : X \to X$
   (c) A free homotopy $h_X : \psi_X^2 \to \psi_X$
2. Morphisms from $(X, \psi_X, h_X) \to (Y, \psi_Y, h_Y)$: continuous basepoint-preserving maps $f : X \to Y$ such that $f \circ \psi_X = \psi_Y \circ f$ and $f \circ h_X = h_Y \circ f$

$$\begin{tikzcd} *_X \arrow[d, "f"'] & \in X \arrow[d, "f"'] \arrow[r, bend left, "\psi_X"] \arrow[r, bend right, "\psi_X^2"'] \arrow[r, phantom, "h_X"] & X \arrow[d, "f"] \\ *_Y & \in Y \arrow[r, bend left, "\psi_Y"] \arrow[r, bend right, "\psi_Y^2"'] \arrow[r, phantom, "h_Y"] & Y \end{tikzcd}$$

Definition (F)**:** The group of order-preserving automorphisms of $[0,1]$ that are :

1. Piecewise-linear
2. Have finitely many singularities ('breakpoints')
3. Each breakpoint is a *dyadic rational*, i.e. of the form $\frac{a}{2^n}$
4. The gradient of each linear region is a power of 2



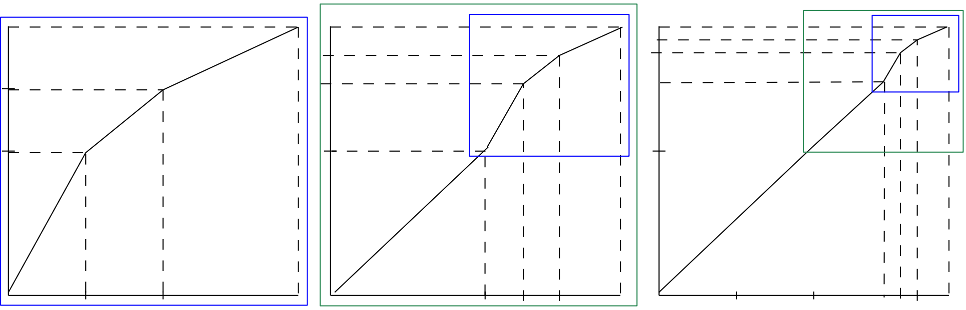
Claim: *Any $f \in F$ maps dyadic rationals to dyadic rationals.*

$$x \mapsto (x - x_i)2^{n_i} + y_i$$

Claim: *Any $f \in F$ gives a bijection on dyadic rationals in $[0,1]$*

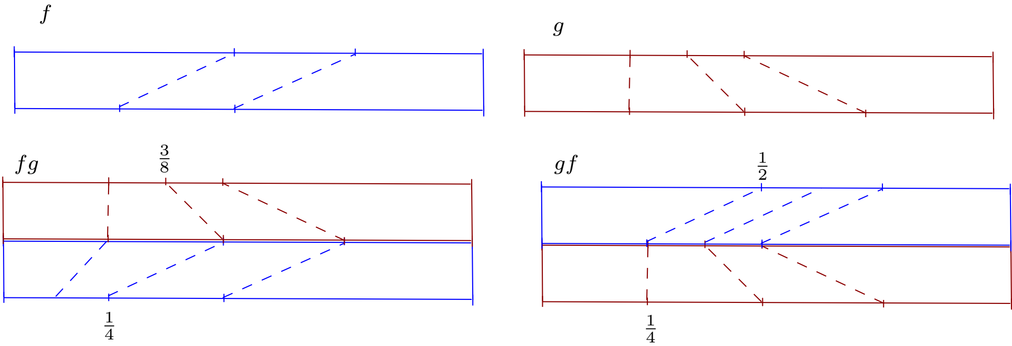
Claim: *Composition defines a group operation on F.*

Is F finite?



No

Is F commutative (abelian)?

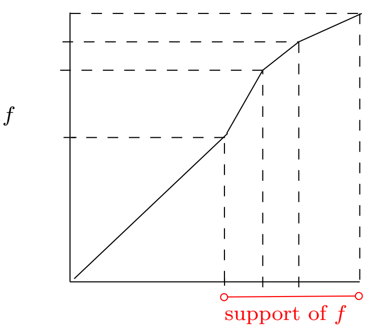


No

Definition (Torsion-free)**:** A *torsion element* g of a group G is an element of finte order, i.e. $g^n = e$ for some n . If G has no torsion elements other than e then we say it is *torsion free*

Is F torsion-free?

Definition (Support)**:** The *support* of $f \in F$ is the set $\{x \mid f(x) \neq x\}$.



Proof. If $f \neq \mathrm{id}_{[0,1]}$ then is has non-empty support. Let a be the infimum of this support. Then $f(a) = a$ and the "derivative on the right" of f at (written $D_a^+(f)$) is not 1. It follows that:

$$D_a^+(f^n) = \left(D_a^+(f)\right)^n \neq 1$$

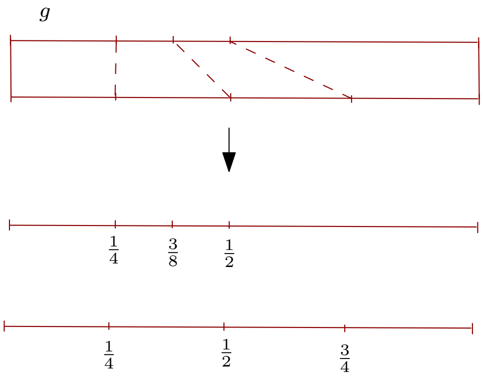
for any $n \neq 0$ and so $f^n \neq \mathrm{id}_{[0,1]}$. $\square$

Definition (Finitely generated)**:** A group G is *generated* by a subset $S \subseteq G$ if every element of G can be expressed in terms of products of elements of S and their inverses. G is *finitely generated* if it has a finite generating set.

Example: $(\mathbb{Z}, +)$ is finitely generated by $\{1\}$. $(\mathbb{Q}, +)$ is generated by $\left\{ \frac{1}{p} \mid p \text{ is prime} \right\}$ but is not finitely generated.

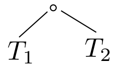
Definition (F)**:** Pairs of rooted binary trees with equal numbers of nodes (with a non-obvious group operation).

Elements of F can be represented by a pair of subidivisions of $[0, 1]$.

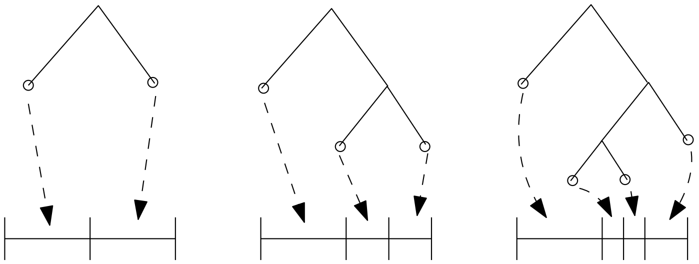


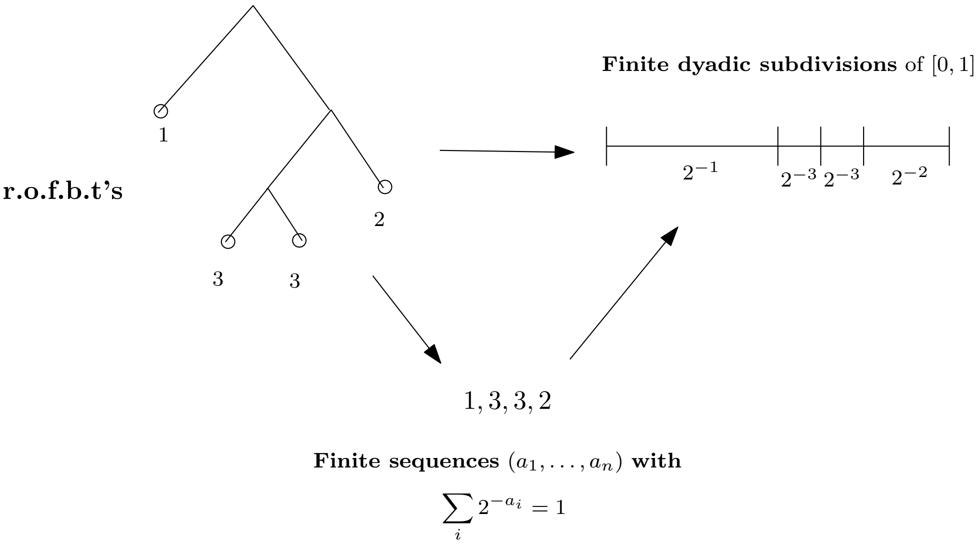
We can think of rooted, ordered, full binary trees as representing subdivisions of $[0, 1]$ (or any compact interval). We can formalise this by an inductive definition.

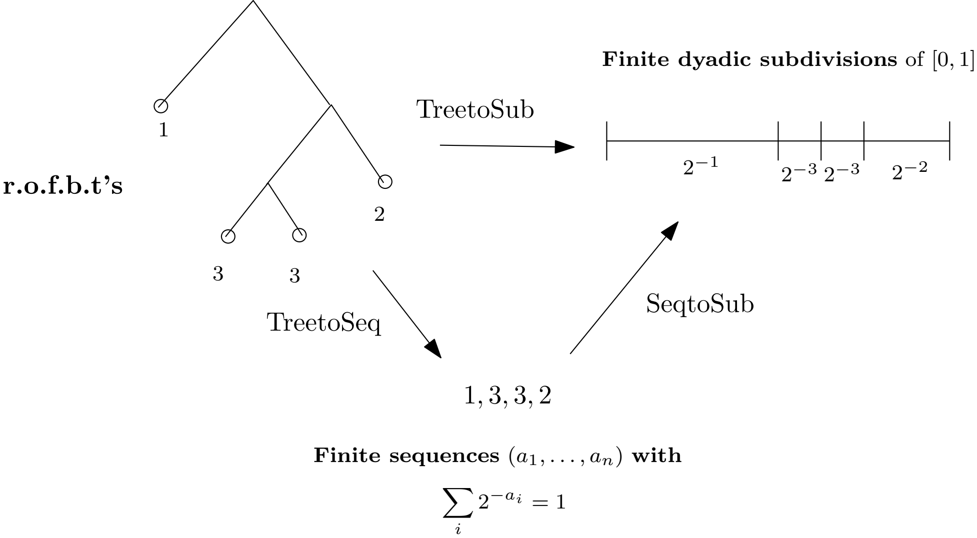
Definition (T-subdivision)**:** Given a r.o.b.t, T, we define the T-subdivision of $[a, b]$, denoted $T^{[a,b]}$, to be:

1. $[a, b]$ for: $T = \circ$
2. $T_1^{[a,c]} | T_2^{[c.b]}$ where $T =$ 

   and c is the midpoint of $[a, b]$.







Definition (Proper subdivision)**:** The image of SeqtoSub

Definition (Leaf sequence)**:** The image of TreetoSeq

Definition (Sequence refinement)**:** A *refinement* of a sequence $(a_1,\ldots,a_n)$ is a new sequence obtained by replacing an a_1 with two a_i+1's (as many times as you like)

$$\begin{tikzcd}
\text{Tree} \arrow[rr, "\text{TreetoSub}"] \arrow[dr, "\text{TreetoSeq}"'] & & \text{Sub} \\
& \text{Seq} \arrow[ur, "\text{SeqtoSub}"'] &
\end{tikzcd}$$

Example:

$$2,1^*,2 \longrightarrow 2,(2,2),2^* \longrightarrow 2,2,2,(3,3)$$

Claim: *Every sequence has a refinement which is a leaf sequence (i.e. obtained from a tree)*

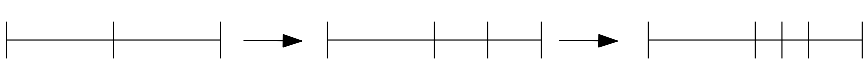
Proof. We can refine any sequence to obtain a new sequence where all the terms are the same. Such a sequence is always a leaf sequence. □

Example:

$$1^*,2,3,3 \longrightarrow 2,2,2^*,3,3 \longrightarrow 2,2^*,3,3,3,3 \longrightarrow 2^*,3,3,3,3,3,3 \longrightarrow 3,3,3,3,3,3,3,3$$

Claim: *A sequence refinement of a leaf sequence is a leaf sequence.*

Definition (Subdivision refinement)**:** A subdivision P is a refinement of Q if its set of breakpoints is a superset of the breakpoints of Q

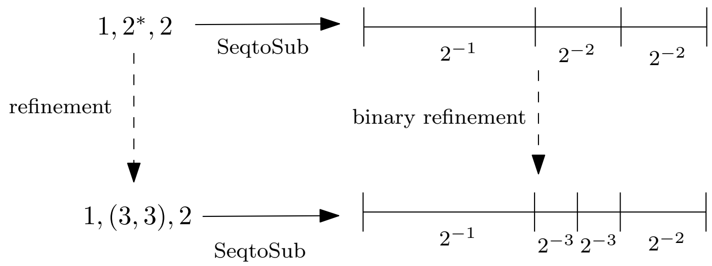


Claim: *Every dyadic subdivision has a refinement which is a proper subdivision.*

Proof. Every inteval of a dyadic subdivision is of the form $[a2^{-n}, b2^{-m}]$ which has length $\frac{b2^n-a2^m}{2^{n+m}}$ and so can be split into $b2^n-a2^m$ intervals of size 2^{-m-n}. We can therefore find a refinement of a dyadic subdivision where all intervals are powers of two. Such a subdivision is proper. □

Definition (Binary refinement)**:** A subdivision P is a binary of Q if it can be obtained from Q by bisecting intervals.

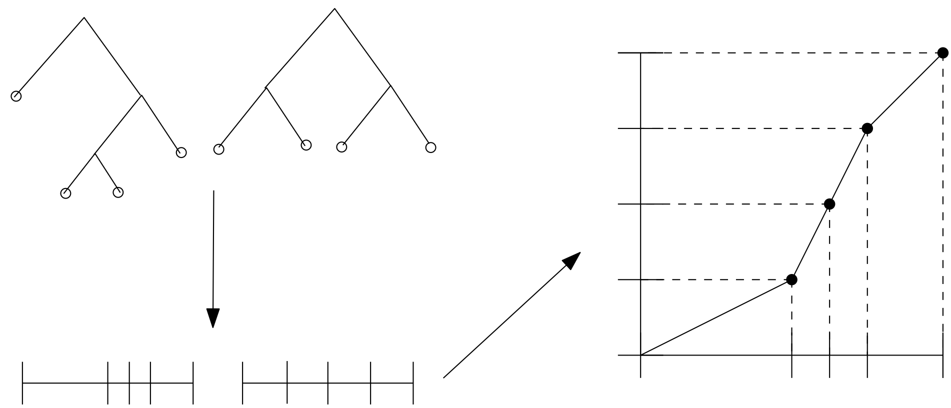
Note. A subdivision Q is a binary subdivision of a proper subdivision P produced by sequence L if and only if it is produced by a refinement of L (and therefore proper).

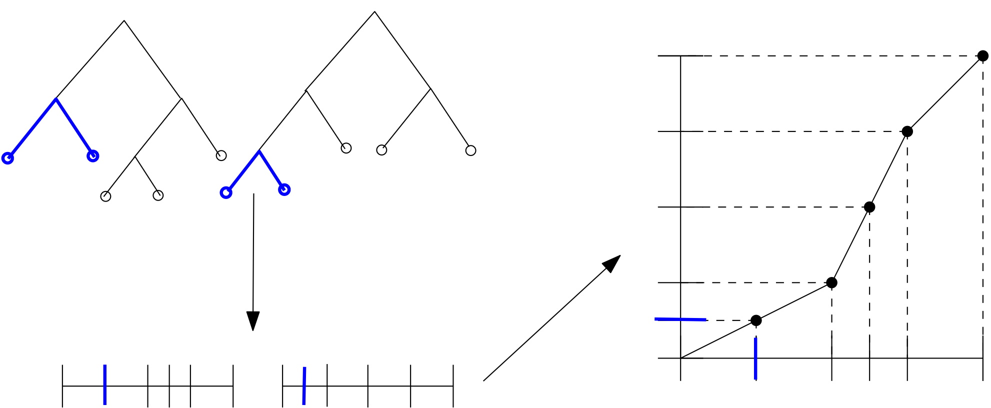


The connection to F

Claim: *A pair of dyadic subdivision with the same number of breakpoints doesn't necessarily interpolate to an element of F, but a pair of* proper *subdivisions always will.*

Claim: *Every pair of Thompson trees with the same number of leaves represents an element of F*





Claim: *If:*

1. $f \in F$ *is the interpolation of subdivisions* P *and* Q
2. P' *refines* P
3. Q' *refines* Q

Then:

1. $f(P) = Q$
2. $f(P')$ *is a refinement of* Q
3. $f^{-1}(Q')$ *is a refinement of* P
4. $(P', f(P'))$ *interpolates to* f
5. $\big(f^{-1}(Q'), Q'\big)$ *interpolates to* f

Claim: *Elements of* F *preserve proper subdivisions which refine their breakpoints, and binary refinements of subdivisions which refine their breakpoints.*

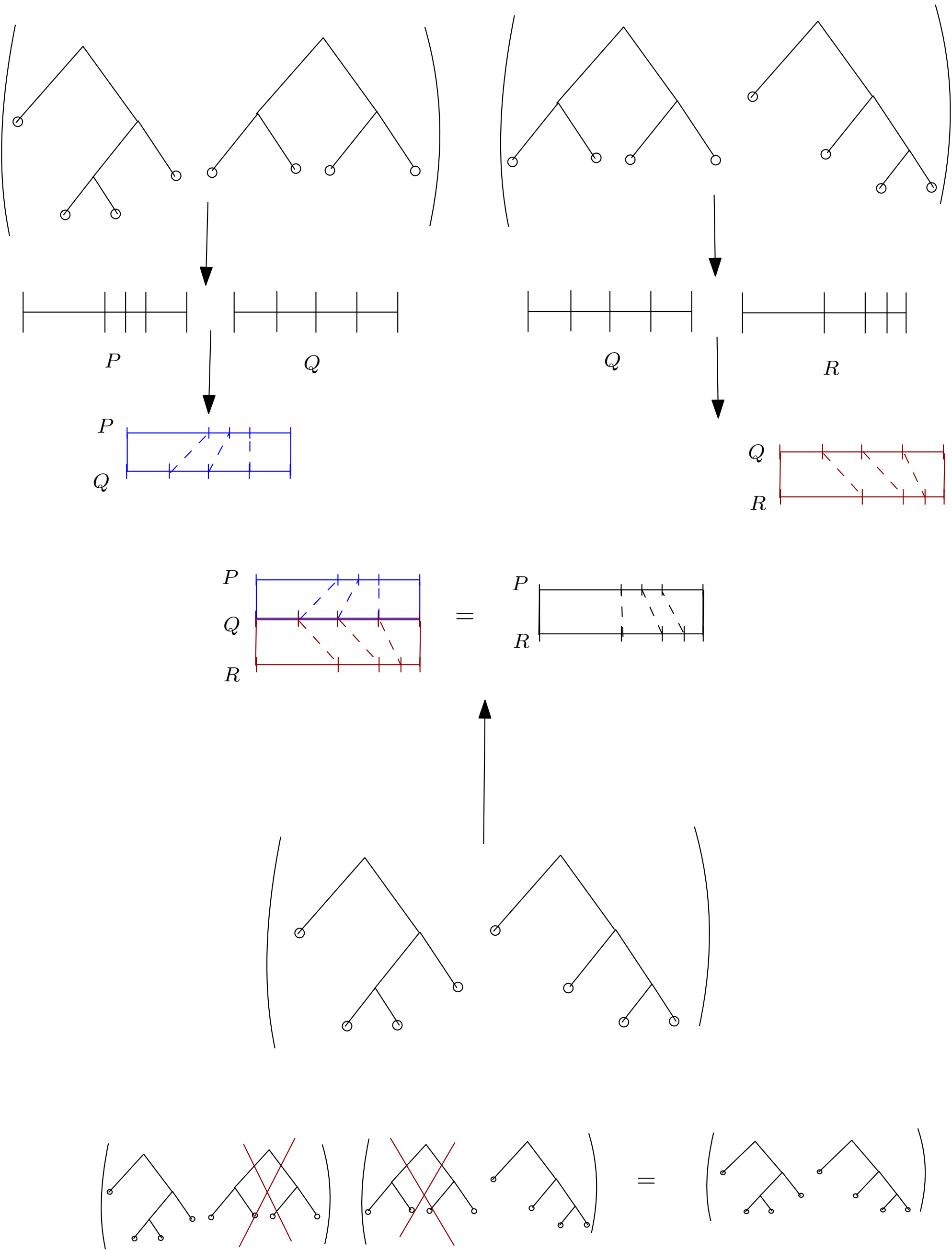
Claim: *Every element of* F *can be represented by a pair of trees.*

Proof. Assume that $f \in F$ is represented by a pair of dyadic rationals, (P, Q). Then:

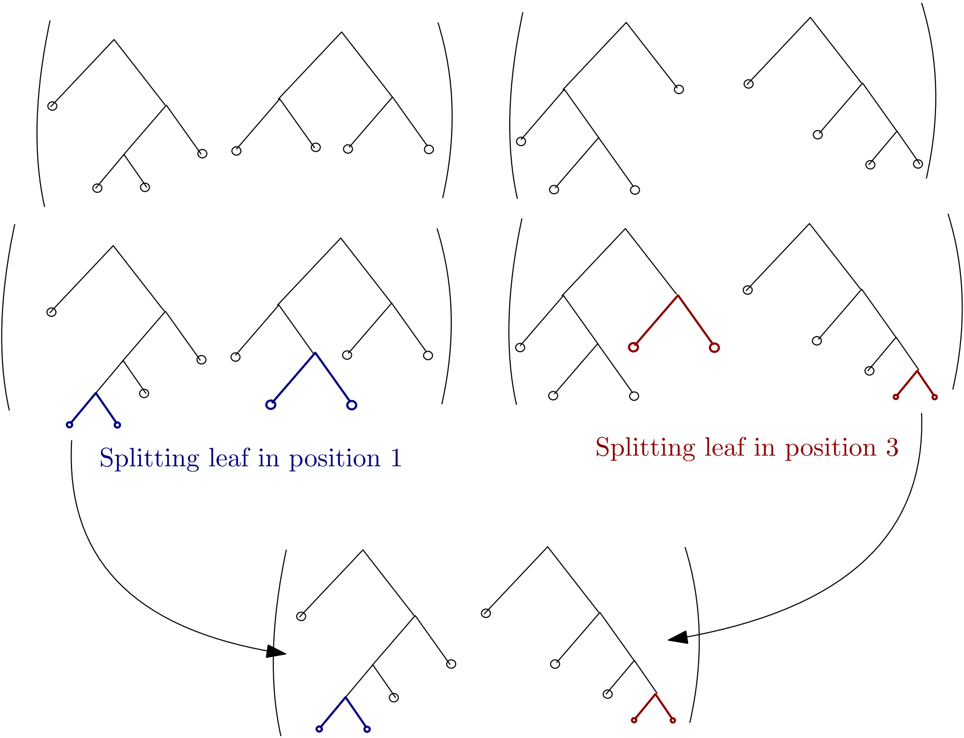
1. P has some refinement P' which is proper, coming from a sequence L
2. L has some refinement L' which is a leaf sequence and so P' has a (binary) refinement P'' which is generated by a tree.
3. f is represented by $(P'', f(P''))$
4. $f(P'')$ is a proper subdivision
5. $f(P'')$ has a binary refinement Q' generated by a tree T_2 .
6. $f^{-1}(Q')$ is a binary refinement of P'' and thus also generated by a tree T_1
7. f is represented by $(f^{-1}(Q'), Q')$ and thus is represented by the pair of trees (T_1, T_2)

□

Tree pair multiplication: the simple case

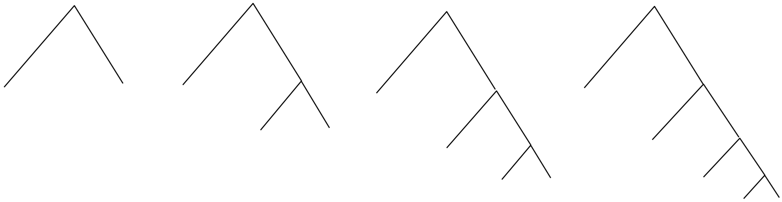


Tree pair multiplication



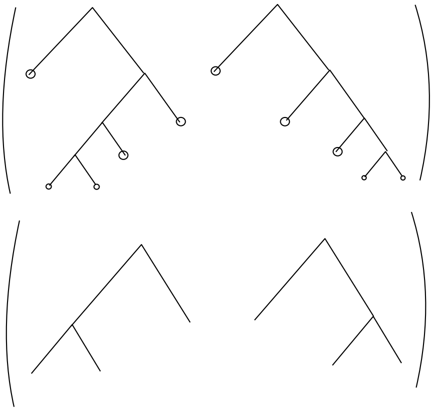
Spine pair factorisation

Definition (Spine)**:**

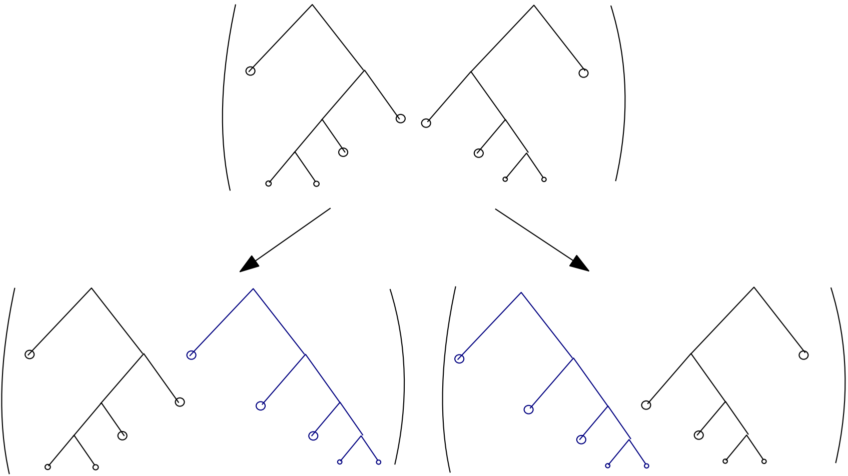


Spines of various lengths

Definition (Spine pair)**:** A *spine pair* is a tree pair where the tree on the right is a spine.

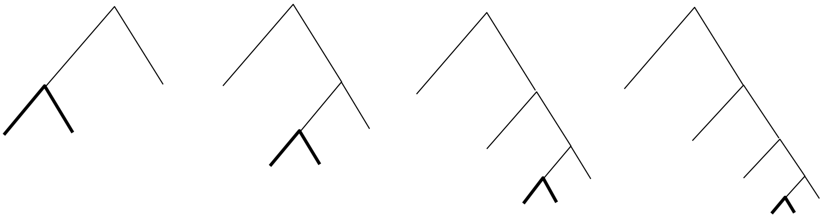


Claim: *Every tree pair can be expressed as the product of a spine pair and the inverse of a spine pair (i.e. spine pairs form a generating set for F.*



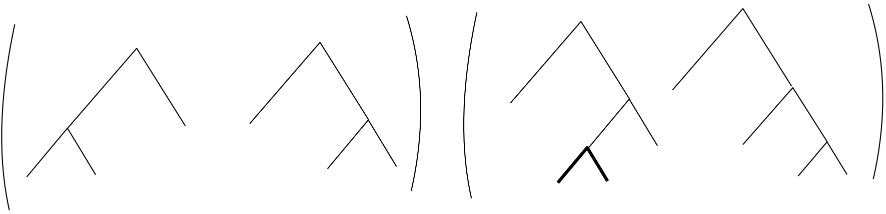
Tree generators

Definition (X_i)**:** For $i = 0, 1, 2, \ldots$ we define X_i to be the tree constructed by starting with a spine with i leaves, and adding a "caret" at leaf $i - 1$.



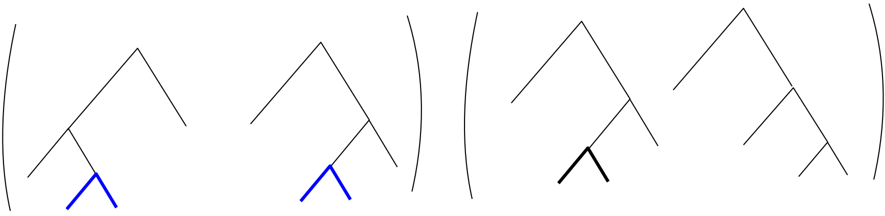
X_0 X_1 X_2 X_3

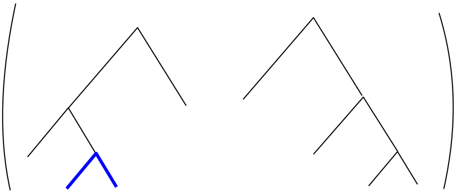
Notice what happens when we multiply a spine pair by a spine pair for an X_i tree:



A spine pair

The spine pair for X_1





We have:

$$\mathrm{SP}(T) \cdot \mathrm{SP}(X_i) = \mathrm{SP}(T')$$

where T' is obtained from T by adding a caret to node i

Let x_i denote the spine pair for X_i

Any tree T can be constructed by starting with a spine S of the appropriate length, and then adding carets to the leaves. Say we start by adding a caret to leaf i_1, then to leaf i_2 then to leaf ... up to leaf i_n to obtain T from S. Then we have:

$$\mathrm{SP}(T) = \mathrm{SP}(S) \cdot x_{i_1} \cdot x_{i_2} \cdot \cdots \cdot x_{i_n}$$

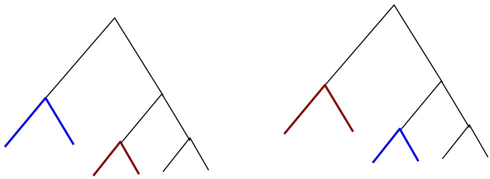
But $\mathrm{SP}(S) = \mathrm{id}_F$ so:

$$\mathrm{SP}(T) = x_{i_1} \cdot x_{i_2} \cdot \cdots \cdot x_{i_n}$$

Corollary 1: *The set* $X = \{x_i \mid i \geq 0\}$ *generates* F

Relations

Multiple ways to construct a tree by adding carets.



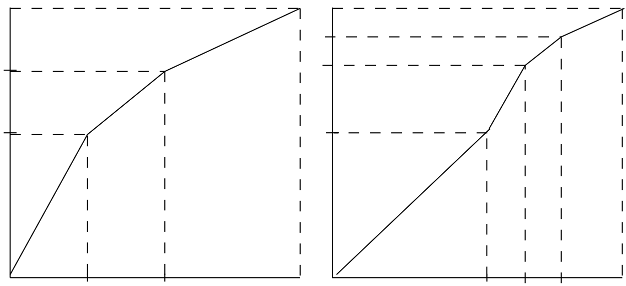
$$x_0 x_2 = x_1 x_0$$

$$x_i x_{j+1} = x_j x_i$$

$$x_{j+1} = x_i^{-1} x_j x_i$$

$$x_{j+1} = x_0^{-j} x_1 x_j$$

Corollary 2: *The set* $\{x_0, x_1\}$ *generates* F.



x_0 x_1

So F is finitely generated.

Presentations

Definition (Presentation for a group)**:** A *group presentation* is a pair consisting of:

1. A set X
2. A set $R \subseteq \mathrm{FreeGroup}(X)$

The group G corresponding to this presentation is the kernel:

$$\mathrm{FreeGroup}(R) \xrightarrow{h} \mathrm{FreeGroup}(X) \longrightarrow G$$

Where h is the map induced by the inclusion of R into $\mathrm{FreeGroup}(X)$

$$\begin{tikzcd}
\mathbf{Pres} \arrow[r, shift left, "\mathrm{PresentedGroup}"] \arrow[d] & \mathbf{Grp} \arrow[l, shift left, "\mathfrak{U}'"] \arrow[d] \\
\mathbf{Set} \arrow[r, shift left, "\mathrm{FreeGroup}"] & \mathbf{Grp} \arrow[l, shift left, "\mathfrak{U}"]
\end{tikzcd}$$

Example:

1. Cyclic group of order n : $\langle a \mid a^n \rangle$
2. Free group of order n : $\langle a_1, \dots, a_n \mid \emptyset \rangle$
3. $\mathbb{Z}^2$: $\langle a, b \mid aba^{-1}b^{-1} \rangle$
4. The Quaternion group Q_8 : $\langle a, b \mid abab^{-1}, baba^{-1} \rangle$

Example:
If G is presented by $\langle X \mid R \rangle$ and H is presented by $\langle Y \mid S \rangle$ then the free product (coproduct) $G * H$ is presented by $\langle X \cup Y \mid R \cup S \rangle$

Note that presentations are not unique. For example, C_6 has the following presentations:

$$\langle a \mid a^6 \rangle \quad \langle a, b \mid a^2,\ b^3,\ aba^{-1}b^{-1} \rangle$$

Claim: *Thompson's group F has the following presentation:*

$$\left\langle x_i\ i \geq 0 \mid x_j x_i x_{j+1}^{-1} x_i^{-1}\ 0 \leq i < j \right\rangle$$

Definition (Finite presentation)**:** A finite presentation is a presentation $\langle X \mid R \rangle$ where both X and R are finite. A group with a finite presentation is said to be *finitely presented.*

Note that if G is presented by $\langle X \mid R \rangle$ then X generates G . So the property of being finitely presented is stronger than that of being finitely generated.

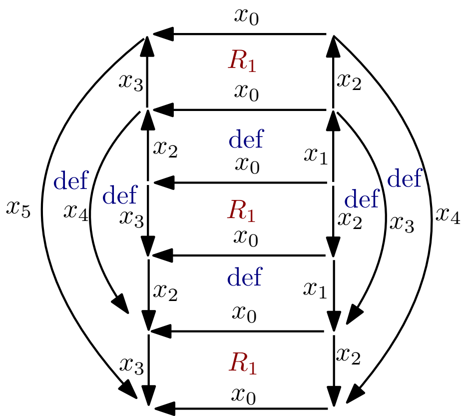
Is F finitely presented?

Claim: *Thompson's group F has the following presentation:*

$$\left\langle x_0, x_1 \mid x_0^{-1}x_2x_0x_3^{-1},\ x_1^{-1}x_3x_1x_4^{-1} \right\rangle$$

where $x_n := x_{n-2}^{-1}x_{n-1}x_{n-2}$ for $n > 1$

Idea of proof: We could try to show that any of the relations for the finite presentation, for example $x_0^{-1}x_4x_0x_5^{-1}$ can be expressed in terms of the relations in the above presentation:



Abelianisation

Definition (Commutator subgroup)**:** The commutator subgroup (a.k.a derived subgroup) of G denoted $[G, G]$ or G' is the normal subgroup generated by (i.e. smallest normal subgroup containing) every element of the form $ghg^{-1}h^{-1}$

Definition (Abelianisation)**:** The *abelianisation* of G is the quotient of G by the commutator subgroup.

Example:

1. The abelianisation of an abelian group is itself
2. The abelianisation of the free group of order n is the free abelian group of order n , i.e. $\mathbb{Z}^n$.
3. The abelianisation of the fundamental group of a connected space is its first homology group.

Note that the abelianisation is, in fact, abelian.

We can obtain a presentation for the abelianisation by "abelianising" the relations in a presentation for the group.

Claim: *Thompson's group F has the following presentation:*

$$\left\langle x_0, x_1 \mid x_0^{-1}x_2x_0x_3^{-1},\ x_1^{-1}x_3x_1x_4^{-1}\right\rangle$$

where $x_n := x_{n-2}^{-1}x_{n-1}x_{n-2}$ for $n > 1$

$$x_n = x_{n-2}^{-1}x_{n-1}x_{n-2} \longrightarrow x_n = -x_{n-2} + x_{n-1} + x_{n-2} = x_{n-1} \quad n > 1$$

$$x_0^{-1}x_2x_0x_3^{-1} \longrightarrow -x_0 + x_1 + x_0 - x_1 \quad \text{(trivial)}$$

$$x_1^{-1}x_3x_1x_4^{-1} \longrightarrow -x_1 + x_1 + x_1 - x_1 \quad \text{(trivial)}$$

Claim: *The abelianisation of F is isomorphic to the free abelian group on two generators, $\mathbb{Z}^2$*

Note. A more explicit definition of the abelianisation map in terms of tree pairs, is that it is the map sending a pair of trees to the differences in height between the first leaves of each tree and the last leaves of each tree.

Definition (Simple group)**:** A group is *simple* if it is non-trivial, and has no non-trivial normal subgroups.

Corollary 3: *F is not simple*

Theorem: *The derived subgroup of F is simple.*

Corollary 4: *$F'' = F'$*

Definition (Solvable group)**:** A group G is *solvable* if you can get to the trivial group after taking the derived subgroup finitely many times.

Corollary 5: *F is not solvable.*

Note. The smallest non-solvable group is A_5 which corresponds to the fact that the lowest degree polynomials which are not solvable by radicals have order 5

Some other properties

Definition (Solvable word problem)**:** A group has solvable word problem if for any finite generating set $X \subseteq G$ it is decidable whether a word over these generators is the identity.

Not all finitely presented groups have solvable word problem

$$\begin{array}{llll}
\langle & a,b,c,d,e,p,q,r,t,k & | & \\
 & p^{10}a = ap, & pacqr = rpcaq, & ra = ar, \\
 & p^{10}b = bp, & p^2adq^2r = rp^2daq^2, & rb = br, \\
 & p^{10}c = cp, & p^3bcq^3r = rp^3cbq^3, & rc = cr, \\
 & p^{10}d = dp, & p^4bdq^4r = rp^4dbq^4, & rd = dr, \\
 & p^{10}e = ep, & p^5ceq^5r = rp^5ecaq^5, & re = er, \\
 & aq^{10} = qa, & p^6deq^6r = rp^6edbq^6, & pt = tp, \\
 & bq^{10} = qb, & p^7cdcq^7r = rp^7cdceq^7, & qt = tq, \\
 & cq^{10} = qc, & p^8ca^3q^8r = rp^8a^3q^8, & \\
 & dq^{10} = qd, & p^9da^3q^9r = rp^9a^3q^9, & \\
 & eq^{10} = qe, & a^{-3}ta^3k = ka^{-3}ta^3 & \rangle
\end{array}$$

(From wikipedia `https://en.wikipedia.org/wiki/Word_problem_for_groups`)

Does F have solvable word problem?

Yes

Theorem (Boone-Higman)**:** *A finitely presented group has solvable word problem if and only if it can be embedded in a simple group that can be embedded in a finitely presented group.*

Group cohomology

Definition (Classifying space)**:** A *classifying space* for a group G (or $K(G,1)$) is a pointed topological space with fundamental group G and trivial higher homotopy groups. Equivalently, it is a weakly contractible space on which G acts freely and properly.

Example:

1. S^1 is a classifying space for $\mathbb{Z}$ (corresponding to the action of $\mathbb{Z}$ on $\mathbb{R}$)
2. T^n is a classifying space for $\mathbb{Z}^n$ (corresponding to the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$)
3. The wedge of n circles is a classifying space for F_n (the free group of order n

Definition (Geometric Dimension)**:** The *geometric dimension* is the smallest n such that there exists an n -dimensional CW complex which is a $K(G,1)$.

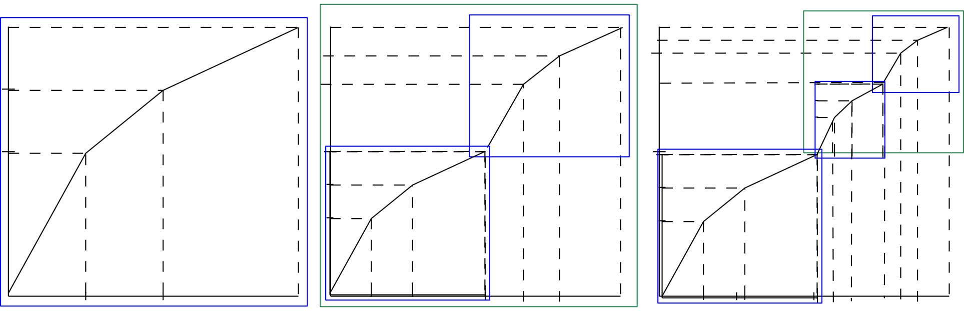
Example:

1. $\mathrm{gd}(\mathbb{Z}) = 1$
2. $\mathrm{gd}(\mathbb{Z}^n) = n$
3. A group is free if and only if its geometric dimension is 1

Theorem: *Thompson's group has infinite dimension.*

Idea of proof:

1. F contains a subgroup isomorphic to $\mathbb{Z}$
2. F contains a subgroup isomorphic to $F \times F$
3. F contains a subgroup isomorphic to $\mathbb{Z}^n$ for any n
4. The geometric dimension of a group is greater than the geometric dimension of any of its subgroups.



Definition (Cohomological Dimension)**:** The *Cohomological dimension* of a group G is the length of the shortest free $\mathbb{Z}G$-resolution of $\mathbb{Z}$.

Theorem: *Thompson's group has infinite cohomological dimension.*

Theorem: *For a group G , if* $\mathrm{cd}(G) > 2$, *then* $\mathrm{gd}(G) = \mathrm{cd}(G)$

Definition (F_∞)**:** A group G has type F_∞ if it has a $K(G,1)$ with finitely many cells in each dimension.

Theorem: *Thompson's group is of type F_∞*

Definition (FP_∞)**:** A group G has type FP_∞ if $\mathbb{Z}$ has a free $\mathbb{Z}G$ resolution by finitely generated modules.

Theorem: $FP_\infty \implies F_\infty$, *but the converse is false.*

F was the first example of a torsion-free, infinite dimensional FP_∞ group.

Definition (Amenability)**:** A (discrete group) is amenable if there exists a finitely additive positive "translation invariant" measure on F

Is F amenable?

Not sure