Weights for Oplax Colimits

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Defn. (Cat-*weighted colimit*): For a Cat-presheaf $W : \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ and 2-functor $F : \mathcal{A} \to \mathcal{B}$, the *$W$-weighted colimit of F* is a representation:

$$\mathcal{B}(W * F, -) \cong \left[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}\right](W, \mathcal{B}(F, -))$$

Defn. (*Oplax colimit*): (For $W : \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$, $F : \mathcal{A} \to \mathcal{B}$, the *$W$-weighted oplax colimit of F* is a representation:

$$\mathcal{B}(W \circledast F, -) \cong \left[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}\right]_{\mathsf{oplax}}(W, \mathcal{B}(F, -))$$

When $W = \Delta\mathbb{1} : \mathcal{A}^{\mathsf{op}} \to \mathcal{B}$ we say the the (oplax) colimit is *conical*.

Examples

Coproducts: $(\Delta\mathbb{1} : X \to \mathbf{Cat}, X \text{ discrete})$

Copowers (aka tensors): for C *locally*-discrete,
$\Delta_C\mathbb{1} \circledast \Delta a \cong C \odot a$ for a in some 2-category $\mathcal{A}$.

Cographs of functors: $\big(W = \Delta\mathbb{1} : (\bullet \to \bullet) \to \mathbf{Cat}\big)$

CoKleisli objects: $\big(W = \Delta\mathbb{1} : \Sigma\big({\Delta_+}^{\mathrm{op}}\big) \to \mathcal{A}\big)$

Grothedieck constructions: $\big(W = \Delta\mathbb{1} : C \to \mathbf{Cat}\big)$

Defn. (*oplax-morphism classifier*): of $W : \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ is a representation:

$$[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]\Big(W^\sharp, -\Big) \cong [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}(W, -)$$

Assuming $W^\sharp$ exists for a given $W : \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$:

$$\begin{aligned} \mathcal{B}(W \circledast F, -) &\cong [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}(W, \mathcal{B}(F, -)) \\ &\cong [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]\Big(W^\sharp, \mathcal{B}(F, -)\Big) \\ &\cong \mathcal{B}\Big(W^\sharp * F, -\Big) \end{aligned}$$

So oplax colimits are special Cat-weighted colimits.

Constructing $W^\sharp$

(Street 1972) for $W : A \to \mathbf{Cat}$ an oplax functor from a 1-category

(Bozapalidès 1980) for arbitrary presheaves $W : \mathcal{A} \to \mathbf{Cat}$ using *(op)lax (co)ends*

(Blackwell, Kelly and A. J. Power 1989) for the general case of oplax morphism classifiers for the algebras of certain 2-monads

(Lack 2002) as *codescent* objects, again for general 2-monad algebras

Idea:

$$
\begin{aligned}
\left[\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}\right]_{\mathrm{oplax}}(W, X) &\cong \oint_{a \in \mathcal{A}^{\mathrm{op}}} [W_a, X_a] \\
&\cong \oint_{a \in \mathcal{A}^{\mathrm{op}}} \left[W_a, \int_{x \in \mathcal{A}} [\mathcal{A}(x, a), X_x] \right] \\
&\cong \int_{x \in \mathcal{A}} \oint_{a \in \mathcal{A}^{\mathrm{op}}} [W_a \times \mathcal{A}(x, a), X_x] \\
&\cong \int_{x \in \mathcal{A}} \left[\oint^{a \in \mathcal{A}} W_a \times \mathcal{A}(x, a), X_x \right] \\
&\cong \left[\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}\right] \left(\oint^{a \in \mathcal{A}} W_a \times \mathcal{A}(-, a), X \right)
\end{aligned}
$$

Explicitly: for $a \in \mathcal{A}$, $W^\sharp_a$ is the category with:

0-cells pairs $(u : a \to b, x \in W_b)$

1-cells from $(u : a \to b, x)$ to $(v : a \to b', y)$ given by pairs:

$$
\begin{tikzcd}
 & b \arrow[d, "f"] \\
a \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & b'
\end{tikzcd}
\qquad x \xrightarrow{\beta} X_f y \in X_b
$$

modulo the equivalence relation generated by:

$$
\begin{tikzcd}
 & b \arrow[d, "w"] \arrow[d, bend left=60, "w'"] \arrow[d, phantom, shift left=3, "\overset{\theta}{\Leftarrow}"] \\
a \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[r, phantom, "\alpha\Downarrow"] & b'
\end{tikzcd}
\; x \xrightarrow{\beta} X_{w'} y \;\sim\;
\begin{tikzcd}
 & b \arrow[d, "w"] \\
a \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[r, phantom, "\alpha\Downarrow"] & b'
\end{tikzcd}
\; x \xrightarrow{\beta} X_{w'} y \xrightarrow{X_\theta y} X_w y
$$

Two questions:

(a) What is the *saturation* of weights of the form $W^\sharp$?

(b) What are the *coalgebras* for the comonad $W \mapsto W^\sharp$?

Saturation

For a class Φ of weights, the *saturation* Φ^* contains all (small) weights $W : \mathcal{A}^{\mathsf{op}} \to \mathbf{Cat}$ such that Φ-complete/continuous $\implies$ W-complete/continuous. If $\Phi = \Phi^*$, the class is said to be *saturated*.

Examples:

$$\{\text{non-empty finite coprods}\} \subseteq \{\text{binary coprods}\}^*$$

$$\{\text{representables}\} = \varnothing^*$$

$$\{\text{small weights}\} = \left\{\begin{matrix}\text{coproducts} \\ \text{coequalisers} \\ \text{tensors by } \mathbb{2}\end{matrix}\right\}^*$$

Coalgebras

$$\begin{tikzcd}[\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}] \arrow[rr, bend right, "\text{forget}"'] \arrow[rr, phantom, "\perp"] && {[\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}]_{\mathrm{oplax}}} \arrow[ll, bend right, "\sharp"']\end{tikzcd}$$

$$[\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}]\left(W^{\sharp}, X\right) \cong [\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}]_{\mathrm{oplax}}(W, X)$$

This is an *oplax-idempotent comonad* (Blackwell, Kelly and A. J. Power 1989; Lack and Shulman 2012).

Aside: $\natural$-coalgebras:

$$\begin{tikzcd} {[\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}]} \arrow[r, bend right, "\text{forget}"'] \arrow[r, phantom, "\perp"] & {[\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}]_{\text{pseudo}}} \arrow[l, bend right, "\natural"'] \end{tikzcd}$$

$$[\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}]\left(W^{\natural}, X\right) \cong [\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}]_{\text{pseudo}}(W, X)$$

$\natural$-coalgebras are precisely the PIE weights, i.e.:

(a) the saturation of {products, inserters, equifiers}

(b) weights W such that $\mathbf{el}(W_o)$ has terminal objects in each connected component.

Coalgebra characterisation: (Lack and Shulman 2012)
$(a) \Leftrightarrow (b)$: (J. Power and Robinson 1991)

Defn. (*2-category of elements*, $\mathbf{el}\,W$): for $W\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Cat}$, the 2-category $\mathbf{el}\,W$ has:

0-cells: pairs $(a \in \mathcal{A}, x \in Wa)$

1-cells: $(a,x) \to (b,y)$ are pairs $(u\colon a \to b, f\colon x \to W_u y)$

2-cells: $(u,f) \Rightarrow (v,g)\colon (a,x) \to (b,y)$ are 2-cells $\sigma\colon u \Rightarrow v$ in $\mathcal{A}$ such that $W_\sigma y\, f = g$:

$$\begin{tikzcd} & W_u y \arrow[d, "W_\sigma y"] \\ x \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[r, phantom, "\circlearrowright"] & W_v y \end{tikzcd}$$

A 2-functor $|W|\colon \mathbf{el}\,W \to \mathcal{A}$ is then given by projection onto the first component, e.g. $|W|(a,x) = a$.

Discrete 2-fibrations

A *discrete 2-fibration* is a split 2-fibration which is a discrete *opfibration* on hom-categories.

Claim: every discrete 2-fibration $F : \mathcal{A} \to \mathcal{B}$ in Cat is isomorphic to $|W| : \mathsf{el}\, W \to \mathcal{B}$ for some $W : \mathcal{B}^{\mathsf{op}} \to \mathsf{Cat}$.

Moreover, $\mathsf{el} : [\mathcal{B}^{\mathsf{op}}, \mathsf{Cat}] \to \mathsf{D2Fib}(\mathcal{B})$ underlies an equivalence of 2-categories, where $\mathsf{D2Fib}(\mathcal{B}) \subseteq \mathsf{2Cat}/\mathcal{B}$ is the locally-full subcategory of discrete 2-fibrations and split-cartesian functors (Lambert 2020).

The equivalence $\mathsf{el} : [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}] \to \mathsf{D2Fib}(\mathcal{A})$ extends to an equivalence from $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}$ to the **full** subcategory of discrete 2-fibrations in $\mathsf{2Cat}/\mathcal{A}$, denoted $\mathsf{2Cat}/_{\mathsf{d2f}}(\mathcal{A})$.

$$\begin{tikzcd}
{[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]} \arrow[rr, bend right, "\mathrm{forget}"'] \arrow[d, "\mathsf{el}"'] \arrow[rr, phantom, "\perp"] & & {[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}} \arrow[ll, bend right, "\sharp"'] \arrow[d, "\mathsf{el}"] \\
\mathsf{D2Fib}(\mathcal{A}) \arrow[rr, bend right, "\mathrm{forget}"'] \arrow[rr, phantom, "\perp"] & & \mathsf{2Cat}/_{\mathsf{d2f}}(\mathcal{A}) \arrow[ll, bend right, dashed, "\mathcal{F}"']
\end{tikzcd}$$

Conclusion: The map $\mathsf{el} : [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}] \to \mathsf{2Cat}/_{\mathsf{d2f}}(\mathcal{A})$ is (up to equivalence) the coKleisli map for $\sharp$.

A "larger" adjunction generates the same comonad:

$$
\begin{tikzcd}
 & \mathbf{2Cat}/\mathcal{A} \arrow[dr, bend left, "\text{free fib}"] \arrow[dr, phantom, "\perp"] & \\
\mathbf{2Cat}/_{\mathbf{d2f}}(\mathcal{A}) \arrow[ur, hook] & \circlearrowright & {[\mathcal{A}^{op}, \mathbf{2Cat}]} \arrow[ul, "\mathrm{el}"] \arrow[dl, bend left, "{[\mathcal{A}^{op}, \pi_1]}"] \arrow[dl, phantom, "\perp"] \\
 & {[\mathcal{A}^{op}, \mathbf{Cat}]} \arrow[ul, "\mathrm{el}"] \arrow[ur, "{[\mathcal{A}^{op}, \mathrm{d}]}"'] &
\end{tikzcd}
$$

Compare with the 1-categorical situation:

$$
\begin{tikzcd}
{[A^{op}, \mathbf{Set}]} \arrow[r, bend right, "\mathrm{el}"'] \arrow[r, phantom, "\perp"] & \mathbf{Cat}/A \arrow[l, bend right, "\mathrm{im}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
{[\mathcal{A}^{op}, \mathbf{Cat}]} \arrow[r, bend right, "\mathrm{el}"'] \arrow[r, phantom, "\perp"] & \mathbf{2Cat}/\mathcal{A} \arrow[l, bend right, "\mathrm{im}_{\mathrm{oplax}}"']
\end{tikzcd}
$$

idempotent $\qquad\qquad$ oplax-idempotent

$\mathrm{im}_{\mathrm{oplax}} F$ is the *oplax image presheaf* of $F : \mathcal{B} \to \mathcal{A}$,

defined as the **oplax** colimit of $\mathcal{B} \xrightarrow{F} \mathcal{A} \xrightarrow{よ} [\mathcal{A}^{\mathrm{op}}, \mathrm{Cat}]$

$$
\begin{aligned}
\mathcal{C}\left(\mathrm{im}_{\mathrm{oplax}} F * G, c\right) &\cong \left[\mathcal{A}^{\mathrm{op}}, \mathrm{Cat}\right]\left(\mathrm{im}_{\mathrm{oplax}} F, \mathcal{C}(G-, c)\right) \\
&\cong \int_{x \in \mathcal{A}} \left[\oint^{b \in \mathcal{B}} \mathcal{A}(x, Fb), \mathcal{C}(Gx, c)\right] \\
&\cong \oint_{b \in \mathcal{B}^{\mathrm{op}}} \int_{x \in \mathcal{A}} [\mathcal{A}(x, Fb), \mathcal{C}(Gx, c)] \\
&\cong \oint_{b \in \mathcal{B}^{\mathrm{op}}} \mathcal{C}(GFb, c) \\
&\cong \mathcal{C}(\Delta \mathbb{1} \circledast GF, c)
\end{aligned}
$$

In particular, $W \circledast G \cong W^{\sharp} * G \cong \mathrm{im}_{\mathrm{oplax}} |W| * G \cong \Delta \mathbb{1} \circledast G\, |W|$.

$$
\begin{tikzcd}
{[\mathcal{A}^{\mathrm{op}}, \mathrm{Cat}]} \arrow[rr, bend right, "\mathrm{el}"'] \arrow[rr, phantom, "\perp"] & & \mathrm{2Cat}/\mathcal{A} \arrow[ll, bend right, "\mathrm{im}_{\mathrm{oplax}}"']
\end{tikzcd}
\quad \sim \quad
\begin{tikzcd}
\mathrm{D2Fib}(\mathcal{A}) \arrow[rr, bend right, "\text{inclusion}"'] \arrow[rr, phantom, "\perp"] & & \mathrm{2Cat}/\mathcal{A} \arrow[ll, bend right, "\text{free d2fib}"']
\end{tikzcd}
$$

For a 2-functor $p : \mathcal{B} \to \mathcal{A}$, the free *split* 2-fibration is given by a lax comma 2-category (λ is lax):

$$
\begin{tikzcd}
\mathcal{A} \Downarrow p \arrow[rr] \arrow[dr, "\pi"'] & \arrow[d, phantom, "\overset{\lambda}{\Longrightarrow}"] & \mathcal{B} \arrow[dl, "p"] \\
& \mathcal{A} &
\end{tikzcd}
$$

The free *discrete* 2-fibration $p^{\star} : \widehat{p} \to \mathcal{A}$ is constructed by quotienting out the π-vertical 2-cells of $\mathcal{A} \Downarrow p$.

Explicitly, for $p : \mathcal{B} \to \mathcal{A}$ the 2-category $\widehat{p}$ has:

0-cells given by pairs $(x \in \mathcal{B}, u\colon a \to px)$

1-cells equivalence classes of lax squares:

$$
\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, "s"'] & px \arrow[d, "pf"] \arrow[dl, phantom, "\overset{\sigma}{\Longleftarrow}"] \\
b \arrow[r, "v"'] & py
\end{tikzcd}
\quad \sim \quad
\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, "s"'] & px \arrow[d, "pf"] \arrow[dl, phantom, "\overset{\sigma}{\Longleftarrow}"] \arrow[d, bend left=60, "pg"] \arrow[d, phantom, shift left=4ex, "\overset{p\alpha}{\Longleftarrow}"] \\
b \arrow[r, "v"'] & py
\end{tikzcd}
$$

2-cells $(s, f, \sigma) \Rightarrow (t, g, \tau)$ are 2-cells $\kappa\colon s \Rightarrow t$ such that:

$$
\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, "s"] \arrow[d, bend right=60, "t"'] \arrow[d, phantom, shift right=4ex, "\overset{\kappa}{\Longleftarrow}"] & px \arrow[d, "pf"] \arrow[dl, phantom, "\overset{\sigma}{\Longleftarrow}"] \\
b \arrow[r, "v"'] & py
\end{tikzcd}
\quad \sim \quad
\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, "t"'] & px \arrow[d, "pg"] \arrow[dl, phantom, "\overset{\tau}{\Longleftarrow}"] \\
b \arrow[r, "v"'] & py
\end{tikzcd}
$$

A $\star$-coalgebra structure $G : p \to p^\star$ on a discrete 2-fibration involves a section of the counit:

$$x \overset{G}{\mapsto} \left(px \xrightarrow{g_x} pG_x\right) \overset{\epsilon}{\mapsto} g_x^*(G_x)$$

which picks for each $x \in \mathcal{B}$ some chosen cartesian arrow

$$x = u^*(G_x) \xrightarrow{\gamma_x = \bar{g}_x(G_x)} G_x$$

Because G preserves chosen cartesian morphisms:

$$\begin{tikzcd} u^*x \arrow[d, "\bar{u}x"'] \\ x \end{tikzcd} \overset{G}{\longmapsto} \begin{tikzcd} pu^*x \arrow[r, "g_{u^*x}"] \arrow[d, "u"'] & pG_{u^*x} \arrow[d, "pG_{\bar{u}x}"] \arrow[l, phantom, "\overset{h_{\bar{u}x}}{\Longleftarrow}"] \\ px \arrow[r, "g_x"'] & pG_x \end{tikzcd} \sim \begin{tikzcd} a \arrow[r, "g_x\, u"] \arrow[d, "u"'] \arrow[dr, phantom, "\circlearrowright"] & pG_x \arrow[d, equal, "p1_{Gx}"] \\ b \arrow[r, "g_x"'] & pG_x \end{tikzcd}$$

so $G_{u^*x} = G_x$ and $\gamma_{u^*x} = \gamma_x\, \bar{u}_x$.

Conclusion: a $\star$-coalgebra structure on $p : \mathcal{B} \to \mathcal{A}$ involves choosing a terminal object in each connected component of the wide sub-1-category $p_{\mathsf{cart}} \subseteq \mathcal{B}$ of chosen cartesian 1-cells.

When $p = |W|$ for some $W : \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$, $p_{\mathsf{cart}} \cong \mathsf{el}\, W_{\mathsf{o}}$, so $\sharp$-coalgebras are PIE weights.

We call objects of the form G_x *generic*, and the restriction of p to the full subcategory of $\mathcal{B}$ containing these objects is the *generic core*, $p_\Gamma : \mathcal{B}_\Gamma \to \mathcal{A}$.

$$
\begin{tikzcd}
\mathsf{2Cat}/A \arrow[rr, "K"'] \arrow[rr, phantom, shift left=2, "\top"] \arrow[dr, "\star"'] & & \star\text{-coalg} \arrow[ll, bend right, "(-)_{\mathsf{G}}"'] \arrow[dl, "\text{forget}"] \\
& \mathsf{D2Fib}(\mathcal{A}) &
\end{tikzcd}
$$

The right adjoint to K must send $G : p \to p^\star$ to the ($\star$-split) equaliser $E \overset{e}{\hookrightarrow} p \rightrightarrows_{\eta_p}^{G} p^\star$

$$
\begin{tikzcd}
x \arrow[d, "f"'] & \overset{G}{\longmapsto} & px \arrow[r, "g_x"] \arrow[d, "pf"'] \arrow[dr, phantom, "\overset{g_f}{\Longleftarrow}"] & pG_x \arrow[d, "pG_f"] \\
y & & py \arrow[r, "g_y"'] & pG_y
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[d, "f"'] & \overset{\eta_p}{\longmapsto} & px \arrow[r, equal] \arrow[d, "pf"'] \arrow[dr, phantom, "\circlearrowleft"] & px \arrow[d, "pf"] \\
y & & py \arrow[r, equal] & pPy
\end{tikzcd}
$$

$$
\begin{tikzcd}
pG_x \arrow[r, equal] \arrow[d, "pf"'] & pG_x \arrow[d, "pG_f"] \arrow[dl, phantom, "\overset{g_f}{\Longleftarrow}"] \\
pG_y \arrow[r, equal] & pG_y
\end{tikzcd}
=
\begin{tikzcd}
pG_x \arrow[r, equal] \arrow[d, "pf"'] \arrow[dr, phantom, "\circlearrowright"] & pG_x \arrow[d, bend right, "pf"'] \arrow[d, bend left, "pG_f"] \arrow[d, phantom, "\overset{\underline{pg_f}}{\Longleftarrow}"] \\
pG_y \arrow[r, equal] & pG_y
\end{tikzcd}
\sim
\begin{tikzcd}
pG_x \arrow[r, equal] \arrow[d, "pf"'] \arrow[dr, phantom, "\circlearrowright"] & pG_x \arrow[d, "pf"] \\
pG_y \arrow[r, equal] & pG_y
\end{tikzcd}
$$

So $p_\Gamma \xhookrightarrow{\mathrm{incl}} p \underset{\eta_p}{\overset{G}{\rightrightarrows}} p^\star$ is an equaliser, and $(-)_\Gamma$ is right-adjoint to $K : \mathbf{2Cat}/\mathcal{A} \to \star\text{-}\mathbf{coalg}$.

The counit of $K \colon \mathbf{2Cat}/\mathcal{A} \rightleftharpoons \star\text{-}\mathbf{coalg} : (-)_\Gamma$ has component at $G \colon p \to p^\star$ given by:

$$
{p_\Gamma}^\star \xrightarrow{\mathrm{incl}^\star} p_\Gamma \xrightarrow{\epsilon_p} p
$$

Proposition: $p_{\Gamma^\star} \xrightarrow{\text{incl}^\star} p_\Gamma \xrightarrow{\epsilon_p} p$ is an isomorphism.

Proof. The coalgebra structure map $G : p \longrightarrow p^\star$ forms an adjunction $\epsilon_p \dashv G$ with identity counit (from the general theory of coalgebras for oplax-idempotent monads). So G is fully-faithful and thus restricts to an isomorphism to its image, which is $p_{\Gamma^\star}$. The restriction of ϵ_p to $p_{\Gamma^\star}$ is a left-inverse, and thus an inverse to this map.

Corollary: $(-)_\Gamma : \star$-**coalg** $\to$ **2Cat**/$\mathcal{A}$ is a reflective subcategory.

In fact, this adjunction underlies a *comprehensive factorisation system* (Berger and Kaufmann 2017) on **2Cat** whose *covering* morphisms (i.e. right class) are "local discrete opfibrations" and whose *connected* morphisms are b.o.o locally initial 2-functors.

This orthogonal factorisation system lifts the comprehensive factorisation system of (Street and Walters 1973) on **Cat** to **2Cat** locally.

$$\begin{tikzcd} \mathcal{B} \arrow[r, "\mathsf{r}_p"'] \arrow[rr, bend left, "p"] & (\mathcal{B}^\star)_\Gamma \arrow[r, "(p^\star)_\Gamma"'] \arrow[u, phantom, "\circlearrowleft"] & \mathcal{A} \end{tikzcd}$$

Returning to presheaves:

$$\begin{tikzcd}
& \mathsf{2Cat}/\mathcal{A} \arrow[dd, bend left, "\mathrm{im}_{\mathrm{oplax}}"] \arrow[dr, bend left, "\mathrm{refl.}"] \arrow[dr, phantom, shift left, "\perp"] & \\
\mathsf{2Cat}/_{\mathrm{d2f}}(\mathcal{A}) \arrow[ur, hook] & & \mathsf{2Cat}/_{\mathrm{ld}}\mathcal{A} \arrow[ul, hook'] \arrow[dl, "\mathrm{im}_{\mathrm{oplax}}"] \\
& {[\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}]} \arrow[ul, "\mathrm{el}"] \arrow[uu, bend left, "\mathrm{el}"] &
\end{tikzcd}$$

A presheaf in $[\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}]$ admits a $\sharp$-coalgebra structure precisely if:

(a) it is the oplax image presheaf of a 2-functor $F : \mathcal{B} \to \mathcal{A}$

(b) it is the oplax image presheaf of a local discrete opfibration $p : \mathcal{B} \to \mathcal{A}$

I.e. $\sharp$-coalgebras are the oplax colimits of representables.

Set-presheaves which are coproducts of representables are those whose category of elements have "enough component-terminal objects".

Cat-presheaves which are PIE weights are those whose category of elements has a enough component-terminal objects in its chosen-cartesian sub-1-category.

For $p : \mathcal{B} \to \mathcal{A}$ a discrete 2-fibration, an object $x \in \mathcal{B}$ is *oplax generic* if it is "cartesian-component-terminal" and:

> for any $f : y \to x$ and chosen-cartesian $g : y \to z$, $(y \Downarrow \mathcal{B})(g, f)$ has a single connected component.

Prop: **Cat**-presheaves are $\sharp$-coalgebras if their category of elements has enough oplax-generic objects.

Saturation

Let δ, θ and Θ denote the classes of $\Delta\mathbb{1}^\sharp$'s, $W^\sharp$'s and $\sharp$-coalgebras respectively. Note: $\delta \subset \theta \subset \Theta$.

Thm: (Kelly and Schmitt 2005) for a class of small weights Φ, the weights in the saturation Φ^* with domain $\mathcal{A}$ are those in the closure of the representables in $[\mathcal{A}^{\mathsf{op}}, \mathbf{Cat}]$ under Φ-colimits (henceforth denoted $\Phi\mathcal{A}$).

Cor: $\Theta \subseteq \delta^*$, and so $\Theta^* \subseteq \delta^*$, and so $\Theta^* = \delta^* = \theta^*$.

Prop: Θ is saturated.

Proof. It suffices to show that $\Theta_{\mathcal{A}} = \Theta^*_{\mathcal{A}}$; i.e. that $\Theta_{\mathcal{A}} \subseteq [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ contains the representables and is closed in $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ under oplax colimits.

Now $\sharp$ is an oplax-idempotent comonad, so $\mathsf{U}\colon \sharp\text{-}\mathsf{coalg}_{\mathsf{oplax}} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ is fully-faithful. The repletion of U's image is $\Theta_{\mathcal{A}}$. Because U creates oplax colimits (Thm. 4.8, Lack 2005) $\Theta_{\mathcal{A}}$ is indeed closed under oplax colimits in $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$.

See also a more general proof for saturation of *rigged weights* in (Lack and Shulman 2012).

Corollary: $\delta^* = \theta^* = \Theta$.

Corollary: $\Theta_{\mathcal{A}} \simeq \sharp\text{-}\mathsf{coalg}_{\mathsf{oplax}}$ is the free cocompletion of $\mathcal{A}$ under oplax colimits.

Other questions

oplax versions of *(semi)-flexible* weights?

small generating class of weights, as for PIE weights?

Thanks

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