Weights for Oplax Colimits

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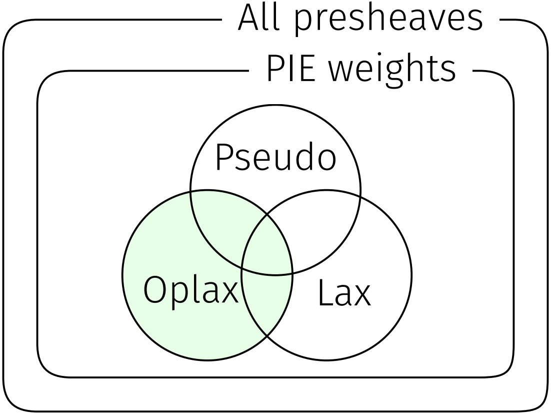
Weights for oplax colimits are…

Informally: …presheaves $W : \mathcal{A}^{\mathrm{op}} \to \mathbf{Cat}$ such that weighted colimits $W * F$ are *oplax* colimits.



Example:

The presheaf on the arrow category:

$$\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \quad \mapsto \quad \begin{array}{c} [0 \to 1] \\ \uparrow 1 \\ [0] \end{array}$$

whose weighted colimits are oplax colimits (cographs) of arrows.

Non-example:

$$\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \quad \mapsto \quad \begin{array}{c} [0] \\ \uparrow 1 \\ [0, 1] \end{array}$$

whose weighted colimits are cokernel pairs.

Objectives

Establish convenient characterisations of oplax weights

Relate these weights to the *oplax morphism classifier* for presheaves

Discuss the *saturation* properties of this class of weights

Observe connections to PIE weights and *multirepresentable* presheaves

Defn. ($\mathbf{Cat}$-*weighted colimit*): For a $\mathbf{Cat}$-presheaf $W : \mathcal{A}^{\mathrm{op}} \to \mathbf{Cat}$ and 2-functor $F : \mathcal{A} \to \mathcal{B}$, the W-*weighted colimit of* F is a representation:

$$\mathcal{B}(W * F, -) \cong \left[\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}\right](W, \mathcal{B}(F, -))$$

Defn. (*Oplax colimit*): The W-weighted *oplax* colimit of F is a representation:

$$\mathcal{B}(W \circledast F, -) \cong \left[\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}\right]_{\mathrm{oplax}}(W, \mathcal{B}(F, -))$$

When $W = \Delta\mathbb{1} : \mathcal{A}^{\mathrm{op}} \to \mathbf{Cat}$ we say the the (oplax or otherwise) colimit is *conical*.

Examples

Indexing cat.	Weight	F	$W * F$	$W \circledast F$
Discrete X	$\Delta\mathbb{1}$	-	$\coprod_{x\in X} Fx$	$\coprod_{x\in X} Fx$
$\mathbb{1}$	$\langle C\rangle$	$\langle a\rangle$	$C \odot a$	$C \odot a$
Loc. disc. C	$\Delta\mathbb{1}$	Δa	$\coprod_{\pi_0(C)} a$	$C \odot a$
Loc. disc. C	$\Delta\mathbb{1}$	presheaf	$\mathrm{colim}(F)$	$\int F$
$\bullet \to \bullet$	$\Delta\mathbb{1}$	$a \xrightarrow{u} b$	b	$\mathrm{coGraph}(u)$
$\Sigma(\Delta_+{}^{\mathrm{op}})$	$\Delta\mathbb{1}$	-	"$\mathrm{Fix}(F)$"	$\mathrm{coKl}(F)$

There is an adjunction:

$$\begin{tikzcd} {[\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}]} \arrow[r, bend right, "\mathrm{forget}"'] \arrow[r, phantom, "\perp"] & {[\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}]_{\mathrm{oplax}}} \arrow[l, bend right, "\sharp"'] \end{tikzcd}$$

$$[\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}]\left(W^\sharp, X\right) \cong [\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}]_{\mathrm{oplax}}(W, X)$$

Assuming $W^\sharp$ exists for a given $W : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$:

$$\begin{aligned} \mathcal{B}(W \circledast F, -) &\cong [\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}]_{\mathrm{oplax}}(W, \mathcal{B}(F, -)) \\ &\cong [\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}]\left(W^\sharp, \mathcal{B}(F, -)\right) \cong \mathcal{B}\left(W^\sharp * F, -\right) \end{aligned}$$

Conclusion: oplax colimits are just $\mathbf{Cat}$-weighted colimits for a special class of weights.

The lax coend construction:

$$
\begin{aligned}
\left[\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}\right]_{\mathrm{oplax}}(W, X) &\cong \oint_{a \in \mathcal{A}^{\mathrm{op}}} [W_a, X_a] \\
&\cong \oint_{a \in \mathcal{A}^{\mathrm{op}}} \left[W_a, \int_{x \in \mathcal{A}} [\mathcal{A}(x, a), X_x]\right] \\
&\cong \int_{x \in \mathcal{A}} \oint_{a \in \mathcal{A}^{\mathrm{op}}} [W_a \times \mathcal{A}(x, a), X_x] \\
&\cong \int_{x \in \mathcal{A}} \left[\oint^{a \in \mathcal{A}} W_a \times \mathcal{A}(x, a), X_x\right] \\
&\cong \left[\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}\right]\left(\oint^{a \in \mathcal{A}} W_a \times \mathcal{A}(-, a), X\right)
\end{aligned}
$$

Explicitly: for $a \in \mathcal{A}$, $W_a^\sharp$ is the category with:

0-cells pairs $(u : a \to b, x \in W_b)$

1-cells from $(u : a \to b, x)$ to $(v : a \to b', y)$ given by pairs:

$$
\begin{tikzcd}
 & b \arrow[d, "f"] \\
a \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & b'
\end{tikzcd}
\qquad x \xrightarrow{\beta} X_f y \in X_b
$$

modulo the equivalence relation generated by:

$$
\begin{tikzcd}
 & b \arrow[d, "w"] \arrow[d, bend left=60, "w'"] \arrow[d, phantom, shift left=3ex, "\overset{\theta}{\Leftarrow}"] \\
a \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[r, phantom, "\alpha\Downarrow"] & b'
\end{tikzcd}
\; x \xrightarrow{\beta} X_{w'} y \;\sim\;
\begin{tikzcd}
 & b \arrow[d, "w"] \\
a \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[r, phantom, "\alpha\Downarrow"] & b'
\end{tikzcd}
\; x \xrightarrow{\beta} X_{w'} y \xrightarrow{X_\theta y} X_w y
$$

Two questions:

So, weights of the form $W^\sharp$ are "oplax weights", since $W^\sharp * F \cong W \circledast F$. Let's call this class of presheaves θ.

Defn. *Oplax weights* are the saturation of the class θ.

Q. What does this saturation look like?

Weights in θ are also (free) $\sharp$-coalgebras.

Q. What are the *general* $\sharp$-coalgebras?

Aside: $\natural$-coalgebras:

$$\begin{tikzcd} {[\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}]} \arrow[r, bend right, "\mathrm{forget}"'] \arrow[r, phantom, "\perp"] & {[\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}]_{\mathrm{pseudo}}} \arrow[l, bend right, "\natural"'] \end{tikzcd}$$

$$[\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}]\left(W^{\natural}, X\right) \cong [\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}]_{\mathrm{pseudo}}(W, X)$$

$\natural$-coalgebras are precisely the PIE weights, i.e.:

(a) the saturation of $\{$products, inserters, equifiers$\}$

(b) weights W such that $\mathbf{el}(W_o)$ has terminal objects in each connected component.

Coalgebra characterisation: (Lack and Shulman 2012)
$(a) \Leftrightarrow (b)$: (Power and Robinson 1991)

Weights as fibrations

Other classes of weights are characterised by their categories of elements:

> Weights $[A^{\mathrm{op}}, \mathsf{Set}]$ in the saturation of coproducts (aka *multi-representables*) are those whose categories of elements have component-terminal objects.

We will use an equivalence $W \mapsto \mathsf{el}(W)$ for Cat-presheaves to understand and characterise $\sharp$-coalgebras.

Defn. (*2-category of elements*, $\mathbf{el}\, W$): for $W\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Cat}$, the 2-category $\mathbf{el}\, W$ has:

0-cells: pairs $(a \in \mathcal{A}, x \in Wa)$

1-cells: $(a, x) \to (b, y)$ are pairs $(u\colon a \to b, f\colon x \to W_u y)$

2-cells: $(u, f) \Rightarrow (v, g)\colon (a, x) \to (b, y)$ are 2-cells $\sigma\colon u \Rightarrow v$ in $\mathcal{A}$ such that $W_\sigma y\, f = g$:

$$\begin{tikzcd} & W_u y \arrow[dd, "W_\sigma y"] \\ x \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[r, phantom, "\circlearrowright"] & \\ & W_v y \end{tikzcd}$$

A 2-functor $|W|\colon \mathbf{el}\, W \to \mathcal{A}$ is then given by projection onto the first component, e.g. $|W|(a, x) = a$.

Discrete 2-fibrations

A *discrete 2-fibration* is a split 2-fibration which is a discrete *opfibration* on hom-categories.

Claim: every discrete 2-fibration $F : \mathcal{A} \to \mathcal{B}$ in Cat is isomorphic to $|W| : \mathsf{el}\, W \to \mathcal{B}$ for some $W : \mathcal{B}^{\mathsf{op}} \to \mathsf{Cat}$.

Moreover, $\mathsf{el} : [\mathcal{B}^{\mathsf{op}}, \mathsf{Cat}] \to \mathsf{D2Fib}(\mathcal{B})$ underlies an equivalence of 2-categories, where $\mathsf{D2Fib}(\mathcal{B}) \subseteq \mathsf{2Cat}/\mathcal{B}$ is the locally-full subcategory of discrete 2-fibrations and split-cartesian functors (Lambert 2020).

The equivalence $\mathsf{el} : [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}] \to \mathsf{D2Fib}(\mathcal{A})$ extends to an equivalence from $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}$ to the **full** subcategory of discrete 2-fibrations in $\mathsf{2Cat}/\mathcal{A}$, denoted $\mathsf{2Cat}/_{\mathsf{d2f}}(\mathcal{A})$.

$$\begin{tikzcd}
{[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]} \arrow[rr, bend right, "\text{forget}"'] \arrow[rr, phantom, "\perp"] \arrow[d, "\mathsf{el}"'] & & {[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}} \arrow[ll, bend right, "\sharp"'] \arrow[d, "\mathsf{el}"] \\
\mathsf{D2Fib}(\mathcal{A}) \arrow[rr, bend right, "\text{forget}"'] \arrow[rr, phantom, "\perp"] & & \mathsf{2Cat}/_{\mathsf{d2f}}(\mathcal{A}) \arrow[ll, bend right, dashed, "\mathcal{F}"']
\end{tikzcd}$$

Conclusion: The map $\mathsf{el} : [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}] \to \mathsf{2Cat}/_{\mathsf{d2f}}(\mathcal{A})$ is (up to equivalence) the coKleisli map for $\sharp$.

A "larger" adjunction generates the same comonad:

$$\begin{tikzcd}
 & \mathsf{2Cat}/\mathcal{A} \arrow[dr, bend left, "\text{free fib}"] \arrow[dr, phantom, "\dashv"] & \\
\mathsf{2Cat}/_{\mathsf{d2f}}(\mathcal{A}) \arrow[ur, hook] & \circlearrowleft & {[\mathcal{A}^{\mathrm{op}}, \mathsf{2Cat}]} \arrow[ul, "\mathsf{el}"] \arrow[dl, bend left, "{[\mathcal{A}^{\mathrm{op}},\pi_1]}"] \\
 & {[\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}]} \arrow[ul, "\mathsf{el}"] \arrow[ur, "{[\mathcal{A}^{\mathrm{op}},\mathsf{d}]}"'] \arrow[ur, phantom, "\dashv"] &
\end{tikzcd}$$

Compare with the 1-categorical situation:

$$\begin{tikzcd}
{[A^{\mathrm{op}}, \mathsf{Set}]} \arrow[r, bend right, "\mathsf{el}"'] \arrow[r, phantom, "\perp"] & \mathsf{Cat}/\mathcal{A} \arrow[l, bend right, "\mathsf{im}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
{[\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}]} \arrow[r, bend right, "\mathsf{el}"'] \arrow[r, phantom, "\perp"] & \mathsf{2Cat}/\mathcal{A} \arrow[l, bend right, "\mathsf{im}_{\mathsf{oplax}}"']
\end{tikzcd}$$

idempotent oplax-idempotent

$\mathrm{im}_{\mathrm{oplax}} F$ is the *oplax image presheaf* of $F : \mathcal{B} \to \mathcal{A}$, defined as the **oplax** colimit of $\mathcal{B} \xrightarrow{F} \mathcal{A} \xrightarrow{よ} [\mathcal{A}^{\mathrm{op}}, \mathrm{Cat}]$

$$\begin{aligned}
\mathcal{C}\left(\mathrm{im}_{\mathrm{oplax}} F * G, c\right) &\cong \left[\mathcal{A}^{\mathrm{op}}, \mathrm{Cat}\right]\left(\mathrm{im}_{\mathrm{oplax}} F, \mathcal{C}(G-, c)\right) \\
&\cong \int_{x \in \mathcal{A}} \left[\oint^{b \in \mathcal{B}} \mathcal{A}(x, Fb), \mathcal{C}(Gx, c)\right] \\
&\cong \oint_{b \in \mathcal{B}^{\mathrm{op}}} \int_{x \in \mathcal{A}} [\mathcal{A}(x, Fb), \mathcal{C}(Gx, c)] \\
&\cong \oint_{b \in \mathcal{B}^{\mathrm{op}}} \mathcal{C}(GFb, c) \\
&\cong \mathcal{C}(\Delta \mathbb{1} \circledast GF, c)
\end{aligned}$$

In particular, $W \circledast G \cong W^{\sharp} * G \cong \mathrm{im}_{\mathrm{oplax}} |W| * G \cong \Delta \mathbb{1} \circledast G\,|W|$.

$$
\begin{tikzcd}
{[\mathcal{A}^{op}, \mathsf{Cat}]} \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\mathsf{el}"'] & \mathsf{2Cat}/\mathcal{A} \arrow[l, bend right, "\mathsf{im}_{\mathsf{oplax}}"']
\end{tikzcd}
\quad \sim \quad
\begin{tikzcd}
\mathsf{D2Fib}(\mathcal{A}) \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\text{inclusion}"'] & \mathsf{2Cat}/\mathcal{A} \arrow[l, bend right, "\text{free d2fib}"']
\end{tikzcd}
$$

For a 2-functor $p : \mathcal{B} \to \mathcal{A}$, the free *split* 2-fibration is given by a lax comma 2-category (λ is lax):

$$
\begin{tikzcd}
\mathcal{A} \Downarrow p \arrow[rr] \arrow[dr, "\pi"'] & \arrow[d, phantom, "\overset{\lambda}{\Longrightarrow}"] & \mathcal{B} \arrow[dl, "p"] \\
& \mathcal{A} &
\end{tikzcd}
$$

The free *discrete* 2-fibration $p^\star : \widehat{p} \to \mathcal{A}$ is constructed by quotienting out the π-vertical 2-cells of $\mathcal{A} \Downarrow p$.

Explicitly, for $p : \mathcal{B} \to \mathcal{A}$ the 2-category $\widehat{p}$ has:

0-cells given by pairs $(x \in \mathcal{B}, u \colon a \to px)$

1-cells equivalence classes of lax squares:

$$
\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, "s"'] & px \arrow[d, "pf"] \\
b \arrow[r, "v"'] & py
\arrow[from=1-2, to=2-1, phantom, "\overset{\sigma}{\Longleftarrow}"]
\end{tikzcd}
\quad \sim \quad
\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, "s"'] & px \arrow[d, "pf"] \arrow[d, bend left=80, "pg"] \\
b \arrow[r, "v"'] & py
\arrow[from=1-2, to=2-1, phantom, "\overset{\sigma}{\Longleftarrow}"]
\end{tikzcd}
$$

2-cells $(s, f, \sigma) \Rightarrow (t, g, \tau)$ are 2-cells $\kappa \colon s \Rightarrow t$ such that:

$$
\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, "s"] \arrow[d, bend right=80, "t"'] & px \arrow[d, "pf"] \\
b \arrow[r, "v"'] & py
\arrow[from=1-2, to=2-1, phantom, "\overset{\sigma}{\Longleftarrow}"]
\end{tikzcd}
\quad \sim \quad
\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, "t"'] & px \arrow[d, "pg"] \\
b \arrow[r, "v"'] & py
\arrow[from=1-2, to=2-1, phantom, "\overset{\tau}{\Longleftarrow}"]
\end{tikzcd}
$$

$\star$-coalgebras and $\sharp$-coalgebras

The weights W which admit $\sharp$-coalgebra structures are those such that $\mathsf{el}(W)$ admits a $\star$-coalgebra structure.

Admitting a $\star$-coalgebra structure is an easier property to describe.

The counit of $\star$ acts as on objects in the domain of a discrete fibration $p : \mathcal{B} \longrightarrow \mathcal{A}$ as:

$$\left(a \xrightarrow{u} px\right) \in \widehat{p} \quad \overset{\epsilon}{\longmapsto} \quad u^* x \in \mathcal{B}$$

A $\star$-coalgebra structure $G : p \longrightarrow p^\star$ on a discrete 2-fibration involves a section of the counit:

$$x \in \mathcal{B} \quad \overset{G}{\longmapsto} \quad \left(px \xrightarrow{g_x} pG_x\right) \in \widehat{p} \quad \overset{\epsilon}{\longmapsto} g_x^*(G_x) = x$$

which corresponds to a choice for each $x \in \mathcal{B}$ of some chosen cartesian arrow out of x

$$x = g_x^*(G_x) \xrightarrow{\gamma_x = \bar{g}_x(G_x)} G_x$$

Because G preserves chosen cartesian morphisms:

$$\begin{tikzcd} u^*x \arrow[d, "\bar{u}x"'] \\ x \end{tikzcd} \quad \overset{G}{\longmapsto} \quad \begin{tikzcd} p(u^*x) \arrow[r, "g_{u^*x}"] \arrow[d, "u"'] & pG_{u^*x} \arrow[d, "pG_{\bar{u}x}"] \arrow[dl, phantom, "\overset{h_{\bar{u}x}}{\Longleftarrow}"] \\ px \arrow[r, "g_x"'] & pG_x \end{tikzcd} \sim \begin{tikzcd} p(u^*x) \arrow[r, "g_x\, u"] \arrow[d, "u"'] \arrow[dr, phantom, "\circlearrowright"] & pG_x \arrow[d, equal, "p1_{Gx}"] \\ px \arrow[r, "g_x"'] & pG_x \end{tikzcd}$$

so $G_{u^*x} = G_x$ and $\gamma_{u^*x} = \gamma_x\, \bar{u}_x$.

Conclusion: a $\star$-coalgebra structure on $p : \mathcal{B} \to \mathcal{A}$ involves choosing a terminal object in each connected component of the wide sub-1-category $p_{\mathsf{cart}} \subseteq \mathcal{B}$ of chosen cartesian 1-cells.

When $p = |W|$ for some $W : \mathcal{A}^{\mathrm{op}} \to \mathbf{Cat}$, $p_{\mathsf{cart}} \cong \mathsf{el}\, W_0$, so $\sharp$-coalgebras are PIE weights.

We call objects of the form G_x *generic*, and the restriction of p to the full subcategory of $\mathcal{B}$ containing these objects is the *generic core*, $p_\Gamma : \mathcal{B}_\Gamma \to \mathcal{A}$. The map $p \mapsto p_\Gamma$ extends to a right adjoint to the comparison functor:

$$\begin{tikzcd}
\mathbf{2Cat}/\mathcal{A} \arrow[rr, "K"'] \arrow[dr, "\star"'] & & \star\text{-}\mathbf{coalg} \arrow[ll, bend right, "(-)_\Gamma"'] \arrow[ll, phantom, shift right=2, "\top"] \arrow[dl, "\text{forget}"] \\
& \mathbf{D2Fib}(\mathcal{A}) &
\end{tikzcd}$$

This follows from the general theory of adjunctions: a right adjoint to K must send $G : p \to p^\star$ to the ($\star$-split) equaliser:

$$E \overset{e}{\hookrightarrow} p \underset{\eta_p}{\overset{G}{\rightrightarrows}} p^\star$$

$$
\begin{tikzcd} x \arrow[d, "f"'] \\ y \end{tikzcd}
\overset{G}{\longmapsto}
\begin{tikzcd} px \arrow[r, "g_x"] \arrow[d, "pf"'] & pG_x \arrow[d, "pG_f"] \\ py \arrow[r, "g_y"'] \arrow[u, phantom, ""] & pG_y \arrow[ul, phantom, "\overset{g_f}{\Longleftarrow}"] \end{tikzcd}
\qquad
\begin{tikzcd} x \arrow[d, "f"'] \\ y \end{tikzcd}
\overset{\eta_p}{\longmapsto}
\begin{tikzcd} px \arrow[r, equal] \arrow[d, "pf"'] \arrow[dr, phantom, "\circlearrowright"] & px \arrow[d, "pf"] \\ py \arrow[r, equal] & pPy \end{tikzcd}
$$

$$
\begin{tikzcd} pG_x \arrow[r, equal] \arrow[d, "pf"'] \arrow[dr, phantom, "\overset{g_f}{\Longleftarrow}"] & pG_x \arrow[d, "pG_f"] \\ pG_y \arrow[r, equal] & pG_y \end{tikzcd}
=
\begin{tikzcd} pG_x \arrow[r, equal] \arrow[d, "pf"'] \arrow[dr, phantom, "\circlearrowright"] & pG_x \arrow[d, bend right=60, "pf"'] \arrow[d, bend left=60, "pG_f"] \arrow[d, phantom, "\overset{pg_f}{\Longleftarrow}"] \\ pG_y \arrow[r, equal] & pG_y \end{tikzcd}
\sim
\begin{tikzcd} pG_x \arrow[r, equal] \arrow[d, "pf"'] \arrow[dr, phantom, "\circlearrowright"] & pG_x \arrow[d, "pf"] \\ pG_y \arrow[r, equal] & pG_y \end{tikzcd}
$$

So $p_\Gamma \xhookrightarrow{\text{incl}} p \underset{\eta_p}{\overset{G}{\rightrightarrows}} p^\star$ is an equaliser, and $(-)_\Gamma$ is right-adjoint to $K : \mathbf{2Cat}/\mathcal{A} \to \star\text{-}\mathbf{coalg}$.

The counit of $K\colon \mathbf{2Cat}/\mathcal{A} \rightleftharpoons \star\text{-}\mathbf{coalg} : (-)_\Gamma$ has component at $G\colon p \to p^\star$ given by $p_\Gamma{}^\star \xrightarrow{\mathrm{incl}^\star} p^\star \xrightarrow{\epsilon_p} p$

Proposition: $p_\Gamma{}^\star \xrightarrow{\mathrm{incl}^\star} p^\star \xrightarrow{\epsilon_p} p$ is an isomorphism.

Idea of Proof. The coalgebra structure map $G : p \to p^\star$ forms an adjunction $\epsilon_p \dashv G$ with identity counit (from the general theory of coalgebras for oplax-idempotent monads). So G is fully-faithful and thus restricts to an isomorphism to its image, which is $p_\Gamma{}^\star$. The restriction of ϵ_p to $p_\Gamma{}^\star$ is a left-inverse, and thus an inverse to this map.

$$\begin{tikzcd}
p \arrow[r, "\cong"] \arrow[rrr, bend left, "1_p"] \arrow[rr, bend right, "G"'] & p_\Gamma{}^\star \arrow[r, "\mathrm{incl}^\star"] & p^\star \arrow[r, "\epsilon_p"] & p
\end{tikzcd}$$

Corollary: $(-)_\Gamma : \star\text{-}\mathbf{coalg} \to \mathbf{2Cat}/\mathcal{A}$ is equivalent to the reflective sub-category of 2-functors which are discrete opfibrations on hom-categories (aka *local discrete opfibrations*).

In fact, this adjunction underlies a *comprehensive factorisation system* (Berger and Kaufmann 2017) on **2Cat** whose *covering* morphisms (i.e. right class) are "local discrete opfibrations" and whose *connected* morphisms are b.o.o locally initial 2-functors — i.e. a "local" lift of the comprehensive factorisation system of (Street and Walters 1973) on **Cat** to **2Cat**.

$$
\begin{tikzcd}
\mathcal{B} \arrow[r, "\text{b.o.o loc. init.}"'] \arrow[rr, bend left, "p"] \arrow[rr, phantom, bend left=20, "\circlearrowleft"] & (\widehat{p})_\Gamma \arrow[r, "{p^\star}_\Gamma", "\text{local d.op.fib}"'] & \mathcal{A}
\end{tikzcd}
$$

In particular, the reflector $K : \mathbf{2Cat}/\mathcal{A} \to \star\text{-}\mathbf{coalg}$ is essentially surjective, so those discrete 2-fibrations which admit $\star$-coalgebra structures are precisely those which are freely generated by a 2-functor (or equivalently, freely generated by a local discrete opfibration).

$$
\begin{tikzcd}
\mathbf{2Cat}/\mathcal{A} \arrow[rr, "K"'] \arrow[rr, phantom, shift left=3, "\top"] \arrow[dr, "\star"'] & & \star\text{-}\mathbf{coalg} \arrow[ll, bend right, "(-)_\Gamma"'] \arrow[dl, "\text{forget}"] \\
& \mathbf{D2Fib}(\mathcal{A}) &
\end{tikzcd}
$$

Returning to presheaves... the "free discrete 2-fibration" functor correspond to $\mathsf{im}_{\mathsf{oplax}} : \mathsf{2Cat}/\mathcal{A} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ under the $\mathsf{D2Fib}(\mathcal{A}) \simeq [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ equivalence:

$$\begin{tikzcd} {[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]} \arrow[r, bend right, "\mathsf{el}"'] \arrow[r, phantom, "\perp"] & \mathsf{2Cat}/\mathcal{A} \arrow[l, bend right, "\mathsf{im}_{\mathsf{oplax}}"'] \end{tikzcd} \quad \simeq \quad \begin{tikzcd} \mathsf{D2Fib}(\mathcal{A}) \arrow[r, bend right, "\text{inclusion}"'] \arrow[r, phantom, "\perp"] & \mathsf{2Cat}/\mathcal{A} \arrow[l, bend right, "\text{free d2fib}"'] \end{tikzcd}$$

so a presheaf in $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ admits a $\sharp$-coalgebra structure precisely if:

(a) it is the oplax image presheaf of a 2-functor $F : \mathcal{B} \to \mathcal{A}$

(b) it is the oplax image presheaf of a local discrete opfibration $p : \mathcal{B} \to \mathcal{A}$

I.e. $\sharp$-coalgebras are the oplax colimits of representables.

Recognising $\sharp$-coalgebras

Recall: The category of elements for a $\sharp$-coalgebra, W, must have component-terminal objects in every component of its chosen-cartesian sub-1-category (i.e. must be a PIE weight).

A PIE weight is an $\sharp$-coalgebra precisely if the component-terminal objects x additionally satisfy:

> for any $f : y \longrightarrow x$ and chosen-cartesian $g : y \longrightarrow z$, $(y \Downarrow \mathrm{el}(W))(g, f)$ has a single connected component.

Examples

Inserters

$$\begin{tikzcd} A \arrow[d, bend right, "a"'] \arrow[d, bend left, "b"] \\ B \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} {[0 \xrightarrow{u} 1]} \\ {[0]} \arrow[u, bend left, "0"] \arrow[u, bend right, "1"'] \end{tikzcd}$$

$$\begin{tikzcd} (A,0) \arrow[rr, "{(1_A,u)}"] \arrow[dr, "{(b,u)}"] \arrow[dr, bend right, "{(a,1_0)}"'] & \arrow[d, phantom, "\circlearrowright"] & (A,1) \arrow[dl, "{(b,1_1)}"] \\ & (B,0) & \end{tikzcd}$$

"Span Inserters"

$$\begin{tikzcd} A \arrow[d, bend right, "a"'] \arrow[d, bend left, "b"] \\ B \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} {[0 \xleftarrow{u} 2 \xrightarrow{v} 1]} \\ {[0]} \arrow[u, bend left, "0"] \arrow[u, bend right, "1"'] \end{tikzcd}$$

$$\begin{tikzcd} (A,0) \arrow[dr, "{(a,1_0)}"'] & (A,2) \arrow[l, "{(1_A,u)}"'] \arrow[r, "{(1_A,v)}"] \arrow[d, shift right] \arrow[d, shift left] \arrow[dl, phantom, "\circlearrowright"] \arrow[dr, phantom, "\circlearrowright"] & (A,1) \arrow[dl, "{(b,1_1)}"] \\ & (B,0) & \end{tikzcd}$$

The saturation of oplax weights

For a class Φ of weights, the *saturation* Φ^* contains all (small) weights $W : \mathcal{A}^{\mathsf{op}} \to \mathbf{Cat}$ such that Φ-complete/continuous $\implies$ W-complete/continuous. If $\Phi = \Phi^*$, the class is said to be *saturated*.

Examples:

$$\{\text{non-empty finite coprods}\} \subseteq \{\text{binary coprods}\}^*$$

$$\{\text{representables}\} = \varnothing^*$$

$$\{\text{all small weights}\} = \left\{\begin{matrix}\text{coproducts, coequalisers,} \\ \text{tensors by } \mathbb{2}\end{matrix}\right\}^*$$

$$\{\text{PIE weights}\} = \left\{\begin{matrix}\text{coproducts, coinserters,} \\ \text{coequifiers}\end{matrix}\right\}^*$$

Saturation

Consider the following classes of weights:

δ, the $\Delta\mathbb{1}^\sharp$'s (conical oplax colimits)

θ, the $W^\sharp$'s (oplax colimits)

Θ, the class of $\sharp$-coalgebras

Note: $\delta \subset \theta \subset \Theta$.

Thm: (Kelly and Schmitt 2005) for a class of small weights Φ, the weights in the saturation Φ^* with domain $\mathcal{A}$ are those in the closure of the representables in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}]$ under Φ-colimits (henceforth denoted $\Phi_{\mathcal{A}}$).

Corollary: $\Theta \subseteq \delta^*$, and so $\Theta^* \subseteq \delta^*$, and so $\Theta^* = \delta^* = \theta^*$.

Proposition: Θ is saturated.

Proof. It suffices to show that $\Theta_{\mathcal{A}} = \delta^*_{\mathcal{A}}$; i.e. that $\Theta_{\mathcal{A}} \subseteq [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ contains the representables and is closed in $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ under conical oplax colimits.

Now $\sharp$ is an oplax-idempotent comonad, so $\mathsf{U} \colon \sharp\text{-}\mathsf{coalg}_{\mathsf{oplax}} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ is fully-faithful. The repletion of U's image is $\Theta_{\mathcal{A}}$. Because U creates oplax colimits (Thm. 4.8, Lack 2005) $\Theta_{\mathcal{A}}$ is indeed closed under oplax colimits in $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$.

Corollary: $\delta^* = \theta^* = \Theta$.

Corollary: $\Theta_{\mathcal{A}} \simeq \sharp\text{-}\mathsf{coalg}_{\mathsf{oplax}}$ is the free cocompletion of $\mathcal{A}$ under oplax colimits.

Corollary: δ and θ are *pre-saturated.*

Some further results:

The class of conical oplax colimits of *oplax* (or *normal oplax*) functors from 1-categories is presaturated. It's saturation is given by weights $W : \mathcal{A}^{\mathrm{op}} \to \mathbf{Cat}$ such that the component-terminal objects in $\mathbf{el}(W)$ have the property that for any $f : y \to x$ and chosen-cartesian $g : y \to z$, $(y \Downarrow \mathbf{el}(W))(g, f)$ has an initial object. This class includes weights for coKleisli objects of comonads.

The class of conical oplax colimits of pseudo or strict functors from 1-categories is *not* presaturated.

Some further questions:

What are the oplax versions of *(semi)-flexible* weights?

Is there a finite class of weights which generates all oplax weights, as for PIE weights?

Is there a similar characterisation of weights for *pseudo*-colimits?

Thanks

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