BOUNCER CHALLENGE

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CONTENTS

Section 1: Calculation

Part 1.1: Introduction

To summarise, the objective of the Bouncer Challenge was to model the trajectory of a small stainless steel projectile (a ball bearing) as it was rolled from rest down a PVC tube and then left the tube to fall under gravity. The eventual aim was to calculate distance D (see below), the horizontal distance from the end of the tube to the ball's second bounce. In order to achieve this, SUVAT-style equations of motion were constructed and applied as needed.

The setup was first represented diagrammatically:

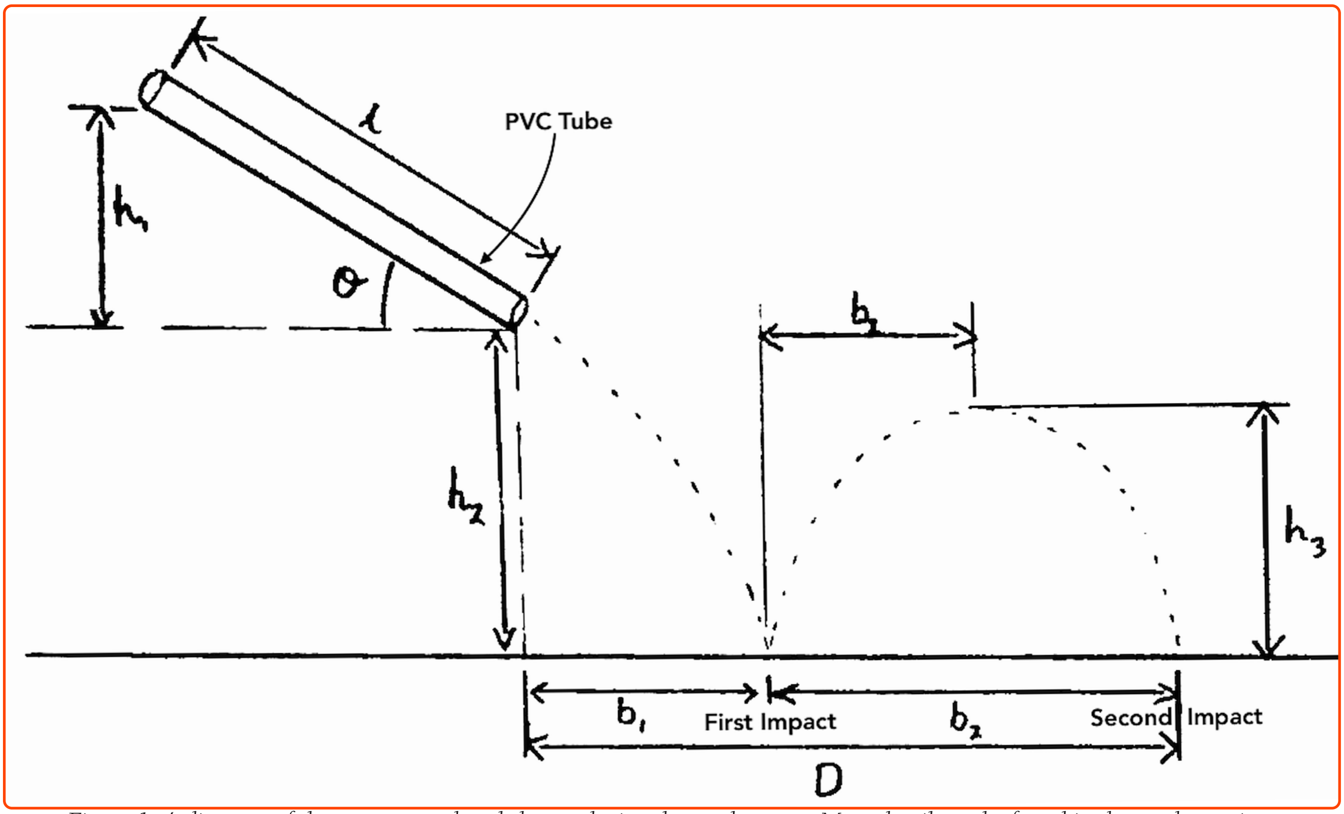


Figure 1: A diagram of the system was hand-drawn during the maths stage. More detail can be found in the maths section.

In order to make the predictions, the following simplifying assumptions were made:

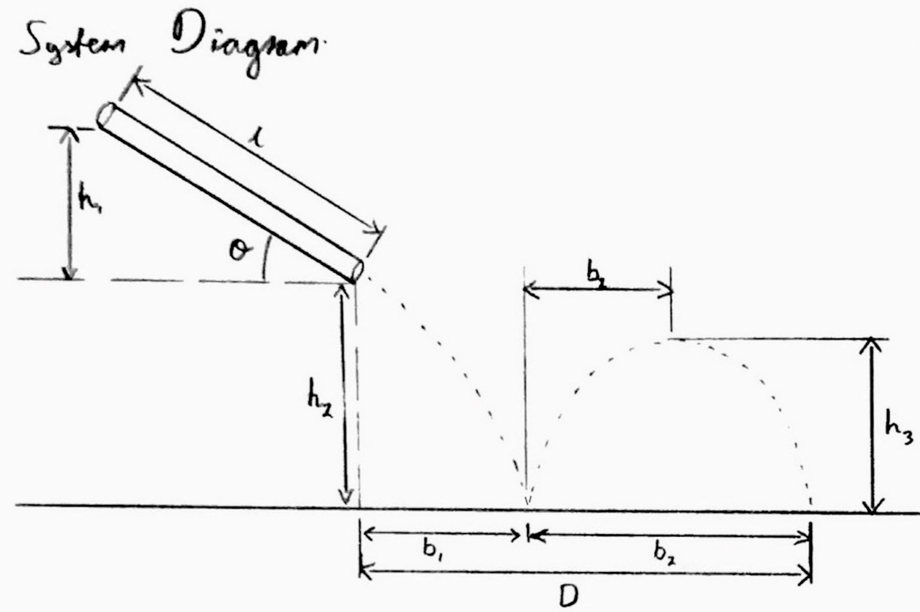
1. The ball did not slide in the tube. Friction between the steel ball and the PVC tube was such that it kept the ball rolling but did not affect its acceleration;
2. An accessory to point 1., the ball was of uniform density throughout, and thus rolled perfectly evenly;
3. The reported angle was accurate to the number of significant figures provided (2s.f.);
4. Point 2. likewise applied to the height h_2, although here to 3s.f.;
5. Coefficient of Restitution was accurate to within the provided uncertainty;
6. Air resistance was negligible;
7. In the tube, 100% of the relative *GPE* at the top was converted to *KE* at the bottom - i.e. no energy was lost producing sound, etc.;
8. Once the ball left the tube, no horizontal forces acted on it;
9. And likewise, the first impact did not cause a change of translational velocity;
10. Acceleration due to gravity g was accurate at $9.81 m/s^2$;

Which of these assumptions applies at which step is indicated below, and assumptions 3., 4., and 5. are investigated below.

Part 1.2: Methods

Here an energy balance was used to calculate the velocity of the steel ball as it left the tube. The main equation used for this, $mg\Delta h = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$, will be derived and explained in more detail in the Derivations & Definitions section. In order to provide context to the mathematics, assumptions that were used to arrive at the answer of $v = 2.87m/s$ are listed below.

System Diagram



Given Parameters

$l = 100cm = 1m$

$\theta = 36°$

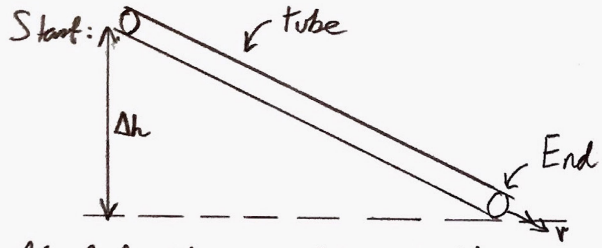
$h_2 = 67.8cm = 0.678m$

$e = 0.6557 \pm 0.02$

Steel Ball

$m = 6g, \ d = 10mm$

Energy Method: (in tube)



At start, $V = 0$, At end $V = v$

$\therefore KE_0 = 0$

Step 1: Find KE balance

So: $\Delta GPE = KE_{end}$

$$mg\Delta h = \frac{1}{2}mv^2 + \underbrace{\frac{1}{2}I\omega^2}$$

$I = \frac{2}{5}mr^2$ (for a sphere)

$\omega = \frac{2\pi}{T}$, $T = \frac{\pi d}{V}$ $(d = 2r)$

$\therefore \omega = \frac{2\pi}{\left(\frac{\pi d}{V}\right)} = \frac{V}{r}$

SO: $\cancel{m}g\Delta h = \frac{1}{2}\cancel{m}v^2 + \frac{1}{2} \times \frac{2}{5}\cancel{mr^2} \times \frac{v^2}{\cancel{r^2}}$

$\therefore 2g\Delta h = v^2 + \frac{2}{5}v^2$

$\Delta h = h_1 = l\sin(\theta) = \sin(36)$

$\therefore 2g\sin(36) = \frac{7}{5}v^2$

So $v = \sqrt{\frac{2g\sin(\theta)}{7/5}} = \sqrt{\frac{10}{7}g\sin(36)}$

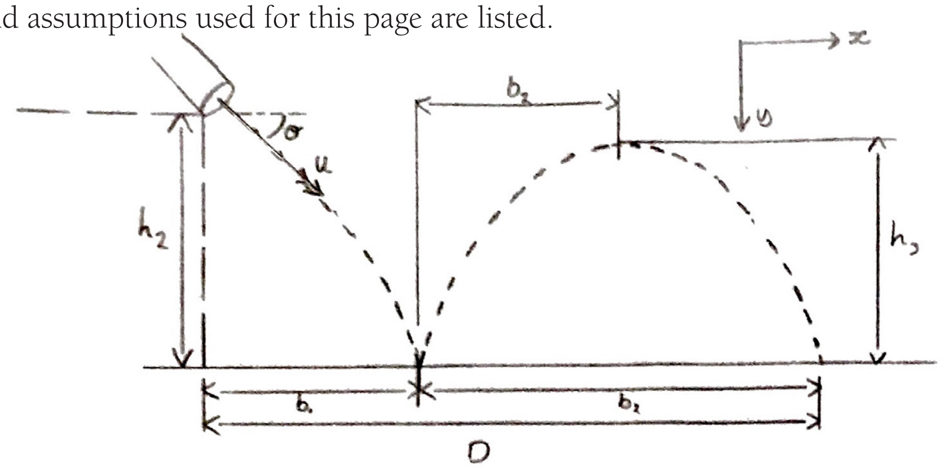
$v = 2.87m/s$

Assumptions used here:

- Ball rolls in tube
- Air resistance and other resistive forces in the tube are negligible
- 100% of the GPE at start (relative to the end height) has been transferred to KE at the end of the tube - i.e. rolling produces no sound etc.
- $g = 9.81m/s^2$
- Angle theta is measured accurately at 36°

The last assumption is challenged in part 1.3.

This page uses simple trigonometry to decompose *v* from above into its horizontal and vertical components, before applying SUVAT equations of motion in these two dimensions to arrive at a final solution. More information (and derivation) is provided on the starred * equations in the Derivations & Definitions section, and assumptions used for this page are listed.



Given Parameters

$\theta = 36^\circ$

$h_2 = 67.8cm = 0.678m$

$e = 0.6557 \pm 0.02$

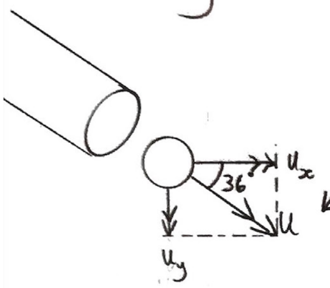
$u = 2.87 m/s$

Steel Ball:

$m = 6g = 6\times10^{-3} kg$

$d = 10mm = 1\times10^{-2} m$

(Final velocity v from above is now initial velocity u)



← u can be broken down into u_x & u_y

Step 2: Find b1

$u_x = u\times\cos(\theta)$, $u_x = \underline{\underline{2.32 m/s}}$

$u_y = u\times\sin(\theta)$, $u_y = \underline{\underline{1.69 m/s}}$

Time t_1 of first impact:

(using $s = ut + \frac{1}{2}at^2$)

$h_2 = u_y t_1 + \frac{1}{2}g t_1^2$, $\therefore 0 = \frac{1}{2}g t_1^2 + u_y t_1 - h_2$ *

$0 = \frac{9.81}{2}t_1^2 + 1.69t_1 - 0.678$

↓

$t_1 = 0.238s$ OR $-0.582s$

$t_1 > 0$,

$\therefore t_1 = \underline{\underline{0.238s}}$

Assumptions used here:

- h_2 is accurate at 0.678m
- e is accurate at 0.6557
- $g = 9.81 m/s^2$
- No other resistive forces are present

The first two assumptions are challenged in part 1.3.

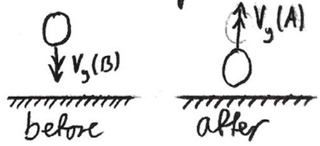
Using $s = vt$,* $b_1 = u_x \times t_1$

$\therefore b_1 = 0.238 \times 2.32$

$b_1 = \underline{\underline{0.552 m}}$

Step 3: Find Vy(B)

At 1^{st} Impact:



Find $V_y(B)$ (using $v^2 = u^2 + 2as$)

$V_y(B) = \sqrt{u_y^2 + 2\times g\times h_2}$ * $= \underline{\underline{4.02 m/s}}$

↙ Change in direction

$V_y(A) = -V_y(B)\times e$ (Restitution)

$V_y(A) = \underline{\underline{-2.63 m/s}}$

Time t_2 from 1^{st} to 2^{nd} impact ($t = \frac{v-u}{a}$)

$t_2 = 2\left(\frac{-V_y(A)}{g}\right)$ * ← i.e. 2× time to apogee

$t_2 = \underline{\underline{0.537s}}$

Step 4: Find b2

Again with $s = vt$, $b_2 = t_2 \times U_x = 1.25m$

$D = b_1 + b_2 = 1.799m = \underline{\underline{179.9cm}}$

(1 d.p.)

Full Calculation with Uncertainty:

Here, the above calculations were repeated - this was partly as a safety net to catch mathematical errors in the previous pages, but mainly to investigate the effects of changing coefficient of restitution e on the final outcome of the experiment. This gave a 6.7cm window of values for D. There is more detail on predicting possible errors, specifically the effects of uncertainty in the angle theta and in height h_2, in part 1.3, and the equations and assumptions here are the same as those used in the preceding pages.

$$\Delta GPE = KE$$

$$\therefore mg\Delta h = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$I = \frac{2}{5}mr^2, \quad \omega = \frac{2\pi}{T}, \quad T = \frac{\pi d}{V}$$

$$\therefore \omega = \frac{V}{r}$$

$$\therefore \cancel{m}g\Delta h = \frac{1}{2}\cancel{m}v^2 + \frac{1}{2}\times\frac{2}{5}\cancel{m}\cancel{r^2}\left(\frac{V}{\cancel{r}}\right)^2$$

$$g\Delta h = \frac{1}{2}v^2 + \frac{1}{2}\times\frac{2}{5}v^2$$

$$2g\Delta h = v^2 + \frac{2}{5}v^2$$

$$2g\Delta h = \frac{7}{5}v^2$$

$$\Delta h = h_1 = l\sin(36)$$

$$\therefore v = \sqrt{\frac{10}{7}g\Delta h}$$

$$\therefore V = \sqrt{\frac{10}{7}gl\sin(36)} = 2.87\,m/s$$

$$u_x = V\times\cos(\theta) = 2.32\,m/s$$

$$u_y = V\times\cos(\theta) = 1.69\,m/s$$

t_1 using $s = ut + \frac{1}{2}at^2$:

$$h_2 = u_y t_1 + \frac{1}{2}gt_1^2$$

$$\therefore t_1 = 0.238\,s$$

$$v = \frac{d}{t}, \quad \therefore b_1 = u_x \times t_1$$

$$b_1 = 0.552\,m$$

V_y, $V\updownarrow$ before 1^{st} impact: $(v^2 = u^2 + 2as)$

$$V_y = \sqrt{u_y^2 + 2\times g\times h_2} = 4.02\,m/s$$

$$e = 0.6557 \pm 0.02$$

V_y Post-Impact $= V_y \times e$ - then, $t = \frac{v-u}{a}$

$$t_2 = 2\left(\frac{V_y\,Post}{g}\right)$$

$$b_2 = t_2 u_x, \quad D = b_1 + b_2$$

@$e_{min} = 0.6357$	@$e = 0.6557$	@$e_{max} = 0.6757$
$t_2 = \frac{V_y \times 0.6357}{g}\times 2$	$t_2 = \frac{V_y\times 0.6557}{g}\times 2$	$t_2 = \frac{V_y\times 0.6757}{g}\times 2$
$t_2 = 0.521$	$t_2 = 0.537$	$t_2 = 0.554$
$\therefore b_2 = 1.21m$	$b_2 = 1.25\,m$	$\therefore b_2 = 1.29m$
$\therefore D_{min} = b_1 + b_2$	$D_{mid} = b_1 + b_2$	$D_{max} = b_1 + b_2$
$D_{min} = 1.7611$	$D_{mid} = 1.79917$	$D_{max} = 1.83722$

$$176.1cm \leq D \leq 183.7cm$$

$$(D_{mid} = 179.9cm)$$

Derivations & Definitions

$$mg\Delta h = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

This equation is relatively simple to derive:

- When the ball is stationary at the top of the slope, its energy is entirely potential (it isn't moving): $E_0 = GPE = mg\Delta h$, where Δh is vertical height above the base of the tube.
- Once the ball has rolled to the bottom of the tube, Δh is 0, so $mg\Delta h = 0$ - but it now has kinetic and rotational energy. The kinetic energy at the end $E_K = \tfrac{1}{2}mv^2$ (where m is the mass of the ball and v is its velocity) and the rotational energy $E_R = \tfrac{1}{2}I\omega^2$ (where I and ω are defined below).
- Energy can neither be created nor destroyed and we are assuming that no energy is lost, so $E_0 = E_K + E_R$, therefore we get the final equation $mg\Delta h = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$.
- I, the moment of inertia of a sphere, is defined as $I = \tfrac{2}{5}mr^2$ (where m is again ball mass and r is its radius).
- ω is angular speed: $\omega = (2\pi)/T$, where T = time period of rotation of the sphere. T can be further refined as C/v, where v is the ball's velocity and C its circumference ($=2\pi r$). Thus, $\omega = v/r$.

SUVAT Equations

Four SUVAT-based equations were used in calculation, although none in their native form - the derivations are explained below, in order of appearance.

1. The first SUVAT to be used, $s = ut + \tfrac{1}{2}at^2$, was rearranged as a quadratic in t, with g substituted for a as we are assuming that gravity causes the only acceleration in the system. The final equation was $0 = \tfrac{1}{2}gt^2 + ut - s$

2. Perhaps the simplest SUVAT used, quoted as $s = vt$, is a simple rearrangement of '*speed = distance / time*', using standard symbols: s = displacement = 'distance', v = velocity = 'speed' and t = time.

3. Third, $v^2 = u^2 + 2as$ was used as $v = (u^2 + 2gs)^{1/2}$, which can be achieved by simply substituting g for a as above and then taking the square-root of both sides.

4. $t = 2(-u/g)$ was used to find the length of time the projectile is in the air between its first and second bounces.
   - SUVAT equation $v = u + at$ is rearranged as $t = (v - u)/g$ (because g is the only acceleration).
   - This is applied to the y-direction, requiring final velocity v to be 0, and using initial velocity $u = V_y(A)$ (which is upwards, and thus defined as negative)
   - The equation now tells us the length of time the projectile takes to reach its apogee (highest point).
   - Since the trajectory is symmetrical, doubling this time will give us the time to impact. Therefore air time $t_{air} = 2(-u/g)$.

Rounding

With the exception of uncertainties leading up to the final answer, all values in the above maths were provided to 3s.f. However, this was only for ease of reading: the accurate values were stored and used for onward calculation. This was to minimise rounding errors introduced by working to 3s.f. when a higher precision was available.

Part 1.3: Results

As on page 6, values could be calculated to within a window using restitutional uncertainty alone. However, this was not the only uncertainty present, as there were uncertainties in both angle theta and height h_2 due to the physical limitations of the measuring equipment (there were also friction forces, but these were deemed to be negligible). The protractor used to measure the angle was accurate to the nearest degree, and thus its true value was 36(±0.5)°, and the height h_2 was accurate to the nearest 1mm, thus its true value was 67.8(±0.05) cm (or 0.678(±0.0005)m, which is equivalent and was used for calculation).

In order to include the uncertainty in calculations, extremes were taken: a maximum, average and minimum value for each parameter were used. The uncertainty was relatively easy to quantify, as it simply involved re-calculating the trajectory using each value. Rather than performing these calculations manually, a simple Python program was used to iterate through input variables, calculating all 27 possible outcomes. This allowed for versatility, as simply changing input parameters allows for the same program to be used for other initial conditions. The plot below was created on Python, and the code used is available on Github.[1]

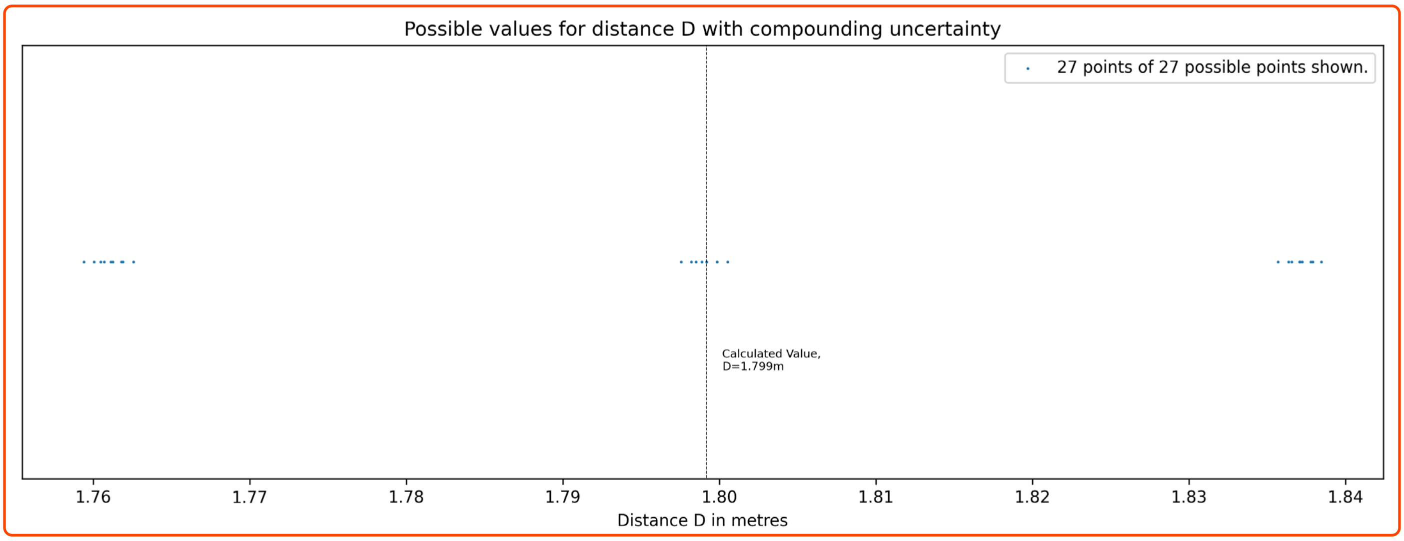


Figure 2: This graph plots all distance D values from three discrete values for each of three uncertain parameters - this gives 3^3 = 27 values.

Once plotted, the Python program was also used to create a summary table of the raw data:

Table 1: Inputs for the Python program, and their results.

	Parameter	Value
e	Minimum	0.6357
	Maximum	0.6757
θ	Minimum	35.5
	Maximum	36.5
h2	Minimum	67.75
	Maximum	67.85
D	Minimum	175.94
	Mean	179.91
	Maximum	183.85
	Standard Deviation	0.031
	Accurate	**179.9**

As can be seen here, relatively small errors in input measurements were found to compound significantly when calculating final distance. It is worth noting that the 'accurate D' is not suggesting this will be the exact value of D found, merely that this is the theoretical value of D if all parameters take their exact, stated values. The Excel spreadsheet used to format this table is likewise available on Github.

$$176.1\text{cm} \leq D \leq 183.7\text{cm}$$

$$D_{mid} = 179.9\text{cm}$$

The minimum and maximum, as well as the predicted, distanced D based on restitutional uncertainty alone.

$$D_1 = 174.9\text{cm} \quad \bullet \quad D_2 = 179.9\text{cm} \quad \bullet \quad D_3 = 184.9\text{cm}$$

The above three values of D were submitted for testing. The reasoning behind this was that, with a 5cm radius bullseye, this would give a central region of 10cm that was covered by two bullseyes, and a further 5cm each way covered by one. This method was chosen due to an incorrect understanding of how 'median bounce' value would be calculated.

Section 2: Analysis

Part 2.1: Target Sheet

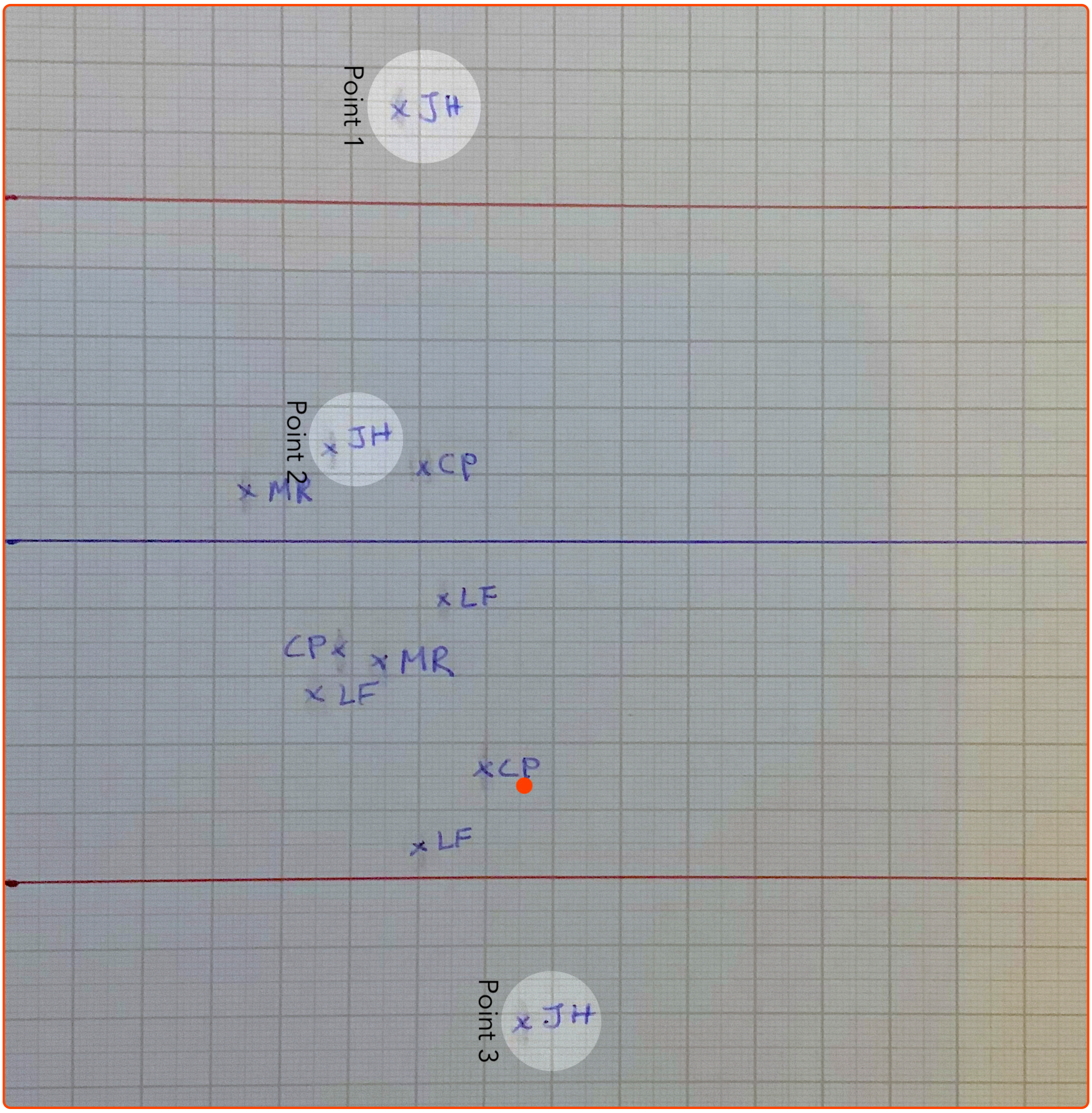


Figure 3: The group target sheet for Group 9, with relevant impacts highlighted.

The above image shows the target sheet for a selection of Group 9's bounces. Those highlighted result from my target distance predictions above.

These three points are within the bands (10), (20), and (10) respectively, from top to bottom. This means that the median (10, 10, 20) = 10. This is unfortunate and, as mentioned, stems from a misunderstanding of the method of calculation of the median: placed in bounce order, the median of (10, 20, 10) is, of course, 20. Perhaps surprisingly, however, even setting this misunderstanding aside, two of the three points *should* still be within the red zone, as there should be exactly 5cm between each. This is true for the uppermost points - the distance between points 1 and 2 ≈ 4.95cm. The distance between points 2 and 3, however, is ≈ 8.55cm. This suggests that this point 3 is anomalous, and a repeat of the experiment would place it closer to the red dot.

Part 2.2: Error Discussion

All members of the group produced at least one calculation within 5cm of 180cm, which is the region in which the experimentally verified answer was - however, there were some noteworthy numerical differences in outcomes, as well as a significant mathematical difference in one method. Different styles for setting out windows of uncertainty were also particularly interesting - and it is likely that other sources of error were also present. These are analysed below.

2.2.1. Different Outcomes - Same Method

Different answers to the same method were caused mainly by rounding: whether values were rounded at each stage or precise values were kept and rounded at the end affected the final value. Both methods gave answers that would be in the central, highest scoring band of the target. Rounding at each stage of the process yielded the following values for the various parameters:

Rounding Regime 1: Ludo, 2d.p.

$v = 2.87m/s$

$u_x = 2.32m/s,\ u_y = 1.69m/s$

$t_1 = 0.24s,\ b_1 = \underline{0.56m}$

$V_y(B) = \underline{4.04m/s},\ V_y(A) = \underline{-2.65m/s}$

$t_2 = 0.54s,\ b_2 = 1.25m$

$D = \underline{1.81m}$

Rounding Regime 2: Chris, 3/4s.f.

$v = 2.870m/s$

$u_x = 2.322m/s,\ u_y = 1.687m/s$

$t_1 = 0.2377s,\ b_1 = 0.5519m$

$V_y(B) = 4.018m/s$

$t_2 = 0.372s,\ b_2 = 1.247m$

$D = 1.799m$

Values different to unrounded values from (1.2) are underlined. It is worth noting that rounding errors only appear in later stages of the process.

As all members of the group used similar methods and found similar values, fine errors must be due to rounding. These were analysed using mean deviation[2] from the target. Our windows of uncertainty (range of provided predictions) were symmetrical, so using an average worked to cancel these out and find the accuracy of the calculations themselves.

- Chris' mean deviation was *(16 + 34 - 11)/3 = 13mm*
- Ludo's was *(8 + 23 + 46)/3 = 25.7mm (3s.f.)*
- Mine was *(72 - 63 -14)/3 = -1.67mm (3s.f.)*
- Differences between Chris and my average deviation are likely caused by anomalous results: both of our submitted value sets had an average of *179.9cm*.

This suggests that Ludo's policy of rounding values throughout was less successful than Chris' using higher numbers of significant figures.

2.2.2 Window Provision

Each member of the group used different logic to provide windows of uncertainty for their calculations, to varying degrees of success.

In order to investigate the effects of windows of uncertainty, mean squared deviations[3] were used. In order:
- Mila's MSD was $(7^2 + (-18)^2)/2 = \underline{186.5mm^2}$ (this excludes the result for when the target sheet was at 1.08m)
- Chris' was $((-11)^2 + 16^2 + 34^2)/3 = \underline{511mm^2}$
- Ludo's was $(8^2 + 23^2 + 46^2)/3 = \underline{903mm^2}$
- Mine was $((-63)^2 + (-14)^2 + 72^2)/3 = \underline{3116mm^2}$.

- This suggests that Mila's placing two target sheets at exactly the calculated value was the most successful.
- Ludo used the same method, but rounding errors meant the values were slightly further from the target.
- Chris' separation of targets by a few cm was likewise successful.
- My method of wide spacing was the least successful. Bounces were, on average, further from their targets.

2.2.3. Differing Methods

Most members of the group used similar methods. Where the methods themselves did differ, they differed in the following ways:

- Specific detail: some methods merged several steps into one, others worked through more algebra. For example, the above use of $t = -2u/g$ to calculate air time of the second bounce was split into several steps in other methods.
- One group member did one calculation omitting rotational energy $E_R = ½I\omega^2$, which caused the following significant discrepancy:

$$v = 3.40m/s$$
$$u_x = 2.75m/s,\ u_y = 2.00m/s$$
$$t_1 = 0.627s,\ b_1 = 1.77m$$
$$V_y(B) = 4.16m/s,\ V_y(A) = -2.77m/s$$
$$b_2 = 1.00m$$
$$D = 2.77m$$

2.2.4. Other Sources of Error

None of the group's bounces came within 5mm of the target. There may be any number of reasons for this, but some are described below, in chronological order:

1. Release
   - Ball might not start from rest: releasing it may have given it a velocity.
   - Releasing the ball might make it rotate in an unexpected direction.
2. Rolling in tube
   - The rolling ball will have lost energy in deforming the PVC pipe etc.
   - The ball had to push air in front of it - like the wind in front of an underground train.
   - Ball may initially have slid (rather than rolled) in the tube.
3. Leaving tube
   - Burrs or imperfections at the end of the tube could have slowed the ball or changed its direction.
   - If the ball was oscillating sideways in the tube (due to any of the above factors), it could have left the tube at an angle.
4. Flying through the air
   - The rolling ball would have generated a slight lift as it flew due to Bernoulli's equation.
   - Vortex shedding would have slowed the ball down - golf balls have dimples to combat this exact problem, which suggests that the effect is detectable.
   - Air resistance would have had an effect, however small.
5. Bounce
   - If the bounce was not instantaneous, ball would have travelled along the ground for a short distance.
   - Due to all the above, the ball would have hit the bounce surface differently each time, affecting it's onward trajectory.
   - As the spinning ball bounced, it would have applied a slight backwards force F to the bounce surface. $F = ma$, so it is likely that the ball accelerated very slightly in the x direction on impact.
6. Lateral Deviation
   - At any point in the trajectory, the ball could have been slightly deflected sideways. This deflection would have an effect on the perpendicular distance D the ball travelled.

2.2.5. Conclusions

After analysis of the various methods used by members of the group, and a discussion of extra sources of error, the following can be concluded:

- Keeping unrounded values was important for final accuracy, although did not detract from point-scoring potential.
- It was extremely important to provide a small window of error for the overall calculation. There was absolutely no upside in giving a large window, as the only way to score full points was for 2/3 bounces to land in the central section. Any window wider than this was worse than useless
- Rotational energy played a very significant role in the trajectory of the ball
- A whole host of other effects could have been taken into account - but given the relatively high accuracy of predictions this would have been unnecessary.

Revised Predictions

The above analyses and conclusions have now provided enough information to adjust the predictions made at the start of the report:

- The mathematics were found experimentally to be fairly accurate: mean deviation of my bounces was less than 2mm away from the target
- The problem with the predictions was found to be in the window of uncertainty provided

Consequently, the revised prediction would be:

$$D_1 = D_2 = D_3 = 179.9\text{cm}$$

Endnotes

1 Github repo with Python code (note that 'Bouncer2.py' is the correct file): https://github.com/Btecpete/Bouncer

2 The average deviation is useful here because it essentially calculates where the average ball would land. This gives us a useful proxy for the overall accuracy of the calculations, and, using information about the calculations themselves, we can analyse the extent to which this error stems from rounding.

3 Mean Squared Deviations were chosen as some of the deviations are positive, some negative - and as we are here studying the of the width of the window of error, positive and negative values are unhelpful. MSDs are also far more sensitive to inaccuracies, as an error of 2mm is $2^2 = 4x$ worse than an error of 1mm. This was important to differentiate between window provision styles as, in reality, deviations at the highest level in the group were small.

All figures, graphs and tables in this document were created by me.