B1-Perturbations stage 2

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First, some definitions. It is important that we all use the same conventions. We should stick to the $(-,+,+,+)$ signature, which is not what Weinberg does. The less minus signs we use, the less we are prone to sign mistakes.

The background metric is

$$ds^2 = S^2\left[-d\eta^2 + \gamma_{ij}(\eta)dx^i dx^j\right] ,$$

where

$$\gamma_{ij} := \mathrm{diag}(e^{2\beta_1}, e^{2\beta_2}, e^{2\beta_3}) , \qquad \sum_{i=1}^{3}\beta_i = 0 .$$

Note that we are working in a very specific coordinate system in which the shear $\sigma_{ij} := (\gamma_{ij})'/2$ has only two independent components. Since σ_{ij} is a symmetric traceless matrix, the other three componentes can be seen as the three Euler angles needed to rotate γ_{ij} to a general coordinate system.

The Lie derivative of the background metric along the vector ξ is

$$\begin{aligned}
\mathscr{L}_\xi \bar{g}_{00} &= -2S^2\left(T' + HT\right)\\
\mathscr{L}_\xi \bar{g}_{0i} &= S^2\left(-\partial_i T + \gamma_{ij}\partial_0\xi^j\right)\\
\mathscr{L}_\xi \bar{g}_{ij} &= S^2\left(2\mathscr{H}T\gamma_{ij} + 2T\sigma_{ij} + 2\partial_{(i}\xi_{j)}\right)
\end{aligned}$$

This is always true, regardless of the splitting.

We will parameterize ξ^μ as

$$\xi^\mu = \left(T, \partial^1 X, \partial^2 Y, \partial^3 Z\right)$$

and the line element as

$$ds^2 = S^2\left[-(1+2A)d\eta^2 + 2B_i dx^i d\eta + (\gamma_{ij} + h_{ij})dx^i dx^j\right] ,$$

where

$$B_i = (\partial_1 E, \partial_2 F, \partial_3 G)$$

and h_{ij} will be built later. Gauge transformations are such that

$$\Delta\delta g_{\mu\nu} = \mathscr{L}_\xi \bar{g}_{\mu\nu}$$

It is thus straightforward to compute the following transformation for A and B_i:

$$\begin{aligned}
A &\to A + T' + \mathscr{H}T = A + (ST)'/S\\
E &\to E - T + X' - 2\beta_1' X = E - T + (\gamma^{11}X)'/\gamma^{11}\\
F &\to F - T + Y' - 2\beta_2' Y = F - T + (\gamma^{22}Y)'/\gamma^{22}\\
G &\to G - T + Z' - 2\beta_3' Z = G - T + (\gamma^{33}Z)'/\gamma^{33}
\end{aligned}$$

In order to parameterize h_{ij}, we need ask how many ways there exist to build a tensor from scalars only (since we want SSS decomposition). There are two ways: either by multiplying a scalar by γ_{ij}, or by taking two derivatives of a scalar. Thus we write

$$h_{ij} = \left(\gamma_{ij} + \frac{\sigma_{ij}}{\mathscr{H}}\right) 2C + \bar{h}_{ij}$$

where C is a scalar and $\bar{h}_{ij}$ is a traceless matrix built out of (two) derivatives of 6 new scalar fields. One possibility is

$$\bar{h}_{ij} = \begin{pmatrix} 2\partial_1^2 B & \partial_1\partial_2 H & \partial_1\partial_3 I \\ & 2\partial_2^2 Q & \partial_2\partial_3 J \\ & & 2\partial_3^2 D \end{pmatrix},$$

with the constraint

$$\partial_1^2 B + \partial_2^2 Q + \partial_3^2 D = 0\,.$$

The gauge transformations are

$$\begin{aligned} C &\to C + \mathscr{H}T \\ B &\to B + X \\ Q &\to Q + Y \\ D &\to D + Z \\ H &\to H + X + Y \\ I &\to I + X + Z \\ J &\to J + Y + Z \end{aligned}$$

The following are Gauge invariant combinations:

$$A + \frac{1}{S}\left[S\left(E - \frac{(\gamma^{11}B)'}{\gamma^{11}}\right)\right]', \tag{1}$$

$$A + \frac{1}{S}\left[S\left(F - \frac{(\gamma^{22}Q)'}{\gamma^{22}}\right)\right]', \tag{2}$$

$$A + \frac{1}{S}\left[S\left(G - \frac{(\gamma^{33}D)'}{\gamma^{33}}\right)\right]', \tag{3}$$

$$C + \mathcal{H}\left[E - \frac{(\gamma^{11}B)'}{\gamma^{11}}\right], \tag{4}$$

$$C + \mathcal{H}\left[F - \frac{(\gamma^{22}Q)'}{\gamma^{22}}\right], \tag{5}$$

$$C + \mathcal{H}\left[G - \frac{(\gamma^{33}D)'}{\gamma^{33}}\right], \tag{6}$$

$$H - B - Q\,, \tag{7}$$

$$I - B - D\,, \tag{8}$$

$$J - D - Q\,, \tag{9}$$

$$H + I + J - 2(B + Q + D)\,, \tag{10}$$

$$H + I - J - 2B\,, \tag{11}$$

$$H + J - I - 2Q\,, \tag{12}$$

$$I + J - H - 2D\,, \tag{13}$$

and so on...

Maybe we want to choose the off diagonal components of $\bar{h}_{ij}$ to be 0. In this case, appropriate GIV's may be

$$\varphi = A + \left(\frac{C}{\mathcal{H}}\right)' - C\,, \tag{14}$$

$$\xi_1 = E + \frac{C}{\mathcal{H}} - \frac{1}{2}\frac{(\gamma^{11}(H+I-J))'}{\gamma^{11}}\,, \tag{15}$$

$$\xi_2 = F + \frac{C}{\mathcal{H}} - \frac{1}{2}\frac{(\gamma^{22}(H-I+J))'}{\gamma^{22}}\,, \tag{16}$$

$$\xi_3 = G + \frac{C}{\mathcal{H}} - \frac{1}{2}\frac{(\gamma^{33}(-H+I+J))'}{\gamma^{33}}\,, \tag{17}$$

$$\zeta_1 = B - \frac{1}{2}(H+I-J)\,, \tag{18}$$

$$\zeta_2 = Q - \frac{1}{2}(H-I+J)\,, \tag{19}$$

$$\zeta_3 = D - \frac{1}{2}(-H+I+J)\,. \tag{20}$$

We should check the following: there exists a choice of gauge such that $C = I = J = H = 0$. Then our 7 variables (which only constitute 6 degrees of freedom via the constraint $\partial_1^2 B + \partial_2^2 Q + \partial_3^2 D = 0$) are

$$\varphi = A\,, \tag{21}$$

$$\xi_1 = E\,, \tag{22}$$

$$\xi_2 = F\,, \tag{23}$$

$$\xi_3 = G\,, \tag{24}$$

$$\zeta_1 = B\,, \tag{25}$$

$$\zeta_2 = Q\,, \tag{26}$$

$$\zeta_3 = D\,. \tag{27}$$

It would be nice, however, if we could set the temporal and mixed components to zero.

In general, we have the following Christoffel symbols.

$$\delta\Gamma^0_{00} = A'\,, \tag{28}$$

$$\delta\Gamma^0_{i0} = \partial_i\left[A + \left(\mathcal{H} + \frac{\sigma_{ii}}{\gamma_{ii}}\right)E_i\right]\,, \tag{29}$$

$$\delta\Gamma^i_{00} = \partial^i\left[\frac{1}{S}\left(SE_i\right)' - A\right]\,, \tag{30}$$

$$\delta\Gamma^0_{ij} = \frac{\left(S^2 h_{ij}\right)' - 2A\left(S^2\gamma_{ij}\right)'}{2S^2} - \partial_i\partial_j\left[\frac{E_i + E_j}{2}\right]\,, \tag{31}$$

$$\delta\Gamma^i_{j0} = \frac{1}{2}\left[h_{ij}\frac{\gamma_{jj}}{\gamma_{ii}}\left(\frac{1}{\gamma_{jj}}\right)' + h'_{ij}\left(\frac{1}{\gamma_{ii}}\right)\right] + \partial_i\partial_j\left[\frac{E_i - E_j}{2}\right]\,, \tag{32}$$

$$\delta\Gamma^i_{jk} = \frac{1}{2\gamma_{ii}}\left(\partial_k h_{ij} + \partial_j h_{ik} - \partial_i h_{jk}\right) - \frac{(S^2\gamma_{jk})'}{2S^2\gamma_{ii}}\partial_i E_i \tag{33}$$

In the above gauge:

$$\delta\Gamma^0_{ij} = \delta_{ij}\frac{\left(S^2 h_{ii}\right)' - 2A\left(S^2\gamma_{ii}\right)'}{2S^2} - \partial_i\partial_j\left[\frac{E_i + E_j}{2}\right], \tag{34}$$

$$\delta\Gamma^i_{j0} = \frac{1}{2}\delta_{ij}\left(\frac{h_{jj}}{\gamma_{jj}}\right)' + \partial_i\partial_j\left[\frac{E_i - E_j}{2}\right], \tag{35}$$

$$\delta\Gamma^i_{jk} = \frac{1}{2\gamma_{ii}}\left[\left(\delta_{ij}\partial_k + \delta_{ik}\partial_j\right)h_{ii} - \delta_{jk}\partial_i\left(h_{jj} + \frac{\left[S^2\gamma_{jj}\right]'}{S^2}E_i\right)\right]. \tag{36}$$

Please double check this. I'm a little suspicious of the $5\mathcal{H}$ term.

$$\delta R_{00} = \sum_i\left[\partial^i(\partial_i + \mathcal{H})\left(A - \frac{(SE_i)'}{S}\right) + \left(\frac{h_{ii}}{\gamma_{ii}}\right)'\frac{\sigma_{ii}}{\gamma_{ii}}\right], \tag{37}$$

$$\delta R_{i0} = -\left[\left(\mathcal{H} + \frac{\sigma_{ii}}{\gamma_{ii}}\right)\partial_i E_i\right]' - \frac{1}{2}\partial_i\left(\frac{h_{ii}}{\gamma_{ii}}\right)' + \frac{1}{2}\partial_i\sum_k\partial_k\partial_k\left(E_k - E_i\right) - \left(4\mathcal{H} + \frac{\sigma_{ii}}{\gamma_{ii}}\right)\partial_i A \tag{38}$$

$$-\left(\mathcal{H} + \frac{\sigma_{ii}}{\gamma_{ii}}\right)(5\mathcal{H})\,\partial_i E_i + \frac{(S\partial_i E_i)'}{S\gamma_{ii}}, \tag{39}$$