B1-Perturbations stage 2

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First, some definitions. It is important that we all use the same conventions. We should stick to the $(-,+,+,+)$ signature, which is not what Weinberg does. The less minus signs we use, the less we are prone to sign mistakes.

The background metric is

$$ds^2 = S^2 \left[-d\eta^2 + \gamma_{ij}(\eta) dx^i dx^j\right] \,,$$

where

$$\gamma_{ij} := \mathrm{diag}(e^{2\beta_1}, e^{2\beta_2}, e^{2\beta_3}) \,, \qquad \sum_{i=1}^{3} \beta_i = 0 \,.$$

Note that we are working in a very specific coordinate system in which the shear $\sigma_{ij} := (\gamma_{ij})'/2$ has only two independent components. Since σ_{ij} is a symmetric traceless matrix, the other three componentes can be seen as the three Euler angles needed to rotate γ_{ij} to a general coordinate system.

The Lie derivative of the background metric along the vector ξ is

$$\begin{aligned}
\mathscr{L}_\xi \bar{g}_{00} &= -2S^2 \left(T' + HT\right) \\
\mathscr{L}_\xi \bar{g}_{0i} &= S^2 \left(-\partial_i T + \gamma_{ij}\partial_0 \xi^j\right) \\
\mathscr{L}_\xi \bar{g}_{ij} &= S^2 \left(2\mathscr{H} T \gamma_{ij} + 2T\sigma_{ij} + 2\partial_{(i}\xi_{j)}\right)
\end{aligned}$$

This is always true, regardless of the splitting.

We will parameterize ξ^μ as

$$\xi^\mu = \left(T, \partial^1 X, \partial^2 Y, \partial^3 Z\right)$$

and the line element as

$$ds^2 = S^2 \left[-(1+2A)d\eta^2 + 2B_i dx^i d\eta + (\gamma_{ij} + h_{ij}) dx^i dx^j\right] \,,$$

where

$$B_i = (\partial_1 E, \partial_2 F, \partial_3 G)$$

and h_{ij} will be built later. Gauge transformations are such that

$$\Delta \delta g_{\mu\nu} = \mathscr{L}_\xi \bar{g}_{\mu\nu}$$

It is thus straightforward to compute the following transformation for A and B_i:

$$\begin{aligned}
A &\to A + T' + \mathscr{H} T = A + (ST)'/S \\
E &\to E - T + X' - 2\beta_1' X = E - T + (\gamma^{11} X)'/\gamma^{11} \\
F &\to F - T + Y' - 2\beta_2' Y = F - T + (\gamma^{22} Y)'/\gamma^{22} \\
G &\to G - T + Z' - 2\beta_3' Z = G - T + (\gamma^{33} Z)'/\gamma^{33}
\end{aligned}$$

In order to parameterize h_{ij}, we need ask how many ways there exist to build a tensor from scalars only (since we want SSS decomposition). There are two ways: either by multiplying a scalar by γ_{ij}, or by taking two derivatives of a scalar. Thus we write

$$h_{ij} = \left(\gamma_{ij} + \frac{\sigma_{ij}}{\mathscr{H}}\right) 2C + \bar{h}_{ij}$$

where C is a scalar and $\bar{h}_{ij}$ is a traceless matrix built out of (two) derivatives of 6 new scalar fields. One possibility is

$$\bar{h}_{ij} = \begin{pmatrix} 2\partial_1^2 B & \partial_1\partial_2 H & \partial_1\partial_3 I \\ & 2\partial_2^2 Q & \partial_2\partial_3 J \\ & & 2\partial_3^2 D \end{pmatrix},$$

with the constraint

$$\partial_1^2 B + \partial_2^2 Q + \partial_3^2 D = 0\,.$$

The gauge transformations are

$$\begin{aligned}
C &\to C + \mathscr{H}T \\
B &\to B + X \\
Q &\to Q + Y \\
D &\to D + Z \\
H &\to H + X + Y \\
I &\to I + X + Z \\
J &\to J + Y + Z
\end{aligned}$$

The following are Gauge invariant combinations:

$$A + \frac{1}{S}\left[S\left(E - \frac{(\gamma^{11}B)'}{\gamma^{11}}\right)\right]', \tag{1}$$

$$A + \frac{1}{S}\left[S\left(F - \frac{(\gamma^{22}Q)'}{\gamma^{22}}\right)\right]', \tag{2}$$

$$A + \frac{1}{S}\left[S\left(G - \frac{(\gamma^{33}D)'}{\gamma^{33}}\right)\right]', \tag{3}$$

$$C + \mathcal{H}\left[E - \frac{(\gamma^{11}B)'}{\gamma^{11}}\right], \tag{4}$$

$$C + \mathcal{H}\left[F - \frac{(\gamma^{22}Q)'}{\gamma^{22}}\right], \tag{5}$$

$$C + \mathcal{H}\left[G - \frac{(\gamma^{33}D)'}{\gamma^{33}}\right], \tag{6}$$

$$H - B - Q\,, \tag{7}$$

$$I - B - D\,, \tag{8}$$

$$J - D - Q\,, \tag{9}$$

$$H + I + J - 2(B + Q + D)\,, \tag{10}$$

$$H + I - J - 2B\,, \tag{11}$$

$$H + J - I - 2Q\,, \tag{12}$$

$$I + J - H - 2D\,, \tag{13}$$

and so on...

Maybe we want to choose the off diagonal components of $\bar{h}_{ij}$ to be 0. In this case, appropriate GIV's may be

$$\varphi = A + \left(\frac{C}{\mathcal{H}}\right)' - C\,, \tag{14}$$

$$\xi_1 = E + \frac{C}{\mathcal{H}} - \frac{1}{2}\frac{(\gamma^{11}(H+I-J))'}{\gamma^{11}}\,, \tag{15}$$

$$\xi_2 = F + \frac{C}{\mathcal{H}} - \frac{1}{2}\frac{(\gamma^{22}(H-I+J))'}{\gamma^{22}}\,, \tag{16}$$

$$\xi_3 = G + \frac{C}{\mathcal{H}} - \frac{1}{2}\frac{(\gamma^{33}(-H+I+J))'}{\gamma^{33}}\,, \tag{17}$$

$$\zeta_1 = B - \frac{1}{2}(H+I-J)\,, \tag{18}$$

$$\zeta_2 = Q - \frac{1}{2}(H-I+J)\,, \tag{19}$$

$$\zeta_3 = D - \frac{1}{2}(-H+I+J)\,. \tag{20}$$

We should check the following: there exists a choice of gauge such that $C = H = I = J = 0$. Then our 7 variables (which only constitute 6 degrees of freedom via the constraint $\partial_1^2 B + \partial_2^2 Q + \partial_3^2 D = 0$) are

$$\varphi = A\,, \tag{21}$$

$$\xi_1 = E\,, \tag{22}$$

$$\xi_2 = F\,, \tag{23}$$

$$\xi_3 = G\,, \tag{24}$$

$$\zeta_1 = B\,, \tag{25}$$

$$\zeta_2 = Q\,, \tag{26}$$

$$\zeta_3 = D\,. \tag{27}$$

It would be nice, however, if we could set the temporal and mixed components to zero.

In general, we have the following Christoffel symbols, where E_i is E, F, G for $i = 1, 2, 3$ respectively.

$$\delta\Gamma^0_{00} = A'\,, \tag{28}$$

$$\delta\Gamma^0_{i0} = \partial_i\left[A + \left(\mathcal{H} + \frac{\sigma_{ii}}{\gamma_{ii}}\right)E_i\right]\,, \tag{29}$$

$$\delta\Gamma^i_{00} = \partial^i\left[\frac{1}{S}\left(SE_i\right)' + A\right]\,, \tag{30}$$

$$\delta\Gamma^0_{ij} = \frac{\left(S^2 h_{ij}\right)' - 2A\left(S^2\gamma_{ij}\right)'}{2S^2} - \partial_i\partial_j\left[\frac{E_i + E_j}{2}\right]\,, \tag{31}$$

$$\delta\Gamma^i_{j0} = \frac{1}{2}\left[h_{ij}\frac{\gamma_{jj}}{\gamma_{ii}}\left(\frac{1}{\gamma_{jj}}\right)' + h'_{ij}\left(\frac{1}{\gamma_{ii}}\right)\right] + \partial^i\partial_j\left[\frac{E_i - E_j}{2}\right]\,, \tag{32}$$

$$\delta\Gamma^i_{jk} = \frac{1}{2\gamma_{ii}}\left(\partial_k h_{ij} + \partial_j h_{ik} - \partial_i h_{jk}\right) - \frac{(S^2\gamma_{jk})'}{2S^2\gamma_{ii}}\partial_i E_i \tag{33}$$

In the above gauge:

$$\delta\Gamma^0_{ij} = \delta_{ij}\frac{\left(S^2 h_{ii}\right)' - 2A\left(S^2\gamma_{ii}\right)'}{2S^2} - \partial_i\partial_j\left[\frac{E_i + E_j}{2}\right], \tag{34}$$

$$\delta\Gamma^i_{j0} = \frac{1}{2}\delta_{ij}\left(\frac{h_{jj}}{\gamma_{jj}}\right)' + \partial^i\partial_j\left[\frac{E_i - E_j}{2}\right], \tag{35}$$

$$\delta\Gamma^i_{jk} = \frac{1}{2\gamma_{ii}}\left[\left(\delta_{ij}\partial_k + \delta_{ik}\partial_j\right)h_{ii} - \delta_{jk}\partial_i\left(h_{jj} + \frac{\left[S^2\gamma_{jj}\right]'}{S^2}E_i\right)\right]. \tag{36}$$

Please double check this. I'm a little suspicious of the $5\mathcal{H}$ term.

$$\delta R_{00} = \sum_i\left[\partial^i\left(\partial_i + \mathcal{H}\right)\left(A - \frac{(SE_i)'}{S}\right) + \left(\frac{h_{ii}}{\gamma_{ii}}\right)'\frac{\sigma_{ii}}{\gamma_{ii}}\right], \tag{37}$$

$$\begin{aligned}\delta R_{i0} = &-\left[\left(\mathcal{H} + \frac{\sigma_{ii}}{\gamma_{ii}}\right)\partial_i E_i\right]' - \frac{1}{2}\partial_i\left(\frac{h_{ii}}{\gamma_{ii}}\right)' + \frac{1}{2}\partial_i\sum_k\partial^k\partial_k\left(E_k - E_i\right) - \left(4\mathcal{H} + \frac{\sigma_{ii}}{\gamma_{ii}}\right)\partial_i A \\ &- \left(\mathcal{H} + \frac{\sigma_{ii}}{\gamma_{ii}}\right)(5\mathcal{H})\,\partial_i E_i + \frac{(S\partial_i E_i)'}{S\gamma_{ii}},\end{aligned} \tag{38}$$

$$\begin{aligned}\delta R_{ij} = \delta_{ij}&\left[\left(\frac{\left(S^2 h_{ii}\right)'}{2S^2}\right)' + 2\left(\mathcal{H} + \frac{\sigma_{ii}}{\gamma_{ii}}\right)\left(-\frac{3}{2}A' - 2\left(\mathcal{H} + \frac{\sigma_{ii}}{\gamma_{ii}}\right)\gamma_{ii}A + h'_{ii} + \left(\mathcal{H} - \frac{\sigma_{ii}}{\gamma_{ii}}\right)h_{ii} - \frac{\gamma_{ii}}{2}\sum_k\partial^k\partial_k E_k\right)\right. \\ &\left.-2\left(\mathcal{H}\gamma_{ii} + \sigma_{ii}\right)' A - 2\mathcal{H}\left(\frac{h_{ii}}{\gamma_{ii}}\right)' - \sum_k\partial^k\partial_k h_{ii}\right] + \partial_i\partial_j\left[A + \mathcal{H}(E_i + E_j) + \frac{1}{2\gamma_{ii}}\left(h_{ii} - 2\sigma_{ii}E_j\right) + \frac{1}{2\gamma_{jj}}\left(h_{jj} - 2\sigma_{jj}E_i\right)\right]\end{aligned} \tag{39}$$

A brief glance at the form of δR_{ij} reveals that it too contains every GIV. Note, however, that A, E, F, G only show up as first derivatives in η. This seems to mean that the dynamic variables are B, Q, D.

I am unaware of any theorems regarding the coupling of modes in one gauge implying the coupling in another, or whether the elimination of the non-dynamic variables may eliminate the coupling. This in mind, we consider another gauge more closely related to the Newtonian gauge.

It is straight forward to write down the components corresponding to the equations for the dynamic variables.

(Undoubtably, I've made a few mistakes typing these out, but the shouldn't be too far off).

$$\begin{aligned}
\delta S_{ii} &= \delta R_{ii}\,, && (40)\\
&= \left[\left(\frac{\left(S^2 h_{ii}\right)'}{2S^2}\right)' + 2\left(\mathcal{H}+\frac{\sigma_{ii}}{\gamma_{ii}}\right)\left(-\frac{3}{2}A' - 2\left(\mathcal{H}+\frac{\sigma_{ii}}{\gamma_{ii}}\right)\gamma_{ii}A + h'_{ii} + \left(\mathcal{H}-\frac{\sigma_{ii}}{\gamma_{ii}}\right)h_{ii} - \frac{\gamma_{ii}}{2}\sum_k \partial^k\partial_k E_k\right)\right.\\
&\left. -2\left(\mathcal{H}\gamma_{ii}+\sigma_{ii}\right)' A - 2\mathcal{H}\left(\frac{h_{ii}}{\gamma_{ii}}\right)' - \sum_k \partial^k\partial_k h_{ii}\right] + \partial_i\partial_i\left[A + 2\mathcal{H}E_i + \frac{1}{\gamma_{ii}}\left(h_{ii} - 2\sigma_{ii}E_i\right)\right], && (41)\\
&= \left(\mathcal{H}' h_{ii} + \mathcal{H} h'_{ii} + \frac{1}{2}h''_{ii}\right) + 2\left(\mathcal{H}+\frac{\sigma_{ii}}{\gamma_{ii}}\right)\left(-\frac{3}{2}A' - 2\left(\mathcal{H}+\frac{\sigma_{ii}}{\gamma_{ii}}\right)\gamma_{ii}A + h'_{ii} + \left(\mathcal{H}-\frac{\sigma_{ii}}{\gamma_{ii}}\right)h_{ii} - \frac{\gamma_{ii}}{2}\sum_k \partial^k\partial_k E_k\right)\\
&- 2\left(\mathcal{H}\gamma_{ii}+\sigma_{ii}\right)' A - 2\mathcal{H}\left(\frac{h_{ii}}{\gamma_{ii}}\right)' - \left(\nabla^2 + \partial^i\partial_i\right)h_{ii} + \partial_i\partial_i\left[A + 2\mathcal{H}E_i - 2\frac{\sigma_{ii}}{\gamma^{ii}}E_i\right], && (42)\\
&= \left(\mathcal{H}' h_{ii} + \mathcal{H} h'_{ii} + \frac{1}{2}h''_{ii}\right)\\
&- 3\left(\mathcal{H}+\frac{\sigma_{ii}}{\gamma_{ii}}\right)A' - 4\left(\mathcal{H}+\frac{\sigma_{ii}}{\gamma_{ii}}\right)^2\gamma_{ii}A + 2\left(\mathcal{H}+\frac{\sigma_{ii}}{\gamma_{ii}}\right)h'_{ii} + 2\left(\mathcal{H}^2 - \left(\frac{\sigma_{ii}}{\gamma_{ii}}\right)^2\right)h_{ii} - \left(\mathcal{H}\gamma_{ii}+\sigma_{ii}\right)\sum_k \partial^k\partial_k E_k\\
&- 2\left(\mathcal{H}\gamma_{ii}+\sigma_{ii}\right)' A - 2\mathcal{H}\left(\frac{h_{ii}}{\gamma_{ii}}\right)' - \left(\nabla^2 + \partial^i\partial_i\right)h_{ii} + \partial_i\partial_i\left[A + 2\mathcal{H}E_i - 2\frac{\sigma_{ii}}{\gamma^{ii}}E_i\right], && (43)\\
&= \left(\mathcal{H}' + 4\mathcal{H}\frac{\sigma_{ii}}{\gamma_{ii}^2} - \nabla^2 - \partial^i\partial_i + 2\mathcal{H}^2 - 2\left(\frac{\sigma_{ii}}{\gamma_{ii}}\right)^2\right)h_{ii} + \left(3\mathcal{H} + 2\frac{\sigma_{ii}}{\gamma_{ii}} - \frac{2}{\gamma_{ii}}\mathcal{H}\right)h'_{ii} + \frac{1}{2}h''_{ii}\\
&- 3\left(\mathcal{H}+\frac{\sigma_{ii}}{\gamma_{ii}}\right)A' + \left(-2\left(\mathcal{H}\gamma_{ii}+\sigma_{ii}\right)' - 4\left(\mathcal{H}+\frac{\sigma_{ii}}{\gamma_{ii}}\right)^2\gamma_{ii} + \partial_i\partial_i\right)A\\
&- \left(\mathcal{H}\gamma_{ii}+\sigma_{ii}\right)\sum_k \partial^k\partial_k E_k - 2(\mathcal{H}\gamma_{ii}-\sigma_{ii})\partial^i\partial_i E_i\,. && (44)
\end{aligned}$$

The units don't work out in some terms. I'm going to redo this calculuation.

Corrected Christoffel symbols in this gauge are:

$$\begin{aligned}
\delta\Gamma^0_{00} &= A'\,, && (45)\\
\delta\Gamma^0_{i0} &= \partial_i A + \mathcal{H}_i\partial_i E_i\,, && (46)\\
\delta\Gamma^i_{00} &= \mathcal{H}\partial^i E_i + \partial^i E'_i + \partial^i A\,, && (47)\\
\delta\Gamma^0_{ij} &= -2A\gamma_{ij}\mathcal{H}_i - \frac{1}{2}\partial_i\partial_j(E_i+E_j) + \frac{(S^2 h_{ij})'}{2S^2}\,, && (48)\\
\delta\Gamma^i_{j0} &= \frac{1}{2}\delta^i_j\left(\frac{h_{ii}}{\gamma_{ii}}\right)' + \frac{1}{2}\partial^i\partial_j(E_i - E_j)\,, && (49)\\
\delta\Gamma^i_{jk} &= \frac{1}{2\gamma_{ii}}\left(\partial_j h_{ik} + \partial_k h_{ij}\right) - \partial^i\left(\frac{1}{2}h_{jk} + \gamma_{jk}\mathcal{H}_j E_i\right). && (50)
\end{aligned}$$

These yield, (with possible mistakes, but the units seem to work out) taking $\mathcal{H}_i := \mathcal{H} + \frac{\sigma_{ii}}{\gamma_{ii}}$,

$$\delta R_{00} = \sum_i \left[\mathcal{H}\partial^i\partial_i E_i + \partial^i\partial_i E_i' + \left(\frac{\sigma_{ii}}{\gamma_{ii}}\right)\left(\frac{h_{ii}}{\gamma_{ii}}\right)' \right] - \nabla^2 A - 3\mathcal{H}A' \,, \tag{51}$$

$$\delta R_{i0} = \left[2\mathcal{H}_i\left(\mathcal{H}_i - 2\mathcal{H}\right) - \mathcal{H}_i'\right]\partial_i E_i - \frac{1}{2}\partial_i \sum_k \left[\partial^k\partial_k(E_k - E_i) + \frac{\sigma_{kk}}{\gamma_{kk}^2}(h_{kk} - \delta_{ik}h_{ii})\right] - \left(3\mathcal{H} - \mathcal{H}_i\right)\partial_i A - \frac{1}{2}\partial^i h_{ii}' \,,$$

$$\begin{aligned}\delta R_{ij} = &\left[\frac{(S^2 h_{ij})'}{2S^2}\right]' + \left(\mathcal{H}_i + \mathcal{H}_j - 4\mathcal{H}\right)\left[\frac{(S^2 h_{ij})'}{2S^2}\right] - \frac{1}{2}\partial_i\partial_j\left[\frac{h_{ii}}{\gamma_{ii}} + \frac{h_{jj}}{\gamma_{jj}}\right] + \frac{1}{2}\nabla^2 h_{ij} + \gamma_{ij}\mathcal{H}_j\left(\frac{h_{jj}}{\gamma_{jj}}\right)' \\ &- \frac{1}{2}\partial_i\partial_j\left[E_i' + E_j' + \left(2\mathcal{H}_j - 4\mathcal{H}\right)E_i + \left(2\mathcal{H}_i - 4\mathcal{H}\right)E_j\right] + \gamma_{ij}\mathcal{H}_i\sum_k \partial^k\partial_k E_k \\ &+ 2\left[(4\mathcal{H} - 2\mathcal{H}_i)\gamma_{ij}\mathcal{H}_i - (\gamma_{ij}\mathcal{H}_i)'\right]A + \partial_i\partial_j A - 3\gamma_{ij}\mathcal{H}_i A' \,.\end{aligned} \tag{52}$$

Hence, defining $\mathscr{H}_i := 2\mathcal{H}_i - 4\mathcal{H}$

$$\delta R_{ii} = \left[\frac{(S^2 h_{ii})'}{2S^2}\right]' + \mathscr{H}_i\left[\frac{(S^2 h_{ii})'}{2S^2}\right] - \partial^i\partial_i h_{ii} + \frac{1}{2}\nabla^2 h_{ii} + \gamma_{ii}\mathcal{H}_i\left(\frac{h_{ii}}{\gamma_{ii}}\right)' \tag{53}$$

$$- \partial_i\partial_i\left[E_i' + \mathscr{H}_i E_i\right] + \gamma_{ii}\mathcal{H}_i\sum_k \partial^k\partial_k E_k - 2\left[\mathscr{H}_i\gamma_{ii}\mathcal{H}_i + (\gamma_{ii}\mathcal{H}_i)'\right]A + \partial_i\partial_i A - 3\gamma_{ii}\mathcal{H}_i A' \,. \tag{54}$$

Rewriting $\delta R_{\mu\nu}$ with $\mathscr{H}_i$, we see that

$$\delta R_{00} = \sum_i \left[\mathcal{H}\partial^i\partial_i E_i + \partial^i\partial_i E_i' + \left(\frac{\sigma_{ii}}{\gamma_{ii}}\right)\left(\frac{h_{ii}}{\gamma_{ii}}\right)' \right] - \nabla^2 A - 3\mathcal{H}A' \,, \tag{55}$$

$$\delta R_{i0} = \left[\mathcal{H}_i\mathscr{H}_i - \mathcal{H}_i'\right]\partial_i E_i - \frac{1}{2}\partial_i \sum_k \left[\partial^k\partial_k(E_k - E_i) + \frac{\sigma_{kk}}{\gamma_{kk}^2}(h_{kk} - \delta_{ik}h_{ii})\right] - \left(3\mathcal{H} - \mathcal{H}_i\right)\partial_i A - \frac{1}{2}\partial^i h_{ii}' \,,$$

$$\begin{aligned}\delta R_{ij} = &\left[\frac{(S^2 h_{ij})'}{2S^2}\right]' + \mathscr{H}_i\left[\frac{(S^2 h_{ij})'}{2S^2}\right] - \frac{1}{2}\partial_i\partial_j\left[\frac{h_{ii}}{\gamma_{ii}} + \frac{h_{jj}}{\gamma_{jj}}\right] + \frac{1}{2}\nabla^2 h_{ij} + \gamma_{ij}\mathcal{H}_j\left(\frac{h_{jj}}{\gamma_{jj}}\right)' \\ &- \frac{1}{2}\partial_i\partial_j\left[E_i' + E_j' + \mathscr{H}_j E_i + \mathscr{H}_i E_j\right] + \gamma_{ij}\mathcal{H}_i\sum_k \partial^k\partial_k E_k - 2\left[\mathscr{H}_i\gamma_{ij}\mathcal{H}_i + (\gamma_{ij}\mathcal{H}_i)'\right]A + \partial_i\partial_j A - 3\gamma_{ij}\mathcal{H}_i A' \,.\end{aligned} \tag{56}$$

Hence for $i \neq j$, we have

$$\delta R_{ij} = \frac{1}{2}\partial_i\partial_j\left[2A - \frac{h_{ii}}{\gamma_{ii}} - \frac{h_{jj}}{\gamma_{jj}} - E_i' - E_j' - \mathscr{H}_j E_i - \mathscr{H}_i E_j\right] \,.$$