B1-Perturbations stage 2

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First, some definitions. It is important that we all use the same conventions. We should stick to the $(-,+,+,+)$ signature, which is not what Weinberg does. The less minus signs we use, the less we are prone to sign mistakes.

The background metric is

$$ds^2 = S^2\left[-d\eta^2 + \gamma_{ij}(\eta)dx^i dx^j\right] ,$$

where

$$\gamma_{ij} := \mathrm{diag}(e^{2\beta_1}, e^{2\beta_2}, e^{2\beta_3}) , \qquad \sum_{i=1}^{3}\beta_i = 0 .$$

Note that we are working in a very specific coordinate system in which the shear $\sigma_{ij} := (\gamma_{ij})'/2$ has only two independent components. Since σ_{ij} is a symmetric traceless matrix, the other three componentes can be seen as the three Euler angles needed to rotate γ_{ij} to a general coordinate system.

The Lie derivative of the background metric along the vector ξ is

$$\begin{aligned}
\mathscr{L}_\xi \bar{g}_{00} &= -2S^2\left(T' + HT\right) \\
\mathscr{L}_\xi \bar{g}_{0i} &= S^2\left(-\partial_i T + \gamma_{ij}\partial_0 \xi^j\right) \\
\mathscr{L}_\xi \bar{g}_{ij} &= S^2\left(2\mathscr{H}T\gamma_{ij} + 2T\sigma_{ij} + 2\partial_{(i}\xi_{j)}\right)
\end{aligned}$$

This is always true, regardless of the splitting.

We will parameterize ξ^μ as

$$\xi^\mu = \left(T, \partial^1 X, \partial^2 Y, \partial^3 Z\right)$$

and the line element as

$$ds^2 = S^2\left[-(1+2A)d\eta^2 + 2B_i dx^i d\eta + (\gamma_{ij} + h_{ij})dx^i dx^j\right] ,$$

where

$$B_i = (\partial_1 E, \partial_2 F, \partial_3 G)$$

and h_{ij} will be built later. Gauge transformations are such that

$$\Delta\delta g_{\mu\nu} = \mathscr{L}_\xi \bar{g}_{\mu\nu}$$

It is thus straightforward to compute the following transformation for A and B_i:

$$\begin{aligned}
A &\to A + T' + \mathscr{H}T = A + (ST)'/S \\
E &\to E - T + X' - 2\beta_1' X = E - T + (\gamma^{11}X)'/\gamma^{11} \\
F &\to F - T + Y' - 2\beta_2' Y = F - T + (\gamma^{22}Y)'/\gamma^{22} \\
G &\to G - T + Z' - 2\beta_3' Z = G - T + (\gamma^{33}Z)'/\gamma^{33}
\end{aligned}$$

In order to parameterize h_{ij}, we need ask how many ways there exist to build a tensor from scalars only (since we want SSS decomposition). There are two ways: either by multiplying a scalar by γ_{ij}, or by taking two derivatives of a scalar. Thus we write

$$h_{ij} = \left(\gamma_{ij} + \frac{\sigma_{ij}}{\mathscr{H}}\right) 2C + \bar{h}_{ij}$$

where C is a scalar and $\bar{h}_{ij}$ is a traceless matrix built out of (two) derivatives of 5 new scalar fields. One possibility is

$$\bar{h}_{ij} = \begin{pmatrix} 2\partial_1^2 B & \partial_1\partial_2 H & \partial_1\partial_3 I \\ & 2\partial_2^2 Q & \partial_2\partial_3 J \\ & & 2\partial_3^2 D \end{pmatrix},$$

with the constraint

$$\partial_1^2 B + \partial_2^2 Q + \partial_3^2 D = 0\,.$$

The gauge transformations are

$$\begin{aligned} C &\to C + \mathscr{H} T \\ B &\to B + X \\ Q &\to Q + Y \\ D &\to D + Z \\ H &\to H + X + Y \\ I &\to I + X + Z \\ J &\to J + Y + Z \end{aligned}$$

The following are Gauge invariant combinations:

$$A + \frac{1}{S}\left[S\left(E - \frac{(\gamma^{11}B)'}{\gamma^{11}}\right)\right]', \tag{1}$$

$$A + \frac{1}{S}\left[S\left(F - \frac{(\gamma^{22}Q)'}{\gamma^{22}}\right)\right]', \tag{2}$$

$$A + \frac{1}{S}\left[S\left(G - \frac{(\gamma^{33}D)'}{\gamma^{33}}\right)\right]', \tag{3}$$

$$C + \mathcal{H}\left[E - \frac{(\gamma^{11}B)'}{\gamma^{11}}\right], \tag{4}$$

$$C + \mathcal{H}\left[F - \frac{(\gamma^{22}Q)'}{\gamma^{22}}\right], \tag{5}$$

$$C + \mathcal{H}\left[G - \frac{(\gamma^{33}D)'}{\gamma^{33}}\right], \tag{6}$$

$$H - B - Q\,, \tag{7}$$

$$I - B - D\,, \tag{8}$$

$$J - D - Q\,, \tag{9}$$

$$I + J + K - 2(B + Q + D)\,, \tag{10}$$

$$H + I - J - 2B\,, \tag{11}$$

$$H + J - I - 2Q\,, \tag{12}$$

$$I + J - H - 2D\,, \tag{13}$$

and so on...

In analogy to The Theory of Cosmological Perturbations in an Anisotropic Universe, we pick the 7 GIV's:

$$\Phi = A + \frac{1}{S}\left[S\left(E - \frac{(\gamma^{11}B)'}{\gamma^{11}}\right)\right]' , \tag{14}$$

$$\frac{\Psi}{\mathcal{H}} = \Phi_1 = E + \frac{C}{\mathcal{H}} - \frac{(\gamma^{11}B)'}{\gamma^{11}} , \tag{15}$$

$$\Phi_2 = F + \frac{C}{\mathcal{H}} - \frac{(\gamma^{22}Q)'}{\gamma^{22}} , \tag{16}$$

$$\Phi_3 = G + \frac{C}{\mathcal{H}} - \frac{(\gamma^{33}D)'}{\gamma^{33}} , \tag{17}$$

$$\Xi_1 = H - B - Q \, , \tag{18}$$

$$\Xi_2 = I - B - D \, , \tag{19}$$

$$\Xi_3 = J - D - Q \, . \tag{20}$$

Perhaps a good choice of Gauge is $E = B = D = Q = 0$. In this case we get

$$\Phi = A \, , \tag{21}$$

$$\Psi = C \, , \tag{22}$$

$$\Phi_2 = F + \frac{C}{\mathcal{H}} , \tag{23}$$

$$\Phi_3 = G + \frac{C}{\mathcal{H}} , \tag{24}$$

$$\Xi_1 = H \, , \tag{25}$$

$$\Xi_2 = I \, , \tag{26}$$

$$\Xi_3 = J \, . \tag{27}$$

Another good choice may be $C = B = D = Q = 0$. We then have

$$\Phi = A + \frac{1}{S}\left(SE\right)' , \tag{28}$$

$$\Phi_1 = E \, , \tag{29}$$

$$\Phi_2 = F \, , \tag{30}$$

$$\Phi_3 = G \, , \tag{31}$$

$$\Xi_1 = H \, , \tag{32}$$

$$\Xi_2 = I \, , \tag{33}$$

$$\Xi_3 = J \, . \tag{34}$$