Current Progress

June 28, 2016

We have the following Fourier transform

$$\phi(\vec{x},t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \Phi(\vec{k},\omega) e^{\imath(\vec{k}\cdot\vec{x}-\omega t)} \,. \tag{1}$$

The wave equation

$$\alpha^2 \nabla^2 \phi = \partial_t^2 \phi \,, \tag{2}$$

imposes the dispersion relation $\alpha k = \omega$ from which we obtain the solution

$$\phi(\vec{x},t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \Phi(\vec{k}) e^{\mathrm{i}(\vec{k}\cdot\vec{x}-\alpha k t)} \,. \tag{3}$$

We can now impose the constraint (where K is the bulk modulus)

$$-p(\vec{x},0) = K \nabla^2 \phi(\vec{x},0) \,, \tag{4}$$

which, in Fourier space, reads as

$$P(\vec{k}) = K k^2 \Phi(\vec{k}) \,. \tag{5}$$

From this, we obtain the displacement field's Fourier coefficients

$$U(\vec{k}) = \mathrm{i} \frac{P(\vec{k})}{K} \frac{\vec{k}}{k^2} \,, \tag{6}$$

and hence

$$u(\vec{x},t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \mathrm{i} \frac{P(\vec{k})}{K} \frac{\vec{k}}{k^2} e^{\mathrm{i}(\vec{k}\cdot\vec{x}-\alpha k t)} \,. \tag{7}$$

From this, we can obtain the total energy

$$E = \frac{1}{2} \int_V \mathrm{d}^3 x \left(\rho |\partial_t u|^2 + (\lambda + 2\mu) |\nabla \cdot u|^2 \right) \,, \tag{8}$$

$$= \frac{(\lambda+2\mu)}{K^2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left| P(\vec{k}) \right|^2 \,, \tag{9}$$

$$= \frac{(\lambda+2\mu)}{K^2} \int \frac{\mathrm{d} k}{(2\pi)^3} 4\pi k^2 |P(k)|^2 \,, \tag{10}$$

$$= 4\pi \frac{(\lambda+2\mu)}{K^2} \int \frac{\mathrm{d}\tilde{\lambda}}{\tilde{\lambda}^4} |P(k)|^2 \,. \tag{11}$$

We now treat an overpressure of p_0 confined to a cylinder of radius r_x and height h.

$$P(\vec{k}) = \frac{4\pi r_x p_0}{k^2 \cos\theta \sin\theta} J_1(k \sin\theta r_x) \sin\left(\frac{hk\cos\theta}{2}\right) , \tag{12}$$

$$\sim_{k\to 0} \frac{2\pi r_x p_0}{k^2 \cos\theta \sin\theta} k \sin\theta r_x \sin\left(\frac{hk\cos\theta}{2}\right) , \tag{13}$$

$$= \frac{2\pi r_x^2 p_0}{k \cos\theta} \sin\left(\frac{hk\cos\theta}{2}\right) . \tag{14}$$

Hence

$$E_{\text{large } \lambda_0} = \frac{(\lambda+2\mu)}{K^2} \int_0^{k_0} \int_0^{\pi} \int_0^{2\pi} \frac{\mathrm{d}^3 k}{(2\pi)^3} \left| \frac{2\pi r_x^2 p_0}{k\cos\theta} \sin\left(\frac{hk\cos\theta}{2}\right) \right|^2 , \tag{15}$$

$$= (r_x^2 p_0)^2 \frac{(\lambda+2\mu)}{K^2} \int_0^{k_0} \int_0^{\pi} \mathrm{d}k\, \mathrm{d}\theta \sin\theta \left[\frac{1}{\cos\theta} \sin\left(\frac{hk\cos\theta}{2}\right) \right]^2 , \tag{16}$$

$$= \left[(r_x^2 p_0)^2 \frac{(\lambda+2\mu)}{K^2} \right] \left[\frac{hk_0 \cos(hk_0) + \sin(hk_0) + hk_0(-2 + hk_0 \mathrm{Si}(hk_0))}{2h} \right] , \tag{17}$$

$$= \left[(r_x^2 p_0)^2 \frac{(\lambda+2\mu)}{K^2} \right] \left[\frac{\pi}{\lambda_0} \cos\left(\frac{2\pi h}{\lambda_0}\right) + \frac{1}{2h} \sin\left(\frac{2\pi h}{\lambda_0}\right) + \frac{\pi}{\lambda_0} \left(-2 + \frac{2\pi h}{\lambda_0} \mathrm{Si}\left(\frac{2\pi h}{\lambda_0}\right) \right) \right] , \tag{18}$$

$$\approx_{h \gg \lambda_0 \gg 1} \left[(\sigma_x p_0)^2 \frac{(\lambda+2\mu)}{K^2} \right] \left[\frac{\pi h}{\lambda_0^2} \right] \tag{19}$$

The real question lies in the factor of $(\sigma_x p_0)$ which I believe is proportional to, if not equal to $\frac{dE}{dx}$. This result, however, has the feature that it only has one power of h, which means that it goes as the energy (if things work as I think they do). Whence, the fraction of energy deposited in the low frequency spectrum is

$$\Xi = \pi \left| \frac{dE}{dx} \right| \frac{\lambda + 2\mu}{K^2 \lambda_0^2} , \tag{20}$$

where we have taken $E_{\text{total}} = \sigma_x h p_0$ and $\left| \frac{dE}{dx} \right| = \sigma_x p_0$.

Our questions are: does this make sense, i.e. can Ξ be greater than 1? It seems as though Ξ is almost proportional to the energy deposition squared. If so, where did we go wrong? If not, how should we proceed? We had a discussion with Professor Taylor the other day, and he was very concerned that we are considering only the linear regime. Clearly a shockwave will form, but will the shockwave to be relegated to the high frequency spectrum? Moreover, Prof. Taylor brought up the importance of the geometry of the projectile. A pointed tip will push matter to the side, while a blunt tip will force matter along its trajectory. Does this matter for such high velocities? I suppose this would correspond to the fraction of energy deposited in the shear versus compressional modes.