

B1-Perturbations stage 2

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First, some definitions. It is important that we all use the same conventions. We should stick to the $(-, +, +, +)$ signature, which is not what Weinberg does. The less minus signs we use, the less we are prone to sign mistakes.

The background metric is

$$ds^2 = S^2 [-d\eta^2 + \gamma_{ij}(\eta)dx^i dx^j] ,$$

where

$$\gamma_{ij} := \text{diag}(e^{2\beta_1}, e^{2\beta_2}, e^{2\beta_3}), \quad \sum_{i=1}^3 \beta_i = 0 .$$

Note that we are working in a very specific coordinate system in which the shear $\sigma_{ij} := (\gamma_{ij})'/2$ has only two independent components. Since σ_{ij} is a symmetric traceless matrix, the other three components can be seen as the three Euler angles needed to rotate γ_{ij} to a general coordinate system.

The Lie derivative of the background metric along the vector ξ is

$$\begin{aligned} \mathcal{L}_\xi \bar{g}_{00} &= -2S^2 (T' + HT) \\ \mathcal{L}_\xi \bar{g}_{0i} &= S^2 (-\partial_i T + \gamma_{ij} \partial_0 \xi^j) \\ \mathcal{L}_\xi \bar{g}_{ij} &= S^2 (2\mathcal{H}T \gamma_{ij} + 2T \sigma_{ij} + 2\partial_{(i} \xi_{j)}) \end{aligned}$$

This is always true, regardless of the splitting.

We will parameterize ξ^μ as

$$\xi^\mu = (T, \partial^1 X, \partial^2 Y, \partial^3 Z)$$

and the line element as

$$ds^2 = S^2 [-(1 + 2A)d\eta^2 + 2B_i dx^i d\eta + (\gamma_{ij} + h_{ij})dx^i dx^j] ,$$

where

$$B_i = (\partial_1 E, \partial_2 F, \partial_3 G)$$

and h_{ij} will be built later. Gauge transformations are such that

$$\Delta \delta g_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu}$$

It is thus straightforward to compute the following transformation for A and B_i :

$$\begin{aligned} A &\rightarrow A + T' + \mathcal{H}T = A + (ST)'/S \\ E &\rightarrow E - T + X' - 2\beta'_1 X = E - T + (\gamma^{11}X)'/\gamma^{11} \\ F &\rightarrow F - T + Y' - 2\beta'_2 Y = F - T + (\gamma^{22}Y)'/\gamma^{22} \\ G &\rightarrow G - T + Z' - 2\beta'_3 Z = G - T + (\gamma^{33}Z)'/\gamma^{33} \end{aligned}$$

In order to parameterize h_{ij} , we need ask how many ways there exist to build a tensor from scalars only (since we want SSS decomposition). There are two ways: either by multiplying a scalar by γ_{ij} , or by taking two derivatives of a scalar. Thus we write

$$h_{ij} = \left(\gamma_{ij} + \frac{\sigma_{ij}}{\mathcal{H}} \right) 2C + \bar{h}_{ij}$$

where C is a scalar and \bar{h}_{ij} is a traceless matrix built out of (two) derivatives of 6 new scalar fields. One possibility is

$$\bar{h}_{ij} = \begin{pmatrix} 2\partial_1^2 B & \partial_1 \partial_2 H & \partial_1 \partial_3 I \\ & 2\partial_2^2 Q & \partial_2 \partial_3 J \\ & & 2\partial_3^2 D \end{pmatrix},$$

with the constraint

$$\partial_1^2 B + \partial_2^2 Q + \partial_3^2 D = 0.$$

The gauge transformations are

$$\begin{aligned} C &\rightarrow C + \mathcal{H}T \\ B &\rightarrow B + X \\ Q &\rightarrow Q + Y \\ D &\rightarrow D + Z \\ H &\rightarrow H + X + Y \\ I &\rightarrow I + X + Z \\ J &\rightarrow J + Y + Z \end{aligned}$$

The following are Gauge invariant combinations:

$$A + \frac{1}{S} \left[S \left(E - \frac{(\gamma^{11}B)'}{\gamma^{11}} \right) \right]', \quad (1)$$

$$A + \frac{1}{S} \left[S \left(F - \frac{(\gamma^{22}Q)'}{\gamma^{22}} \right) \right]', \quad (2)$$

$$A + \frac{1}{S} \left[S \left(G - \frac{(\gamma^{33}D)'}{\gamma^{33}} \right) \right]', \quad (3)$$

$$C + \mathcal{H} \left[E - \frac{(\gamma^{11}B)'}{\gamma^{11}} \right], \quad (4)$$

$$C + \mathcal{H} \left[F - \frac{(\gamma^{22}Q)'}{\gamma^{22}} \right], \quad (5)$$

$$C + \mathcal{H} \left[G - \frac{(\gamma^{33}D)'}{\gamma^{33}} \right], \quad (6)$$

$$H - B - Q, \quad (7)$$

$$I - B - D, \quad (8)$$

$$J - D - Q, \quad (9)$$

$$H + I + J - 2(B + Q + D), \quad (10)$$

$$H + I - J - 2B, \quad (11)$$

$$H + J - I - 2Q, \quad (12)$$

$$I + J - H - 2D, \quad (13)$$

and so on...

Maybe we want to choose the off diagonal components of \bar{h}_{ij} to be 0. In this case, appropriate GIV's may be

$$\varphi = A + \left(\frac{C}{\mathcal{H}} \right)' - C, \quad (14)$$

$$\xi_1 = E + \frac{C}{\mathcal{H}} - \frac{1}{2} \frac{(\gamma^{11}(H + I - J))'}{\gamma^{11}}, \quad (15)$$

$$\xi_2 = F + \frac{C}{\mathcal{H}} - \frac{1}{2} \frac{(\gamma^{22}(H - I + J))'}{\gamma^{22}}, \quad (16)$$

$$\xi_3 = G + \frac{C}{\mathcal{H}} - \frac{1}{2} \frac{(\gamma^{33}(-H + I + J))'}{\gamma^{33}}, \quad (17)$$

$$\zeta_1 = B - \frac{1}{2}(H + I - J), \quad (18)$$

$$\zeta_2 = Q - \frac{1}{2}(H - I + J), \quad (19)$$

$$\zeta_3 = D - \frac{1}{2}(-H + I + J). \quad (20)$$

We should check the following: there exists a choice of gauge such that $C = I = J = H = 0$. Then our 7 variables (which only constitute 6 degrees of freedom via the constraint $\partial_1^2 B + \partial_2^2 Q + \partial_3^2 D = 0$) are

$$\varphi = A, \quad (21)$$

$$\xi_1 = E, \quad (22)$$

$$\xi_2 = F, \quad (23)$$

$$\xi_3 = G, \quad (24)$$

$$\zeta_1 = B, \quad (25)$$

$$\zeta_2 = Q, \quad (26)$$

$$\zeta_3 = D. \quad (27)$$

It would be nice, however, if we could set the temporal and mixed components to zero.

In general, we have the following Christoffel symbols.

$$\delta\Gamma_{00}^0 = A', \quad (28)$$

$$\delta\Gamma_{i0}^0 = \partial_i \left[A + \left(\mathcal{H} + \frac{\sigma_{ii}}{\gamma_{ii}} \right) E_i \right], \quad (29)$$

$$\delta\Gamma_{00}^i = \partial^i \left[\frac{1}{S} (SE_i)' - A \right], \quad (30)$$

$$\delta\Gamma_{ij}^0 = \frac{(S^2 h_{ij})' - 2A(S^2 \gamma_{ij})'}{2S^2} - \partial_i \partial_j \left[\frac{E_i + E_j}{2} \right], \quad (31)$$

$$\delta\Gamma_{j0}^i = \frac{1}{2} \left[h_{ij} \frac{\gamma_{jj}}{\gamma_{ii}} \left(\frac{1}{\gamma_{jj}} \right)' + h'_{ij} \left(\frac{1}{\gamma_{ii}} \right) \right] + \partial_i \partial_j \left[\frac{E_i - E_j}{2} \right], \quad (32)$$

$$\delta\Gamma_{jk}^i = \frac{1}{2\gamma_{ii}} (\partial_k h_{ij} + \partial_j h_{ik} - \partial_i h_{jk}) - \frac{(S^2 \gamma_{jk})'}{2S^2 \gamma_{ii}} \partial_i E_i \quad (33)$$

In the above gauge:

$$\delta\Gamma_{ij}^0 = \delta_{ij} \frac{(S^2 h_{ii})' - 2A (S^2 \gamma_{ii})'}{2S^2} - \partial_i \partial_j \left[\frac{E_i + E_j}{2} \right], \quad (34)$$

$$\delta\Gamma_{j0}^i = \frac{1}{2} \delta_{ij} \left(\frac{h_{jj}}{\gamma_{jj}} \right)' + \partial_i \partial_j \left[\frac{E_i - E_j}{2} \right], \quad (35)$$

$$\delta\Gamma_{jk}^i = \frac{1}{2\gamma_{ii}} \left[(\delta_{ij} \partial_k + \delta_{ik} \partial_j) h_{ii} - \delta_{jk} \partial_i (h_{jj} + [S^2 \gamma_{jj}]' E_i) \right]. \quad (36)$$

The following is rather disappointing if it is correct. Please double check this.

$$\delta R_{00} = \sum_i \left[\partial^i (\partial_i + \mathcal{H}) \left(A - \frac{(SE_i)'}{S} \right) + (h_i^i)' \sigma_i^i \right] \quad (37)$$