

Symbol	Meaning
$i$	Individual or case (Level-1)
$j$	Group or cluster (Level-2)
$\gamma$	Represents a fixed effect
$\gamma_{00}$	Grand mean
$\beta_{0j}$	Random intercept
$\beta_{1j}$	Random slope
$\epsilon_{ij}$	Within-group variability (Level 1)
$v_{0j}$	Between-group variability (Level 2)
$\tau_{00}$	Res. var. of Level-2 random intercepts
$\sigma^2$	Res. var. of Level-1 random intercepts

Concept	Symbol	Level
Raw data	$x_{ij}$	Level 1 and 2
Grand mean centered	$\ddot{x}_{ij}$	Level 1 and 2
Group mean centered	$\dot{x}_{ij}$	Level 1
Group means	$\bar{x}_{.j}$	Level 2
Centered group means	$\bar{x}'_{.j}$	Level 2

IMPORTANT

Independent variables:  $x_1, x_2, x_3 \dots$

Dependent variable:  $Y$

Level 1: **within-group** variation, represented by  $i$   
Level 2: **between-group** variation, represented by  $j$

Note that Level 1 variables can carry both Level 1 and 2 variation. In  $x_{ij}$ , the subscript 0 represents the intercept and the 1 represents the slope. E.g.,  $\gamma_{01}$  denotes the first predictor of the intercept and  $\gamma_{12}$  represents the second predictor of the slope.

The *fixed effects* of the model are constant for all cases in the population and do not carry  $i$  or  $j$  subscripts. Effects that vary across Level-2 units are denoted by  $\gamma$  and  $\epsilon$ .

The *random effects* of the model vary across Level-1 and Level-2 units. Effects that vary across Level-2 units are denoted by  $v$  and  $\tau$ .

Intercept only Regression

$$Y_i = \beta_0 + \epsilon_i$$

Simple linear regression

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

Multiple linear regression

$$Y_i = \beta_0 + \beta_1 x'_i + \epsilon_i$$

Moderation

$$Y_i = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2$$

Moderation can be re-written as:

$$Y_i = \beta_0 + \beta_1 x_1 + (\beta_2 + \beta_3 x_1) x_2 + \epsilon_i$$

Fixed effects model

$$Y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + ... + \beta_p x_{pi} + \epsilon_i$$

$$\epsilon_i \sim_{iid} N(0, \sigma^2)$$

Random effects ANOVA

Level-1:

$$Y_{ij} = \beta_{0j} + \epsilon_{ij}$$

$$\epsilon_i \sim N(0, \sigma^2)$$

Level-2:

$$\beta_{0j} = \gamma_{00} + v_{0j}$$

$$v_{0j} \sim_{iid} N(0, \tau_{00})$$

Reduced:

$$Y_{ij} = \gamma_{00} + v_{0j} + \epsilon_{ij}$$

Random intercept regression model

Level-1:

$$Y_{ij} = \beta_{0j} + \beta_{1j} x_{ij} + \epsilon_{ij}$$

$$\epsilon_i \sim N(0, \sigma^2)$$

Level-2:

$$\beta_{0j} = \gamma_{00} + v_{0j}$$

$$\beta_{1j} = \gamma_{10}$$

$$v_{0j} \sim_{iid} N(0, \tau_{00})$$

Reduced:

$$Y_{ij} = \gamma_{00} + \gamma_{10} x_{ij} + v_{0j} + \epsilon_{ij}$$

Means as outcomes

Level-1:

$$Y_{ij} = \beta_{0j} + \epsilon_{ij}$$

$$\epsilon_i \sim N(0, \sigma^2)$$

Level-2:

$$\beta_{0j} = \gamma_{00} + \gamma_{01} w_j + v_{0j}$$

$$v_{0j} \sim_{iid} N(0, \tau_{00})$$

Reduced:

$$Y_{ij} = \gamma_{00} + \gamma_{01} w_j + v_{0j} + \epsilon_{ij}$$

Intercepts as outcomes

Level-1:

$$Y_{ij} = \beta_{0j} + \beta_{1j} x_{ij} + \epsilon_{ij}$$

$$\epsilon_i \sim N(0, \sigma^2)$$

Level-2:

$$\beta_{0j} = \gamma_{00} + \gamma_{01} w_j + v_{0j}$$

$$\beta_{1j} = \gamma_{10}$$

$$v_{0j} \sim_{iid} N(0, \tau_{00})$$

Reduced:

$$Y_{ij} = \gamma_{00} + \gamma_{01} w_j + \gamma_{10} x_{ij} + v_{0j} + \epsilon_{ij}$$

Slopes as outcomes

Level-1:

$$Y_{ij} = \beta_{0j} + \beta_{1j} x_{ij} + \epsilon_{ij}$$

Level-2:

$$\beta_{0j} = \gamma_{00} + \gamma_{01} w_j + v_{0j}$$

$$\beta_{1j} = \gamma_{10} + \gamma_{11} w_j + v_{1j}$$

$$\left( \begin{array}{c} v_{0j} \\ v_{1j} \end{array} \right) \stackrel{iid}{\sim} \left( \begin{array}{c} \left[ 0 \right] \\ \left[ 0 \right] \end{array}, \begin{array}{cc} \left[ \tau_{00} & \tau_{10} \right] \\ \left[ \tau_{10} & \tau_{11} \right] \end{array} \right)$$

Reduced:

$$Y_{ij} = \gamma_{00} + \gamma_{01} w_j + \gamma_{10} x_{ij} + \gamma_{11} w_j x_{ij} + v_{0j} + v_{1j} x_{ij} + \epsilon_{ij}$$

Simple Intercepts and Simple Slopes

$$\hat{y} = \gamma_{00} + \gamma_{01} w + \gamma_{10} x + \gamma_{11} w x$$

And can be re-written as:

$$\hat{y} = (\gamma_{00} + \gamma_{01} w) + (\gamma_{10} + \gamma_{11} w) x$$

Random Intercepts and Random Slopes Model

Level-1:

$$Y_{ij} = \beta_{0j} + \beta_{1j} x_{ij} + \epsilon_{ij}$$

Level-2:

$$\beta_{0j} = \gamma_{00} + v_{0j}$$

$$\beta_{1j} = \gamma_{10} + v_{1j}$$

$$\left( \begin{array}{c} v_{0j} \\ v_{1j} \end{array} \right) \stackrel{iid}{\sim} \left( \begin{array}{c} \left[ 0 \right] \\ \left[ 0 \right] \end{array}, \begin{array}{cc} \left[ \tau_{00} & \tau_{10} \right] \\ \left[ \tau_{10} & \tau_{11} \right] \end{array} \right)$$

Reduced:

$$Y_{ij} = \gamma_{00} + \gamma_{10} x_{ij} + v_{0j} + v_{1j} x_{ij} + \epsilon_{ij}$$

Calculations

Correlation between random slopes and random intercepts:

$$r = \frac{\tau_{10}}{\sqrt{\tau_{00}} \sqrt{\tau_{11}}}$$

Intraclass correlation:

$$ICC = \frac{\tau_{00}}{\tau_{00} + \sigma^2}$$

**Interpretation**

Regression coefficient:  $\beta_i$  represents the predicted change in  $Y$  for a 1-unit increase in  $x_1$ , holding  $x_2$  constant.

Centered regression coefficient:  $\beta_i$  is the influence if  $x_1'$  on  $Y$  when  $x_2'$  is zero or for an individual at the mean of  $x_2$ .

Regression coefficient when an interaction is present:  $\beta_i$  is the change in  $Y$  for a 1-unit increase in  $x_1$  when  $x_2$  is zero.

**Fixed effects**

The *fixed effects* of the model are constant for all cases in the population and do not carry  $i$  or  $j$  subscripts. The Greek symbol gamma ( $\gamma$ ) denotes fixed effects. E.g.,  $\gamma_{00}$  has no subscript and does not vary.

$$Y_{ij} = \gamma_{00} + \gamma_{10}Female_{ij} + \gamma_{20}Minority_{ij} + \gamma_{30}Female_{ij} * Minority_{ij} + v_{0j} + \epsilon_{ij}$$

The average language score for non-minority males is equal to  $\gamma_{00}$ . Female non-minorities scored  $\gamma_{10}$  points higher/lower on the language test compared to male non-minorities. Male minorities scored  $\gamma_{20}$  higher/lower on the language test compared to male non-minorities. Finally, female minorities scored  $\gamma_{30}$  points higher/lower compared to male minorities.

**Covariance parameter estimates**

$\tau_{00}$  can be interpreted as between-group variance or the covariance within cluster  $j$ , e.g., how much do students in the same class vary, or what percentage of the total variance is attributable to the differences between classrooms?

**Confidence interval**

$$\beta_1 = -0.11, 95\% CI = [-0.22, -0.01]$$

The unknown population effect of mother's IQ on the number of children born to her ( $\beta_1$ ) is expected to lie within 95% of similarly constructed intervals as [-0.22, -0.01].

**Intraclass correlation**

Within a cluster, how correlated are the Level-1 units? Of the total variation of your outcome, what percentage is between-cluster variation?

**p-value and test-statistic**

$$\beta_1 = -0.11, t(8) = -2.48, p = .038.$$

There is a significant negative effect of mother's IQ on the number of children born to her. A 1-unit increase in mother's IQ is associated with a decrease of 1.1 children born to her.

Over repeated samples, the probability of observing a test statistic

of -2.48 or more extreme, given that the null hypothesis of no effect of mother's IQ on children born to her is true, is 3.8%.