γ_5 Schemes in d-Dimensions

I Regularization and Renormalization

I.i Universal Features of Dimensional Regularization

As we know, because many of the Feynman integrals appearing in loop calculations are divergent, it is important to regulate these poles; in modern calculations, the most common regularization scheme is dimensional regularization, where we perform

$$\int \frac{d^4 k_i}{(2\pi)^4} \to \mu^{2\epsilon} \int \frac{d^d k_i}{(2\pi)^d}. \tag{1.1}$$

In other words, we work in d-dimensions; divergences in the integrals will then be manifested as poles in $\epsilon = (4-d)/2$.

Because we work in d-dimensions, note that contracting two metrics yields

$$g^{\mu\nu} g_{\mu\nu} = d, \tag{1.2}$$

and so

$$\begin{aligned} \gamma^\mu \gamma_\mu &= \frac{1}{2} g^{\mu\nu} \{\gamma_\mu, \gamma_\nu\} \\ &= d; \end{aligned} \tag{1.3}$$

in general, γ matrices can be contracted according to

$$\gamma^\mu \gamma_{\nu_1} \dots \gamma_{\nu_n} \gamma_\mu = 2 \sum_{i=1}^{n} (-1)^{i+n} \gamma_{\nu_i} \gamma_{\nu_1} \dots \gamma_{\nu_{i-1}} \gamma_{\nu_{i+1}} \dots \gamma_{\nu_n} + (-1)^n d \gamma_{\nu_1} \gamma_{\nu_n}. \tag{1.4}$$

As we know, in four-dimensions, γ_5 is defined by

$$\gamma_5 = -i \gamma_0 \gamma_1 \gamma_2 \gamma_3; \tag{1.5}$$

however, there is no clear definition for this matrix in d-dimensions. Furthermore, it is not clear that the anticommutation relation

$$\{\gamma_\mu, \gamma_5\} = 0 \tag{1.6}$$

will translate directly into d-dimensions while traces of Dirac matrix chains maintain their cyclic property. There are two different schemes for handling γ_5 in dimensional regularization: in the 't Hooft-Veltman-Breitenlohner-Maison scheme, the anticommutation relation for γ_5 is modified while the cyclic property of the trace is maintained; on the other hand, in Kreimer's scheme, the anticommutation relation is maintained while the cyclic property of the trace is abandoned.

There are no d-dimensional analogues for either γ_5 or the Levi-Civita tensor, $\varepsilon_{\mu\nu\rho\sigma}$; we simply accept that the definition in Equation 1.5 along with the trace

$$\mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5] = -4i\varepsilon_{\mu\nu\rho\sigma} \tag{1.7}$$

and the additional properties

$$\gamma_5^2 = \mathbf{1}, \tag{1.8a}$$
$$\gamma_5^\dagger = \gamma_5 \tag{1.8b}$$

are continued to d-dimensions. However, since γ_5 and $\varepsilon_{\mu\nu\rho\sigma}$ are intrinsically four-dimensional objects, it is common to separate the four- and -2ϵ-dimensional subspaces of the Lorentz indices; we write this as

$$k^\mu = \bar{k}^\mu + \hat{k}^\mu, \tag{1.9a}$$
$$\gamma^\mu = \bar{\gamma}^\mu + \hat{\gamma}^\mu, \tag{1.9b}$$

where the bar denotes the four-dimensional components and the hat denotes the -2ϵ-components. This allows us to write down the usual four-dimensional identities for the contractions of Levi-Civita tensors, e.g.

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} = -24, \tag{1.10a}$$
$$\varepsilon^{\mu\nu\rho}{}_{\alpha}\varepsilon_{\mu\nu\rho\beta} = -6\bar{g}_{\alpha\beta}, \tag{1.10b}$$
$$\varepsilon^{\mu\nu}{}_{\alpha\delta}\varepsilon_{\mu\nu\beta\eta} = -2(\bar{g}_{\alpha\beta}\bar{g}_{\delta\eta} - \bar{g}_{\alpha\eta}\bar{g}_{\beta\delta}), \tag{1.10c}$$

where $\bar{g}_{\mu\nu}$ denotes the four-dimensional metric.

When contracting indices of d-dimensional four-vectors, we must explicitly separate out the parts as

$$\bar{g}_{\mu\nu}k_i^\mu k_j^\nu = k_i \cdot k_j - \hat{k}_i \cdot \hat{k}_j; \tag{1.11}$$

this separation can cause some difficulties in loop calculations, e.g. numerators of the form $\hat{k}_i \cdot \hat{k}_j$ can appear. This, however, is a solved problem at both one- and two-loops.

II The Naïve Approach

In this prescription, γ_5 satisfies the anticommutation relation

$$\{\gamma_5, \gamma^\mu\} = 0; \tag{2.1}$$

in addition to this, in order to study diagrams with odd numbers of γ_5's, we must include the additional criterion that

$$\mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4i\varepsilon^{\mu\nu\rho\sigma} + \mathcal{O}(d-4), \tag{2.2}$$

where, all indices take values $0, 1, 2, 3$, since the Levi-Civita tensor is inherently a 4-dimensional object.