MAE 5930 Optimization
Homework 2

Purpose: The problems assigned help develop your ability to • implement numerical algorithms for unconstrained optimization. • solve unconstrained problems using MATLAB's `pinv`, `quadprog`, and `fminunc`. • solve unconstrained problems using gradient descent and Newton's Method. • calculate derivatives of vector functions. • solve linear programming problems graphically and with MATLAB's `linprog`. NOTE: You are welcome to program in MATLAB or Python.

Problem 1: Program the backtracking algorithm as described by Boyd on page 464. (Note that Boyd considers gradient vectors to be column vectors instead of row vectors.)

(a) Inputs to the function are the objective function, gradient function evaluated at x, the point x, and the descent direction v.

(b) Within the function, set $\alpha = 0.20$ and $\beta = 0.50$.

Problem 2. Program the gradient descent method as described by Boyd on page 466. (Note that Boyd considers gradient vectors to be column vectors instead of row vectors.)

(a) Inputs to the function are the objective, gradient, and starting point.

(b) Within the function, set $\eta = $ 1e–6.

Problem 3. Program Newton's Method as described by Boyd on page 487. (Note that Boyd considers gradient vectors to be column vectors instead of row vectors.)

(a) Inputs to the function are the objective, gradient, Hessian, and starting point.

(b) Within the function, set $\epsilon = $ 1e–6.

Problem 4. Consider the least squares estimation problem wherein we are trying to find the $x \in \mathbb{R}^n$ that minimizes the error in the linear equation $Ax = b$. The matrix $A \in \mathbb{R}^{m \times n}$ and the vector $b \in \mathbb{R}^m$. Assume that $m > n$ and that $\text{rank}(A) = n$.

After defining the error as $e = Ax - b$, we can write the optimization problem as

$$\min_x f(x) = \tfrac{1}{2} e^T e = \tfrac{1}{2}(Ax - b)^T (Ax - b)$$

(a) Expand the objective function so that it has three terms. Identify the quadratic term, linear term, and constant term.

(b) Calculate the gradient vector with respect to x.

(c) Calculate the Hessian matrix with respect to x.

(d) Use the first-order necessary condition to find all candidates for a local minimum. Check if the second-order necessary condition is satisfied. Explain.

(e) Use the first and second-order sufficient condition to check if the candidate is indeed a local minimum. Explain.

Problem 5. In MATLAB, create a random A and b for the problem above.

```
>> A = rand(100,3);
>> b = rand(100,1);
```

(a) Solve the problem using your analytical solution from Problem 4.

(b) Solve the problem using MATLAB's `pinv` command.

(c) Solve the problem using MATLAB's `quadprog` command.

(d) Solve the problem using MATLAB's `fminunc` command.

(e) Solve the problem using your gradient descent program from Problem 2.

(f) Solve the problem using your Newton's method program from Problem 3.

(g) Try different initial guesses and document your observations/thoughts.

Problem 6. Consider the following function $f(x, y) = (\alpha - x)^2 + \beta(y - x^2)^2$ with $\alpha = 1$ and $\beta = 100$.

(a) Calculate the gradient vector.

(b) Calculate the Hessian matrix.

(c) Use the first-order necessary condition to find all candidates for a local minimum.

(d) Compute the eigenvalues of the Hessian matrix evaluated at the candidate points using MATLAB's `eig` command. Is the matrix positive semidefinite or positive definite or something else?

(e) Use the first and second-order sufficient condition to check if the candidate is indeed a local minimum. Explain.

Problem 7. For the optimization problem described in Problem 6:

(a) Solve the problem using MATLAB's `fminunc` command.

(b) Solve the problem using your gradient descent program from Problem 2.

(c) Solve the problem using your Newton's method program from Problem 3.

(d) Try the initial guesses $[1 \ \ -1]^T$ and $[100 \ \ -1]^T$. Document your observations.

Problem 8. Consider the linear programming (LP) problem with objective

$$\begin{aligned} \text{minimize} \quad & x_1 - x_2 \\ \text{subj. to} \quad & x_1 + x_2 \leq 1 \\ & -x_1 + 2x_2 \leq 2 \\ & x_1 \geq -1 \\ & -x_1 + 3x_2 \geq -3 \end{aligned}$$

(a) Plot the level curves (lines) of the objective on an x_1-x_2 graph. Identify the direction of decreasing objective value.

(b) Identify the feasible region by plotting the constraints on the same graph.

(c) Graphically identify the minimum point using your graph.

(d) Convert the problem to standard form and solving using MATLAB's `linprog`.