MAE 5930 - Optimization

Fall 2019

Homework 3

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Due: 7:00 AM, Thursday October 3, 2019

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Purpose: the problems assigned help develop your ability to

- formulate linear programs (LPs).
- solve LPs analytically using the necessary and sufficient optimality conditions.
- solve LPs numerically using MATLAB's `linprog`,

NOTE: you are welcome to program in MATLAB or Python.

1. You are the project manager for a house building company. Building a house requires the following steps (and completion time):

   - B: building the foundation (takes 3 weeks)
   - F: framing (2 weeks)
   - E: electrical (3 weeks)
   - P: plumbing (4 weeks)
   - D: dry wall (1 week)
   - L: landscaping (2 weeks)

   Some tasks can start only after another task is complete. The following sequencing rules must be followed: F after B, L after B, E after F, P after F, D after E, D after P. You are allowed to do tasks in parallel. Your job as project manager is to build the schedule to complete the project as quickly as possible.

(a) Formulate this as a linear programming problem by defining the variables, objective, and constraints.

Problem 1

1a. Formulate this as a linear programming problem by defining the variables, objective, and constraints.

- **Defining variables:**

$$\underset{\sim}{x} = \begin{bmatrix} x_B \\ x_F \\ x_E \\ x_P \\ x_D \\ x_L \\ y \end{bmatrix} \longrightarrow$$

let x_B be the time at which we begin B (building foundation)
let x_F " " " F (framing)
let x_E " " " E (electrical)
let x_P " " " P (plumbing)
let x_D " " " D (dry wall)
let x_L " " " L (landscaping)
let y be the time at which we have completed all tasks

Therefore, our set of decision variables is $\{x_B, x_F, x_E, x_P, x_D, x_L, y\}$.

- **Objective:** our objective is to complete the house in the least time possible.

Expressed mathematically: $\left[\min\limits_{(\underset{\sim}{x})} \; y\right]$ subject to the constraints below

$$\left(\min_{\underset{\sim}{x}} \; [0,0,0,0,0,0,1][\underset{\sim}{x}]\right) \longrightarrow$$

- **Constraints:** we have three sets of constraints: set (i) is that the final completion is no sooner than the completion of each task, set (ii) is that we must sequence tasks according to the rules provided, set (iii) is that all of our task completion times must be positive.

In mathematical notation:

SET i

$y \geq x_B + 3$, $y \geq x_P + 4$,
$y \geq x_F + 2$, $y \geq x_D + 1$,
$y \geq x_E + 3$, $y \geq x_L + 2$

SET ii

$x_F \geq x_B + 3$, $x_D \geq x_E + 3$
$x_L \geq x_B + 3$, $x_D \geq x_P + 4$
$x_E \geq x_F + 2$
$x_P \geq x_F + 2$

SET iii

$x_B \geq 0$, $x_P \geq 0$
$x_F \geq 0$, $x_D \geq 0$
$x_E \geq 0$, $x_L \geq 0$

1a. (continued)

- To put our LP into standard form let's rewrite our inequalities from set i and set ii to generate the $Ax \le b$ constraint.

But first let's write the objective, as well as the bounds on x:

* Objective: $\left[\min_x \; y\right] \underset{\approx}{\text{equivalent to}} \left[\min_x \; c^T x\right]$ where $c^T = [0,0,0,0,0,0,1]$

We will define the vector $x = [x_B, x_F, x_E, x_P, x_D, x_L, y]^T$

* Bounds on x: this is the constraint of form $[l \le x \le u]$ where l and u are vectors bounding the values of x. The "set iii" constraints are used to specify l, and we'll say that $y \ge 0$ as well. Also, also all values can range to ∞, giving: $[0,0,0,0,0,0,0]^T \le [x_B, x_F, x_E, x_P, x_D, x_L, y]^T \le [\infty,\infty,\infty,\infty,\infty,\infty,\infty]^T$

* $Ax \le b$: we determine this system from set i and set ii constraints. We have to rewrite all those constraints in the form:

$(a)(x_B) + (b)(x_F) + (c)(x_E) + (d)(x_P) + (e)(x_D) + (f)(x_L) + (g)(y) \le K$

where $\{a, b, \dots, g\}$ are integer coefficients, and K is the constant term.

For example: $[x_P + 4 \le x_D]$ becomes $[(0)(x_B) + (0)(x_F) + (0)(x_E) + (1)(x_P) + (-1)(x_D) + 0(x_L) + 0(y) \le -4]$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_B \\ x_F \\ x_E \\ x_P \\ x_D \\ x_L \\ y \end{bmatrix} \le \begin{bmatrix} -3 \\ -2 \\ -3 \\ -4 \\ -1 \\ -2 \\ -3 \\ -3 \\ -2 \\ -2 \\ -3 \\ -4 \end{bmatrix}$$

$\rightarrow 1x_B + 0x_F + 0x_E + 0x_P + 0x_D + 0x_L - 1y \le -3$

$\vdots$

$\rightarrow 0x_B + 0x_F + 0x_E + 1x_P - 1x_D + 0x_L + 0y \le -4$

(b) Solve the linear program using MATLAB's `linprog`.

```
%======================================================================
%======================================================================
%== PROBLEM 1
%======================================================================
%======================================================================

% Coefficients for objective function where x = [xb,xf,xe,xp,xd,xl,y] is
% our vector of decision variables. (We only care about the last
% element of the vector, y=completion time, so we set all other
% coefficients to 0).
f = [0, 0, 0, 0, 0, 0, 1];
% The matrix for inequality constraints
A(1,:)  = [1, 0, 0, 0, 0, 0, -1];
A(2,:)  = [0, 1, 0, 0, 0, 0, -1];
A(3,:)  = [0, 0, 1, 0, 0, 0, -1];
A(4,:)  = [0, 0, 0, 1, 0, 0, -1];
A(5,:)  = [0, 0, 0, 0, 1, 0, -1];
A(6,:)  = [0, 0, 0, 0, 0, 1, -1];
A(7,:)  = [1,-1, 0, 0, 0, 0,  0];
A(8,:)  = [1, 0, 0, 0, 0,-1,  0];
A(9,:)  = [0, 1,-1, 0, 0, 0,  0];
A(10,:) = [0, 1, 0,-1, 0, 0,  0];
A(11,:) = [0 ,0, 1, 0,-1, 0,  0];
A(12,:) = [0, 0, 0, 1,-1, 0,  0];
% The column vector for inequality constraints
b = [-3; -2; -3; -4; -1; -2; -3; -3; -2; -2; -3; -4];
% The vector bounding x on the low side
lb = [0; 0; 0; 0; 0; 0; 0];
% The vector bounding x on the high side
ub = [inf; inf; inf; inf; inf; inf; inf];
% Solve the problem using linprog
linprog_soln = linprog(f, A, b, [], [], lb, ub);
disp("Problem 1 solution (construction):")
linprog_soln
```

Figure 1: Here is my code for solving the construction sequencing problem using MATLAB's `linprog` command.

```
Problem 1 solution (construction):

linprog_soln =

     0
     3
     5
     5
     9
     3
    10
```

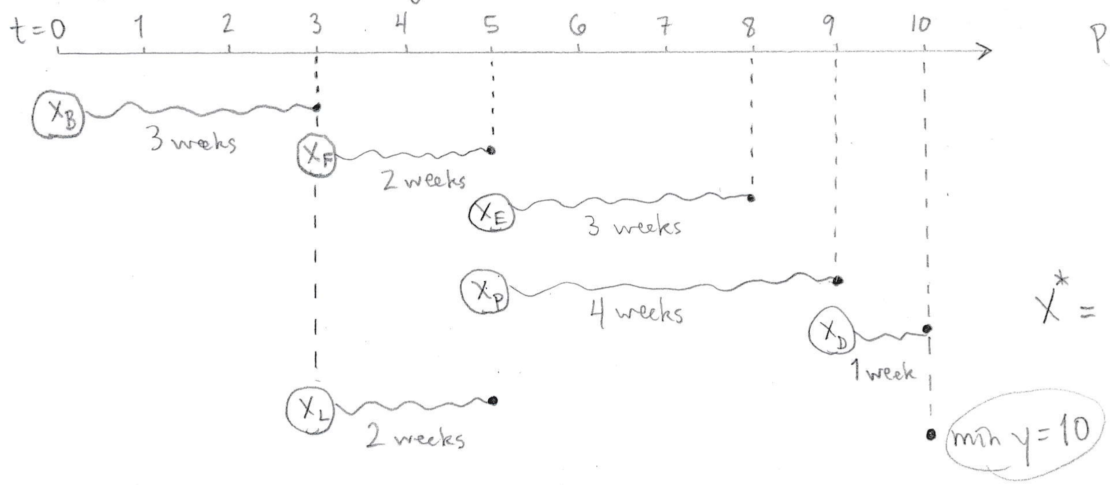
Figure 2: Here is the output of my code, and the solution to the problem.

SOLUTION: $\left(\begin{bmatrix} x_B{}^* & x_F{}^* & x_E{}^* & x_P{}^* & x_D{}^* & x_L{}^* & y^* \end{bmatrix}^T = \begin{bmatrix} 0 & 3 & 5 & 5 & 9 & 3 & 10 \end{bmatrix}^T \right)$

(c) Comment on any observations you have.

1c. Comment on any observations that you have.

- First, Let's create a graphical representation of the LP's solution:



Per MATLAB:

$$X^* = \begin{bmatrix} X_B^* \\ X_F^* \\ X_E^* \\ X_P^* \\ X_D^* \\ X_L^* \\ y^* \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 5 \\ 5 \\ 9 \\ 3 \\ 10 \end{bmatrix}$$

- Based on the MATLAB output and our graphical representation we can say:
  - The sequencing constraint that task x_L (landscaping) take place after the completion of task x_B (building foundation) has no effect on total time y.
  - The same can be said for requiring x_E to commence after completing x_F.
  - In words, the tasks, and sequencing that constrains (and minimizes) total time y is:
    - First, start x_B (foundation), finish x_B. Only after that... (3 weeks total)
    - Next, start x_F (framing), finish x_F. Only after that... (5 weeks total)
    - Next, start x_P (plumbing), finish x_P. Only after that... (9 weeks total)
    - Finally, start x_D (dry wall), finish x_D. Now we are at 10 weeks total after starting x_B (foundation), and we have completed all six tasks in the minimum amount of time $y = 10$.
  - A final note: we still get $y=10$ and obey constraints if $[3 \le x_L \le 8]$ and if $[5 \le x_E \le 6]$.

2. You are a facilities engineer at a manufacturing plant. A by-product of the manufacturing process is a hazardous chemical. Only 1,000 liters (L) of this chemical are allowed on site at any given time, and none can be store overnight.
Fortunately your facility is connected by pipeline to a reprocessing plant, which can take the by-product at a cost. The table below shows your producting and the cost of piping to the reprocessing plant on an hourly basis.

	9-10 AM	10-11 AM	11-12 PM	12-1 PM	1-2 PM	2-3 PM
Production (L)	300	240	600	200	300	900
Cost ($/L)	30	40	35	45	38	50

The work day is from 9 AM to 3 PM. As facility engineer, your job is to determine how much chemical by-product should be sent each hour to the reprocessing plant to minimize cost and meet environmental regulations.

(a) Formulate this as a linear programming problem by defining the variables, objective, and constraints.

$\boxed{2a.}$ Formulate the problem as a LP problem by defining variables, objective, and constraints.

- **Defining variables:**

let r_{10} be the amount sent for reprocessing at 10:00 AM

let r_{11} be " " " 11:00 AM

let r_{12} be " " " 12:00 PM

Let r_1 be " " " 1:00 PM

let r_2 be " " " 2:00 PM

Let r_3 be " " " 3:00 PM

let s_{10} be (the amount of chemical on-site prior to sending it out for reprocessing at) 10:00 AM

let s_{11} be (" " ") 11:00 AM

let s_{12} be (" " ") 12:00 PM

let s_1 be (" " ") 1:00 PM

let s_2 be (" " ") 2:00 PM

Let s_3 be (" " ") 3:00 PM

- **Define objective:** minimize the cost of piping out chemical (while meeting environmental regulations (constraints)).

$$\min_{\underset{\sim}{x}} \; c^T \underset{\sim}{x} = \begin{bmatrix} 30, 40, 35, 45, 38, 50 \end{bmatrix} \begin{bmatrix} r_{10} \\ r_{11} \\ r_{12} \\ r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

← find $\underset{\sim}{x}$ that minimizes the total cost of piping out chemical.

2a. (continued)

- Constraints:

(i.) All of the variables are non-negative: $\left[r_i, S_i \geq 0 \ \forall i \in \{10,11,12,1,2,3\}\right]$

(ii.) Only 1000 L on site at any given time: $\left[S_i \leq 1000 \ \forall i \in \{10,11,12,1,2,3\}\right]$

(iii.) The volume of chemical that can be reprocessed at a given time cannot be more than what is stored: $\left[r_i \leq S_i \ \forall i \in \{10,11,12,1,2,3\}\right]$

(iv.) Equality constraints arise from excess chemical left over being added to new production (contributes to the storage of the following hour):

$$S_{10} = 300$$

$$S_{11} = \left[(S_{10} - r_{10}) + 240\right] = \left[(300 - r_{10}) + 240\right] = \left[(300 + 240) - (r_{10})\right]$$

$$S_{12} = (S_{11} - r_{11}) + 600 = \left[(300 + 240 + 600) - (r_{10} + r_{11})\right]$$

$$S_{1} = (S_{12} - r_{12}) + 200 = \left[(300 + 240 + 600 + 200) - (r_{10} + r_{11} - r_{12})\right]$$

$$S_{2} = (S_{1} - r_{1}) + 300 = \left[(300 + 240 + 600 + 200 + 300) - (r_{10} + r_{11} + r_{12} + r_{1})\right]$$

$$S_{3} = (S_{2} - r_{2}) + 900 = \left[(300 + 240 + 600 + 200 + 300 + 900) - (r_{10} + r_{11} + r_{12} + r_{1} + r_{2})\right]$$

$$S_3 - r_3 = 0$$

all chemical must be piped out by 3:00 PM since the facility closes at 3:00 PM and none can be stored overnight.

$$= \left[(300 + 240 + 600 + 200 + 300 + 900) - (r_{10} + r_{11} + r_{12} + r_{1} + r_{2} + r_{3})\right] = 0$$

(v.) It must be that all the chemicals produced during the day are shipped out by end-of-day. Thus we have $\left[\sum r_i = \sum (\text{production 9AM-3PM})\right]$

$$= \left[r_{10} + r_{11} + r_{12} + r_{1} + r_{2} + r_{3}\right] = \left[300 + 240 + 600 + 200 + 300 + 900 = 2540 \text{ L}\right]$$

2a. (continued)

- To put our LP into standard form let's rewrite our inequalities to specify: $Ax \le b$, $\bar{A}x \le \bar{b}$, and $l \le x \le u$

* **Bounds on x:** using constraints (i.) and (iii.) from the previous page,

$$[l \le x \le u] \rightarrow [0,0,0,0,0,0]^T \le [r_{10}, r_{11}, r_{12}, r_1, r_2, r_3] \le [s_{10}, s_{11}, s_{12}, s_1, s_2, s_3]$$

* **$Ax \le b$:** these constraints come from (ii.) and (iii.) on the previous page.

(ii). constraints:

$$[s_{10} \le 1000] = [300 \le 1000]$$

$$[s_{11} \le 1000] = [(300+240) - (r_{10}) \le 1000]$$

$$[s_{12} \le 1000] = [(300+240+600) - (r_{10}+r_{11}) \le 1000]$$

$$\vdots$$

$$\leadsto \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} r_{10} \\ r_{11} \\ r_{12} \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} \le \begin{bmatrix} 700 \\ 460 \\ -140 \\ -340 \\ -640 \\ -1540 \end{bmatrix}$$

(iii.) constraints:

$$[r_{10} \le s_{10}] = [r_{10} \le 300]$$

$$[r_{11} \le s_{11}] = [r_{11} \le (300+240) - r_{10}]$$

$$= [r_{10} + r_{11} \le (300+240)]$$

$$[r_{12} \le s_{12}] = \left[r_{12} \le \begin{pmatrix} 300+240 \\ +600 \end{pmatrix} - (r_{10}+r_{11})\right]$$

$$= [r_{10}+r_{11}+r_{12} \le (300+240+600)]$$

$$\vdots$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{bmatrix} \le \begin{bmatrix} 300 \\ 540 \\ 1140 \\ 1340 \\ 1640 \\ 2540 \end{bmatrix}$$

PUTTING TOGETHER (ii.) and (iii) WE HAVE OUR $Ax \le b$ CONSTRAINT:

$$[Ax \le b] \text{ is } \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} r_{10} \\ r_{11} \\ r_{12} \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} \le \begin{bmatrix} 700 \\ 460 \\ -140 \\ -340 \\ -640 \\ -1540 \\ 300 \\ 540 \\ 1140 \\ 1340 \\ 1640 \\ 2540 \end{bmatrix}$$

$\boxed{\text{2a. (continued)}}$

- Still working to put our LP into standard form.

* $\boxed{\bar{A}x = \bar{b}:}$ First, let's note something interesting: the so-called "eq. constraints" in group (iv.) two pages up aren't really equality constraints. Rather, they provide definitions of r_{10}, r_{11}, r_{12}, r_1, r_2, and r_3 that are recursive in nature.

The only equality constraints that we have on our vector $\underset{\sim}{x} = [r_{10}, r_{11}, r_{12}, r_1, r_2, r_3]^T$ is the (v.) constraint two pages up. Namely, $\|\underset{\sim}{x}\| = \sum_i r_i = (r_{10} + r_{11} + r_{12} + r_1 + r_2 + r_3) =$ the total production during the day $= (300 + 240 + 600 + 200 + 300 + 900 = 2540\ L)$.

Mathematically, $[1, 1, 1, 1, 1, 1]\begin{bmatrix} r_{10} \\ r_{11} \\ r_{12} \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = 2540$

- See MATLAB code & output below.

(b) Solve the linear program using MATLAB's `linprog`.

```
clear all; close all; clc;
%===========================================================================
%== PROBLEM 2
%===========================================================================
%===========================================================================

% Coefficients for the objective function (piping costs during each
% hour of the facility's operation)
f = [30, 40, 35, 45, 38, 50];
% The matrix for inequality constraints
A(1,:)  = [ 0,  0,  0,  0,  0, 0];
A(2,:)  = [-1,  0,  0,  0,  0, 0];
A(3,:)  = [-1, -1,  0,  0,  0, 0];
A(4,:)  = [-1, -1, -1,  0,  0, 0];
A(5,:)  = [-1, -1, -1, -1,  0, 0];
A(6,:)  = [-1, -1, -1, -1, -1, 0];
A(7,:)  = [ 1,  0,  0,  0,  0, 0];
A(8,:)  = [ 1,  1,  0,  0,  0, 0];
A(9,:)  = [ 1,  1,  1,  0,  0, 0];
A(10,:) = [ 1,  1,  1,  1,  0, 0];
A(11,:) = [ 1,  1,  1,  1,  1, 0];
A(12,:) = [ 1,  1,  1,  1,  1, 1];
% The column vector for inequality constraints
b = [700; 460; -140; -340; -640; -1540; 300; 540; 1140; 1340; 1640; 2540];
% The matrix for equality constraint
Aeq(1,:) = [1, 1, 1, 1, 1, 1];
% The column vector for the equality constraint
beq = [2540];
% The vector bounding x on the low side
lb = [0; 0; 0; 0; 0; 0];
% The vector bounding x on the high side
ub = [300; 1000; 1000; 1000; 1000; 1000];
% Solve the problem using linprog
linprog_soln_2 = linprog(f, A, b, Aeq, beq, lb, ub);
disp("Problem 2 solution (chemicals):")
linprog_soln_2
```

Figure 3: Here is my code for solving the chemical piping problem using MATLAB's `linprog` command.

```
Optimal solution found.

Problem 2 solution (chemicals):

linprog_soln_2 =

   300
     0
   840
     0
   500
   900
```

Figure 4: Here is the output of my code, and the solution to the problem.

SOLUTION: $\left(\begin{bmatrix} r_{10}^* & r_{11}^* & r_{12}^* & r_1^* & r_2^* & r_3^* \end{bmatrix}^T = \begin{bmatrix} 300 & 0 & 840 & 0 & 500 & 900 \end{bmatrix}^T \right)$

(c) Comment on any observations you have.

2c. Comment on any observations you have.

- First, let's create a graphical representation of the LP's solution:

9AM 10 11 12 1 2 3:00PM

Prod: 300L	Prod: 240 L	Prod: 600 L	Prod: 200L	Prod: 300L	Prod: 900L
r_{10} : 300L	r_{11} : 0 L	r_{12} : 840L	r_1 : 0L	r_2 : 500L	r_3 : 900L
S_{10} : 300L	S_{11} : 240L	S_{12} : 840L	S_1 : 200L	S_2 : 500L	S_3 : 900L

- From this, we observe things that make us confident in our answer:

i- All r_i, s_i values are ≥ 0

ii- We never exceed (or even) meet the storage limit of 1000 L ($s_i \leq 1000$).

iii- We never process more chemical than is stored ($r_i \leq s_i$)

iv- Our s_i and r_i values follow the definitions in (iv.) a few pages up.

v- We ship out all chemical produced, e.g. $\|\underset{\sim}{x}\| = (r_{10} + r_{11} + \dots + r_3) = 2540\,L$

$= (300 + 0 + 840 + 0 + 500 + 900) = 2540\,L$

vi- At the end of the day, 3:00 PM, we have $[s_3 - r_3] = [900 - 900] = [0\,L]$, leaving no chemical & satisfying the regulation of no overnight storage.

- Why our answer intuitively makes sense: to minimize piping costs we'd want to pipe as much volume as possible during the cheapest hours (between 9-10AM, 11-12PM, and 1-2PM). We can observe that this happens. However, we get to the last hour of the day, 2-3 PM, and have no choice but to pipe out all 900 L produced since we can't store any chemical overnight.

3. Consider the linear programming (LP) problem with objective:

<u>minimize</u>:
$f = x_1 - x_2$

<u>subject to</u>:

$$
\begin{aligned}
x_1 + x_2 &\leq 1 \\
-x_1 + 2x_2 &\leq 2 \\
x_1 &\geq -1 \\
-x_1 + 3x_2 &\geq -3
\end{aligned}
$$

You solved this problem graphically in Homework 2.

(a) Write the linear program in inequality form by defining c, A, and b.

Problem 3 Consider the linear programming (LP) problem with objective:

$$\underset{x}{\text{minimize}}\ f = x_1 - x_2, \quad \text{subject to:} \quad \begin{array}{ll} (i.) & x_1 + x_2 \leq 1 \\ (ii.) & -x_1 + 2x_2 \leq 2 \\ (iii.) & x_1 \geq -1 \\ (iv.) & -x_1 + 3x_2 \geq -3 \end{array}$$

(Solved this problem graphically in HW2.)

3a. Write the LP in inequality form by defining c, A, and b.

- First, let's recall that inequality form for an LP is: $\min_x f = c^T x$, s.t. $Ax \leq b$
- From HW2, we had the general form for this LP as:

$$\left\{ [Ax \leq b] \rightarrow \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \text{ and } [l \leq x \leq u] \rightarrow \begin{bmatrix} -1 \\ -\infty \end{bmatrix} \leq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} \infty \\ \infty \end{bmatrix} \right\}$$

$$\left(\underset{x}{\text{minimize}}\ c^T x = \min_x f = [1, -1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \text{ subject to this set of constraints.}$$

- For the remainder of this problem, let $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & -3 \end{bmatrix}$ from HW2 be A_{orig}, and let $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ from HW2 be b_{orig}.

- Let's rewrite $(l \leq x \leq u)$ from HW2 as $(-x \leq -l)$ and $(x \leq u)$:

$$\begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} \leq \begin{bmatrix} 1 \\ \infty \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} \infty \\ \infty \end{bmatrix}$$

- Next, we'll combine these constraints into one system $(A_{new})(x) \leq (b_{new})$ as:

$$\underbrace{\begin{bmatrix} A_{orig} \\ -I \\ I \end{bmatrix}}_{(A_{new})} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{(x)} \leq \underbrace{\begin{bmatrix} b_{orig} \\ -l \\ u \end{bmatrix}}_{(b_{new})}$$

since this still gives us: ① $(A_{orig})(x) \leq (b_{orig})$

② $(-x \leq -l) = (l \leq x)$

③ $(x \leq u)$

Which are the same constraints that we have in standard LP form.

3a (continued)

- Combining these constraints into our new constraints system gives:

$$
\begin{array}{r}
A_{orig} = \\ \\ \\ -I = \\ \\ I = \\ \\
\end{array}
\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & -3 \\ \hline -1 & 0 \\ 0 & -1 \\ \hline 1 & 0 \\ 0 & 1 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\leq
\begin{bmatrix} 1 \\ 2 \\ 3 \\ \hline 1 \\ \infty \\ \hline \infty \\ \vdots \\ \infty \end{bmatrix}
\begin{array}{l}
b_{orig} \\ \\ \\ -\ell \\ \\ u \\ \\
\end{array}
$$

Which is of the form

$$A_{new}\, x \leq b_{new}$$

- Writing everything together, our final inequality formulation of this LP is:

$\min_x f = c^T x$, subject to $Ax \leq b$ where:

$$
\boxed{c = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & -3 \\ -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ +1 \\ +\infty \\ \infty \\ \infty \end{bmatrix}}
$$

final answer

(b) Use the optimality conditions to find the minimum point. (See **Theorem 1** below in the description for problem 4.)

(c) Is this point a local minimum, global minimum, both, or neither?

$\boxed{3b.}$ Use Optimality Conditions to find the minimum point.

- Optimality conditions for LP in inequality form: the LP in inequality form has a minimum at x iff the following system is solvable:

  (1.) $Ax \leq b$, (2.) $\lambda \geq 0$, (3.) $\lambda^T(Ax-b) = 0$, (4.) $A^T\lambda + c = 0$.

- Initially I just tried to solve (1.) and (4.) but couldn't. I've forgotten the linear algebra terminology, but both (1.) and (4.) are "too ambiguous" to nail down a specific $\underset{\sim}{\lambda}$ or $\underset{\sim}{x}$.

- After confirming this with Dr. Harris, he said it's okay to work backwards, e.g. using our graphically-deduced $x^* = \begin{bmatrix} x_1^* = -1.0 \\ x_2^* = 0.5 \end{bmatrix}$ and then finding λ such that the system is solvable.

- Let's examine the condition $[\lambda^T(Ax-b) = 0]$:

$$\left([\lambda_1, \lambda_2, \ldots, \lambda_7] \left(\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & -3 \\ -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0.5 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ \infty \\ \infty \\ \infty \end{bmatrix} \right) = 0 \right) \rightarrow \left([\lambda_1, \lambda_2, \ldots, \lambda_7] \begin{bmatrix} -3/2 \\ 0 \\ -11/2 \\ 0 \\ -\infty \\ -\infty \\ -\infty \end{bmatrix} = 0 \right)$$

$$\rightarrow \left[(-3/2)(\lambda_1) + (0)(\lambda_2) + (-11/2)(\lambda_3) + (0)(\lambda_4) + (-\infty)(\lambda_5) + (-\infty)(\lambda_6) + (-\infty)(\lambda_7) = 0 \right]$$

* Assume that $(-\infty)(0) = 0$: this equation tells us $\lambda_5 = \lambda_6 = \lambda_7 = 0$ since $\lambda_i \geq 0$ and the only way to make $(\lambda_i)(-\infty) \geq 0$ is to let $\lambda_i = 0$.

* This equation simplifies to (when letting $\lambda_5 = \lambda_6 = \lambda_7 = 0$): $\left[-\frac{3}{2}\lambda_1 - \frac{11}{2}\lambda_3 = 0 \right]$

$\rightarrow [-3\lambda_1 = 11\lambda_3] \rightarrow [\lambda_1 = -\frac{11}{3}\lambda_3] \Rightarrow [\lambda_3 = \lambda_1 = 0$ to satisfy $\lambda_i \geq 0$ & this equality.$]$

* This equation doesn't tell us what λ_2 or λ_4 are. Let's use the condition $[A^T\lambda + c = 0]$ along with knowing $[\lambda_1 = \lambda_3 = \lambda_5 = \lambda_6 = \lambda_7 = 0]$ to find λ_2, λ_4.

3b. (continued)

- Examining the condition $[A^T\lambda + c = 0]$ while knowing $[\lambda_1 = \lambda_3 = \lambda_5 = \lambda_6 = \lambda_7 = 0]$:

$$\left(\begin{bmatrix} 1, & -1, & 1, & -1, & 0, & 1, & 0 \\ 1, & 2, & -3, & 0, & -1, & 0, & 1 \end{bmatrix}\begin{bmatrix} 0 \\ \lambda_2 \\ 0 \\ \lambda_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \longrightarrow \begin{bmatrix} -\lambda_2 - \lambda_4 = -1 \\ 2\lambda_2 + 0\lambda_4 = 1 \end{bmatrix} \Rightarrow \lambda_2 = 1/2$$

$$(-\lambda_2 - \lambda_4 = -1) \longrightarrow (\lambda_4 = 1 - \lambda_2) = 1/2$$

$$\Rightarrow \lambda_4 = 1/2$$

- We have found a minimizer x^* and $\underset{\sim}{\lambda}$ such that the LP optimality conditions system is solveable:

$$\boxed{x^* = \begin{bmatrix} x_1^* = -1.0 \\ x_2^* = 0.5 \end{bmatrix}, \quad \underset{\sim}{\lambda} = \left[0, \frac{1}{2}, 0, \frac{1}{2}, 0, 0, 0\right]^T}$$

3c. Is this point a local min, global min, both, or neither?

- **Answer:** the point $x^* = \begin{bmatrix} x_1^* = -1.0 \\ x_2^* = 0.5 \end{bmatrix}$ is both a local & global minimizer.

  This yields that $f(x^*) = \min f = -1.5$.

- Reasoning: from the optimality conditions, since they are necessary & sufficient (iff), we know $x^* = [-1, .5]^T$ is a global minimizer since we found $\underset{\sim}{\lambda}$ such that the system of optimality conditions is satisfied.

  Also, since $\{\text{local minimizers}\} \subseteq \{\text{global minimizers}\} \Longrightarrow$ all global minimizers are also local minimizers $\Longrightarrow$ [global minimizer $x^* = [-1, .5]^T$] is also a local minimizer.

4. Consider the following objective function:
<u>minimize</u>:
$f = x_1 + x_2$

For each of the constraints below, solve the problem:

(i) graphically by plotting the lines of constant cost and the constraints

(ii) by using the optimality conditions. To do this part, write each problem in inequality form by defining c, A, and b and apply the optimality conditions.

Your graphical and analytical answers should agree.

Theorem 1 (Optimality Conditions for LP in Inequality Form).

The linear program in inequality form has a minimum at x if and only if the following system is solvable:

- $\left[Ax \leq b\right]$
- $\left[\lambda \geq 0\right]$
- $\left[\lambda^T(Ax - b) = 0\right]$
- $\left[A^T\lambda + c = 0\right]$

(a) $\left[x_1 \geq 0 \text{ and } x_2 \geq 0\right]$

Solving the problem:

(i) graphically by plotting the lines of constant cost and the constraints

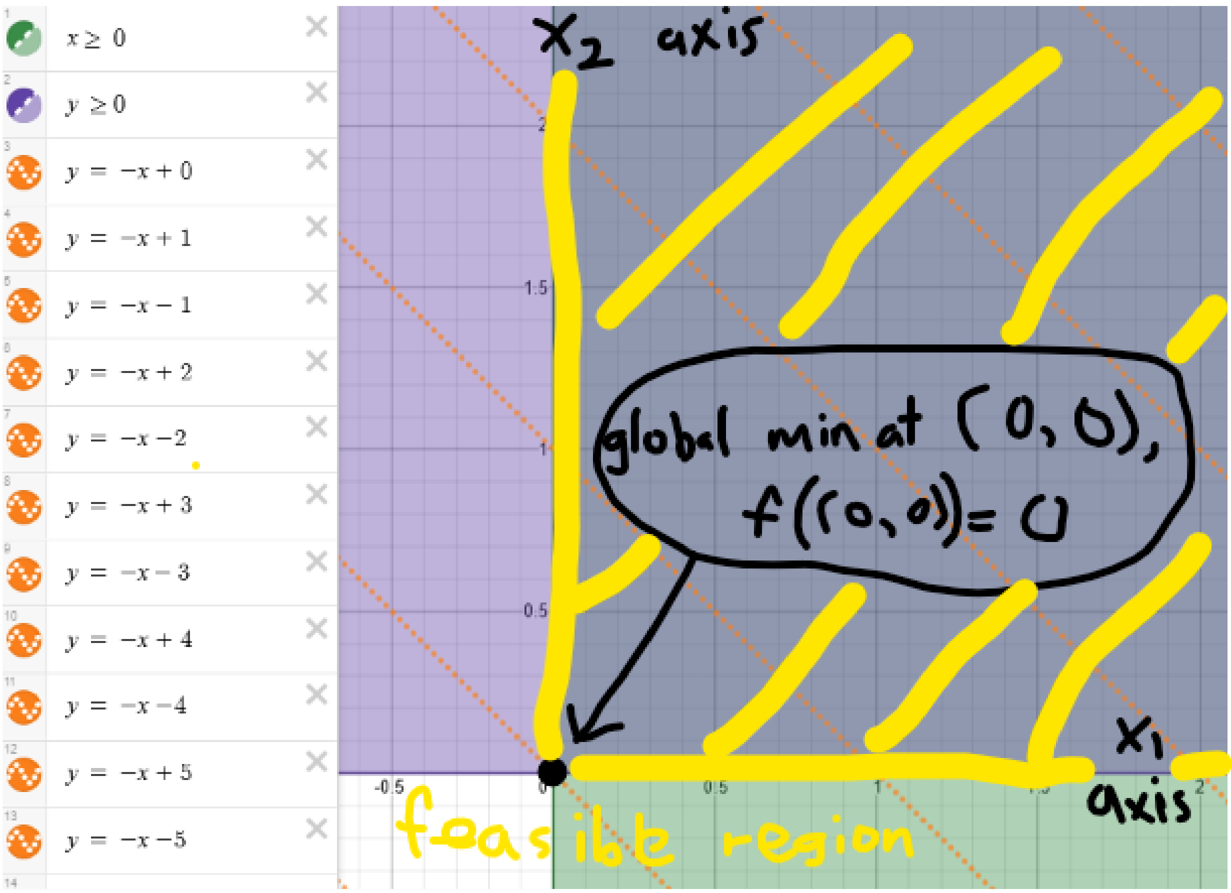


Figure 5: Here is the graphical solution for 4a.

- We can see that the point at which the "least" level curve of the objective function (e.g. value of f for that level set, here $f = 0$) intersects the feasible region is at $x^* = \begin{bmatrix} {x_1}^* \\ {x_2}^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
- Now let's verify this analytically.

(ii) by using the optimality conditions. To do this part, write each problem in inequality form by defining c, A, and b and apply the optimality conditions.

$\boxed{\text{4a. (analytical)}}$

- From our graphical solution we assume $x^* = \begin{bmatrix} x_1^* = 0 \\ x_2^* = 0 \end{bmatrix}$.

- For LP inequality form we have: $\min_x c^T x$ s.t. $Ax \leq b$.

- Here, $f = x_1 + x_2 = [1, 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow c^T = [1, 1]$

- Let's write our constraints as $\leq$ and put them in matrix form:

$[x_1 \geq 0] \rightarrow [-x_1 \leq 0]$ and $[x_2 \geq 0] \rightarrow [-x_2 \leq 0]$. Together this gives

$(Ax \leq b) \rightarrow \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- Now let's find λ such that the Optimality Conditions are all satisfied.

Using condition $[A^T\lambda + c = 0] \rightarrow \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -\lambda_1 \\ -\lambda_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$\Rightarrow \lambda_1 = \lambda_2 = 1$

- Now let's see if the other conditions hold:

$(Ax^* \overset{?}{\leq} b) \rightarrow \left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} \overset{?}{\leq} \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \rightarrow \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark\right)$

$(\lambda \overset{?}{\geq} \underset{\sim}{0}) \rightarrow \left(\lambda = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark\right)$

$(\lambda^T(Ax^* - b) \overset{?}{=} 0) \rightarrow \left([1, 1]\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \rightarrow \left([1, 1]\begin{bmatrix} 0 \\ 0 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \rightarrow \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark\right)$

- Since all optimality conditions (necessary & sufficient) are satisfied for $\lambda = [1, 1]^T$ and $x^* = [0, 0]^T \Rightarrow x^* = [0, 0]^T$ is a global minimizer of f, $f(x^*) = 0$. This agrees with our graphical solution.

(b) $\left[x_1 \geq 0\right]$

Solving the problem:

(i) graphically by plotting the lines of constant cost and the constraints

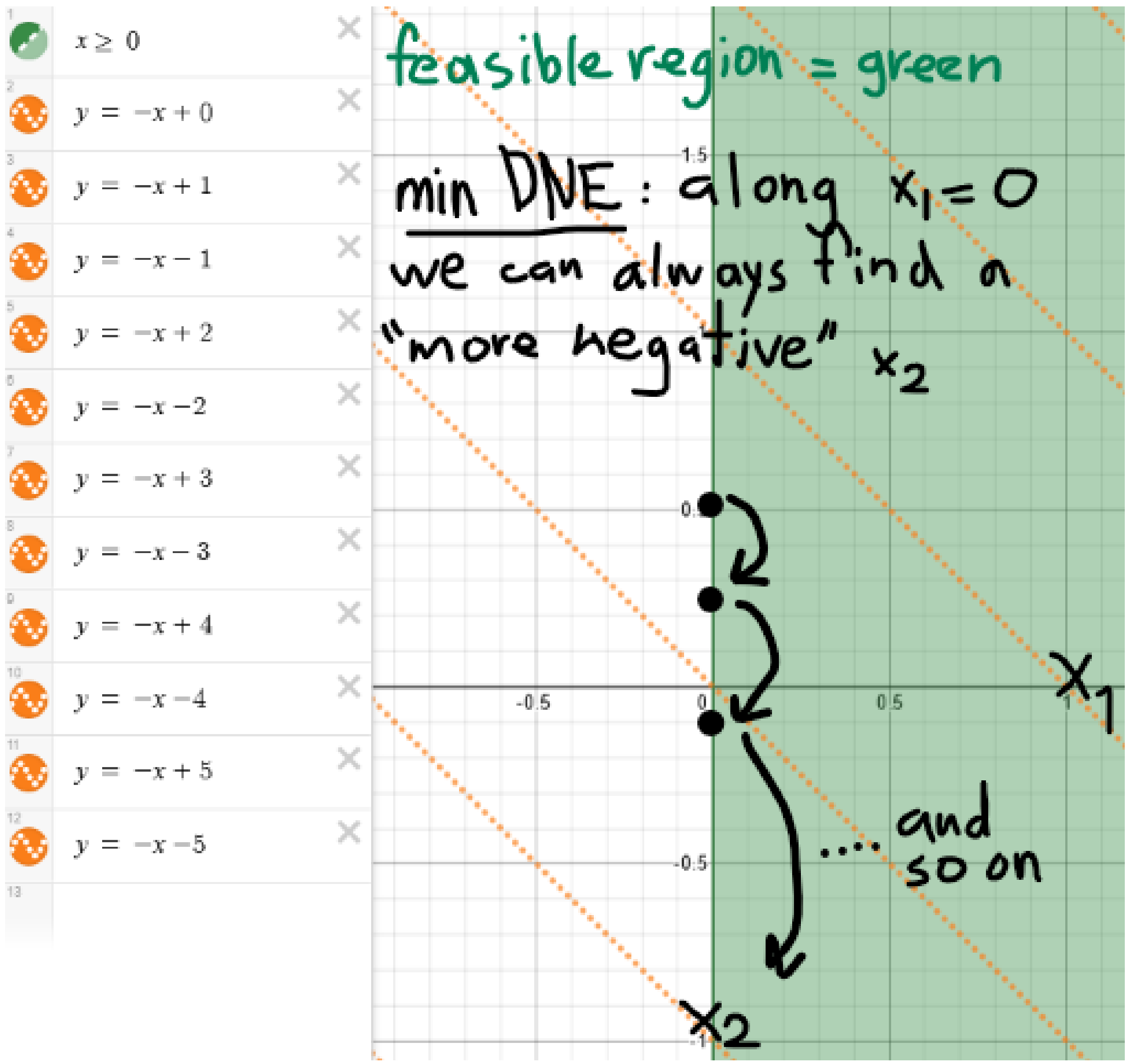


Figure 6: Here is the graphical solution for 4b.

- We can see that there is no point at which a "least" level curve of the objective function (e.g. value of f for that level set) intersects the feasible region. In other words, we can always find a "more negative" level set that intersects the boundary of the feasible region, always finding a "more minimizing" solution. Therefore, there is no solution x^* that minimizes the function f for this constraint.
- This conclusion comes from the fact that the set of x_2 values, which is $(-\infty, \infty) = \mathbb{R}$, has no minimum.
- Now let's verify this analytically.

(ii) by using the optimality conditions. To do this part, write each problem in inequality form by defining c, A, and b and apply the optimality conditions.

$\boxed{\text{4b. (analytical)}}$

- Let's write our program, [$\min_x (x_1+x_2)$, s.t. $x_1 \geq 0$] in inequality form:

$$\left(\min c^T x\right) = [1, 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 + x_2 \implies c = [1, 1]^T$$

$$(x_1 \geq 0) \longrightarrow (-x_1 \leq 0) \longrightarrow \left([-1, 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq [0] \quad \text{is } Ax \leq b\right)$$

- To show that we can't solve the system of optimality condition equations

Let's examine $[A^T\lambda + c = 0]$ condition: $\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}[\lambda] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$

$\rightarrow \left(\begin{bmatrix} -\lambda \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) \rightarrow$ while this says that $\lambda = 1$, which seems legal, the bottom equality of $[0 = -1]$ is mathematically impossible, implying that this condition $[A^T\lambda + c = 0]$ has no solution, which also implies that the entire system of optimality conditions has no solution.

- Since we have shown that the system of optimality conditions has no solution $\implies \nexists$ an x that minimizes the LP [$\min (x_1+x_2)$, s.t. $x_1 \geq 0$]. This agrees with our graphical solution (or lack thereof) that there is no solution $x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$ that solves the LP and satisfies the optimality conditions. (NOTE: since the optimality conditions are both necessary AND sufficient, since $\nexists$ an x that satisfies them we know $\nexists$ x that minimizes $f = (x_1+x_2)$ s.t. $x_1 \geq 0$.)

(c) $\left[-x_1 - x_2 \leq 0\right]$

Solving the problem:

(i) graphically by plotting the lines of constant cost and the constraints

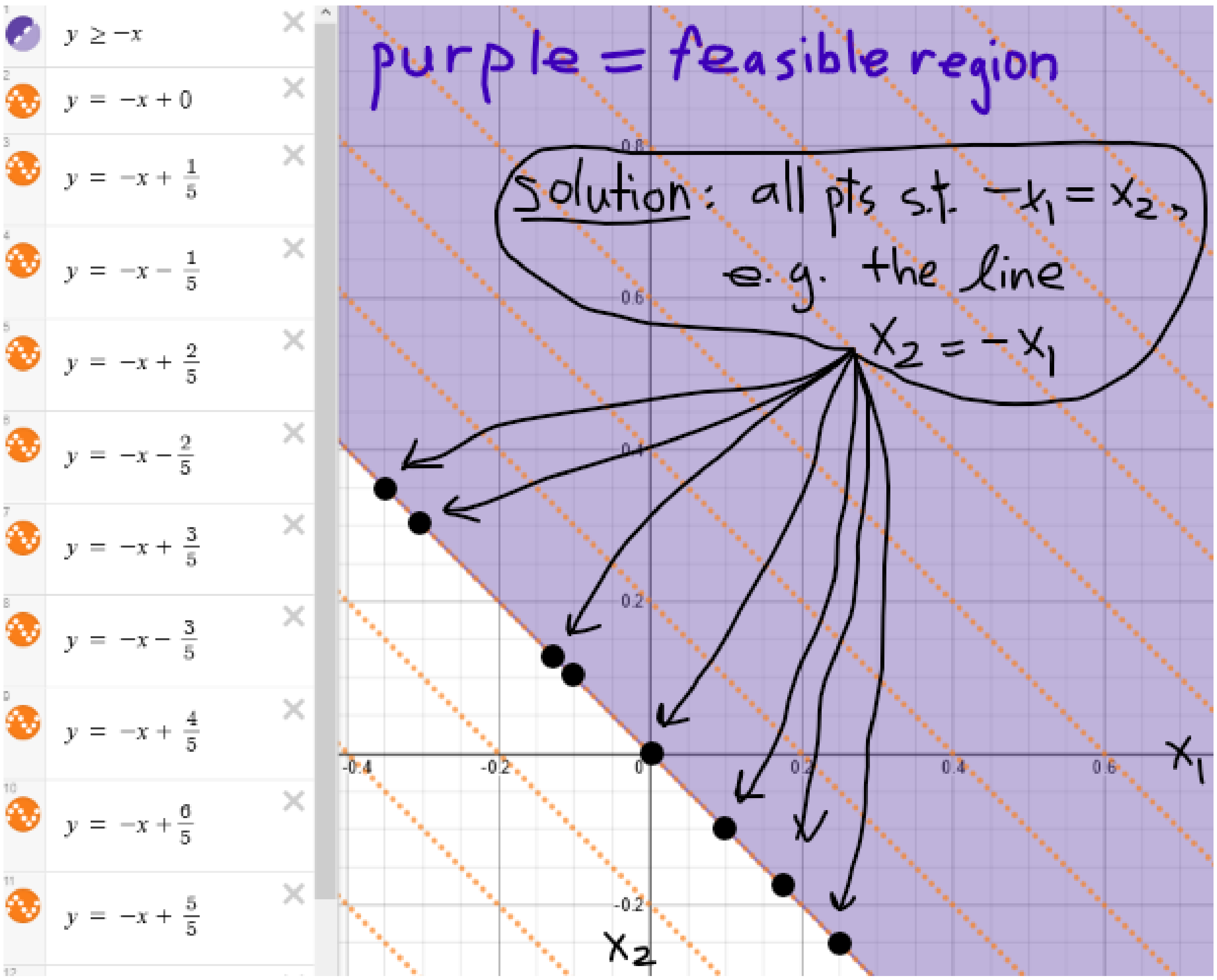


Figure 7: Here is the graphical solution for 4c.

- We can see that there are infinitely many points at which the "least" level curve of the objective function (e.g. value of f for the level set where $x_2 = -x_1$) intersects the feasible region because these two are one and the same.
- In other words, the "lower bound" of the feasible region is identical to the "least level set" of the objective function: the line $[-x_1 - x_2 = 0] = [x_2 = -x_1]$. The points that fall along this line are all points (x_1, x_2) such that $x_2 = -x_1$. And since we have no constraints on the values of x_1 and x_2 individually, this makes the number of minimizing points infinite.
- For example, giving this constraint all of the following are minimizers of f: $\left\{\ldots, (-2,2), (-1,1), (0,0), (1,-1), (1.768, -1.768), (400.33, -400.33), \ldots\right\}$
- Thus, the minimizer for this LP is $x^* = \begin{bmatrix} {x_1}^* \\ {x_2}^* \end{bmatrix} = \begin{bmatrix} {x_1}^* \\ -{x_1}^* \end{bmatrix}$
- Now let's verify this analytically.

(ii) by using the optimality conditions. To do this part, write each problem in inequality form by defining c, A, and b and apply the optimality conditions.

4c. (analytical)

- From our graphical solution we assume $x^* = \begin{bmatrix} x_1^* \\ -x_1^* \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \dots$ etc.

- Put the LP in inequality form by rewriting our constraint as $[Ax \leq b]$:

$[-x_1 - x_2 \leq 0] \rightarrow [-1, -1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq [0] \rightarrow A = [-1, -1], \; b = [0].$

- Solve $[A^T\lambda + c = 0]$ for λ: $\left(\begin{bmatrix} -1 \\ -1 \end{bmatrix} [\lambda] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \rightarrow \left(\begin{bmatrix} -\lambda \\ -\lambda \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right) \rightarrow \lambda = 1$
and $\lambda \geq 0$ ✓

- Can we satisfy $[\lambda^T(Ax^* - b) = 0]^?$: $\left([1]\left([-1, -1]\begin{bmatrix} x_1^* \\ -x_1^* \end{bmatrix} - [0] \right) \stackrel{?}{=} [0] \right)$

$\rightarrow \left((-x_1^* + + x_1^*) - 0 \stackrel{?}{=} 0 \right) \rightarrow [0 = 0]$ ✓

- Can we satisfy $[Ax^* \stackrel{?}{\leq} b]$: $\left([-1, -1]\begin{bmatrix} x_1^* \\ -x_1^* \end{bmatrix} \stackrel{?}{\leq} [0] \right) \rightarrow \left((-x_1^* + + x_1^* = 0) \leq 0 \right)$ ✓

- Since all optimality conditions (together are necessary & sufficient) are satisfied for $[\lambda = 1)$ and $x^* = \begin{bmatrix} x_1^* \\ -x_1^* \end{bmatrix} \Rightarrow x^* = \begin{bmatrix} x_1^* \\ -x_1^* \end{bmatrix}$ is a global minimizer of $f = x_1 + x_2$. $[f(x^*) = x_1^* - x_1^* = 0]$

  This agrees with our graphical solution, which shows that the line of constant cost $x_2 = -x_1$ is the same as "lower bound" of the feasible region. In other words, every point on this line (e.g. every ordered pair for which $(x_1, x_2) = (x_1, -x_1)$) is a global minimizer of $[f = x_1 + x_2]$ for the constraint $[-x_1 - x_2 \leq 0]$.