- First let's do some analysis of the problem. Is it convex? Yes, it is.
  = We know from problem #1 that $f(x) = x_1^2 + x_2^2$ is convex.
  = $g_1(x) = x_1 - 10$ is linear and thus convex, $g_2(x) = -x_1 + x_2^2 + 4$ has $\nabla^2 g_2(x) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ which is PSD making $g_2(x)$ convex also. So the inequality constraints are convex.
  = $h(x) = 0$ is trivially affine.
  = Thus, if $\exists$ a candidate that satisfies optimality conditions it is a guaranteed global min.
- $L(x) = \lambda_0 x_1^2 + \lambda_0 x_2^2 + \lambda_1(x_1 - 10) + \lambda_2(-x_1 + x_2^2 + 4)$
- $\partial L/\partial x_1 = 2\lambda_0 x_1 + \lambda_1 - \lambda_2 \overset{set}{=} 0$ $\boxed{eq1}$

  $\partial L/\partial x_2 = 2\lambda_0 x_2 + 2\lambda_2 x_2 \overset{set}{=} 0$ $\boxed{eq2}$
- Complimentarity: $\lambda_1(x_1 - 10) = 0$ and $\lambda_2(-x_1 + x_2^2 + 4) = 0$

- CASE 1: $\lambda_0 = 0$
- eq1 becomes $\lambda_1 = \lambda_2$, eq2 becomes $2\lambda_2 x_2 = 0 \Rightarrow$ either $\lambda_2 = 0$ or $x_2 = 0$. We can't let $\lambda_2 = 0$ since that makes $\lambda_1 = 0$ which gives $\lambda_0 = \lambda_1 = \lambda_2 = 0$ which is trivial. Thus let $x_2 = 0$.
- With $x_2 = 0$ let's look at complimentarity: $\lambda_1(x_1 - 10) = 0$ and $\lambda_2(-x_1 + 4) = 0 \Rightarrow$ $\lambda_1(x_1 - 10) = 0$ and $\lambda_2(-x_1 + 4) = 0$. Since λ_1 and λ_2 cannot $= 0$ this means that $x_1 - 10 = 0 \Rightarrow x_1 = 10$ and $-x_1 + 4 = 0 \Rightarrow x_1 = 4$. Since x_1 cannot $= 4$ and 10 we know that $\lambda_0 = 0$ won't work, and any feasible candidates must be normal ($\lambda_0 = 1$).

- CASE 2: $\lambda_0 = 1$
- eq1 becomes: $2x_1 + \lambda_1 - \lambda_2 = 0$, eq2 becomes: $2x_2 + 2\lambda_2 x_2 = 0 \rightarrow 2x_2(1 + \lambda_2) = 0$ means either $x_2 = 0$ or $\lambda_2 = -1$, but $\lambda \geq 0$ is required, so we must let $x_2 = 0$.
- Now the complimentarity conditions: $\lambda_2(-x_1 + 0^2 + 4) = 0$ and $\lambda_1(x_1 - 10) = 0$

  These conditions will give us 4 sub-cases to check when $\lambda_0 = 1$:

  (A) $x_1 = 4$, (B) $x_1 = 10$, (C) $\lambda_1 = 0$, (D) $\lambda_2 = 0$

CASE 2A: $(\lambda_0 = 1)$, $x_1 = 4$

- If $x_1 = 4 \Rightarrow x_1 \neq 10 \Rightarrow \lambda_1 = 0$ (from complimentarity)
- Using eq1: $2x_1 + \lambda_1 - \lambda_2 = 0 \rightarrow 2(4) + 0 = \lambda_2 \rightarrow \lambda_2 = 8$

- This candidate, $(x_1 = 4, x_2 = 0, \lambda_0 = 1, \lambda_1 = 0, \lambda_2 = 8)$ satisfies all conditions. Since the problem is convex, we know $\begin{bmatrix} x_1^* = 4 \\ x_2^* = 0 \end{bmatrix}$ is a global optimizer of f in this case. We'll check the remaining 3 cases to see if $\exists$ any other optima that achieve $f([4,0]) = 16$ optimum given the constraints.

CASE 2B: $(\lambda_0 = 1)$, $x_1 = 10$

- If $x_1 = 10 \Rightarrow x_1 \neq 4 \Rightarrow \lambda_2 = 0$ (from complementarity). Using eq1: $2x_1 + \lambda_1 = 0 \Rightarrow 20 = -\lambda_1$ $\Rightarrow \lambda_1 = -20$ which violates $\lambda \geq 0 \Rightarrow x_1 \neq 10$.

CASE 2C: $(\lambda_0 = 1)$, $\lambda_1 = 0$

- If $\lambda_1 = 0$, eq1 says $2x_1 = \lambda_2 \rightarrow$ plugged into compl. conds gives $2x_1(-x_1 + 4) \Rightarrow x_1$ is either 0 or 4. We've already looked at $x_1 = 4$ (see CASE 2A), so let $x_1 = 0$.
- If $x_1 = 0$, in the other compl. cond we have $\lambda_2\ (0 + 4) = 0 \Rightarrow \lambda_2 = 0$. But is $x_1 = x_2 = 0$ feasible? No. Consider $g_2(x) = x_1 - x_2^2 - 4 \geq 0 \rightarrow g_2([0,0]) = -4 \not\geq 0 \Rightarrow \lambda_1 = 0$ isn't feasible here.

CASE 2D: $(\lambda_0 = 1)$, $\lambda_2 = 0$

- If $\lambda_2 = 0$, eq1 gives $2x + \lambda_1 = 0 \Rightarrow \lambda_1 = -2x_1$. Plugging into compl. cond 1 gives: $-2x_1(x_1 - 10) = 0$. From CASE 2C we know $x_1 = x_2 = 0$ isn't feasible, so let $x_1 = 10$ to satisfy this.
- As above in CASE 2B, if $x_1 = 10$ we have $\lambda_1 = -2(10) = -20$, and $\lambda_1 = -20$ violates $\lambda \geq 0$.

- <u>FINAL</u>: based on all of these cases, and the fact that our problem is convex, we can guarantee that the point $[(x_1^* = 4,\ x_2^* = 0)]$ is a global optimizer of $f(x) = x_1^2 + x_2^2$ for given constraints. This x^* gives $f(x^*) = 16$ as the global optimum, and satisfies optimality conditions with $\lambda_0 = 1$, $\lambda^T = [\lambda_1 = 0, \lambda_2 = 8]$.