MAE 5930 Optimization
Homework 6

Purpose: The problems assigned help develop your ability to

- understand and recognize convexity.
- implement numerical algorithms for linearly constrained optimization.
- solve convex programs using MATLAB's `quadprog`.
- solve convex programs using Newton's Method.

NOTE: Please write or type your formulations clearly so that a reader can understand what you are doing. You are welcome to use the equivalent functions in Python.

Definition 1 (Convex Function)**.** A function $f : D \to \mathbb{R}$ is convex if

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$

for all $x, y \in D$ and all $\theta \in [0, 1]$.

Definition 2 (Convex Set)**.** A set C is convex if

$$\theta x + (1-\theta)y \in C$$

for all $x, y \in C$ and all $\theta \in [0, 1]$.

Theorem 1 (Hessian Test)**.** Let $f : D \to \mathbb{R}$ be a twice continuously differentiable function defined on a convex domain D. The function f is convex on D if and only if the Hessian matrix of f is positive semidefinite for all $x \in D$.

Problem 1: Let Q be an n-by-n symmetric matrix. Use the definition of a convex function to show under what conditions the quadratic function $f(x) = x^\top Qx$ is convex.

Problem 2: Let S be the set of points contained in the unit square, i.e.,

$$S = \{(x_1, x_2) : -1 \leq x_1 \leq 1 \text{ and } -1 \leq x_2 \leq 1\},$$

which reads, "S is the set of all points x_1 and x_2 such that $-1 \leq x_1 \leq 1$ and $-1 \leq x_2 \leq 1$." Use the definition of a convex set to show that S is convex. (You know that the filled-in square is convex because it does not have any holes or indentations. I am asking you to formalize this mathematically.)

Problem 3: Let D be the set of points contained in an annulus (or donut) with inner radius R_i and outer radius R_o, i.e.,

$$D = \{(x_1, x_2) : R_i^2 \leq x_1^2 + x_2^2 \leq R_o^2\}.$$

Use the definition of a convex set to show that D is non-convex. (Again, you can draw a picture and see that the set is non-convex because it has a hole. I am asking you to formalize this mathematically.)

Problem 4: For each of the following functions, use the Hessian test to determine if the function is convex.

(a) $f : \mathbb{R} \to \mathbb{R}, f(x) = -8x^2$

(b) $f : \mathbb{R}^3 \to \mathbb{R}, f(x) = 4x_1^2 + 3x_2^2 + 5x_3^2 + 6x_1x_2 + x_1x_3 - 3x_1 - 2x_2 + 15$

(c) $f : \mathbb{R}^2 \to \mathbb{R}, f(x) = -2x_1x_2 + x_1^2 + x_2^2$

(d) $f : \mathbb{R}^n \to \mathbb{R}, f(x) = \alpha_1 x_1 + \ldots + \alpha_n x_n$

(e) $f : (0, \pi/2) \to \mathbb{R}, f(x) = \sin(x)$

Problem 5: Program Newton's Method with Equality Constraints as described by Boyd on page 528. (Note that Boyd considers gradient vectors to be column vectors instead of row vectors.)

(a) Inputs to the function are the objective, gradient, Hessian, A matrix, and feasible starting point.

(b) Within the function, set $\epsilon = 1e - 6$.

(c) Solve the following problem using MATLAB's `quadprog` and your code. You should get the same answer. Problem data is shown on the next page.

$$\min ||x|| \quad \text{subject to} \quad Ax = b$$

```
A=[  4      4      4      9     10     10      3      6      0      1;
     7      5      6      9      5      7      1      1      5      8;
     0      4      1      1      7      3      0      6      7      4;
     3      7      2      0      3      8      7      7      5      2;
     1      2      8      2      7      1      2      1      9      9;
     1      9     10      9      8      4      3      4      6      3;
     2      0      3      1      0      9      5      7      9      8;
     3      7      7      4      8      3      1      4      1      7];
b = [9      6      8      3      3      9      4     10]';
```