- First, is this problem a convex program like # 1 and # 2 were? No. Look at $g_1(x) \le 0 \rightarrow 4 - x_1 - x_2^2 \le 0$. Is $g_1(x)$ convex? No. $\nabla_x g_1 = \begin{bmatrix} -1 \\ -2x_2 \end{bmatrix}$ and $\nabla_x^2 g_1 = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow \nabla_x^2 g_1$ is not PSD (since evals $\{0,-2\} \not\geq 0$).

- Since not all inequality constraints are convex, we won't have the strong guarantee of any optimality-condition-satisfying points being global minimizers.

- $L(x) = \lambda_0 x_1^2 + \lambda_0 x_2^2 + \lambda_1(4 - x_1 - x_2^2) + \lambda_2(-x_1 + 3x_2) + \lambda_3(-x_1 - 3x_2)$

$\partial L/\partial x_1 = 2\lambda_0 x_1 - \lambda_1 - \lambda_2 - \lambda_3 \overset{\text{set}}{=} 0$ $\boxed{eq\,1}$

$\partial L/\partial x_2 = 2\lambda_0 x_2 - 2\lambda_1 x_2 + 3\lambda_2 - 3\lambda_3 \overset{\text{set}}{=} 0$ $\boxed{eq\,2}$

- Complimentarity: $\lambda_1(4 - x_1 - x_2^2) = 0$ $\boxed{comp\,1}$, $\lambda_2(-x_1 + 3x_2)$ $\boxed{comp\,2}$, $\lambda_3(-x_1 - 3x_2)$ $\boxed{comp\,3}$

- CASE 1: $\lambda_0 = 0$:
  - This yields eq1: $-\lambda_1 - \lambda_2 - \lambda_3 = 0 \rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 0$. Since $\underset{\sim}{\lambda} \geq 0 \ \forall \lambda_i \Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 0$ only holds if $\lambda_1 = \lambda_2 = \lambda_3 = 0$. BUT we can't have this, since it gives $\lambda_0 = 0$, $\underset{\sim}{\lambda} = \underset{\sim}{0}$ which violates the non-triviality condition.
  - Therefore, if $\exists$ a solution it must be "normal", i.e. $\lambda_0 = 1$.

- When $\lambda_0 = 1$: we have eq1: $x_1 = \frac{\lambda_1 + \lambda_2 + \lambda_3}{2}$, eq2: $x_2 = \frac{3}{2}\left(\frac{-\lambda_2 + \lambda_3}{1 - \lambda_1}\right)$
  - We will have numerous cases to check, generated by $\boxed{comp\ i}$ conditions. We can start by checking $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0$, also checking when the other terms in the $\boxed{comp\ i}$ conditions are $= 0$, allowing $\lambda_1, \lambda_2, \lambda_3$ to be some value > 0.

- Since λ_0 must $= 1$, we now must check all combos of $\lambda_1, \lambda_2, \lambda_3$ being $= 0$.

- CASE 2: $\lambda_0 = 1$, while $\lambda_1 = \lambda_2 = \lambda_3 = 0$
  - Leaves eq1: $x_1 = (0+0+0)/2 \rightarrow x_1 = 0$, eq2: $x_2 = \frac{3}{2}\left(\frac{0}{1}\right) = 0$, but is $(x_1 = x_2 = 0)$ feasible?

    No: see $g_1(x) \not\leq 0 \rightarrow g_1([0,0]) = 4 - 0 - 0 = 4 \not\leq 0 \Rightarrow \underset{\sim}{\lambda} \neq \underset{\sim}{0}$

- CASE 3: $\lambda_0=1$, while $\lambda_2=\lambda_3=0$ and $(4-x_1-x_2^2)=0$

- Eq1 is now: $2x_1 - \lambda_1 - 0 - 0 = 0 \Rightarrow x_1 = \lambda_1/2$
- Eq2 is now: $2x_2 - 2\lambda_1 x_2 + 0 - 0 = 0 \Rightarrow x_2(2-2\lambda_1) = 0 \Rightarrow x_2 = 0$ or $\lambda_1 = 1$
- If $x_2=0$, $\boxed{\text{comp1}}$ is $(4-x_1-0^2)=0 \Rightarrow x_1=4 \Rightarrow \lambda_1=8$. Is $(x_1=4, x_2=0)$ feasible?

  (i) $g_1(x) = 4-4-0=0 \leq 0$ ✓

  (ii) $g_2(x) = 3(0)-4=-4 \leq 0$ ✓

  (iii) $g_3(x) = -3(0)-4 = -4 \leq 0$ ✓

  So $(x_1=4, x_2=0)$ with $\lambda_0=1$, $\lambda^T=[\lambda_1=8, \lambda_2=\lambda_3=0]$ is feasible & satisfies optimality conditions. (Not necessarily a local or global min though.)

- What if $\lambda_1=1$? Eq1 is: $2x_1-1=0 \rightarrow x_1=1/2$. Then, to satisfy $\boxed{\text{comp1}}$ we need $(4-x_1-x_2^2)=0 \rightarrow \left(\frac{8}{2}-\frac{1}{2}=x_2^2\right) \Rightarrow x_2 = \pm\sqrt{7/2}$. Is $\left(x_1=\frac{1}{2}, x_2=\pm\sqrt{\frac{7}{2}}\right)$ feasible?

  No for both points. It obviously satisfies $g_1(x)\leq 0$ by construction, but we can't satisfy $g_2(x)\leq 0$ and $g_3(x)\leq 0$.

  $g_2\left(\left[x_1=1/2, x_2=\sqrt{\frac{7}{2}}\right]\right) = 3\left(\sqrt{\frac{7}{2}}\right)-\frac{1}{2} = 5.11 \not\leq 0 \Rightarrow \left(x_1=\frac{1}{2}, x_2=\sqrt{\frac{7}{2}}\right)$ isn't feasible.

  $g_3\left(\left[x_1=1/2, x_2=-\sqrt{\frac{7}{2}}\right]\right) = {+}{+}\,3\left(\sqrt{\frac{7}{2}}\right)-\frac{1}{2} = 5.11 \not\leq 0 \Rightarrow \left(x_1=\frac{1}{2}, x_2=-\sqrt{\frac{7}{2}}\right)$ isn't feasible.

- CASE 4: $\lambda_0=1$, while $\lambda_1=\lambda_2=0$ and $(-x_1-3x_2=0)$

- Eq1 is now: $2x_1-\lambda_3=0 \rightarrow x_1=\lambda_3/2$, Eq2 is now: $2x_2-3\lambda_3 \rightarrow x_2=\frac{3}{2}\lambda_3$
- Sub in to $\boxed{\text{comp3}}$ cond: $\left(-\frac{\lambda_3}{\cancel{2}}=3\left(\frac{3}{\cancel{2}}\lambda_3\right)\right) \rightarrow -\lambda_3=9\lambda_3 \rightarrow$ iff $\lambda_3=0 \Rightarrow x_1=x_2=0$
- But $x_1=x_2=0$ violates $g_1(x)\leq 0 \rightarrow 4-0-0^2=4 \not\leq 0 \Rightarrow$ this is infeasible.

- CASE 5: $\lambda_0=1$, while $\lambda_1=\lambda_3=0$ and $(-x_1+3x_2=0)$

- Eq1 is now: $2x_1-\lambda_2=0$ and Eq2 is now: $2x_2+3\lambda_2=0 \rightarrow x_1=\frac{\lambda_2}{2}$, $x_2=\frac{-3\lambda_2}{2}$
- Plugging in to $\boxed{\text{comp2}}$ condition gives: $\frac{\lambda_2}{\cancel{2}} = 3\left(\frac{-3\lambda_2}{\cancel{2}}\right) \rightarrow \lambda_2=-9\lambda_2 \rightarrow$ iff $\lambda_2=0$

  $\Longrightarrow x_1=x_2=0$, which again violates $g_1(x)\leq 0 \rightarrow 4-0-0^2=4 \not\leq 0$.

  Thus, this route is infeasible.

• CASE 6: $\lambda_0=1$, $\lambda_1=0$, $(-x_1+3x_2=0)$, $(-x_1-3x_2=0)$

• We can see that ($-x_1+3x_2$ must $= -x_1-3x_2$) $\rightarrow -x_1+x_1 = 3x_2 ++ 3x_2$
$\rightarrow 6x_2=0 \rightarrow$ iff $x_2=0 \Rightarrow x_1=0$, but we know $x_1=x_2=0$ violates $g_1(x)\le 0$, since $4-0-0^2 = 4 \nleq 0$. Thus, this solution can't work.

• CASE 7: $\lambda_0=1$, $\lambda_2=0$, $(4-x_1-x_2^2=0)$, $(-x_1-3x_2)=0$

• Using the last equation in the CASE, $\Rightarrow x_1=-3x_2$. Plugging in to the other CASE equation gives: $(4++3x_2-x_2^2=0) \equiv x_2^2-3x_2-4=0 \rightarrow (x_2-4)(x_2+1)=0$
This gives $x_2=4$ and $x_2=-1 \rightarrow (x_1=3, x_2=-1), (x_1=-12, x_2=4)$. Feasible?

• $\boxed{(x_1=3, x_2=-1)\text{: this is a feasible point.}}$ Looking at eq1: $-2-\lambda_1-\lambda_3=0$ and eq2: $-2+2\lambda_1-3\lambda_3=0$. Gives $\lambda_1=-2-\lambda_3 \rightarrow$ into eq2: $-2-4-2\lambda_3-3\lambda_3=0$
$\rightarrow -5\lambda_3=6 \rightarrow \lambda_3=-6/5$ which violates $\lambda_i \ge 0 \ \forall i \in \{1,2,3\}$, so this solution won't work. $\boxed{\text{In short, } (x_1=3, x_2=-1) \text{ is feasible, but doesn't satisfy FJ conditions here.}}$

• $(x_1=-12, x_2=4)$: this is a feasible point. However, it has a much greater objective value than the candidate from CASE 3, so it's not worth finding λ_1 & λ_3.

• CASE 8: $\lambda_0=1$, $\lambda_3=0$, $(4-x_1-x_2^2=0)$, $(-x_1+3x_2=0)$

• Using the equations in the case: $x_1=3x_2 \rightarrow [4-3x_2-x_2^2=0] \equiv [x_2^2+3x_2-4=0]$
$\equiv [(x_2+4)(x_2-1)=0] \Rightarrow$ either $(x_2=1, x_1=3)$ or $(x_2=-4, x_1=-12)$.

• $(x_1=-12, x_2=-4)$: doesn't matter, much larger objective value than case 3 candidate.

• $(x_1=3, x_2=1)$: feasible point. Our new eq1: $6-\lambda_1-\lambda_2=0$, and our new eq2: $2-2\lambda_1+3\lambda_2=0$. $\lambda_1=6-\lambda_2 \rightarrow 2-2(6-\lambda_2)+3\lambda_2=0 \rightarrow 5\lambda_2=10 \rightarrow \lambda_2=2$
$\Rightarrow \lambda_1=4$

Thus, $(x_1=3, x_2=1)$ with $\lambda_0=1$, $\lambda^T=[\lambda_1=4, \lambda_2=2, \lambda_3=0]$ is our best candidate with $f(x)=3^2+1^2=10$. We can't guarantee it to be a global max, but it is the best solution using the FJ conditions. $(x_1=3, x_2=-1)$ also achieves $f(x)=10$, but doesn't satisfy FJ conditions (although it IS a feasible point).