- First, let's do some thinking that will help us see if an answer exists & what it might be: we know $x_1^2 \geq 0$ and $x_2^2 \geq 0$ $\Rightarrow$ the lowest $x_1^2 + x_2^2$ can be (unconstrained) is $0+0=0$, achieved by $(x_1=0, x_2=0.)$
- Is $(x_1=0, x_2=0)$ feasible? $0+0-2 \leq 0$ and $0^2-0-4 \leq 0$, so yes, it is.
- Therefore, our process should lead us to $(x_1^* = 0$ and $x_2^* = 0)$.
- Also, note that this is a convex program. $\nabla_x f = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \rightarrow \nabla_x^2 f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ which is PD $\Rightarrow$ objective is convex. We can see $g_1(x)$ is linear, and $\nabla_x^2 g_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ which is PSD $\Rightarrow g_2(x)$ is convex. $h(x)=0$ is trivially affine.
- Since this is a convex program, if we find a solution candidate such that the optimality conditions system is satisfied, these are also sufficient (for CP's) so it is guaranteed to be a global minimum.
- $L(x) = \lambda_0 x_1^2 + \lambda_0 x_2^2 + \lambda_1 (x_1 + x_2 - 2) + \lambda_2 (x_1^2 - x_2 - 4)$
- $\left[\nabla_x L \overset{set}{=} 0\right] \rightarrow$ $\partial L / \partial x_1 = 2\lambda_0 x_1 + \lambda_1 + 2\lambda_2 x_1 \overset{set}{=} 0$ $\boxed{eq. 1}$

  $\partial L / \partial x_2 = 2\lambda_0 x_2 + \lambda_1 - \lambda_2 \overset{set}{=} 0$ $\boxed{eq. 2}$
- Complimentarity: $\lambda_1 (x_1 + x_2 - 2) = 0$ and $\lambda_2 (x_1^2 - x_2 - 4) = 0$
- CASE 1: $\lambda_0 = 0$
  - $\boxed{eq 1}$ becomes $0 + \lambda_1 + 2\lambda_2 x_1 = 0 \Rightarrow \lambda_1 = -2\lambda_2 x_1$
  - $\boxed{eq 2}$ becomes $0 + \lambda_1 - \lambda_2 = 0 \Rightarrow \lambda_1 = \lambda_2$
  - Thus $\boxed{eq 1}$ now becomes $\lambda_1 = -2\lambda_1 x_1 \rightarrow x_1 = -1/2$
  - Now solve for x_2: $\lambda_1(-1/2 + x_2 - 2) = 0$ and $\lambda_1((-1/2)^2 - x_2 - 4) = 0$

    $\Rightarrow -5/2 + x_2 = -x_2 - 15/4 \Rightarrow x_2 = -5/8$
  - To satisfy compl. cond's, since $(x_1 + x_2 - 2) \neq 0 \Rightarrow \lambda_1 = 0$ and $(x_1^2 - x_2 - 4) \neq 0 \Rightarrow \lambda_2 = 0$
  - This can't work since it requires $\lambda_0 = \lambda_1 = \lambda_2 = 0$ which violates the non-triviality condition. Therefore, if $\exists$ a solution, $\lambda_0 = 1$.

- CASE 2: $\lambda_0 = 1$. with $\lambda_1 = 0$

- $\boxed{eq1}$ becomes $2x_1 + 2\lambda_2 x_1 = 0 \longrightarrow 2x_1(1+\lambda_2) = 0 \Rightarrow x_1 = 0$ or $\lambda_2 = -1$, but since $\underset{\sim}{\lambda} \geq 0 \Rightarrow \lambda_2 \neq -1 \Rightarrow x_1 = 0$ here.

- $\boxed{eq2}$ becomes $2x_2 - \lambda_2 = 0 \Rightarrow \lambda_2 = 2x_2$ or $x_2 = \lambda_2/2$

- To satisfy complim. conds. $\lambda_2(x_1^2 - x_2 - 4) = 0 \longrightarrow$ sub in $\lambda_2 = 2x_2$ and $x_1 = 0$, giving $2x_2(-x_2 - 4) = 0 \Rightarrow$ either $x_2 = 0$ or $x_2 = -4$. If $x_2 = 0 \Rightarrow \lambda_2 = 0$, if $x_2 = -4 \Rightarrow \lambda_2 = -8$ which isn't allowed due to $\lambda \geq 0$ constraint.

- Therefore we have a candidate minimizer of:

$x_1 = 0,\ x_2 = 0,\ \lambda_0 = 1,\ \lambda_1 = 0,\ \lambda_2 = 0$

- Since we showed above that this is a convex program, we know that a candidate that satisfies the optimality condition system is a global minimizer. Also, we showed that our intuition is correct, verifying that $[x_1 = 0, x_2 = 0]$ with $\lambda_0 = 1$ and $\lambda^T = [\lambda_1 = 0, \lambda_2 = 0]$ satisfies the opt. conditions and achieves the unconstrained optimal objective value of 0.

Since $(0,0)$ is a unique point & uniquely achieves $f(x) = 0$ & satisfies the conditions, we can say that $\begin{bmatrix} x_1^* = 0 \\ x_2^* = 0 \end{bmatrix}$ globally minimizes $f(x) = x_1^2 + x_2^2$ for the constraints $g_1(x) \leq 0$ and $g_2(x) \leq 0$.