- First, is the problem convex? <u>No.</u>
- $\nabla_x g_3 = \begin{bmatrix} 3(x_1-1)^2 \\ 1 \end{bmatrix} \longrightarrow \nabla_x^2 g_3 = \begin{bmatrix} 6(x_1-1) & 0 \\ 0 & 0 \end{bmatrix}$ Is this a PSD matrix? No. Let $x_1 = 0$, gives eigvals $\{-6, 0\} \not\geq 0 \Rightarrow \nabla_x^2 g_3$ is not PSD $\Rightarrow$ Ineq. constraint $g_3(x): \; x_2 + (x_1-1)^3 \leq 0$ is not convex $\Rightarrow$ this problem as a whole is not convex.
- Since this isn't a CP we can't make guarantees on candidate points being global optima or not.
- Let's solve the problem without using the FJ conditions to make sure we get the right answer when we do so:
- In general, to minimize $-x_1$ we must make x_1 a positive value as large as possible, relative to the given constraints.
- $g_1(x): \; -x_1 \leq 0$ This fits into the desire to make $-x_1$ as small (x_1 as large) as possible. <u>BUT</u> may leave x_1 unbounded? (Taken care of in constraint $g_3(x)$.)
- $g_2(x): \; -x_2 \leq 0$ We know that $x_2 \geq 0$.
- $g_3(x): \; x_2 + (x_1-1)^3 \leq 0 \longrightarrow (x_1-1)^3 \leq -x_2$ and $g_2(x)$ stipulates $-x_2 \leq 0$ $\Rightarrow (x_1-1)^3 \leq 0$ How do we satisfy this? With some $x_1 \leq 1$. BUT, our means of minimizing $-x_1$ is to make x_1 as large as possible. Therefore, if it must be that $x_1 \leq 1$ we'd choose $x_1 = 1$ to minimize $-x_1$.

- So using a rough approach (that won't work in many cases, but does here) we see $(x_1^* = 1, \; x_2^* = 0)$ which we'll verify using FJ-conditions. (Technically, we may only be showing that $(x_1 = 1, x_2 = 0)$ is the best candidate that comes from applying the FJ conditions. Since the program is non-convex, we can't mathematically guarantee it's the global optimum. Rather, our deductive method helps us with this.)

- $L(x) = -\lambda_0 x_1 - \lambda_1 x_1 - \lambda_2 x_2 + \lambda_3(x_2 + (x_1-1)^3)$

  $\partial L/\partial x_1 = -\lambda_0 - \lambda_1 + 3\lambda_3 (x_1-1)^2 \overset{set}{=} 0$ $\boxed{eq\,1}$

  $\partial L/\partial x_2 = -\lambda_2 + \lambda_3 = 0$ $\boxed{eq\,2}$ $\Rightarrow \lambda_3 = \lambda_2$

- Complimentarity conditions:

  $\boxed{c1}$: $-\lambda_1 x_1 = 0 \longrightarrow$ either $\lambda_1 = 0$ or $x_1 = 0$

  $\boxed{c2}$: $-\lambda_2 x_2 = 0 \longrightarrow$ either $\lambda_2 = 0$ or $x_2 = 0$

  $\boxed{c3}$: $\lambda_3 (x_2 + (x_1-1)^3) = 0 \longrightarrow$ either $\lambda_3 = 0$ or $(x_1-1)^3 = -x_2$

- Since $\boxed{eq\,2}$ requires that $\lambda_3 = \lambda_2$ we need only check cases where $\lambda_3 = \lambda_2$.

- CASE 1: $\lambda_0 = 0,\ \lambda_1 \neq 0,\ \lambda_2 = \lambda_3 = 0$

  - Eq 1: $-\lambda_1 + 0 + 0(x_1-1)^2 = 0 \Rightarrow \lambda_1 = 0 \Rightarrow \lambda_0 = 0$ and $\underset{\sim}{\lambda} = \underset{\sim}{0}$ which violates the non-triviality condition. Therefore, this case is not feasible.

- CASE 2: $\lambda_0 = 0,\ \lambda_1 = 0,\ \lambda_2 \neq 0,\ \lambda_3 \neq 0$

  - Eq 1: $3\lambda_3(x_1-1)^2 = 0$, λ_3 must be $\neq 0 \Rightarrow (x_1 - 1) = 0 \Rightarrow x_1 = 0$
  - $\boxed{c2}$: since $\lambda_2 \neq 0$ here $\Rightarrow \lambda_2 x_2 = 0$ iff $x_2 = 0$
  - $\boxed{c3}$: since $\lambda_3 \neq 0$ we have $\lambda_3(0 + (1-1)^3) = 0 \rightarrow \lambda_3(0) = 0 \Rightarrow \lambda_3 \in \mathbb{R}$

    and since $\lambda_3 \in \mathbb{R}$ and $\lambda_3 = \lambda_2 \Rightarrow \boxed{\lambda_2 \in \mathbb{R}}$

- Thus, $(x_1 = 1,\ x_2 = 0)$ with $\lambda^T = [\lambda_1 = 0,\ \lambda_2 \geq 0,\ \lambda_3 \geq 0]$ with $\lambda_2 = \lambda_3$ is a candidate according to the FJ conditions. Strictly speaking, we can't declare this to be a global minimizer, but from our procedural reasoning above (at problem's outset) we know that this is a best solution (there may exist others, we'll continue checking other cases).

- CASE 3: $\lambda_0 = 1$, $\underset{\sim}{\lambda} = \underset{\sim}{0}$ e.g. $\lambda_1 = \lambda_2 = \lambda_3 = 0$

- Eq 1: $-1 - 0 + 0 = 0 \Rightarrow -1 = 0$ which isn't true. Therefore, having these λ values fails to satisfy FJ conditions ("FJ infeasible").

- CASE 4: $\lambda_0 = 1$, $\lambda_1 \neq 0$, $\lambda_2 = \lambda_3 = 0$

- Eq 1: $-1 - \lambda_1 + 0 = 0 \Rightarrow \lambda_1 = -1$ which can't be, since all elements of $\underset{\sim}{\lambda}$ must be ≥ 0, $\Rightarrow$ this case isn't feasible under the FJ conditions.

- CASE 5: $\lambda_0 = 1$, $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$, ($\lambda_2 = \lambda_3$ still required by $\boxed{\text{eq 2}}$)

- Eq 1: $-1 - 0 + 3\lambda_3 (x_1 - 1)^2 = 0 \rightarrow 3\lambda_3 (x_1 - 1)^2 = 1$
- c2: Since we've said $\lambda_2 \neq 0$, for $\lambda_2 x_2 = 0 \Rightarrow x_2 = 0$
- c3: Since we've said $\lambda_3 \neq 0$, $\lambda_3 (x_1 - 1)^2 = 0 \Rightarrow x_1 - 1 = 0 \Rightarrow x_1 = 1$
- Back to Eq 1: if $3\lambda_3 (x_1 - 1)^2 = 1 \rightarrow 3\lambda_3 (1 - 1)^2 = 1 \Rightarrow 3\lambda_3 (0) = 0 = 1$ which isn't possible $\Rightarrow$ no feasible solution under FJ conditions for $\lambda_0 = 1$, $\lambda_1 = 0$, $[\lambda_2 = \lambda_3] \neq 0$.

- CONCLUSION: we have demonstrated the existence of one viable candidate, $(x_1 = 1, x_2 = 0)$ with $\lambda_0 = 0$ and $\lambda^T = [\lambda_1 = 0, \lambda_2 \geq 0, \lambda_3 \geq 0]$ and $\lambda_2 = \lambda_3$. Since the program isn't convex, we can't declare this to be a global minimizer based on the FJ conditions being satisfied. However, based on the procedural explanation at the problem's outset, we concluded the best solution is $(x_1 = 1, x_2 = 0)$. (Done via some logic and combination of constraints $g_2(x)$ and $g_3(x)$.) This is substantiated by our findings in CASE 2, showing this candidate has a satisfactory formulation for the FJ conditions. Thus, based on some not-typical reasoning, we can conclude $x^* = (x_1^* = 1, x_2^* = 0)$ achieving $f(x^*) = -1$.