B-SPLINE CURVES

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Improving over Bézier curves

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Bézier curves have some drawbacks:

- Degree is proportional to number of control points
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To overcome this: **B-splines**

- Developed by Riesenfeld and others in 1970s
- B-splines $=$ Basis splines
- Several flavors: uniform, non-uniform, rational non-uniform (NURBs)...

Deriving the formula for the quadratic B-splines

Setting:

- Input: $n+1$ control points $P_0, \ldots, P_n$
- Output: **spline curve** where each segment $P_i(t)$ is a **quadratic** parametric polynomial based on P_{i-1},P_i and P_{i+1}

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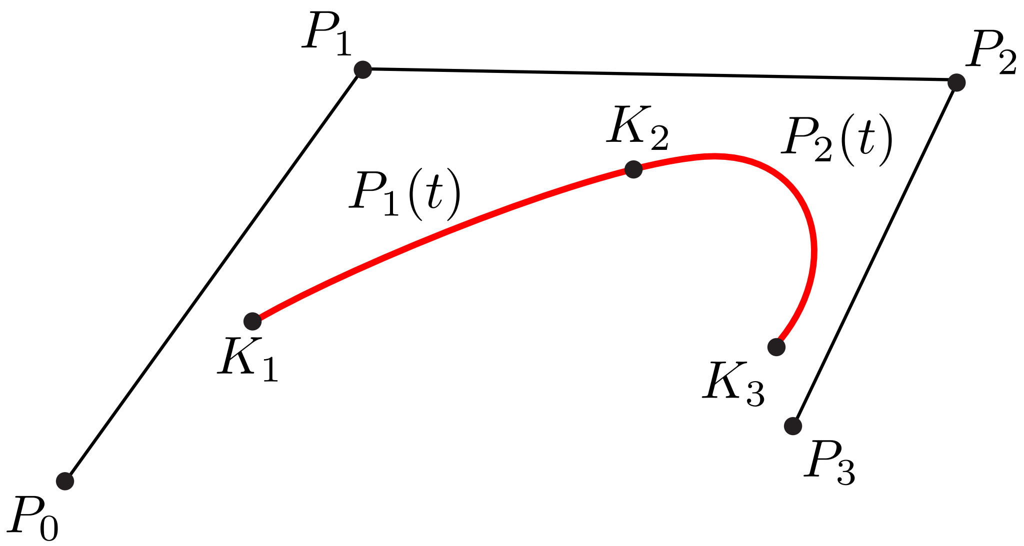

sketch of setting for $n = 3$ (not accurate!)

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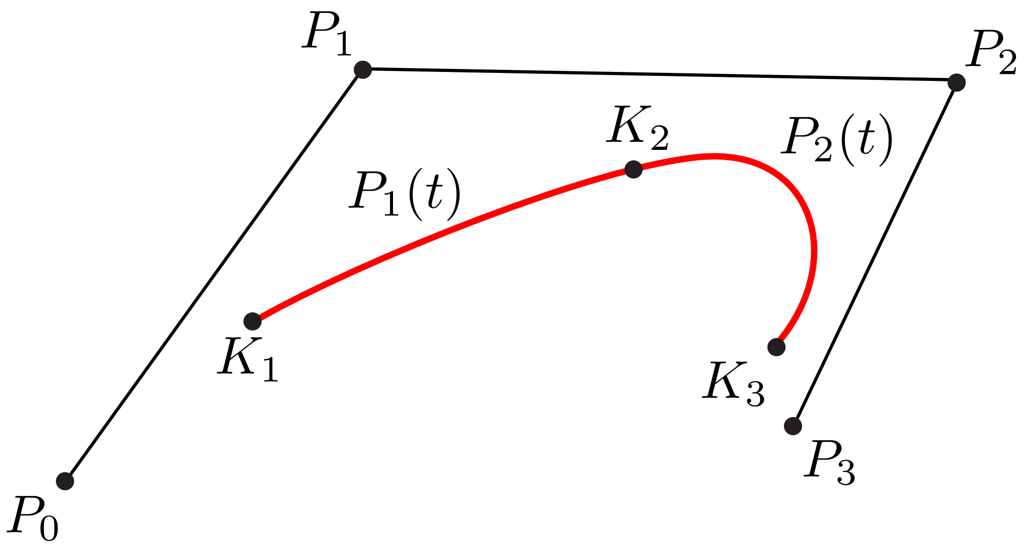

$$P_i(t) = (t^2, t, 1) \begin{pmatrix} & & \\ & ? & \\ & & \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \end{pmatrix}$$

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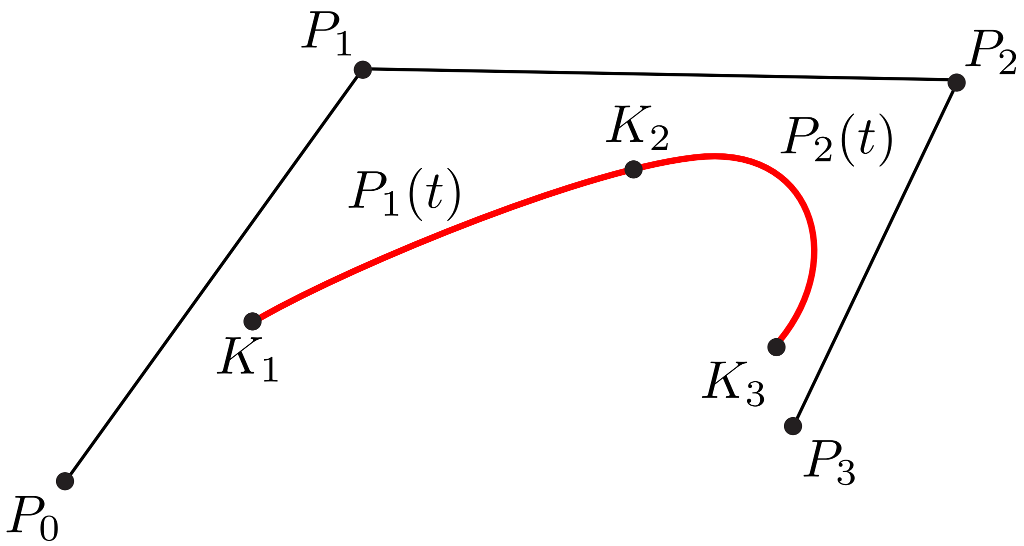

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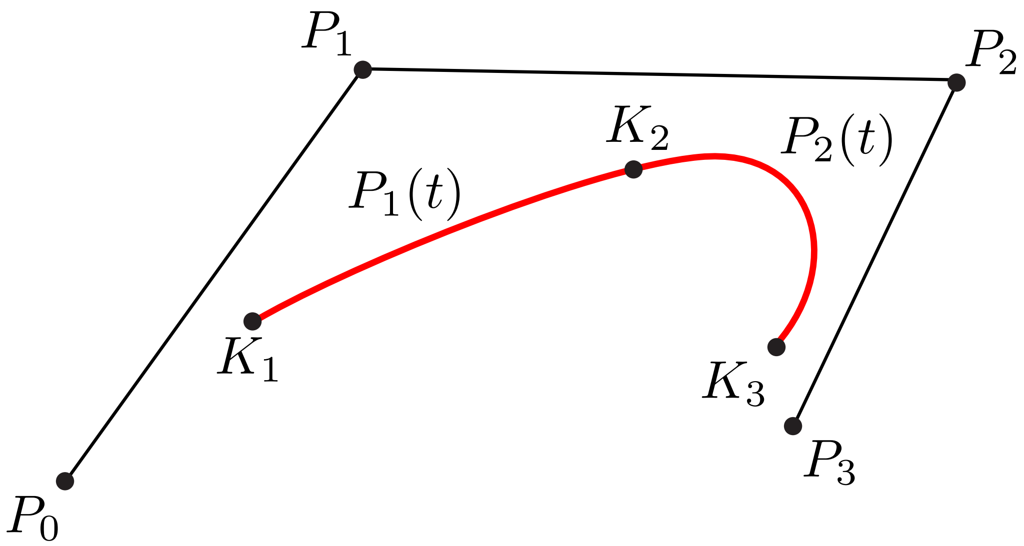

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Requirements:

1. $P_1(t)$ and $P_2(t)$ meet smoothly at common point
2. Affine combination of control points

Question: what is the matrix?

QUADRATIC UNIFORM B-SPLINES

Deriving the formula for the quadratic B-splines

$$
\begin{aligned}
P_i(t) &= \frac{1}{2}(t^2, t, 1)\begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \end{pmatrix}, i = 1, 2 \\
&= \frac{1}{2}(t^2 - 2t + 1)P_{i-1} + \frac{1}{2}(-2t^2 + 2t + 1)P_i + \frac{t^2}{2}P_{i+1}
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  Since $K_i = P_i(0)$ and $K_{i+1} = P_i(1)$, we have:

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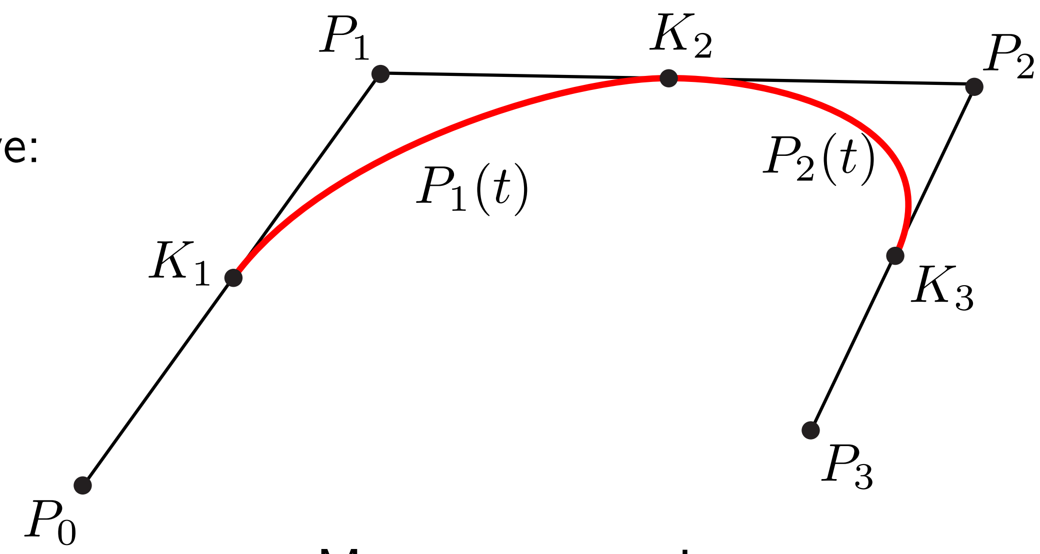

More accurate picture

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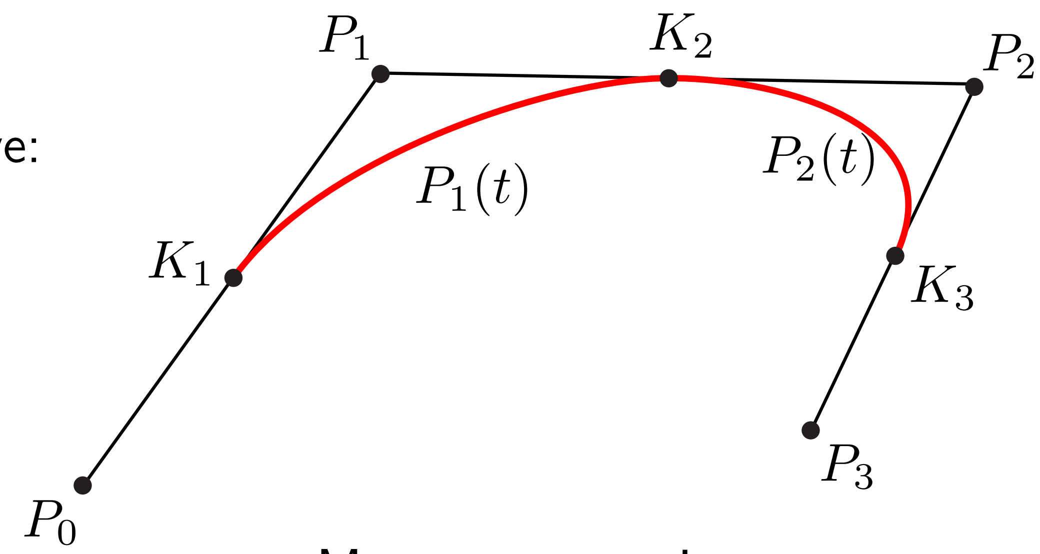

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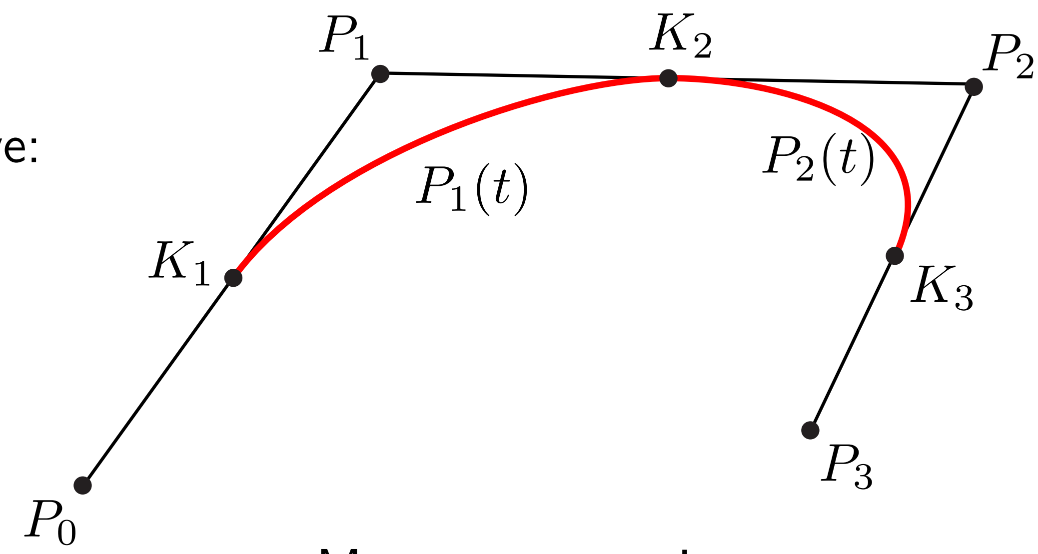

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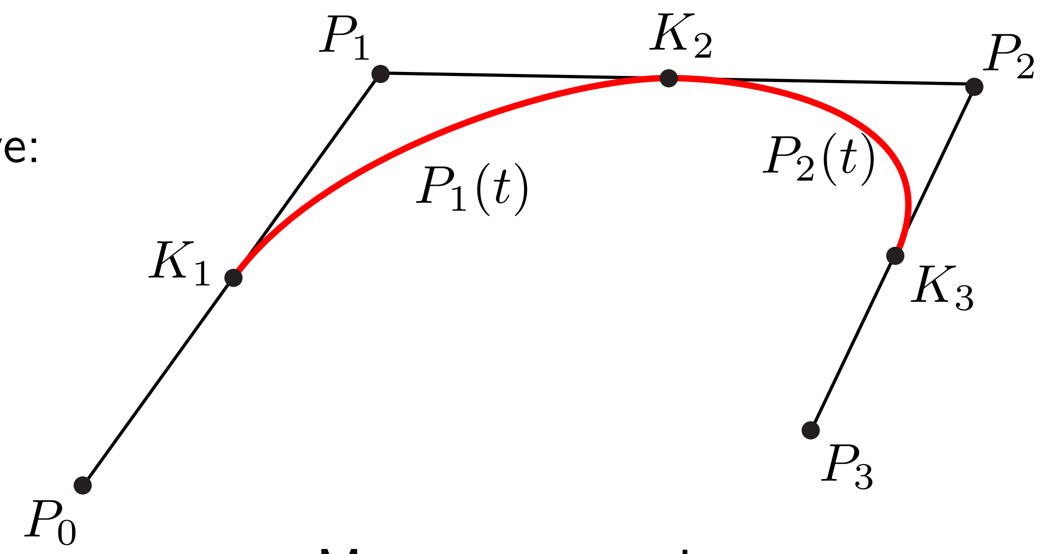

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Example: use control points $\{(1,0),(1,1),(2,1),(2,0)\}$

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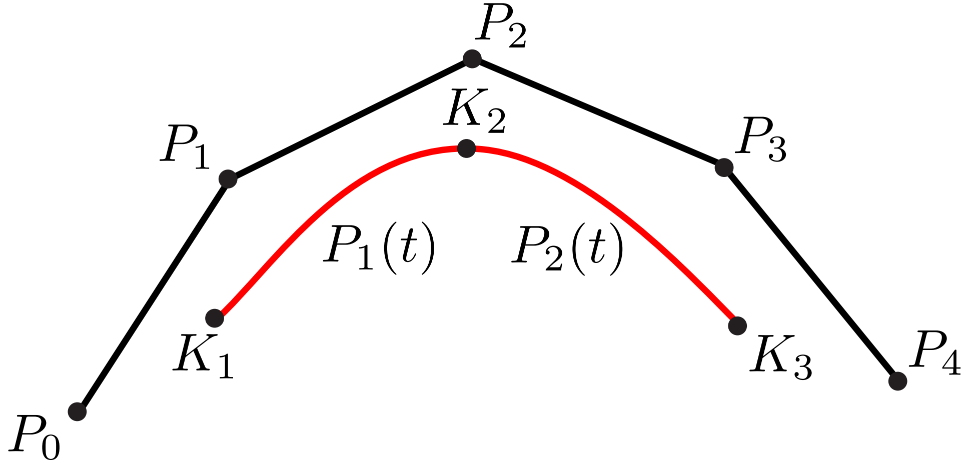

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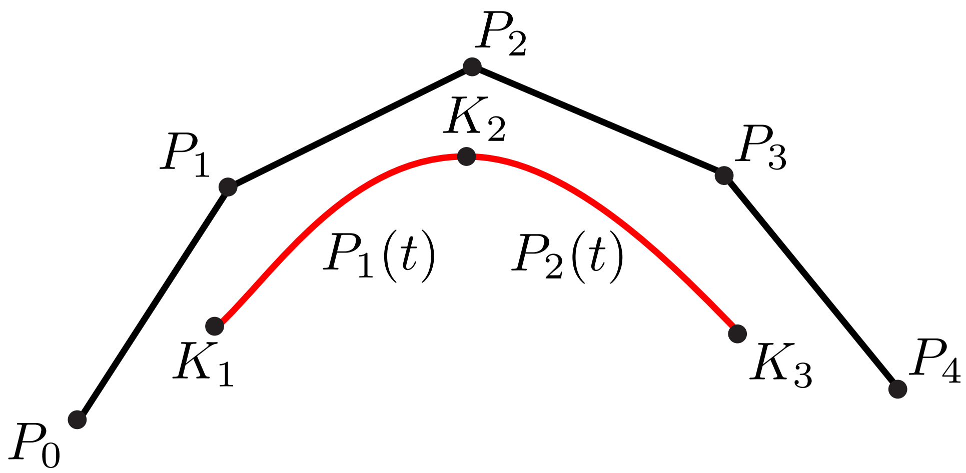

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1. Consecutive segments meet with C^2-continuity
2. Entire curve is affine combination of control points

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16 of them are independent $\Longrightarrow$ unique solution

Formula for the cubic B-splines

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$$P_i(t) = \frac{1}{6}(t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

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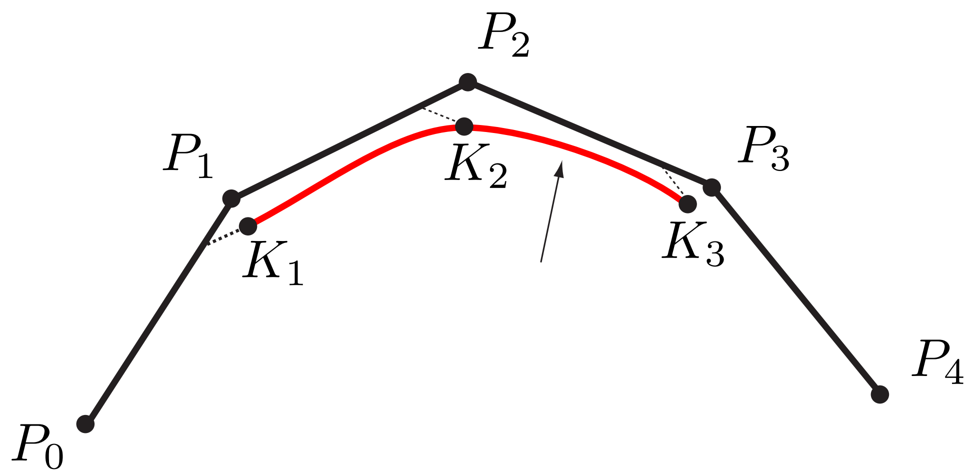

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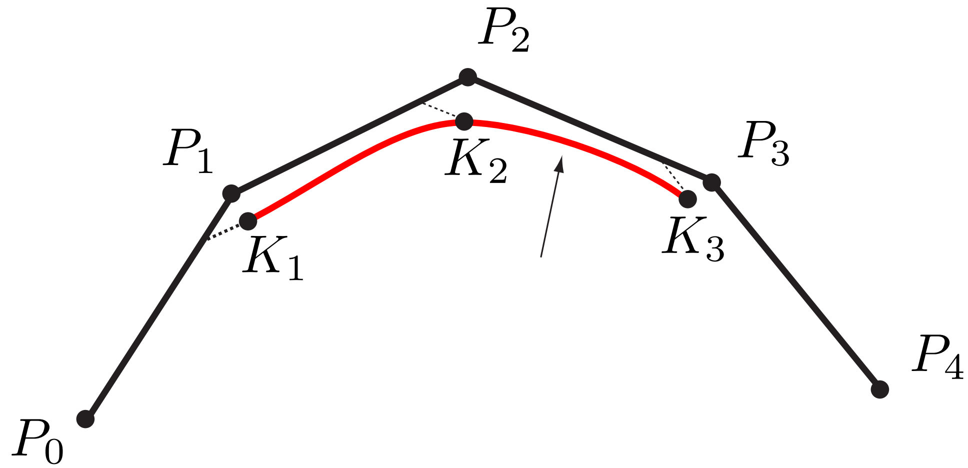
The two endpoints of each curve segment are



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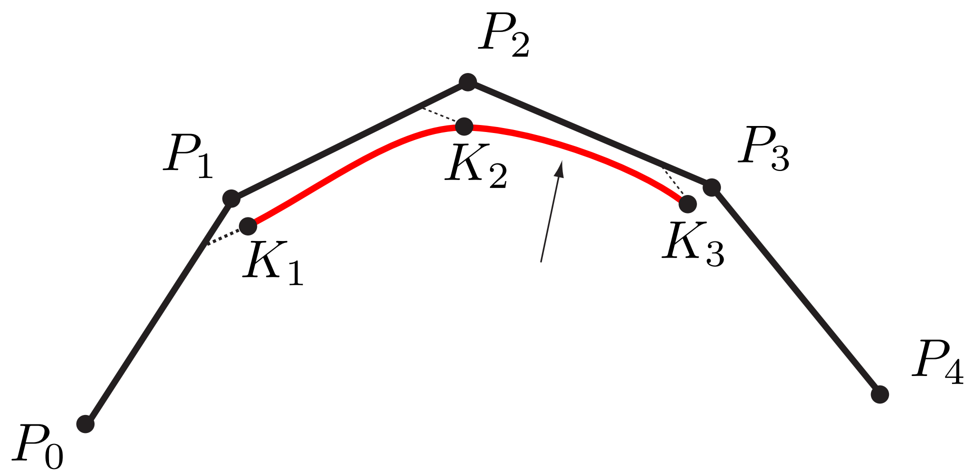

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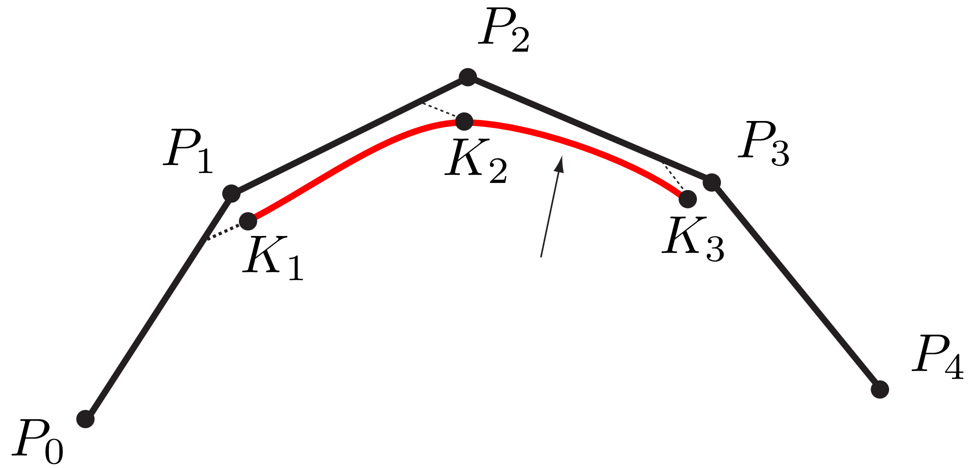
Geometrically, it makes more sense to rewrite as

$$K_i = \left(\tfrac{1}{6}P_{i-1} + \tfrac{5}{6}P_i\right) + \tfrac{1}{6}\left(P_{i+1} - P_i\right)$$
$$K_{i+1} = \left(\tfrac{1}{6}P_i + \tfrac{5}{6}P_{i+1}\right) + \tfrac{1}{6}\left(P_{i+2} - P_{i+1}\right)$$



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$$P_i(t) = \frac{1}{6}(t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

$$= \frac{1}{6}(-t^3+3t^2-3t+1)P_{i-1} + \frac{1}{6}(3t^3-6t^2+4)P_i + \frac{1}{6}(-3t^3+3t^2+3t+1)P_{i+1} + \frac{t^3}{6}P_{i+2}$$

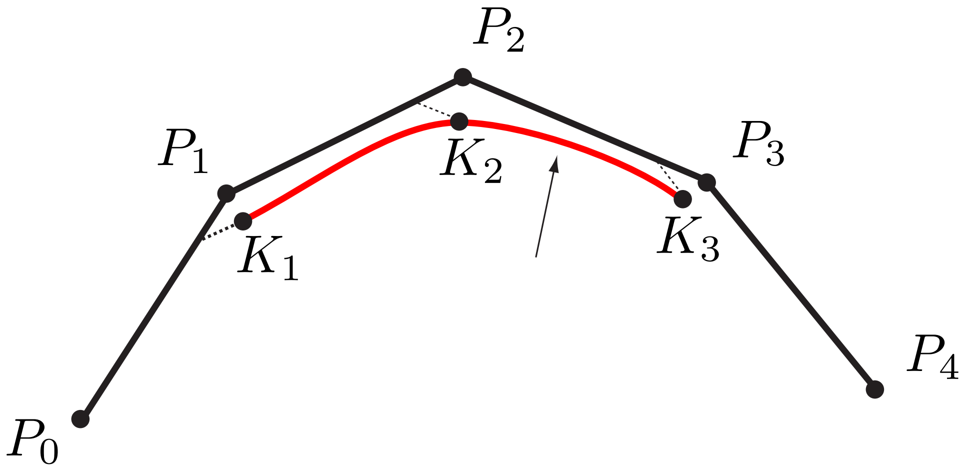
The two endpoints of each curve segment are

$$K_i = P_i(0) = \tfrac{1}{6}(P_{i-1} + 4P_i + P_{i+1})$$
$$K_{i+1} = P_i(1) = \tfrac{1}{6}(P_i + 4P_{i+1} + P_{i+2})$$

Geometrically, it makes more sense to rewrite as

$$K_i = \left(\tfrac{1}{6}P_{i-1} + \tfrac{5}{6}P_i\right) + \tfrac{1}{6}\left(P_{i+1} - P_i\right)$$
$$K_{i+1} = \left(\tfrac{1}{6}P_i + \tfrac{5}{6}P_{i+1}\right) + \tfrac{1}{6}\left(P_{i+2} - P_{i+1}\right)$$

Other geometric interpretations exist (e.g., $\frac{2}{3}$ rule)



Making the curve go from P_0 to P_n

How can we force the curve go through P_0 and P_n?

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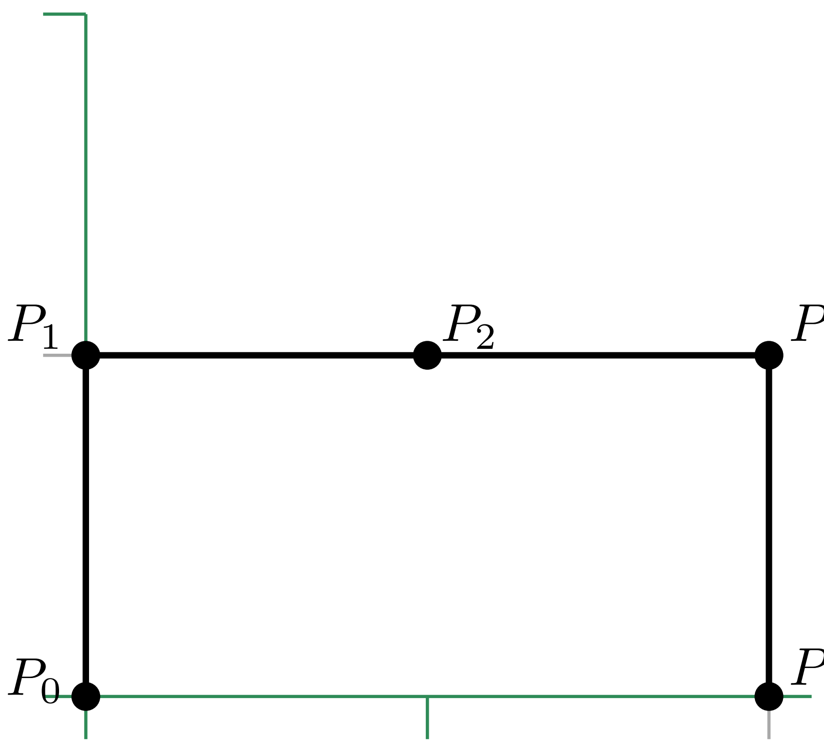
Example: use control points $\{(0,0),(0,1),(1,1),(2,1),(2,0)\}$.
(i) Draw the first segment.
(ii) Add P_{-1} to fix beginning of curve at P_0.



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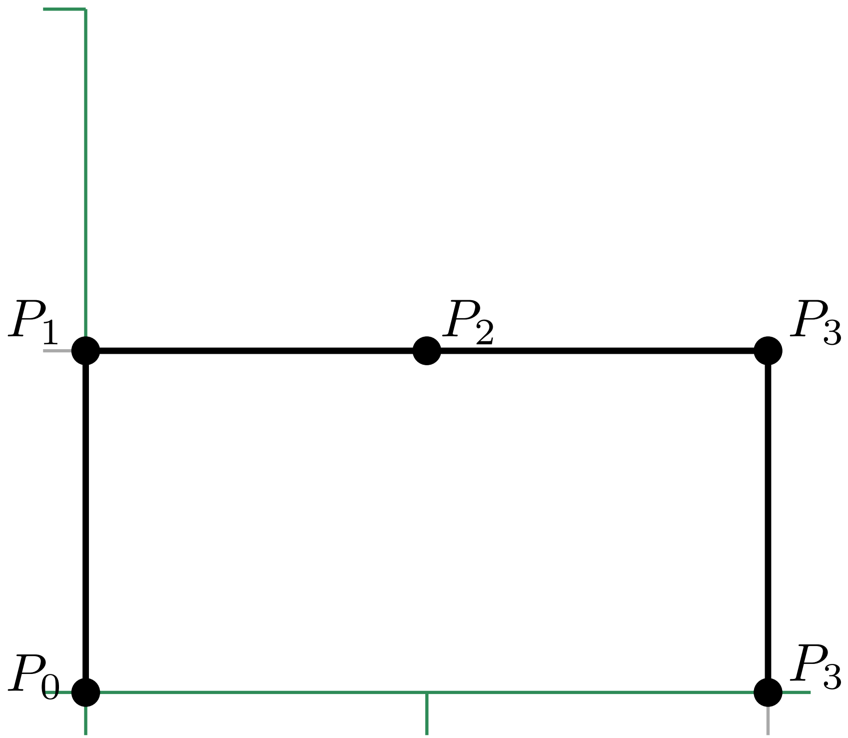

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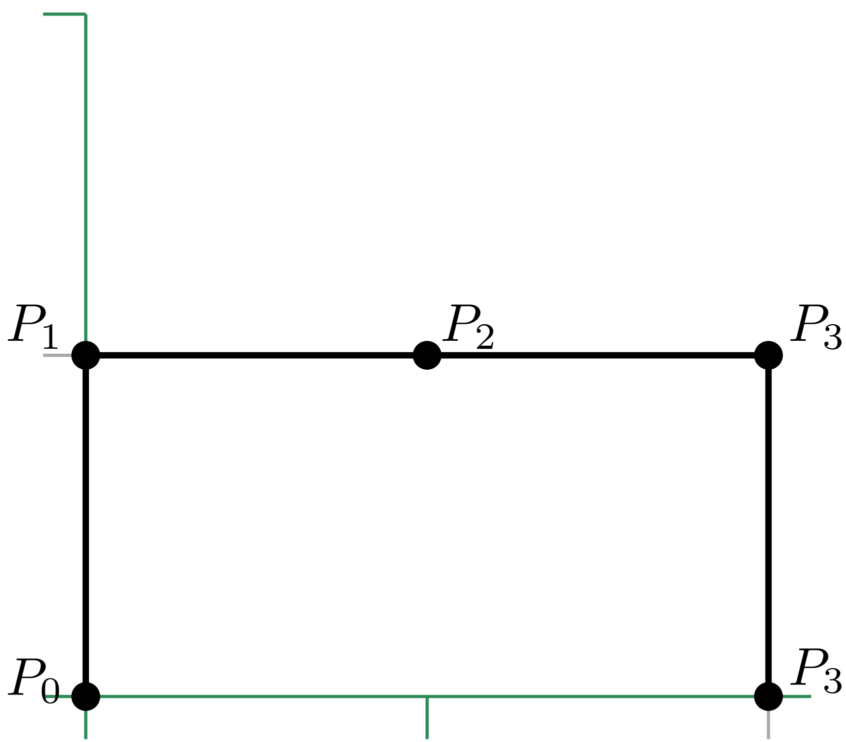

QUADRATIC VS. CUBIC

You can choose the order (=degree+1)

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As opposed to Bézier curves, in B-splines the same $n + 1$ points can be used to construct curve of order 2 (linear), 3 (quadratic), 4 (cubic) ...

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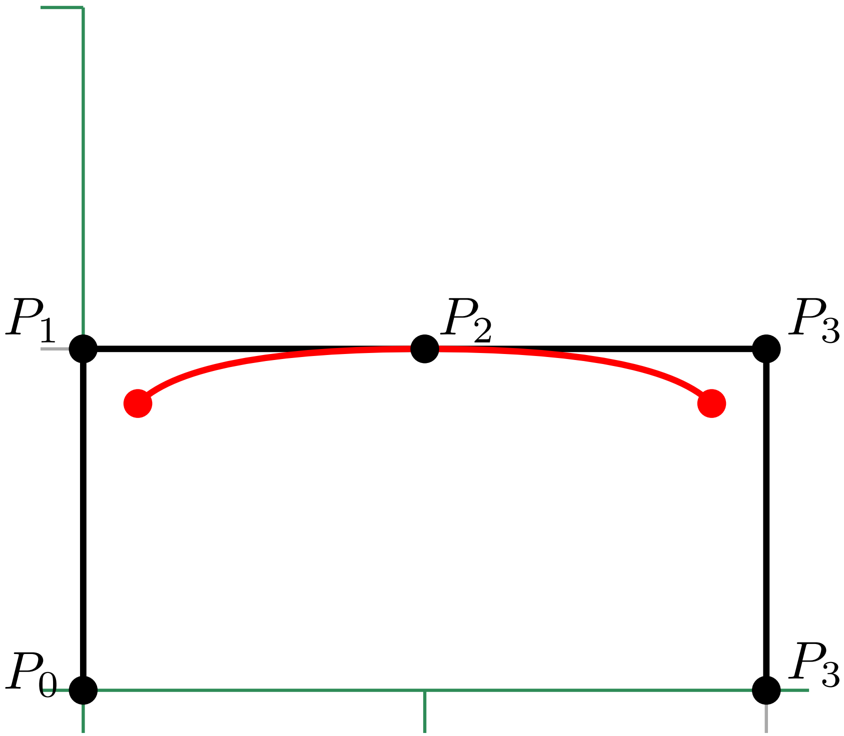

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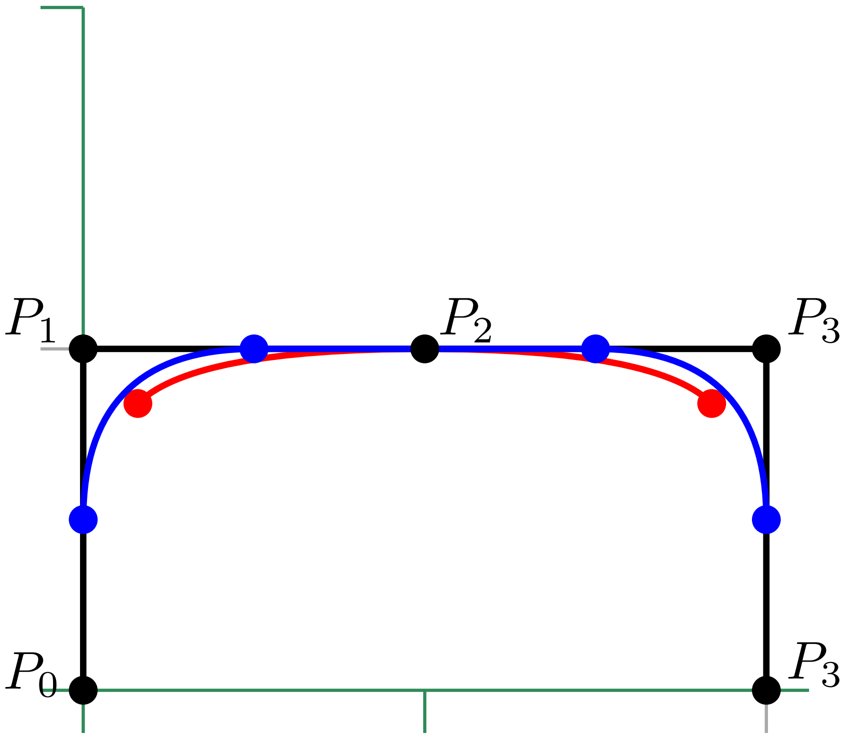

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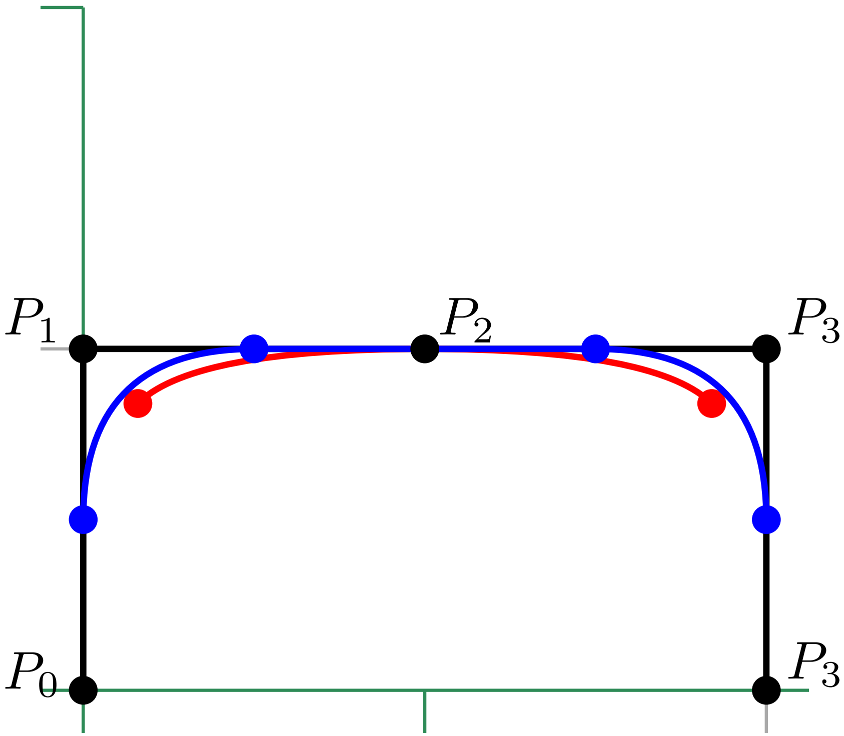

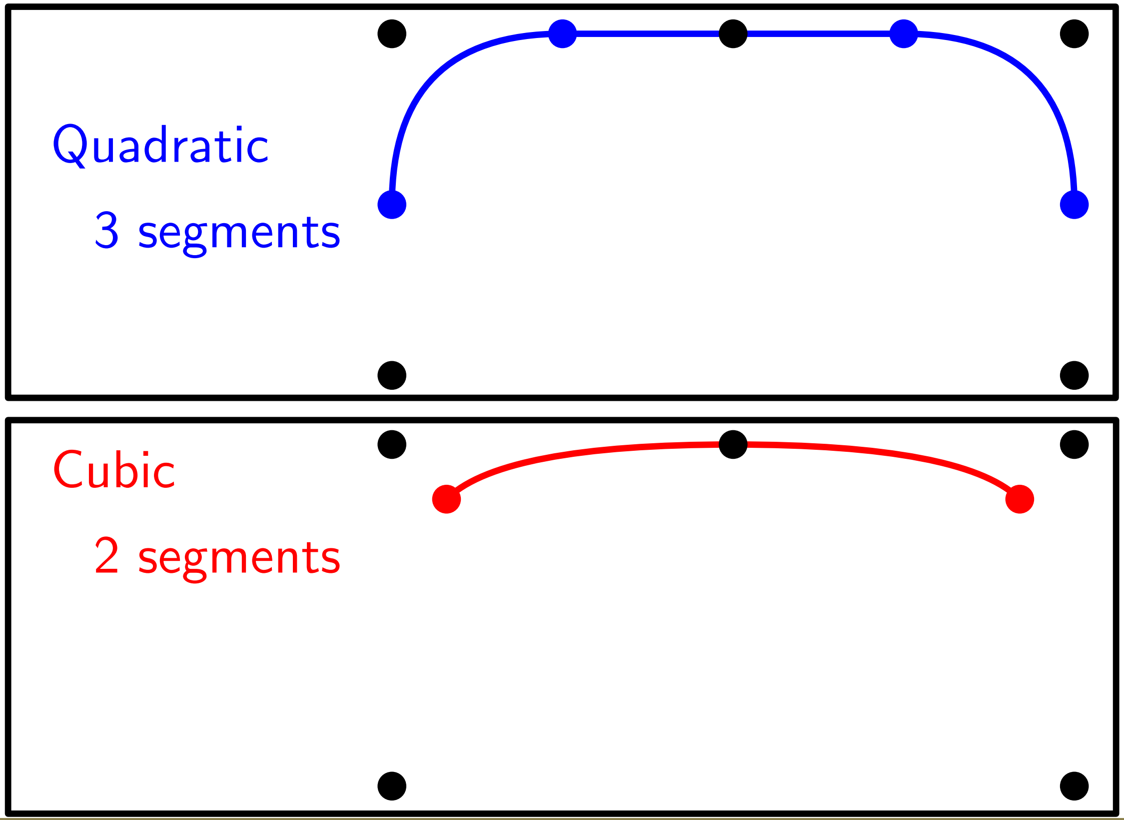

Higher order, better fit

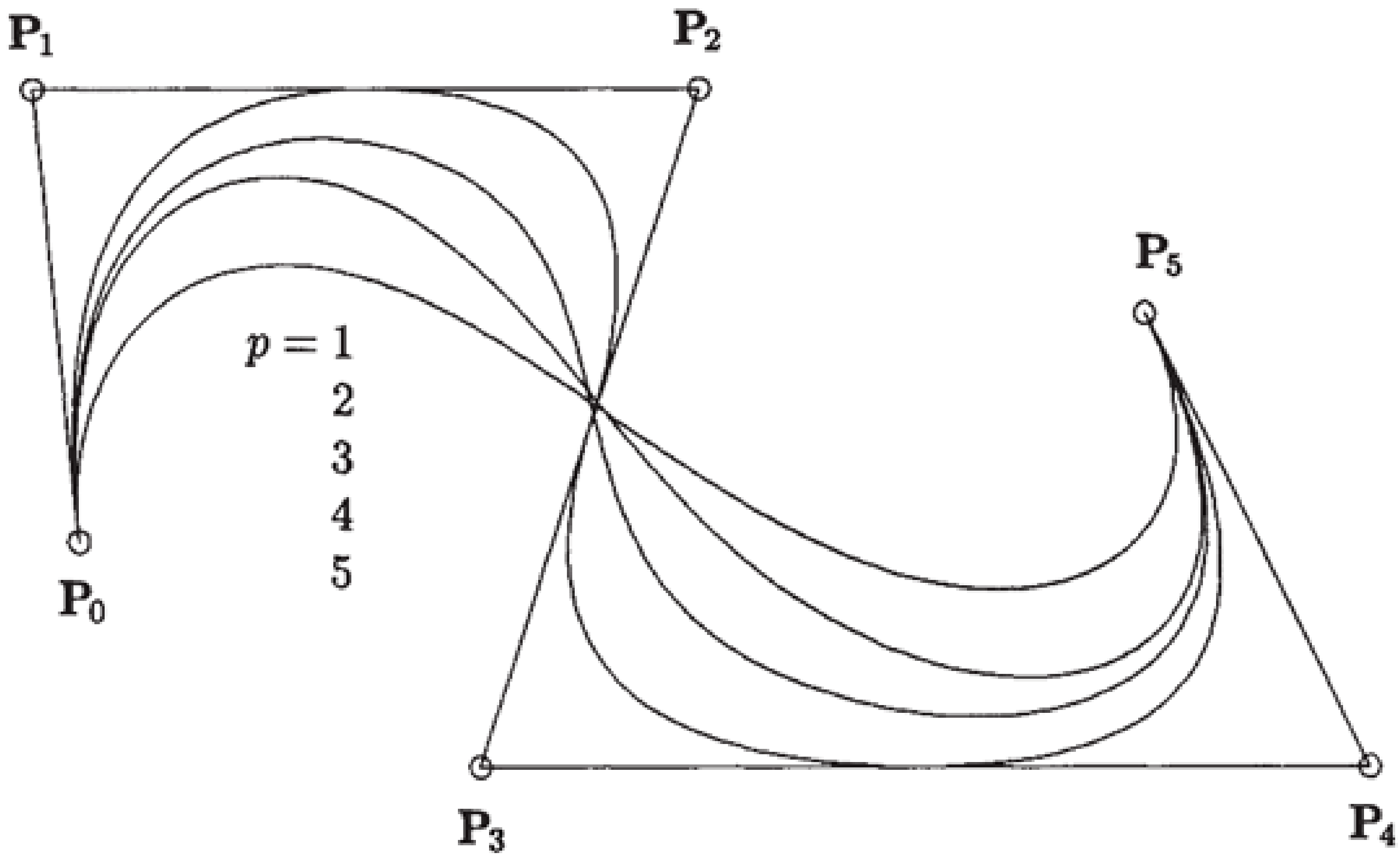

Figure 3.9. B-spline curves of different degree, using the same control polygon.

Where p is the degree of curve (i.e., $p = k - 1$)

Figure from [Piegl and Tiller]

From a cubic B-Spline to a cubic Bézier curve

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Given a cubic B-spline segment $P(t)$ based on control points $P_0, \cdots, P_3$, how can we find a Bézier curve $C(t)$ with the same shape?

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Matrix formulation of cubic B-spline

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Result:

$$Q_0 = \tfrac{1}{6}(P_0 + 4P_1 + P_2),$$
$$Q_1 = \tfrac{1}{6}(4P_1 + 2P_2),$$
$$Q_2 = \tfrac{1}{6}(2P_1 + 4P_2),$$
$$Q_3 = \tfrac{1}{6}(P_1 + 4P_2 + P_3)$$

B-splines of order higher than four

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Matrix formulation of cubic B-spline segment $P_i(t)$

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Matrix formulation of cubic B-spline segment $P_i(t)$

$$P(t) = (t^n, \dots, t^2, t, 1)M_n \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ \vdots \\ P_{i+n-1} \end{pmatrix}$$

Matrix formulation of degree-n B-spline segment $P_i(t)$

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$$m_{ij} = \frac{1}{n!} \binom{n}{i} \sum_{k=j}^{n} (n-k)^i (-1)^{k-j} \binom{n+1}{k-j}$$

HIGHER ORDER B-SPLINE CURVES

B-splines of order higher than four

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Matrix for
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$$M_3 = \frac{1}{6}\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}$$

$$M_4 = \frac{1}{24}\begin{pmatrix} -1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -6 & -6 & 6 & 0 \\ -4 & -12 & 12 & 4 & 0 \\ 1 & 11 & 11 & 1 & 0 \end{pmatrix}$$

$$M_5 = \frac{1}{120}\begin{pmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 20 & 0 & -20 & 10 & 0 \\ 10 & 20 & -60 & 20 & 10 & 0 \\ -5 & -50 & 0 & 50 & 5 & 0 \\ 1 & 26 & 66 & 26 & 1 & 0 \end{pmatrix}$$

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$$P_i(1) = \frac{1}{6}(P_i + 4P_{i+1} + P_{i+2}) = K_{i+1}$$

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We need $n+3$ equations, so we need two more: we fix tangent vectors T_1 and T_n at ends

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How to build an interpolating cubic B-spline curve

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System with $n+3$ unknowns and $n+3$ equations that (one can check) is nonsingular

KNOT VECTOR-BASED APPROACH

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in our case, since there are two segments, the parameter for the first one lives in $[0, 1]$ and for the other in $[1, 2]$

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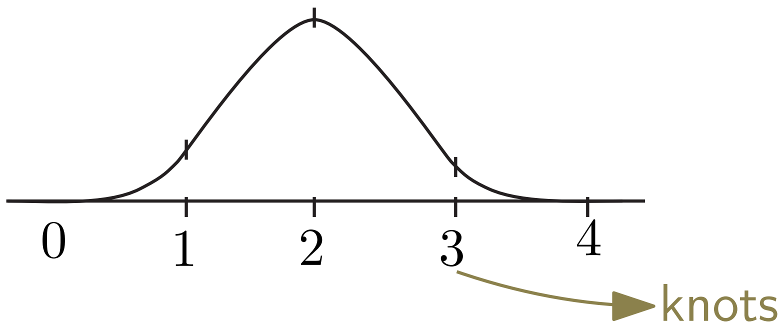

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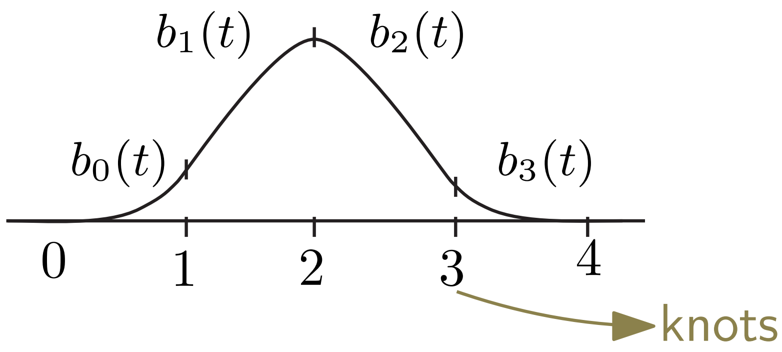

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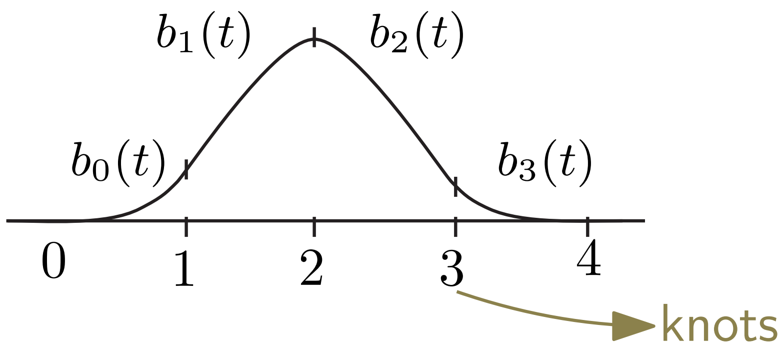

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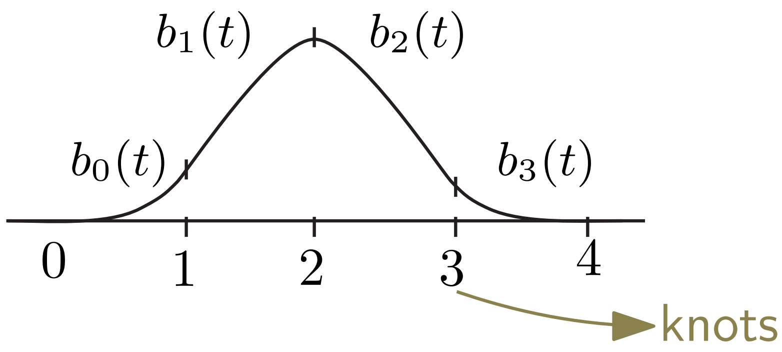

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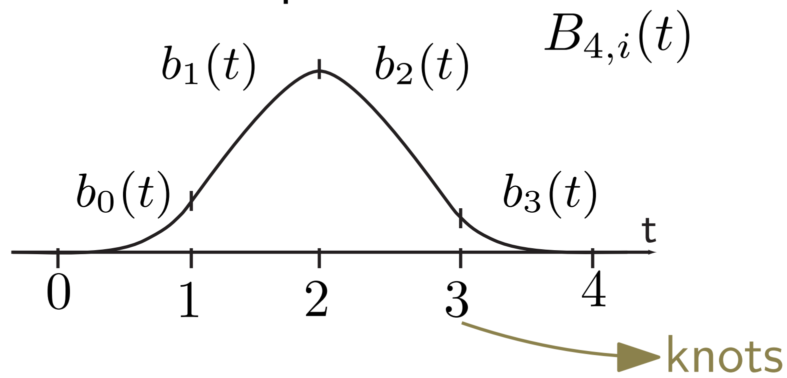

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How many equations on the coefficients of the $b_i(t)$ functions do we obtain from these conditions?

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Solution to the equations:

$$b_0(t) = \frac{1}{6}t^3$$

$$b_1(t) = \frac{1}{6}(1 + 3t + 3t^2 - 3t^3)$$

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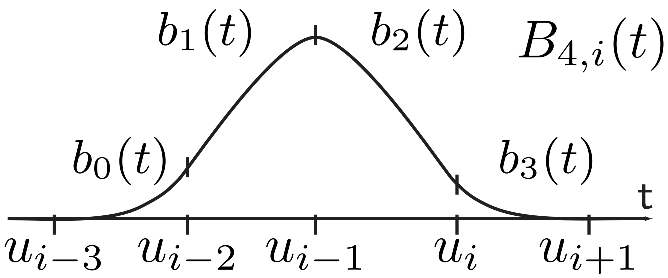
Solution to the equations:

$$N_{i,4}(t) = B_{4,i}(t) = \left[\begin{array}{ll} b_0(t) = \frac{1}{6}t^3 & u_{i-3} \leq t \leq u_{i-2} \\ b_1(t) = \frac{1}{6}(1 + 3t + 3t^2 - 3t^3) & u_{i-2} \leq t \leq u_{i-1} \\ b_2(t) = \frac{1}{6}(4 - 6t^2 + 3t^3) & u_{i-1} \leq t \leq u_i \\ b_3(t) = \frac{1}{6}(1 - 3t + 3t^2 - t^3) & u_i \leq t \leq u_{i+1} \end{array}\right.$$



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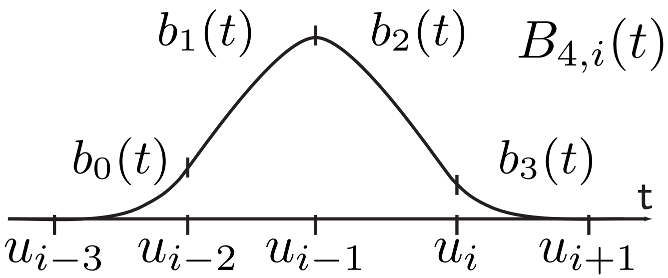

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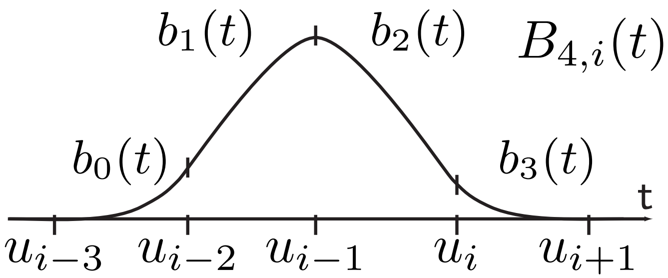

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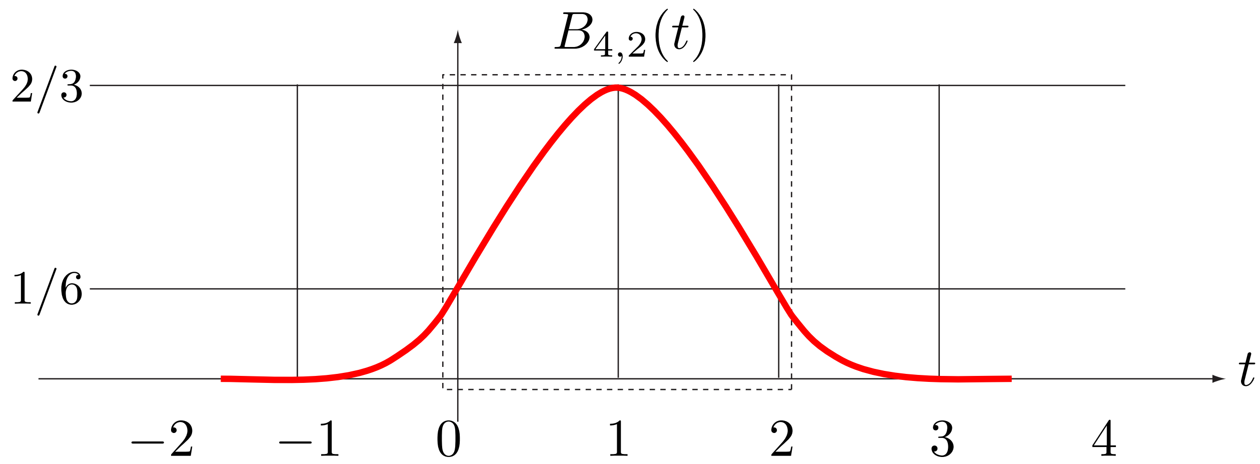

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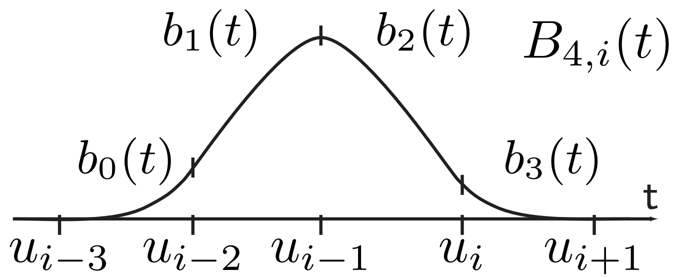

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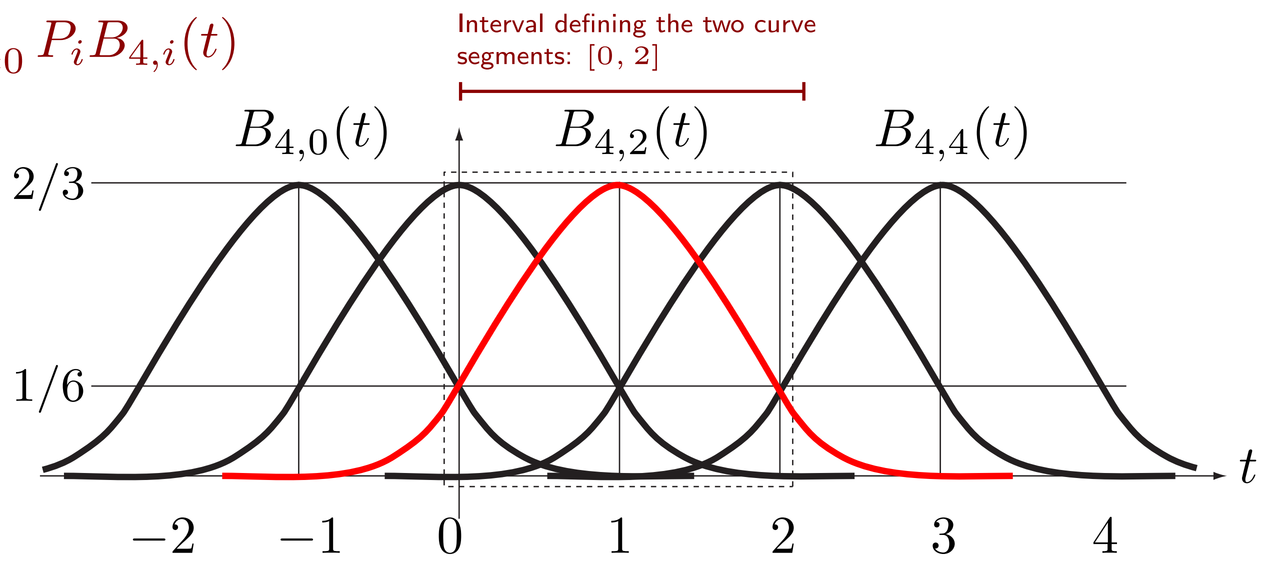

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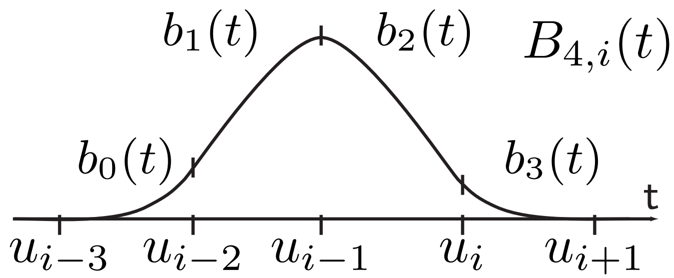

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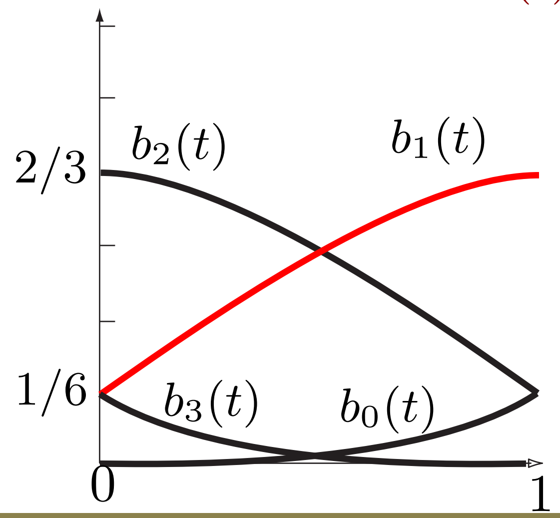

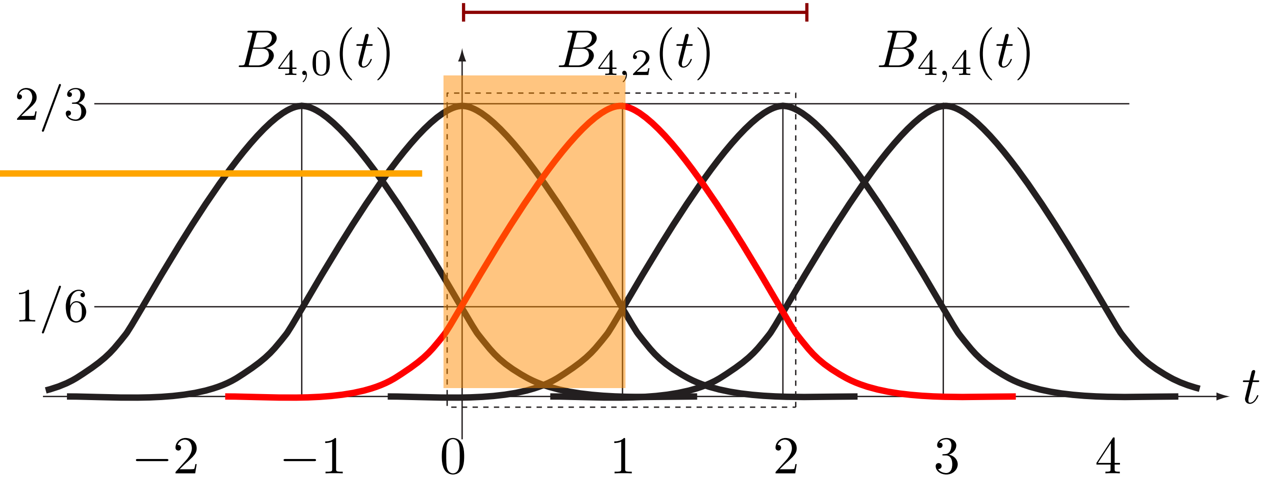

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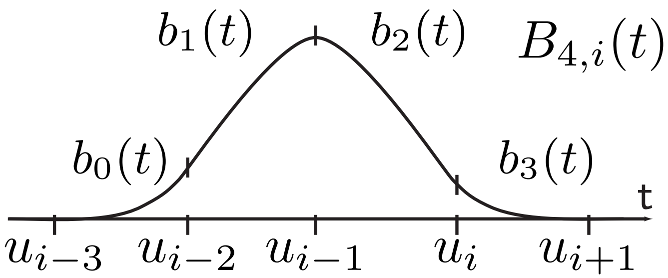

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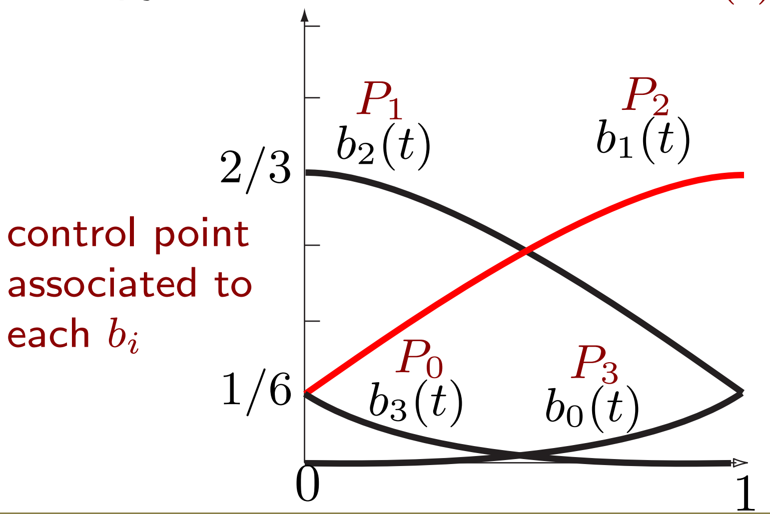

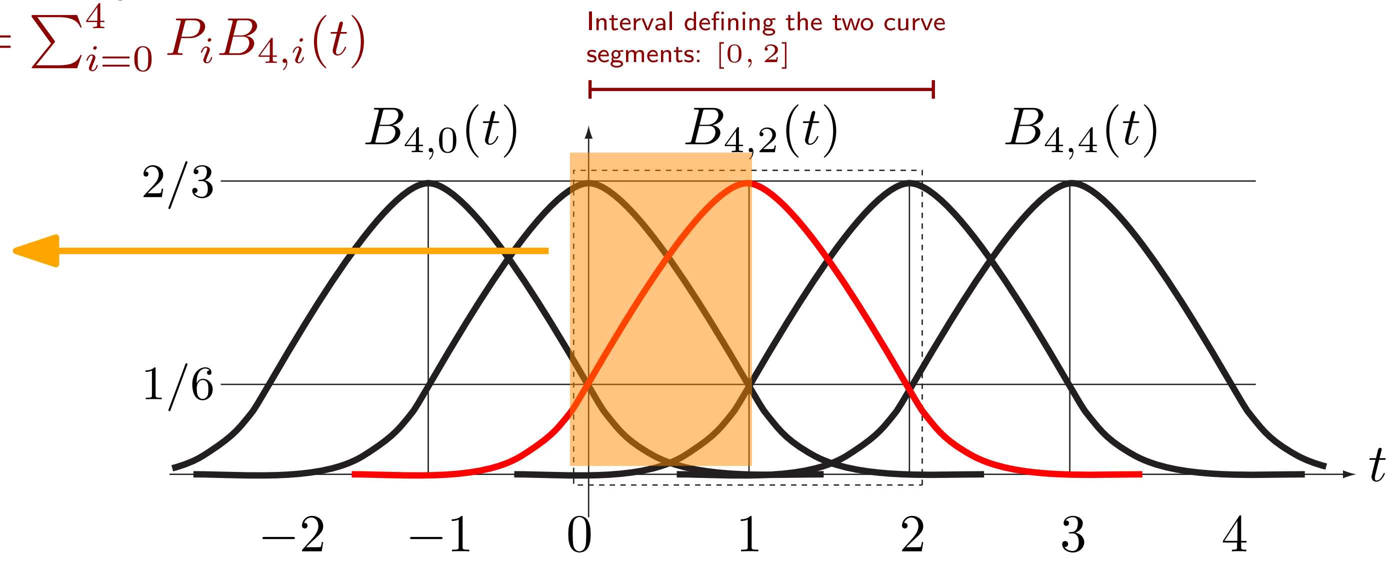

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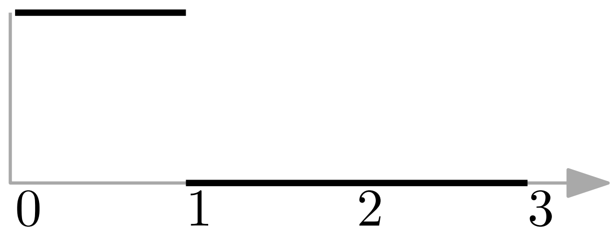

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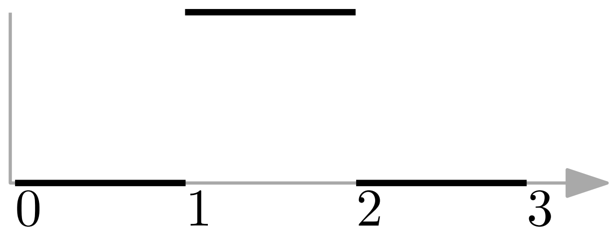

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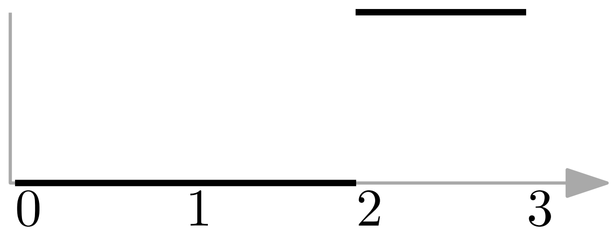

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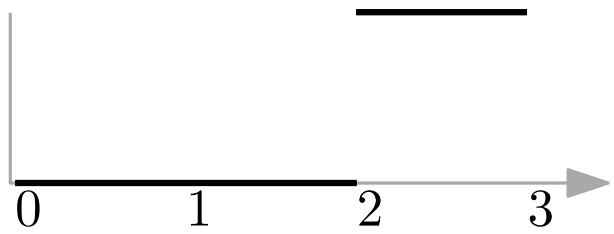

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$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

(taking $0/0$ as 0)

Examples of B-spline basis functions

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Question: how do the basis functions look for order 2 ($k = 2$)? (for uniform knots)

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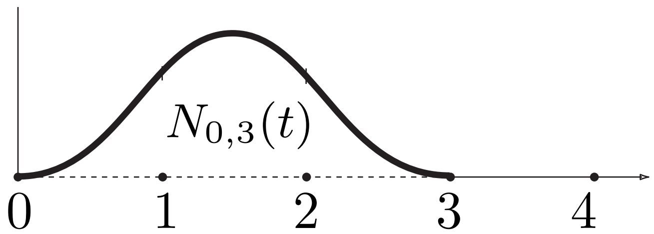

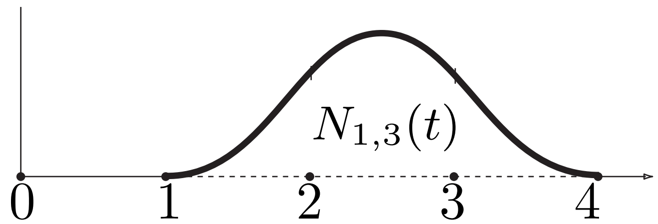

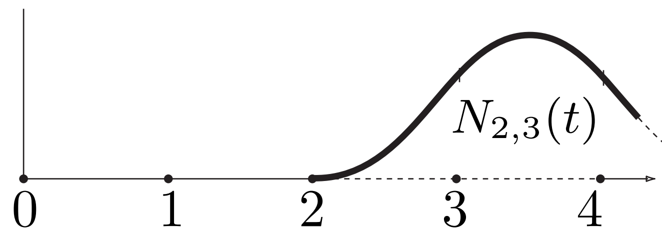

Some important properties

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

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B-SPLINE BASIS FUNCTIONS

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6) Continuity

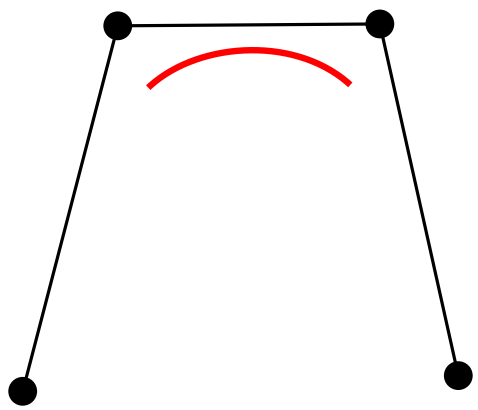
For uniform knots, the curve and its $k-1$ derivatives are continuous
(Non-uniform B-Splines can have discontinuities at knot values!)

UNDERSTANDING KNOT VECTORS

Open (or clampled) uniform B-Splines

Open (or clampled) uniform B-Splines

Uniform knot vector except at ends: at the beginning and end knot values are repeated k times



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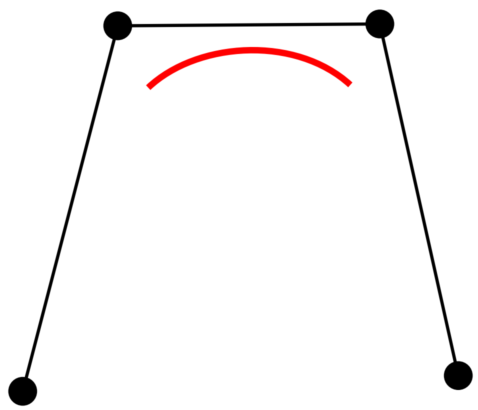

$n = 3$ (4 control points)

$k = 4$ (cubic B-spline)

uniform knot vector: $(0, 1/7, 2/7, 3/7, 4/7, 5/7, 6/7, 1)$

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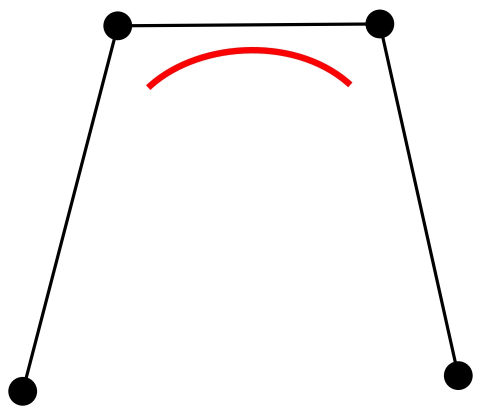

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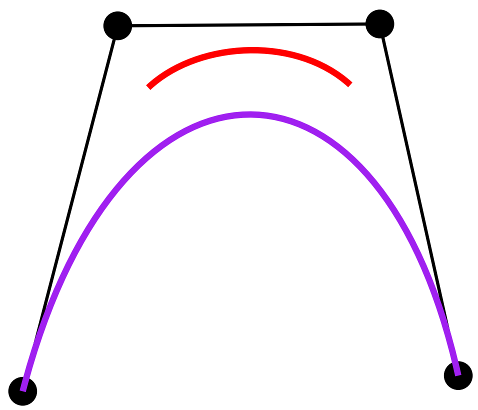

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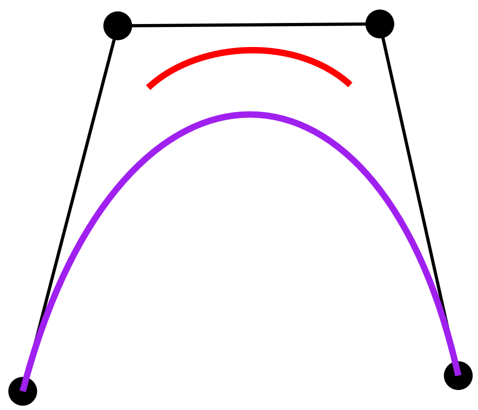

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Cubic Bézier curve! $\longrightarrow$ Always the case when $k = n + 1$

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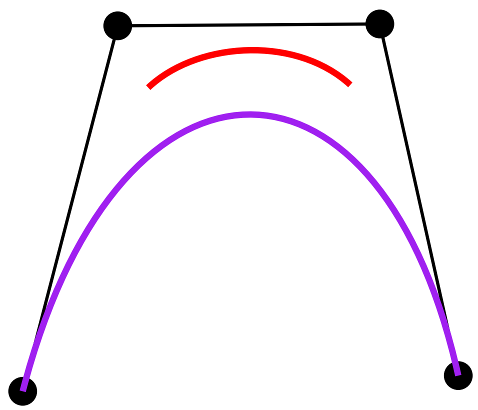

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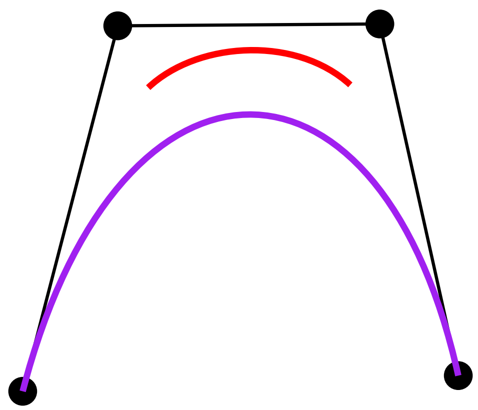
Cubic Bézier curve! $\longrightarrow$ Always the case when $k = n + 1$

Example: compute open basis functions for $n = 2$ and $k = 3$ (quadratic) B-splines

Open (or clampled) uniform B-Splines

Uniform knot vector except at ends: at the beginning and end knot values are repeated k times



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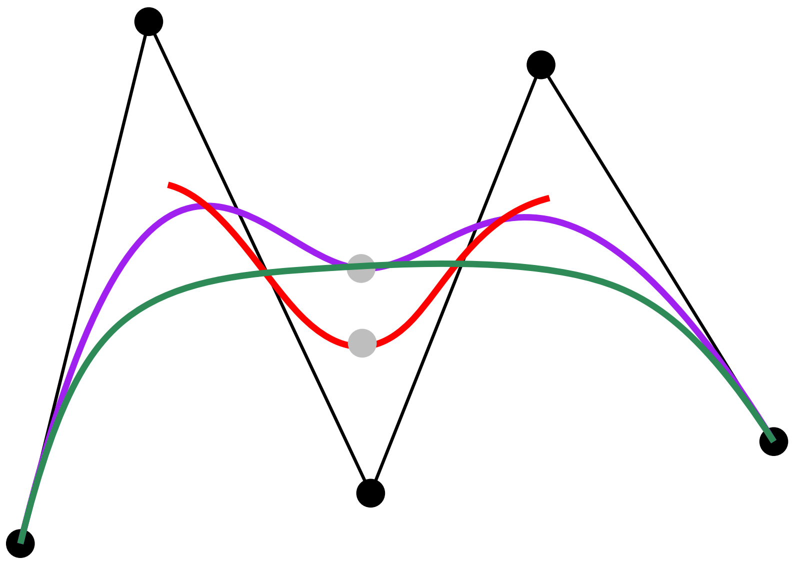

$n = 4$ (5 control points)

$k = 4$ (cubic B-spline)

uniform knot vector: $(0, 1/8, 2/8, 3/8, 4/8, 5/8, 6/8, 7/8, 1)$

"open" knot vector: $(0, 0, 0, 0, 0.5, 1, 1, 1, 1)$

degree-4 Bézier—knot vector: $(0, 0, 0, 0, 0, 1, 1, 1, 1, 1)$

Open uniform B-spline curves always start at P_0 and end at P_n. Tangents are also like in Bézier curves

Example for quadratic open B-Splines

Example: compute basis functions for 5 control points ($n = 4$) and $k = 3$ (i.e., quadratic open B-splines)

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Recall:

- knot vector: $(0, 0, 0, 1, 2, 3, 3, 3)$
- t goes from $t_{k-1} = t_2 = 0$ to $t_{n+1} = t_5 = 3$
- need to compute 5 bases: $N_{0,3}(t)$ to $N_{4,3}(t)$

$$N_{i,1} = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

More examples

Bézier vs open B-Spline of order 3
where $n = 9$ and $k = 3$

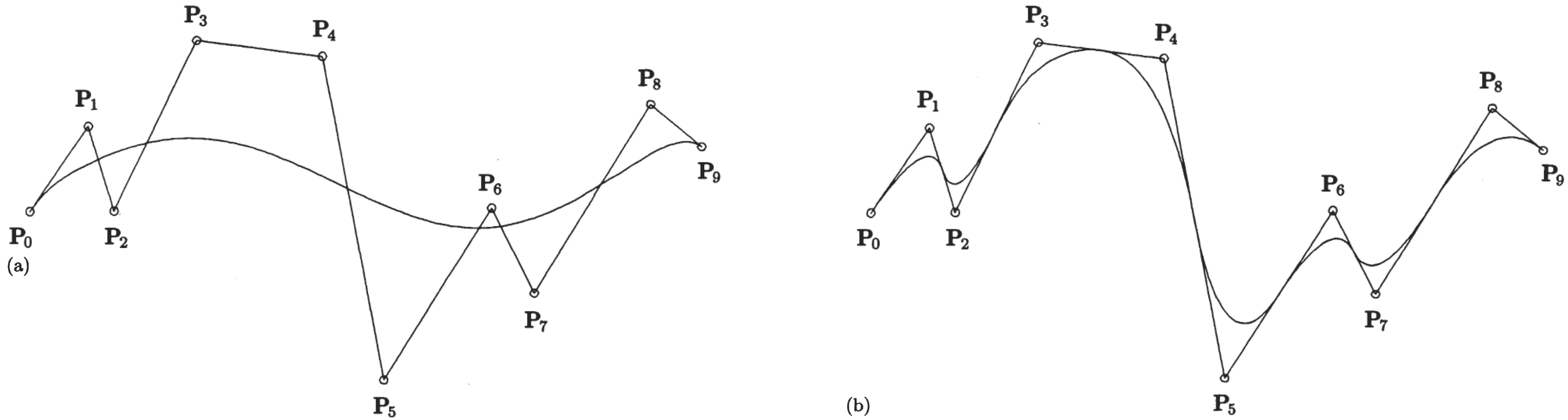

Figure 3.8. B-spline curves. (a) A ninth-degree Bézier curve on the knot vector $U = \{0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,1,1\}$; (b) a quadratic curve using the same control polygon defined on $U = \{0,0,0,\ ^1/_8,\ ^2/_8,\ ^3/_8,\ ^4/_8,\ ^5/_8,\ ^6/_8,\ ^7/_8, 1,1,1\}$.

Figure from [Piegl and Tiller]

NON-UNIFORM B-SPLINES

When knots are *not* equally spaced

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ignoring the first and last k knots, in case of open b-splines

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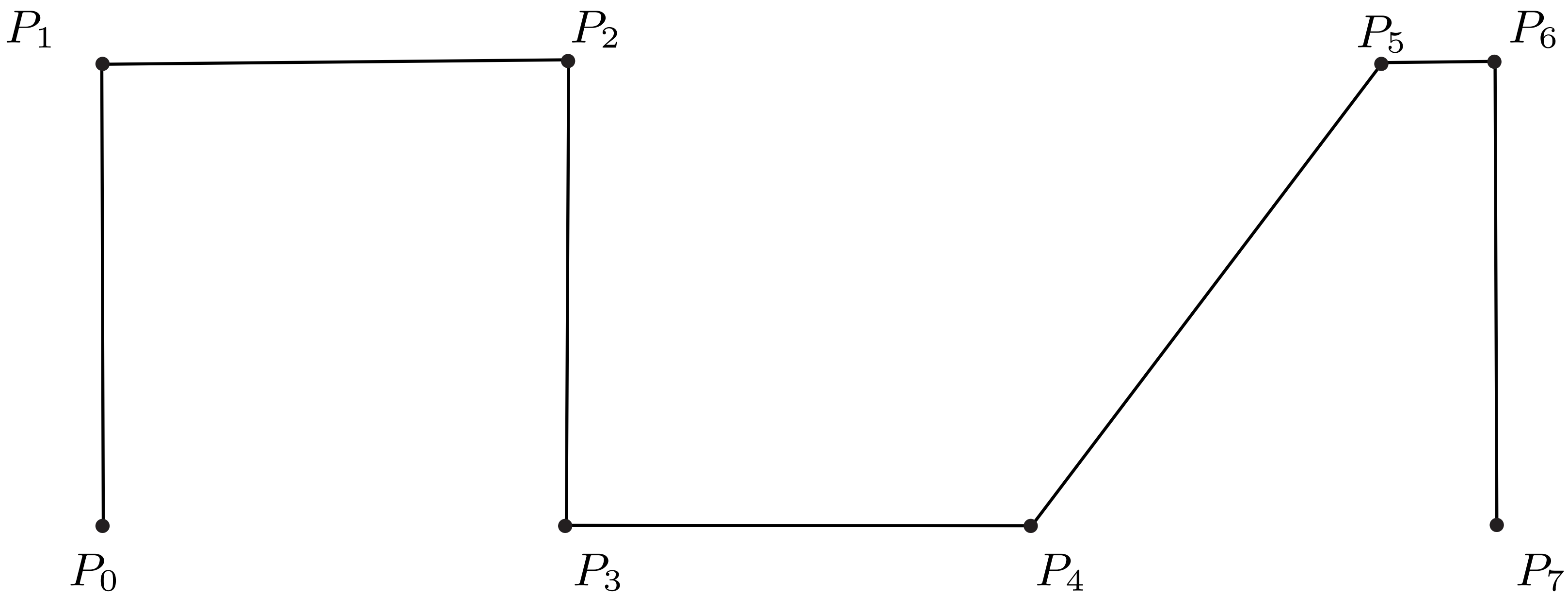

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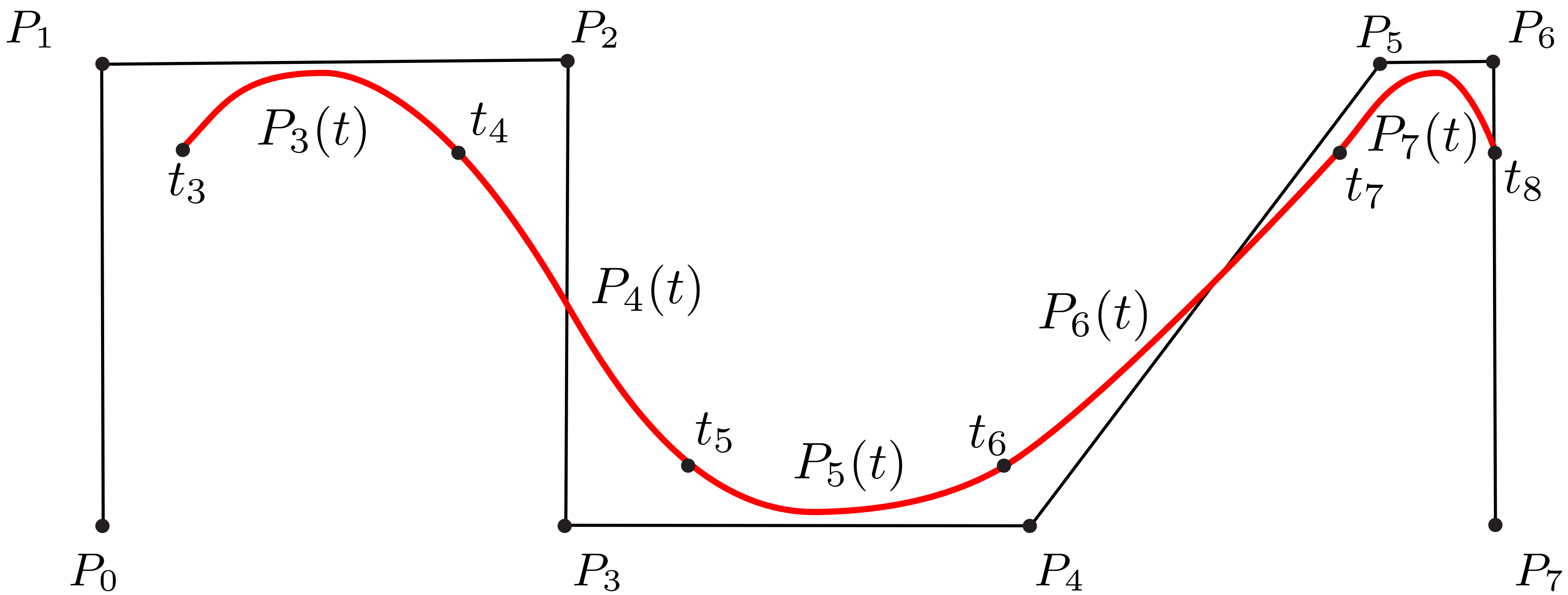

Knot vector:
$(-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8)$

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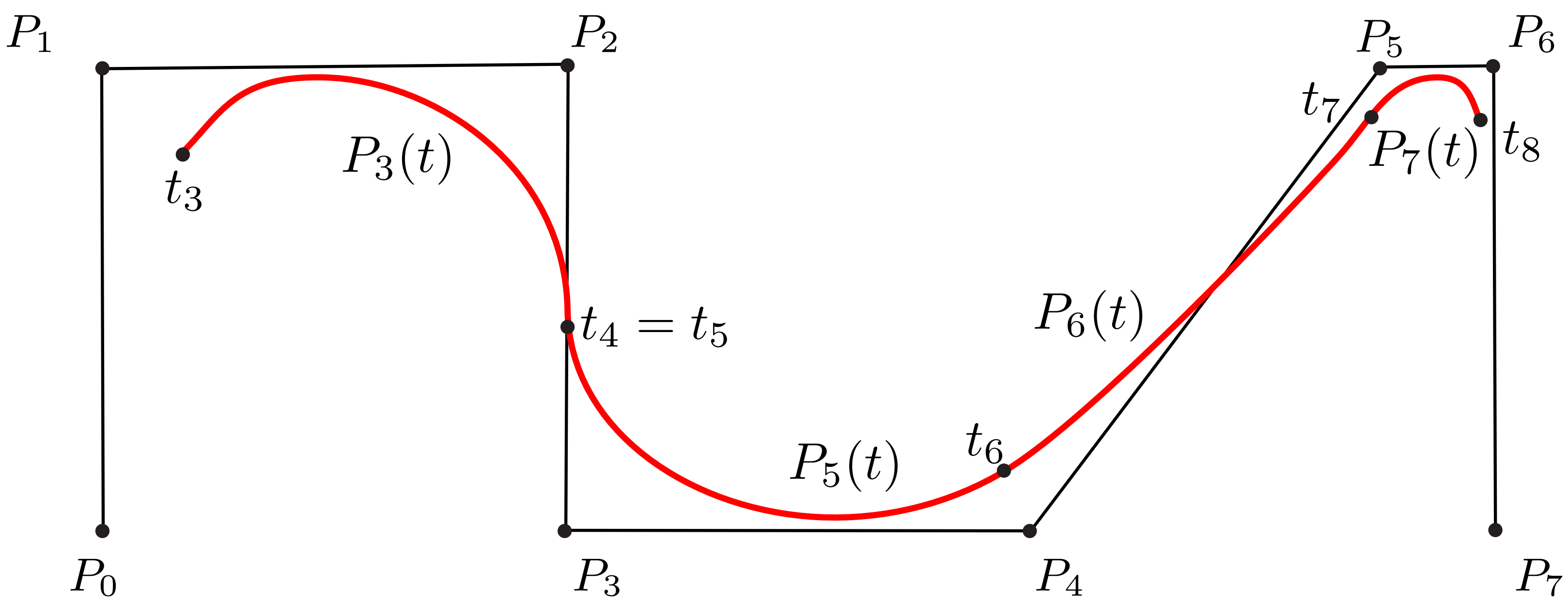

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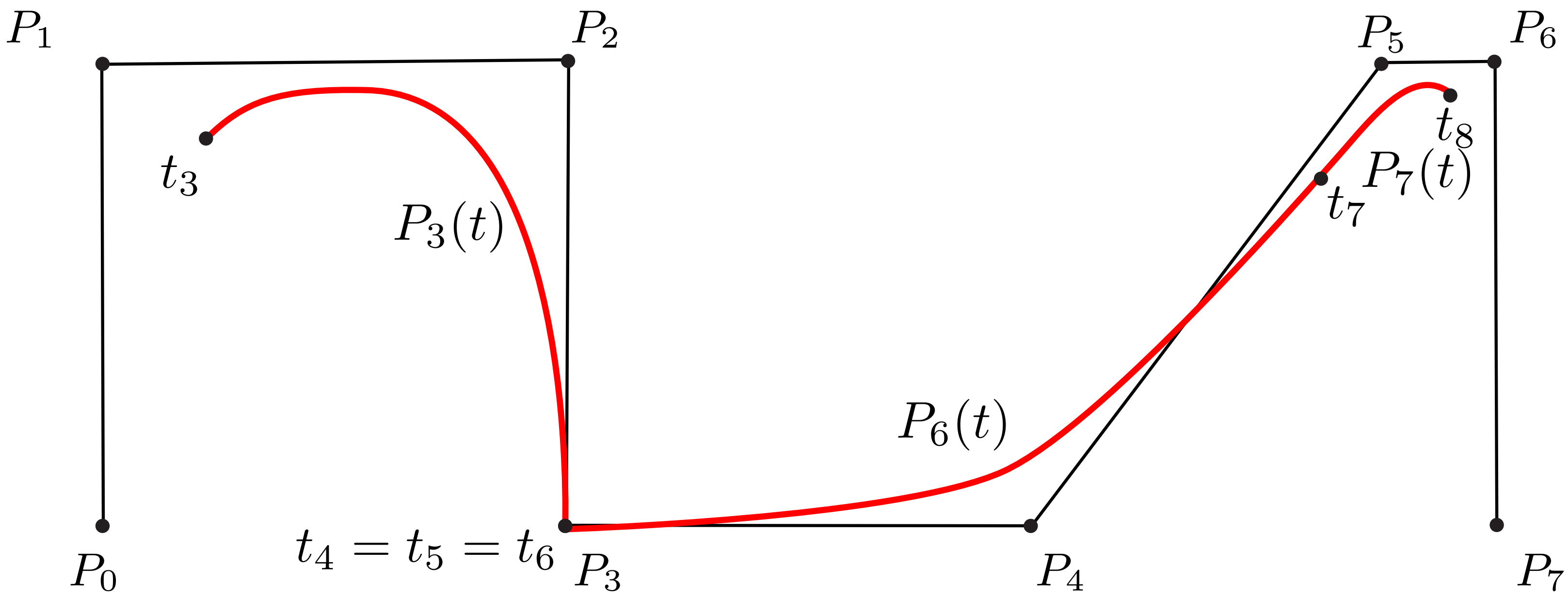

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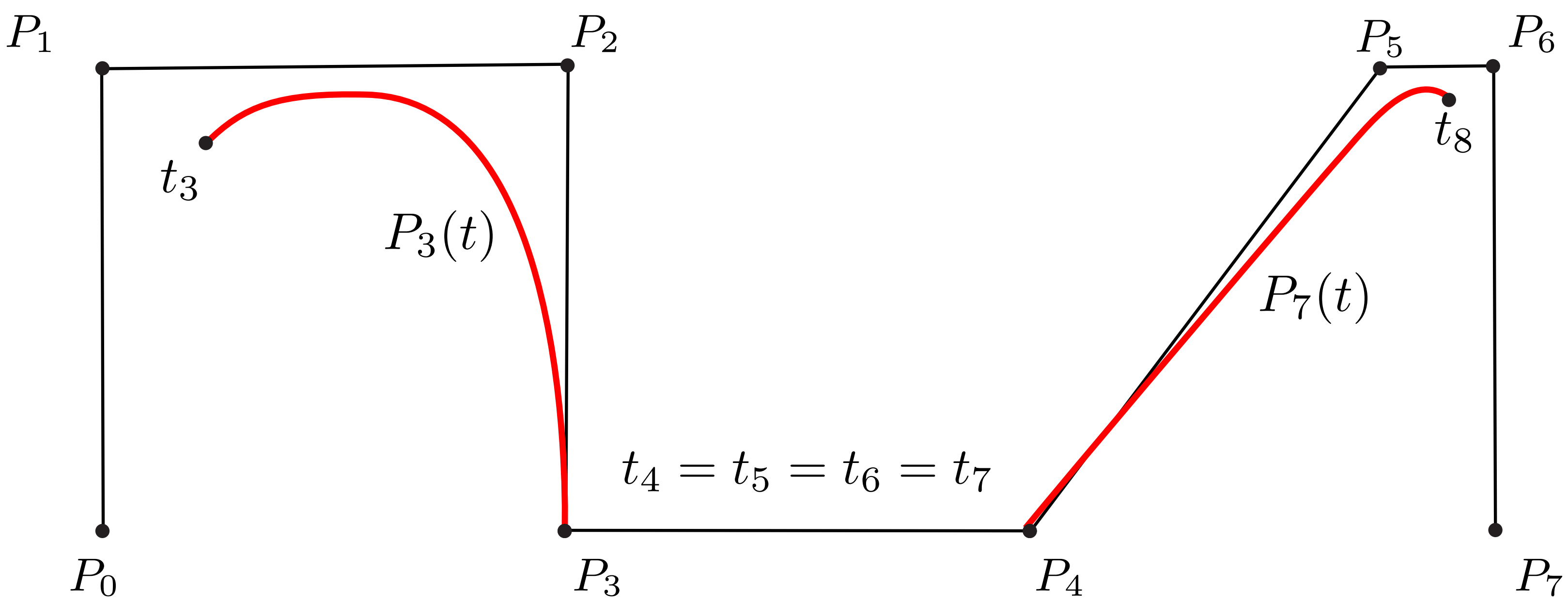

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Understanding knot vectors

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Open uniform B-splines interpolate the first and last control points due to the knot multiplicity

In general: continuity at the knots depends on multiplicity

$N_{i,k}(t)$ is $(k-m-1)$ times continuously differentiable, where m is the multiplicity of the knot

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- If a knot appears twice, the cubic B-spline will be only C^1-continuous there
- If a knot appears three times, the cubic B-spline will be only C^0-continuous there

See example is
http://geometrie.foretnik.net/files/NURBS-en.swf

Matrix form

Matrix-based expressions to compute non-uniform B-splines also exist

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Linear case ($k = 2$)

$$N_{i2} = \frac{u - u_i}{u_{i+1} - u_i} N_{i1}(u) + \frac{u_{i+2} - u}{u_{i+2} - u_{i+2}} N_{i+1,1}(u)$$

$$= \begin{cases} \dfrac{u - u_i}{u_{i+1} - u_i} & \text{for } u \in [u_i, u_{i+1}), \\ \dfrac{u_{i+2} - u}{u_{i+2} - u_{i+1}} & \text{for } u \in [u_{i+1}, u_{i+2}), \\ 0 & \text{otherwise.} \end{cases}$$

For $i = 0$, this becomes

$$N_{02} = \begin{cases} \dfrac{u - u_0}{u_1 - u_0} & \text{for } u \in [u_0, u_1), \\ \dfrac{u_2 - u}{u_2 - u_1} & \text{for } u \in [u_1, u_2), \\ 0 & \text{otherwise.} \end{cases}$$

$$\Delta = u_2 - u_1$$

$$t = \frac{u - u_1}{\Delta} = \frac{u - u_1}{u_2 - u_1}.$$

$$\boxed{\mathbf{P}(t) = (t, 1) \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \end{pmatrix}}$$

$t \in [0, 1]$

$N_{12}(u)$ is obtained by incrementing all the indices

Matrix form

Matrix-based expressions to compute non-uniform B-splines also exist

Quadratic case $(k = 3)$

$N_{13}(u)$ and $N_{23}(u)$ are obtained by incrementing all the indices

$$N_{03}(u) = \begin{cases} \frac{u-u_0}{u_2-u_0} \cdot \frac{u-u_0}{u_1-u_0} & \text{for } u \in [u_0, u_1), \\ \frac{u-u_0}{u_2-u_0} \cdot \frac{u_2-u}{u_2-u_1} + \frac{u_3-u}{u_3-u_1} \cdot \frac{u-u_1}{u_2-u_1} & \text{for } u \in [u_1, u_2), \\ \frac{u_3-u}{u_3-u_1} \cdot \frac{u_3-u}{u_3-u_2} & \text{for } u \in [u_2, u_3), \\ 0 & \text{otherwise.} \end{cases}$$

over subinterval $[u_2, u_3)$

$$N_{03}(u) = \frac{u_3-u}{u_3-u_1} \cdot \frac{u_3-u}{u_3-u_2},$$
$$N_{13}(u) = \frac{u-u_1}{u_3-u_1} \cdot \frac{u_3-u}{u_3-u_2} + \frac{u_4-u}{u_4-u_2} \cdot \frac{u-u_2}{u_3-u_2},$$
$$N_{23}(u) = \frac{u-u_2}{u_4-u_2} \cdot \frac{u-u_2}{u_3-u_2}.$$

Need notation for difference between consecutive knots:

$$\Delta_1 = u_2 - u_1, \quad \Delta_2 = u_3 - u_2, \quad \Delta_3 = u_4 - u_3.$$

We also define $t = (u - u_2)/\Delta_2$, which implies

$$\begin{aligned} u - u_1 &= t\Delta_2 + \Delta_1, \\ u - u_2 &= t\Delta_2, \\ u - u_3 &= (t-1)\Delta_2, \\ u - u_4 &= t\Delta_2 - (\Delta_2 + \Delta_3). \end{aligned}$$

$$\boxed{\mathbf{P}(t) = (t^2, t, 1) \begin{pmatrix} a & -a-b & b \\ -2a & 2a & 0 \\ a & 1-a & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \end{pmatrix} \qquad a = \frac{\Delta_2}{\Delta_1 + \Delta_2}, \quad b = \frac{\Delta_2}{\Delta_2 + \Delta_3},}$$

$t \in [0, 1]$

Matrix form

Matrix-based expressions to compute non-uniform B-splines also exist

Cubic case $(k = 4)$

$$N_{04}(u) = \begin{cases} \frac{u-u_0}{u_3-u_0}\cdot\frac{u-u_0}{u_2-u_0}\cdot\frac{u-u_0}{u_1-u_0} & \text{for } u\in[u_0,u_1),\\ \frac{u-u_0}{u_3-u_0}\cdot\frac{u-u_0}{u_2-u_0}\cdot\frac{u_2-u}{u_2-u_1} & \\ \quad +\frac{u-u_0}{u_3-u_0}\cdot\frac{u_3-u}{u_3-u_1}\cdot\frac{u-u_1}{u_2-u_1} & \\ \quad +\frac{u_4-u}{u_4-u_1}\cdot\frac{u-u_1}{u_3-u_1}\cdot\frac{u-u_1}{u_2-u_1} & \text{for } u\in[u_1,u_2),\\ \frac{u-u_0}{u_3-u_0}\cdot\frac{u_3-u}{u_3-u_1}\cdot\frac{u_3-u}{u_3-u_2} & \\ \quad +\frac{u_4-u}{u_4-u_1}\cdot\frac{u-u_1}{u_3-u_1}\cdot\frac{u_3-u}{u_3-u_2} & \\ \quad +\frac{u_4-u}{u_4-u_1}\cdot\frac{u_4-u}{u_4-u_2}\cdot\frac{u-u_2}{u_3-u_2} & \text{for } u\in[u_2,u_3),\\ \frac{u_4-u}{u_4-u_1}\cdot\frac{u_4-u}{u_4-u_2}\cdot\frac{u_4-u}{u_4-u_3} & \text{for } u\in[u_3,u_4),\\ 0 & \text{otherwise.} \end{cases}$$

$N_{14}(u), N_{24}(u)$ and $N_{34}(u)$ are obtained by incrementing all the indices

Only in $[u_3, u_4)$ all four are nonzero, with values:

$$N_{04}(u) = \frac{u_4-u}{u_4-u_1}\cdot\frac{u_4-u}{u_4-u_2}\cdot\frac{u_4-u}{u_4-u_3},$$

$$N_{14}(u) = \frac{u-u_1}{u_4-u_1}\cdot\frac{u_4-u}{u_4-u_2}\cdot\frac{u_4-u}{u_4-u_3} + \frac{u_5-u}{u_5-u_2}\cdot\frac{u-u_2}{u_4-u_2}\cdot\frac{u_4-u}{u_4-u_3} + \frac{u_5-u}{u_5-u_2}\cdot\frac{u_5-u}{u_5-u_3}\cdot\frac{u-u_3}{u_4-u_3},$$

$$N_{24}(u) = \frac{u-u_2}{u_5-u_2}\cdot\frac{u-u_2}{u_4-u_2}\cdot\frac{u_4-u}{u_4-u_3} + \frac{u-u_2}{u_5-u_2}\cdot\frac{u_5-u}{u_5-u_3}\cdot\frac{u-u_3}{u_4-u_3} + \frac{u_6-u}{u_6-u_3}\cdot\frac{u-u_3}{u_5-u_3}\cdot\frac{u-u_3}{u_4-u_3},$$

$$N_{34}(u) = \frac{u-u_3}{u_6-u_3}\cdot\frac{u-u_3}{u_5-u_3}\cdot\frac{u-u_3}{u_4-u_3}.$$

take
$$\Delta_1 = u_2-u_1,\quad \Delta_2 = u_3-u_2,\quad \Delta_3 = u_4-u_3,$$
$$\Delta_4 = u_5-u_4,\quad \Delta_5 = u_6-u_5,\quad t = (u-u_3)/\Delta_3.$$

Matrix form

Matrix-based expressions to compute non-uniform B-splines also exist

Cubic case $(k = 4)$

$$N_{04}(u) = \begin{cases} \frac{u-u_0}{u_3-u_0} \cdot \frac{u-u_0}{u_2-u_0} \cdot \frac{u-u_0}{u_1-u_0} \\ \frac{u-u_0}{u_3-u_0} \cdot \frac{u-u_0}{u_2-u_0} \cdot \frac{u_2-u}{u_2-u_1} \\ \quad + \frac{u-u_0}{u_3-u_0} \cdot \frac{u_3-u}{u_3-u_1} \cdot \frac{u-u_1}{u_2-u_1} \\ \quad + \frac{u_4-u}{u_4-u_1} \cdot \frac{u-u_1}{u_3-u_1} \cdot \frac{u-u_1}{u_2-u_1} \\ \frac{u-u_0}{u_3-u_0} \cdot \frac{u_3-u}{u_3-u_1} \cdot \frac{u_3-u}{u_3-u_2} \\ \quad + \frac{u_4-u}{u_4-u_1} \cdot \frac{u-u_1}{u_3-u_1} \cdot \frac{u_3-u}{u_3-u_2} \\ \quad + \frac{u_4-u}{u_4-u_1} \cdot \frac{u_4-u}{u_4-u_2} \cdot \frac{u-u_2}{u_3-u_2} \\ \frac{u_4-u}{u_4-u_1} \cdot \frac{u_4-u}{u_4-u_2} \cdot \frac{u_4-u}{u_4-u_3} & \text{for } u \in [u_3, u_4), \\ 0 & \text{otherwise.} \end{cases}$$

$$N_{34}(u) = \frac{\cdots}{u_6-u_3} \cdot \frac{\cdots}{u_5-u_3} \cdot \frac{\cdots}{u_4-u_3}.$$

We obtain:

$$\mathbf{P}(t) = (t^3, t^2, t, 1) \begin{pmatrix} -a & a+b+c & -b-c-d & d \\ 3a & -3a-3b & 3b & 0 \\ -3a & 3a-3e & 3e & 0 \\ a & 1-a-f & f & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix},$$

where

$$a = \frac{\Delta_3^2}{(\Delta_1+\Delta_2+\Delta_3)(\Delta_2+\Delta_3)}, \qquad d = \frac{\Delta_3^2}{(\Delta_3+\Delta_4+\Delta_5)(\Delta_3+\Delta_4)},$$

$$b = \frac{\Delta_3^2}{(\Delta_2+\Delta_3+\Delta_4)(\Delta_2+\Delta_3)}, \qquad e = \frac{\Delta_2\Delta_3}{(\Delta_2+\Delta_3+\Delta_4)(\Delta_2+\Delta_3)},$$

$$c = \frac{\Delta_3^2}{(\Delta_2+\Delta_3+\Delta_4)(\Delta_3+\Delta_4)}, \qquad f = \frac{\Delta_2^2}{(\Delta_2+\Delta_3+\Delta_4)(\Delta_2+\Delta_3)}.$$

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The most general parametric curve

Same idea as for rational Bézier: each control point P_i has a weight, $w_i \geq 0$. This gives even more flexibility to shape the curve

NON-UNIFORM RATIONAL B-SPLINES (NURBS)

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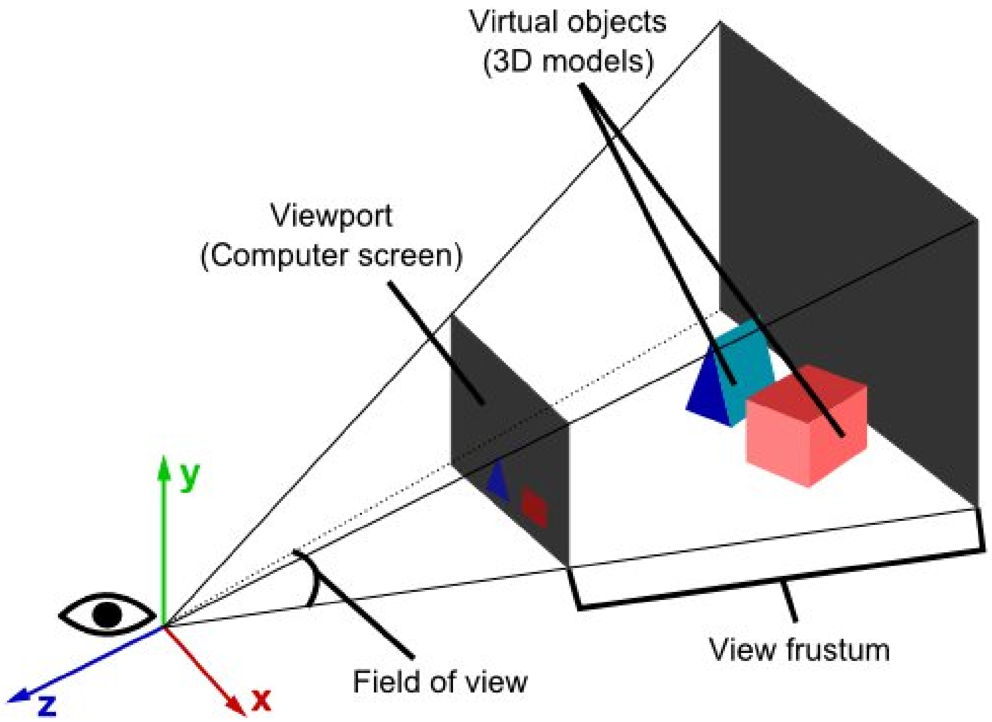

Figure from real3dtutorials.com

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Advantages

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- It can represent conic curves exactly (e.g., segments of circles, ellipses, hyperbolas and parabolas)
- It is more general, so it includes as particular cases all other B-splines and Bézier curves

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Any 2D point (x, y) is equivalent to a 3D point: (wx, wy, w)

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→ e.g., each point $P_i = (x_i, y_i)$, becomes $Q_i = (w_i x_i, w_i y_i, w_i)$, for some $w_i \geq 0$

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$$P_{\mathsf{r}}(t) = \frac{\sum_{i=0}^{n} w_i P_i N_{i,k}(t)}{\sum_{j=0}^{n} w_j N_{j,k}(t)} = \sum_{i=0}^{n} P_i \left(\frac{w_i N_{i,k}(t)}{\sum_{j=0}^{n} w_j N_{j,k}(t)} \right) = \sum_{i=0}^{n} P_i R_{i,k}(t)$$

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Rational curves as curves in projective space

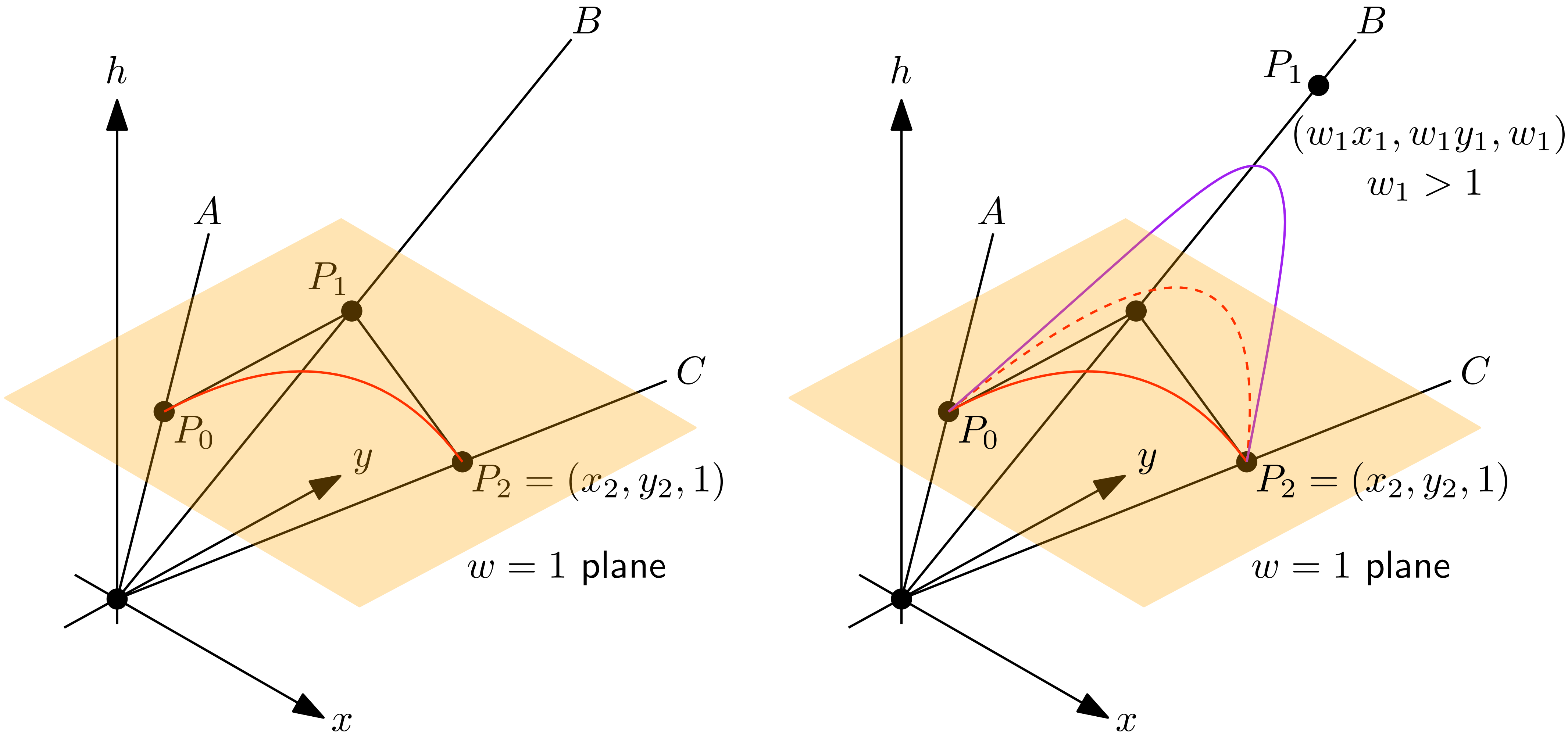

Figure adapted from book by Mortenson

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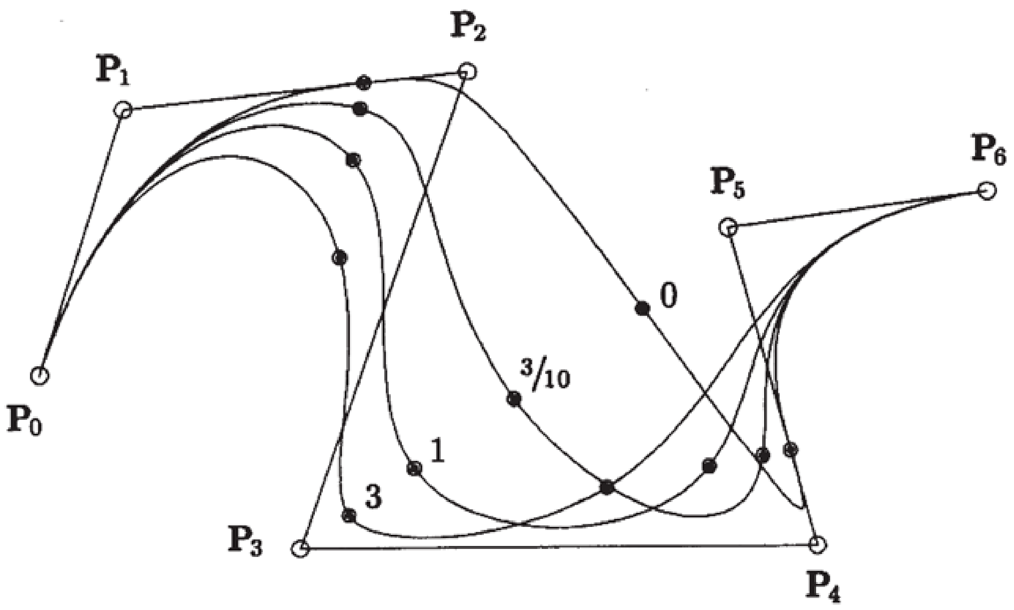

Figure 4.2. Rational cubic B-spline curves, with w_3 varying.

Figure from [Piegl and Tiller]

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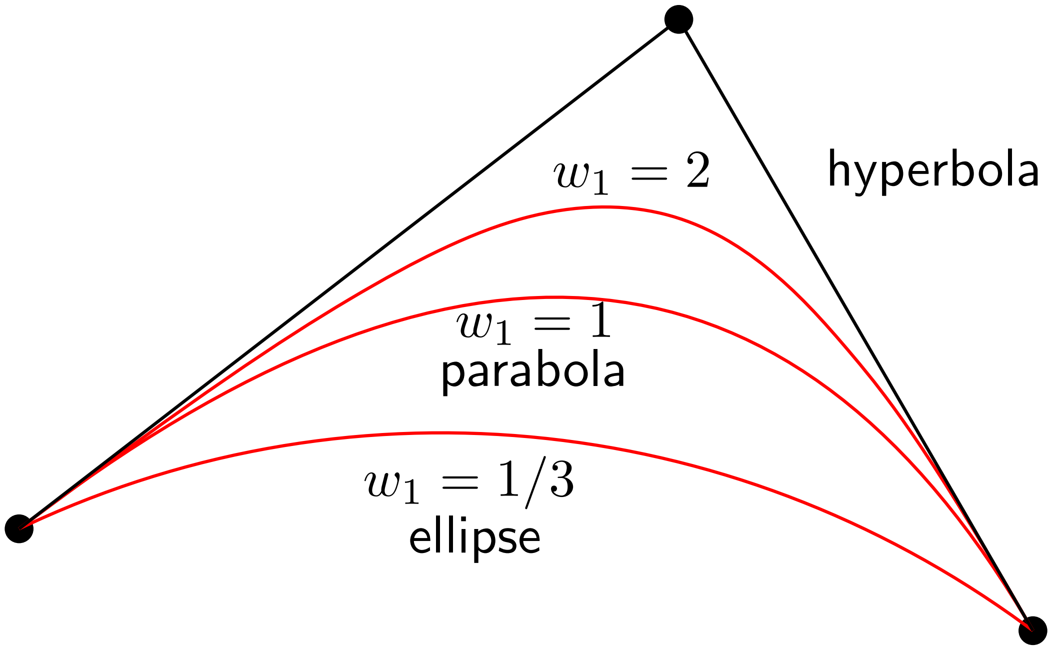

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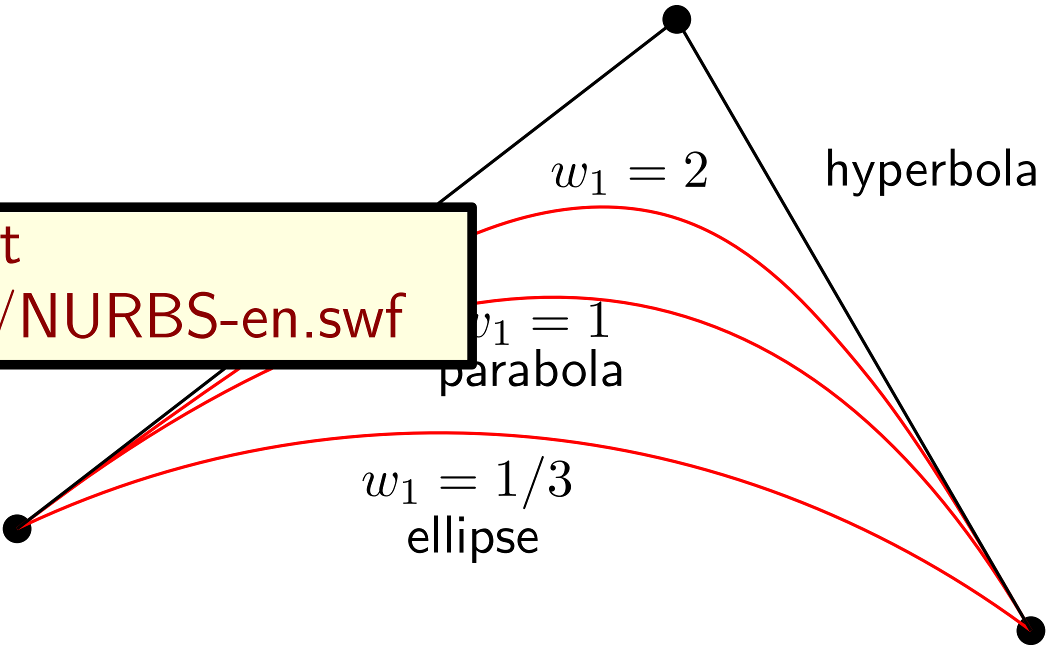

Play with NURBS at
http://geometrie.foretnik.net/files/NURBS-en.swf