Chapter 2

Projectile Drag

Let's talk about air (or liquid) drag!

- Is a resistive force
  - Usually assume that it is directly opposite the direction of motion
  - Not **always** the case though (Lift on airplane wings)

We'll focus on the case where it is opposite the direction of motion.

Definition 2.1: Drag Force

The resistive drag force can thus be defined as:

$$\vec{\boldsymbol{f}} = -f(v)\,\hat{\boldsymbol{v}}$$

where the negative and unit vector in the direction of velocity are what point the force in the direction opposite the motion.

The exact form of $f(v)$ can take on very complicated forms, so we can simplify by writing as a Taylor expansion:

$$f(v) = bv + cv^2 + \mathcal{O}(3) \tag{2.1}$$

This leaves us two terms to consider:

- The linear term: bv
  - Originates from the viscous drag of the air/liquid
  - $b = \beta D$
- The quadratic term: cv^2
  - Originates from having to push along the air/liquid in front
  - $c = \gamma D^2$

where, D is the diameter of a sphere and, at STP,

$$\beta = 1.6 \times 10^{-4}\,\mathrm{Ns/m^2}$$
$$\gamma = 0.25\,\mathrm{Ns^2/m^4}$$

Remark 2.1

The two terms are generally not equally weighted!

Example 2.1

Take a baseball traveling at 100 mph. Thus we have that

$$D = 74.68\,\mathrm{mm}$$
$$v = 100\,\mathrm{mph} = 44.7\,\mathrm{m/s}$$

And thus:

$$b = \beta D = \left(1.6 \times 10^{-4}\right)\left(74.68 \times 10^{-3}\right) = 1.19 \times 10^{-5}$$
$$c = \gamma D^2 = (0.25)\left(74.68 \times 10^{-3}\right)^2 = 1.39 \times 10^{-3}$$

Each term would then have a value of:

$$bv = \left(1.19 \times 10^{-5}\right)(44.7) = 0.0623\,\mathrm{N}$$
$$cv^2 = \left(1.39 \times 10^{-3}\right)(44.7)^2 = 124.5\,\mathrm{N}$$

And so quadratic drag clearly dominates a baseballs flight!

It is useful to be able to quickly get an idea for which terms dominate in different situations:

$$\frac{f_{quad}}{f_{lin}} = \frac{cv^2}{bv} = \frac{\gamma D^2 v^2}{\beta D v} = \frac{\gamma}{\beta} D v$$

So that

- Quadratic term dominates if
  - D or v is large
- Linear term dominates if
  - D or v is tiny

Example 2.2

Take Millikans drop of oil:

$$D = 1.5\,\mu\mathrm{m}$$
$$v = 5 \times 10^{-5}\,\mathrm{m/s}$$

$$\Rightarrow \frac{f_q}{f_l} = \frac{0.25}{1.6 \times 10^{-4}}\left(1.5 \times 10^{-6}\right)\left(5 \times 10^{-5}\right) = 1.17 \times 10^{-7}$$

So so the linear term clearly dominates.

For an instance where both are comparable, consider something like a falling raindrop, where

$$\frac{f_q}{f_l} \approx 1$$

Definition 2.2: Reynold's Number

In terms of classic fluid terms, $\frac{f_q}{f_l}$ is on the same order as a fluid's **Reynold's Number**, defined as:

$$R = \frac{Dv\rho}{\eta}$$

where

$$\rho = \text{ density}$$
$$\eta = \text{ viscosity}$$

2.1 Linear Drag

We'll look at the linear dominating case first. So think tiny droplets or super slow movement through something like molasses.

Consider at object in free-fall:

$$\Rightarrow \text{ 2 forces} \begin{cases} m\vec{\boldsymbol{g}}, & \text{Gravity} \\ -bv\,\hat{\boldsymbol{v}}, & \text{Drag} \end{cases}$$

In Cartesian coordinates, then:

$$\begin{aligned} m\ddot{\vec{\boldsymbol{r}}} &= m\vec{\boldsymbol{g}} - bv\,\hat{\boldsymbol{v}} \\ &= m\vec{\boldsymbol{g}} - b\vec{\boldsymbol{v}} \\ m\dot{\vec{\boldsymbol{v}}} &= m\vec{\boldsymbol{g}} - b\vec{\boldsymbol{v}} \\ &= mg\,\hat{\boldsymbol{y}} - b\left(v_x\,\hat{\boldsymbol{x}} + v_y\,\hat{\boldsymbol{y}}\right) \end{aligned}$$

Breaking things up by components gives us:

$$m\dot{v}_x = -bv_x \tag{2.2}$$
$$m\dot{v}_y = mg - bv_y \tag{2.3}$$

Remark 2.2

Notice that we were able to break the equations up such that each is purely in the x or y direction. We will *not* be able to do this in the quadratic case!

2.1.1 The Horizontal Case

We'll look in the horizontal case first, so rearranging Eq 2.2 gives us.

$$\begin{aligned}\dot{v}_x &= -\frac{b}{m}v_x \\ &= -kv_x\end{aligned}$$

Or, in other words, a function whose derivative is some multiple of itself. We know of functions with this property though: e^{stuff}! So we'll consider a *general solution* of:

$$v_x(t) = Ae^{-kt}$$

If we say that $v_x(0) = v_{x0}$:

$$v_x(t) = v_{x0}\, e^{-kt}$$

Definition 2.3: Characteristic Time

This is often written as:

$$v_x(t) = v_{x0}e^{-t/\tau}$$

where

$$\tau = \frac{m}{b} = \frac{1}{k}$$

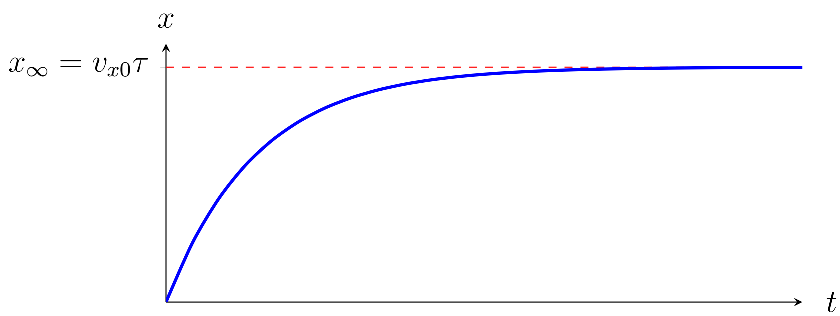
and τ is called the **characteristic time** or the **time constant**.

Note that this solution only gives us the velocity. Since we are normally interested in the position, we'll need to integrate:

$$\begin{aligned}x(t) &= \int v_{x0}e^{-t/\tau}\, dt \\ &= -v_{x0}\tau e^{-t/\tau} + C_1\end{aligned}$$

If $x(0) = 0$, then $C_1 = v_{x0}\tau$:

$$\begin{aligned} &= -v_{x0}\tau e^{-t/\tau} + v_{x0}\tau \\ x(t) &= \underbrace{\boxed{v_{x0}\tau}}_{x_\infty}\left(1 - e^{-t/\tau}\right)\end{aligned}$$



Remark 2.3

We like to put things in terms of a time constant τ because it gives us an easy method to discern how close an object is to its final (and steady) value.

$$t = \tau = \frac{m}{b} \quad \Rightarrow \quad x = x_\infty \left(1 - e^{-1}\right) \approx 0.63 x_\infty$$

t	x
0	0
τ	63% x_∞
2τ	86% x_∞
3τ	95% x_∞
4τ	98% x_∞
5τ	99% x_∞

This might save you a lot of effort if you only care about the objects current position!

2.1.2 The Vertical Case

Recall from Eq 2.3 that we have:

$$m\dot{v}_y = mg - bv$$

Definition 2.4: Terminal Velocity

As the speed increases, it will reach a point where the forces balance:

$$bv = mg$$

This speed in called the **terminal velocity**, as it will stop changing once it reaches this point.

$$v_{term} = \frac{mg}{b} = \frac{mg}{\beta D}$$

Remark 2.4

Note that larger objects will have smaller terminal velocities! (The parachute effect)

We'll rewrite Eq 2.3 in terms of the terminal velocity:

$$m\dot{v}_y = m\left(\frac{bv_{term}}{m}\right) - bv = -b(v - v_{term})$$

How do we go about solving this? Remembering that v_{term} is a *constant*, let:

$$\begin{aligned}
u &= v - v_{term} \\
\Rightarrow \dot{u} &= \dot{v} - \dot{v}_{term} \\
\dot{u} &= \dot{v} \\
\dot{u} &= -\frac{b}{m}(v - v_{term}) \\
\dot{u} &= -\frac{b}{m}u
\end{aligned}$$

But we already know the solution to this DE!

$$u(t) = Ae^{-t/\tau}$$

where τ is defined the same way as before.

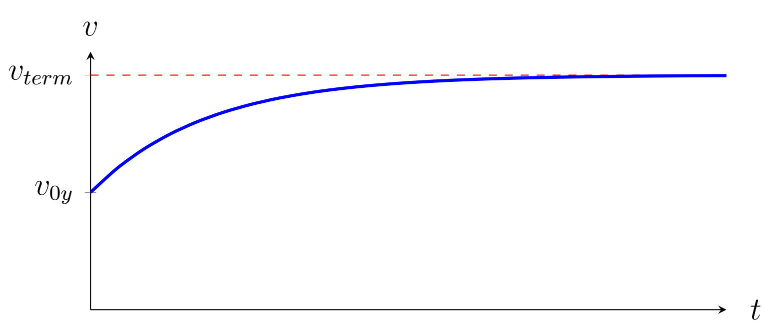
$$v(t) - v_{term} = Ae^{-t/\tau}$$

If $v(0) = V_{y0}$ then $A = v_{y0} - v_{term}$

$$v(t) = (v_{y0} - v_{term})\, e^{-t/\tau} + v_{term}$$

Regrouping terms:

$$v(t) = \underbrace{\boxed{v_{y0}e^{-t/\tau}}}_{\substack{\to 0 \text{ as} \\ \text{t increases}}} + v_{term}\left(1 - e^{-t/\tau}\right)$$



Remark 2.5

Note that if $v_{y0} = 0$ then
$$v(t) = v_{term}\left(1 - e^{-t/\tau}\right)$$

Again, we need to integrate to get back to our position:

$$\begin{aligned}
y(t) &= \int_0^t v_{0y}e^{-t/\tau} + v_{term}\left(1 - e^{-t/\tau}\right)\,dt \\
&= \left(v_{0y}(-\tau)e^{-t/\tau} + v_{0y}\tau\right) + v_{term}t + \left(v_{term}\tau e^{-t/\tau} - v_{term}\tau\right) \\
&= \tau\left(v_{term} - v_{0y}\right)e^{-t/\tau} + \tau(v_{0y} - v_{term}) + v_{term}t \\
&= \tau\left(v_{0y} - v_{term}\right)\left(1 - e^{-t/\tau}\right) + v_{term}t
\end{aligned}$$

2.1.3 Range

So we have expressions for both the x and the y positions of our projectile! Yea! Often times we want to find the trajectory though, which eliminates the time dependence. For non-drag projectiles this looked like:

$$\begin{aligned}
x &= v_{0x}t \\
y &= v_{0y}t - \frac{1}{2}gt^2 \\
\Rightarrow t &= \frac{x}{v_{0x}} \\
\Rightarrow y &= v_{0y}\frac{x}{v_{0x}} - \frac{1}{2}g\frac{x^2}{v_{0x}^2}
\end{aligned}$$

which is a parabola as we'd anticipate. We can do the *same thing* for our linear drag situation, but the math is just a lot uglier, and I won't take the time to show it.

$$\begin{aligned}
x(t) &= v_{0x}\tau\left(1 - e^{-t/\tau}\right) \\
y(t) &= \left(v_{0y} + v_{term}\right)\tau\left(1 - e^{-t/\tau}\right) - v_{term}t \\
\Rightarrow y(x) &= \frac{v_{0y} + v_{term}}{v_{0x}}x + v_{term}\tau\ln\left(1 - \frac{x}{v_{0x}\tau}\right)
\end{aligned}$$

Which is probably not the most useful for forms.

Remark 2.6

We swapped the signs of v_{term} in the y-expression to swap our axis back to being positive when pointing upwards!

One of the reasons we write out the trajectories though is because it makes it easy to find the range of a projectile shot by setting $y = 0$. For non-drag cases:

$$\begin{aligned}
0 &= \frac{v_{0y}}{v_{0x}} x - \frac{1}{2} g \frac{x^2}{v_{0x}^2} \\
\frac{1}{2} g \frac{x^2}{v_{0x}^2} &= \frac{v_{0y}}{v_{0x}} x \\
x &= \frac{2 v_{0x} v_{0y}}{g}
\end{aligned}$$

We can **not** do this for the linear drag case! You can't solve for x and get an analytic function. So we are left with two options:

- Use a computer to calculate the range for a single set of parameters
- Assume air resistance is small to simplify the expression
  - Implies b is small
    - Implies v_{term} and τ are BIG!

The first case is pretty straightforword, so we'll look more at the approximation that air resistance is small. In that case, our second term in the trajectory expression looks like:

$$\ln\left(1 - \frac{x}{v_{x0}\tau}\right) \approx \ln\left(1 - \text{ tiny}\right)$$

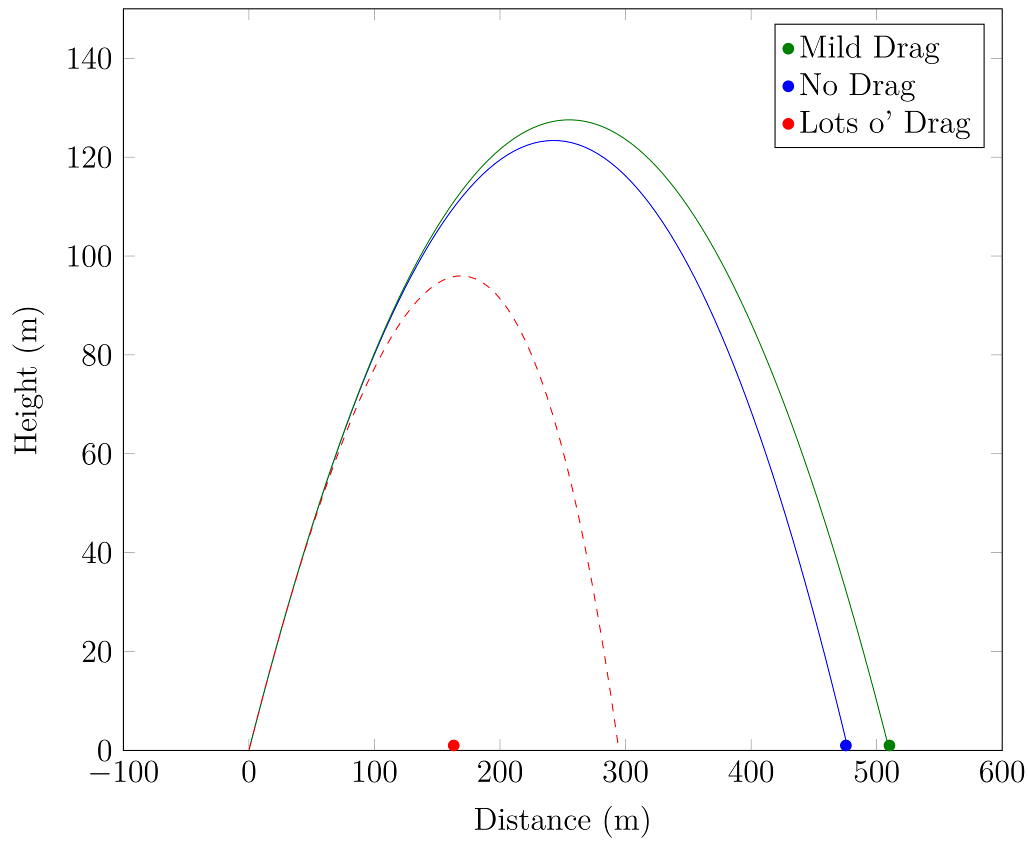
We can Taylor expand this about 0:

$$= -\left(\text{tiny} + \frac{\text{tiny}^2}{2} + \frac{\text{tiny}^3}{3} + \cdots\right)$$

So our trajectory will look like:

$$\begin{aligned}
0 &= \frac{v_{0y} + \cancel{v_{term}}}{\cancel{v_{0x}}} x - v_{term}\tau \left(\frac{x}{\cancel{v_{0x}}\tau} + \frac{1}{2}\left(\frac{x}{v_{0x}\tau}\right)^2 + \frac{1}{3}\left(\frac{x}{v_{x0}\tau}\right)^3 + \cdots\right) \\
&= \frac{v_{0y}}{v_{0x}} x - \frac{1}{2} v_{term} \frac{x^2}{v_{0x}^2 \tau} - \frac{1}{3} v_{term} \frac{x^3}{v_{x0}^3 \tau^2} \\
&= \frac{v_{0y}}{v_{0x}} x - \frac{1}{2} \tau g \frac{x^2}{v_{0x}^2 \tau} - \frac{1}{3} \tau g \frac{x^3}{v_{x0}^3 \tau^2} \\
\frac{1}{2} g \frac{x}{v_{0x}^2} &= \frac{v_{0y}}{v_{0x}} - \frac{1}{3} \frac{x^3}{v_{0x}^3 \tau} \\
x &= \frac{2 v_{0y} v_{0x}}{g} - \frac{2}{3} \frac{1}{v_{0x} v_{term}} x^2 \\
\Rightarrow R &\approx R_{vac} - \frac{2}{3} \frac{1}{v_{0x} v_{term}} R_{vac}^2 \\
&= R_{vac}\left(1 - \frac{4}{3} \frac{v_{0y}}{v_{term}}\right)
\end{aligned}$$

By way of comparison:



2.2 Quadratic Drag

Now we'll focus on the quadratic form of the drag:

$$\vec{\boldsymbol{f}} = -b\vec{\boldsymbol{v}} - \underbrace{\boxed{cv^2\,\hat{\boldsymbol{v}}}}_{\text{Focusing here now}}$$

Looking at our sum of the forces, we again have:

$$\begin{aligned} m\dot{\vec{\boldsymbol{v}}} &= m\vec{\boldsymbol{g}} + \vec{\boldsymbol{f}} \\ &= m\vec{\boldsymbol{g}} - cv^2\,\hat{\boldsymbol{v}} \\ &= m\vec{\boldsymbol{g}} - cv\vec{\boldsymbol{v}} \\ &= m(g_y\,\hat{\boldsymbol{y}}) - c\underbrace{\boxed{\sqrt{v_x^2 + v_y^2}}}_{\text{Mixes Terms!}}\,(v_x\,\hat{\boldsymbol{x}} + v_y\,\hat{\boldsymbol{y}}) \end{aligned}$$

Remark 2.7

We now have *mixed* terms! It is impossible to solve for equations purely in either the x or the y direction. As such, we either need to solve things numerically or hope that either v_x or v_y equals 0.

2.2.1 Horizontal Motion ($v_y = 0$)

In this case we have that

$$\begin{aligned} m\dot{\vec{\boldsymbol{v}}} &= -cv_x^2\,\hat{\boldsymbol{x}} \\ m\dot{v}_x &= -cv_x^2 \\ m\frac{dv_x}{dt} &= -cv_x^2 \end{aligned}$$

This can be solved via separation of variables!

$$\begin{aligned} \frac{dv_x}{v_x^2} &= -\frac{c}{m}dt \\ \int_{v_{0x}}^{v_x} \frac{1}{v_x^2}\,dv_x &= -\int_0^t \frac{c}{m}\,dt \\ -v_x^{-1}\Big|_{v_{0x}}^{v_x} &= -\frac{c}{m}t\Big|_0^t \\ -\left(\frac{1}{v_x} - \frac{1}{v_{0x}}\right) &= -\frac{c}{m}t \\ \frac{1}{v_{0x}} - \frac{1}{v_x} &= -\frac{c}{m}t \\ \frac{1}{v_{0x}} + \frac{ct}{m} &= \frac{1}{v_x} \\ \frac{m + v_{0x}ct}{mv_{0x}} &= \frac{1}{v_x} \\ \Rightarrow v_x &= \frac{mv_{0x}}{m + v_{0x}ct} \end{aligned}$$

Letting $\tau = \frac{m}{v_{0x}c}$:

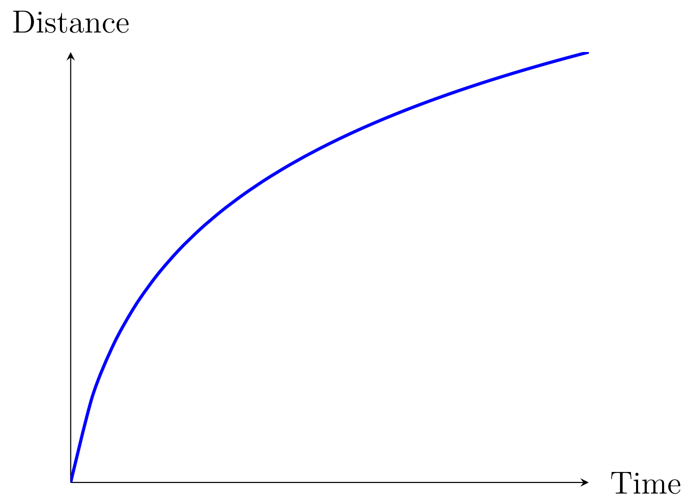
$$\begin{aligned} &= \frac{mv_{0x}}{m + mt/\tau} \\ \Rightarrow v_x(t) &= \frac{v_{0x}}{1 + t/\tau} \end{aligned}$$

Now we'll need to integrate to return to the position:

$$\begin{aligned} x(t) &= \int_0^t \frac{v_{0x}}{1+t/\tau}\, dt \\ &= v_{0x}\left[\tau \ln\left(1+\frac{t}{\tau}\right)\right]_0^t \\ &= v_{0x}\tau \ln\left(1+\frac{t}{\tau}\right) \end{aligned}$$

Remark 2.8

This is actually a much slower decrease in speed than we see in the linear case!



Remark 2.9

This never actually stops the object!! It will just slow the object to the point where linear drag would dominate, and then THAT slowly the object to a stop.

2.2.2 Vertical Motion ($v_x = 0$)

In the vertical direction we have

$$m\dot{v}_y = mg - cv_y^2$$

If the acceleration is 0, we can then define a terminal velocity:

$$\begin{aligned} mg - cv_y^2 &= 0 \\ \Rightarrow v_{term} &= \sqrt{\frac{mg}{c}} \\ \Rightarrow c &= \frac{mg}{v_{term}^2} \end{aligned}$$

Thus we have that

$$
\begin{aligned}
m\dot{v}_y &= mg - \frac{mg}{v_{term}^2} v_y^2 \\
\dot{v}_y &= g\left(1 - \frac{v_y^2}{v_{term}^2}\right)
\end{aligned}
$$

Again, we can actually solve this via separation of variables:

$$
\begin{aligned}
\frac{dv_y}{1 - \frac{v_y^2}{v_{term}^2}} &= g\,dt \\
\int_0^{v_y} \frac{dv_y}{1 - \frac{v_y^2}{v_{term}^2}} &= \int_0^t g\,dt \\
v_{term} \tanh^{-1}\left(\frac{v_y}{v_{term}}\right)\Bigg|_0^{v_y} &= gt \\
\Rightarrow v_y(t) = v_{term} \tanh\left(\frac{gt}{v_{term}}\right)
\end{aligned}
$$

And integrating gets us:

$$
y(t) = v_{term}\left[\frac{\ln\left(\cosh\left(\frac{gt}{v_{term}}\right)\right)}{\frac{g}{v_{term}}}\right] = \frac{v_{term}^2}{g} \ln\left[\cosh\left(\frac{gt}{v_{term}}\right)\right]
$$

Remark 2.10

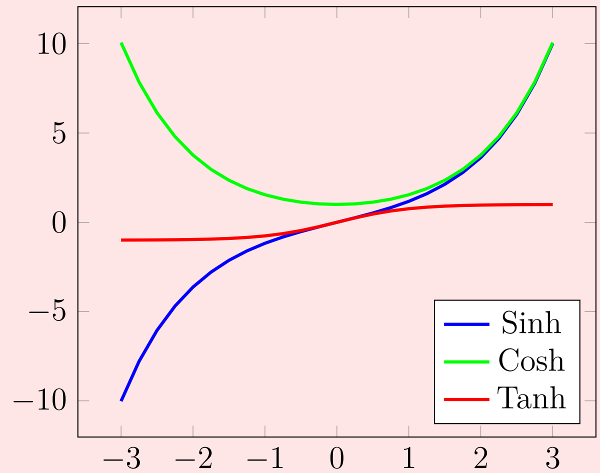
Recall that the hyperbolic trig terms are comparable to the normal trig terms when written as exponents:

$$
\begin{aligned}
\sin(x) &= \frac{e^{ix} - e^{-ix}}{2i} \\
\cos(x) &= \frac{e^{ix} + e^{-ix}}{2}
\end{aligned}
$$

versus:

$$
\begin{aligned}
\sinh(x) &= \frac{e^{x} - e^{-x}}{2} \\
\cosh(x) &= \frac{e^{x} + e^{-x}}{2}
\end{aligned}
$$

And $\tanh(x)$ is defined as per normal as $\frac{\sinh(x)}{\cosh(x)}$. Graphically:



Example 2.3

Suppose a bowling ball slides down a frictionless lane, but is effected by air resistance. If the 7.26 kg, 22 cm diameter ball initially has a velocity of 5 m/s, how fast is it moving 10 s later? How far has it traveled in this span?

Well,

$$\tau = \frac{m}{cv_0} = \frac{m}{\gamma D^2 v_0} = \frac{7.26}{(0.25)\,(0.22)^2\,(5)} = 120$$

$$v(10) = \frac{v_0}{1 + t/\tau} = \frac{5}{1 + 10/120} = 4.615\,\mathrm{m/s}$$

$$x(10) = v_0 \tau \ln(1 + t/\tau) = 5 \cdot 120 \ln(1 + 10/120) = 48\,\mathrm{m}$$

Example 2.4

Suppose we drop a basketball with a mass of 0.625 kg and a diameter of 24.26 cm. How fast is it traveling 5 s later?

The ball has a terminal velocity of:

$$v_{term} = \sqrt{\frac{mg}{\gamma D^2}} = \sqrt{\frac{(.625)\,(9.8)}{(0.25)\,(.2425)^2}} = 20.4\,\mathrm{m/s}$$

Thus the ball is traveling at:

$$v(10) = v_{term} \tanh\left(\frac{gt}{v_{term}}\right) = (20.4) \tanh\left(\frac{(9.8)\,(5)}{20.4}\right) = 20.068\,\mathrm{m/s}$$

2.2.3 Movement in both directions

If the object is free to move in both dimensions, we don't get to simplify, so:

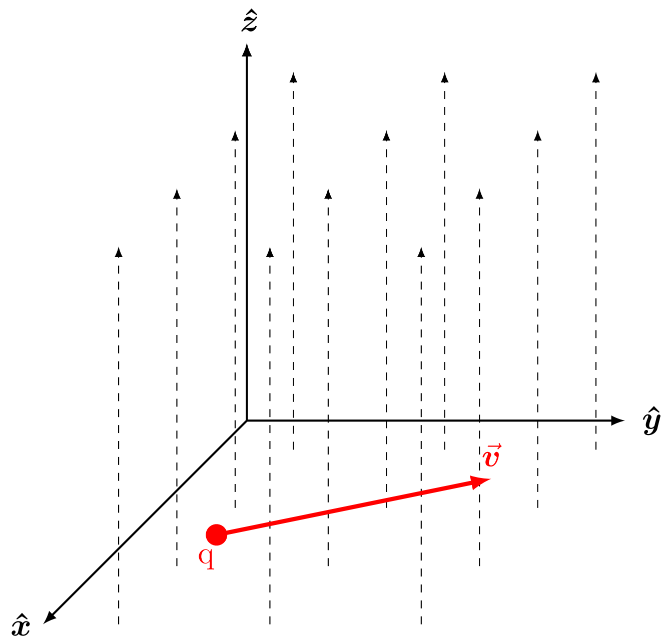
$$\left.\begin{aligned} m\dot{v}_x &= -c\sqrt{v_x^2+v_y^2}\,v_x \\ m\dot{v}_y &= -mg - c\sqrt{v_x^2+v_y^2}\,v_y \end{aligned}\right\} \quad \text{Coupled ODE's. Need to solve numerically}$$

Remark 2.11

When solving this system numerically, note that you will have *FOUR* 1st order differential equations, and you'll need *FOUR* initial conditions!

2.3 Motion of a Charged Particle in a Magnetic Field

Besides drag, another interesting velocity dependence arises from a charged particle moving in a uniform magnetic field:



$$\begin{aligned} \vec{\boldsymbol{F}} &= q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}} \\ m\ddot{\vec{\boldsymbol{r}}} &= q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}} \\ m\dot{\vec{\boldsymbol{v}}} &= q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}} \end{aligned}$$

If our initial velocity and magnetic field are given by:

$$\begin{aligned} \vec{\boldsymbol{v}} &= (v_x, v_y, v_z) \\ \vec{\boldsymbol{B}} &= (0, 0, B) \end{aligned}$$

And thus:

$$\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} = B\left(v_y\,\hat{\boldsymbol{x}} - v_x\,\hat{\boldsymbol{y}}\right) = (Bv_y, -Bv_x, 0)$$

So writing out our system of equations:

$$\begin{aligned} m\dot{v}_x &= qBv_y \\ m\dot{v}_y &= -qBv_x \\ m\dot{v}_z &= 0 \quad \Rightarrow v_z \text{ is constant} \end{aligned}$$

Definition 2.5: Cyclotron Frequency

Let

$$\frac{qB}{m} = \omega$$

be the **cyclotron frequency**. (It does indeed have units of 1/s if you check!)

Therefore, we get that:

$$\begin{aligned} \dot{v}_x &= \omega v_y \\ \dot{v}_y &= -\omega v_x \end{aligned}$$

which, again, are a set of couped differential equations. There are two main ways we can solve this system:

2.3.1 Method 1: No Complex Numbers

$$\begin{aligned} \dot{v}_x &= \omega v_y \\ \frac{d}{dt}\left[v_x\right] &= \frac{d}{dt}\left[\omega v_y\right] \\ \ddot{v}_x &= \omega \dot{v}_y \\ \ddot{v}_x &= \omega\left(-\omega v_x\right) \\ \ddot{v}_x &= -\omega^2 v_x \end{aligned}$$

which is purely our differential equation of SHM! So we can write down the solutions:

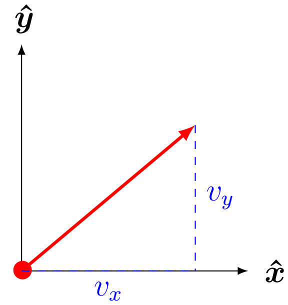
$$v_x(t) = A\sin(\omega t) + b\cos(\omega t) \quad \text{or} \quad v_x(t) = A\sin(\omega t + \phi)$$

2.3.2 Method 2: With Complex Numbers

Let's transport our vector v into imaginary land. Let

$$\eta = v_x + iv_y$$

so that



Therefore:

$$
\begin{aligned}
\dot{\eta} &= \dot{v}_x + i\dot{v}_y \\
&= \omega v_y + i\left(-\omega v_x\right) \\
&= -i\omega\left(i v_y + v_x\right) \\
&= -i\omega\eta
\end{aligned}
$$

This is a function whose derivative equals itself! So again:

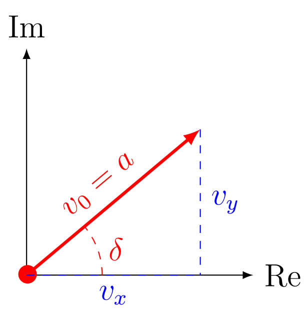
$$\eta = Ae^{-i\omega t}$$

where A is a complex constant. Since $\eta = v_x + iv_y$, then

$$\eta(0) = A = v_{0x} + iv_{0y}$$

We can write this in complex polar form as:

$$A = ae^{i\delta}$$



where

$$
\begin{aligned}
a &= v_0 \\
\delta &= \tan^{-1}\left(\frac{v_{0y}}{v_{0x}}\right)
\end{aligned}
$$

Or a describes the initial magnitude and δ describes the initial angle. Thus we have that:

$$
\begin{aligned}
\eta &= ae^{i\delta}e^{-i\omega t} \\
&= ae^{-i(\omega t - \delta)}
\end{aligned}
$$

Remembering Euler's Theorem:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

and applying it to the above gives us:

$$\begin{aligned}\eta &= a\left[\cos(\delta - \omega t) + i\sin(\delta - \omega t)\right] \\ &= v_x + iv_y\end{aligned}$$

Splitting into the real and imaginary parts gives us:

$$\begin{aligned}v_x &= a\cos(\delta - \omega t) \\ v_y &= a\sin(\delta - \omega t)\end{aligned}$$

If I let ε be our complex position:

$$\varepsilon = x + iy$$

then

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \eta = \dot{\varepsilon}$$

And thus:

$$\begin{aligned}\varepsilon &= \int \tau \, dt \\ &= \int Ae^{i(-\omega t)} \, dt \\ &= \frac{A}{-i\omega} e^{i(-\omega t)} + \underset{\substack{\text{Rotate} \\ \text{about} \\ \text{origin}}}{\boxed{C}} \\ &= \frac{Ai}{\omega} e^{i(-\omega t)}\end{aligned}$$

Remark 2.12

The constant C basically is giving the location in the complex plane about which our oscillations are happening. But we don't care *where we are* in the complex plane, we just care about our orientation in the plane. So we can let our $C = 0$ without loss of information.

Letting $\frac{Ai}{\omega} = D$:

$$
\begin{aligned}
\varepsilon &= De^{i(-\omega t)} \\
&= (x_0 + iy_0)\, e^{i(-\omega t)} \\
&= (x_0 + iy_0)\, (\cos(-\omega t) + i \sin(-\omega t)) \\
&= \underbrace{\boxed{x_0 \cos(-\omega t) - y_0 \sin(-\omega t)}}_{\text{x}} + i \left(\underbrace{\boxed{x_0 \sin(-\omega t) + y_0 \cos(-\omega t)}}_{\text{y}} \right)
\end{aligned}
$$