Chapter 4

Energy

We'll now start our review/study of energies, starting off with the most common: kinetic energy.

Definition 4.1: Kinetic Energy

We define kinetic energy to be

$$T = \frac{1}{2}mv^2$$

where m is the object's mass and v its velocity.

Let's consider then the change in kinetic energy:

$$\begin{aligned}
\frac{\mathrm{d}T}{\mathrm{d}t} &= \frac{1}{2}m\frac{\mathrm{d}}{\mathrm{d}t}\left(v^2\right) \\
&= \frac{1}{2}m\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{v}}\right] \\
&= \frac{1}{2}m\left[\dot{\vec{\boldsymbol{v}}}\cdot\vec{\boldsymbol{v}}+\vec{\boldsymbol{v}}\cdot\dot{\vec{\boldsymbol{v}}}\right] \\
&= m\left[\dot{\vec{\boldsymbol{v}}}\cdot\vec{\boldsymbol{v}}\right] \\
&= \left[\underbrace{m\dot{\vec{\boldsymbol{v}}}}_{\dot{\vec{\boldsymbol{P}}}}\cdot\vec{\boldsymbol{v}}\right] \\
&= \vec{\boldsymbol{F}}\cdot\vec{\boldsymbol{v}}
\end{aligned}$$

Therefore, if we take the integral of both sides:

$$\begin{aligned}\int_{T_i}^{T_f} \mathrm{d}T &= \int_{t_i}^{t_f} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}\ \mathrm{d}t \\ &= \int_{r_i}^{r_f} \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}} \\ \Rightarrow T_f - T_i &= \underbrace{\boxed{\int_{r_i}^{r_f} \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}}}}_{\text{Work done by force } \vec{\boldsymbol{F}}}\end{aligned}$$

4.1 A Reminder on Line Integrals

The easy way is if you can write $\vec{\boldsymbol{F}}$ and $\mathrm{d}\vec{\boldsymbol{r}}$ as vectors. By default, in Cartesian coordinates:

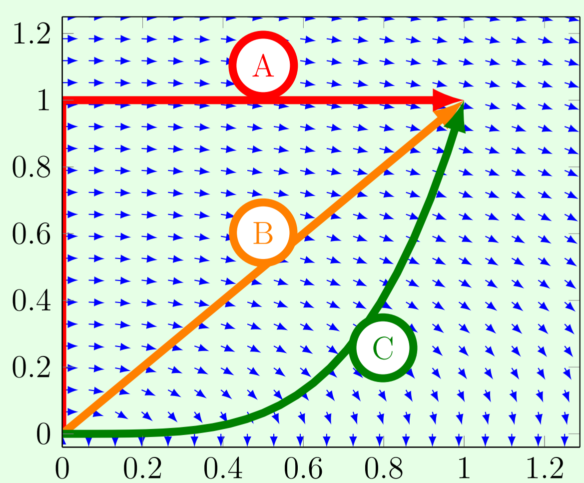
$$\mathrm{d}\vec{\boldsymbol{r}} = \mathrm{d}x\ \hat{\boldsymbol{x}} + \mathrm{d}y\ \hat{\boldsymbol{y}}$$

Example 4.1

Say we have a force:

$$\vec{\boldsymbol{F}} = 3y\,\hat{\boldsymbol{x}} - x\,\hat{\boldsymbol{y}}$$

And we want to know the work done when moving from the origin to the point (1,1) by three different paths.



Path A: For A we break the path up into two separate paths, one in which $x = 0$

and then one in which $y = 1$:

$$\begin{aligned}
W_A &= \int_{A_1} \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}} + \int_{A_2} \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}} \\
&= \int_{A_1} (3y\,\hat{\boldsymbol{x}} - x\,\hat{\boldsymbol{y}}) \cdot (0\,\hat{\boldsymbol{x}} + \mathrm{d}y\ \hat{\boldsymbol{y}}) + \int_{A_2} (3y\,\hat{\boldsymbol{x}} - x\,\hat{\boldsymbol{y}}) \cdot (\mathrm{d}x\ \hat{\boldsymbol{x}} + 0\,\hat{\boldsymbol{y}}) \\
&= \int_{A_1} x\,\mathrm{d}y + \int_{A_2} 3y\,\mathrm{d}x \\
&= \int_0^1 0\,\mathrm{d}y + \int_0^1 3\,\mathrm{d}x \\
&= 3
\end{aligned}$$

Path B: Path B is along the equation of

$$y = x$$

and thus

$$\mathrm{d}y = \mathrm{d}x$$

Plugging in accordingly:

$$\begin{aligned}
W_B &= \int_B \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}} \\
&= \int_B (3y\,\hat{\boldsymbol{x}} - x\,\hat{\boldsymbol{y}}) \cdot (\mathrm{d}x\ \hat{\boldsymbol{x}} + \mathrm{d}y\ \hat{\boldsymbol{y}}) \\
&= \int_B (3x\,\hat{\boldsymbol{x}} - x\,\hat{\boldsymbol{y}}) \cdot (\mathrm{d}x\ \hat{\boldsymbol{x}} + \mathrm{d}x\ \hat{\boldsymbol{y}}) \\
&= \int_0^1 3x\,\mathrm{d}x - x\,\mathrm{d}x \\
&= \int_0^1 2x\,\mathrm{d}x \\
&= x^2\Big|_0^1 \\
&= 1
\end{aligned}$$

Path C: Path C is along a line with the equation

$$y = x^4$$

and thus

$$\mathrm{d}y = 4x^3\,\mathrm{d}x$$

Following the same steps as previously:

$$
\begin{aligned}
W_C &= \int_C \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}} \\
&= \int_C (3y\,\hat{\boldsymbol{x}} - x\,\hat{\boldsymbol{y}}) \cdot (\mathrm{d}x\ \hat{\boldsymbol{x}} + \mathrm{d}y\ \hat{\boldsymbol{y}}) \\
&= \int_C \Big(3(x^4)\,\hat{\boldsymbol{x}} - x\,\hat{\boldsymbol{y}}\Big) \cdot \Big(\mathrm{d}x\ \hat{\boldsymbol{x}} + 4x^3\,\mathrm{d}x\ \hat{\boldsymbol{y}}\Big) \\
&= \int_0^1 3x^4\,\mathrm{d}x - 4x^4\,\mathrm{d}x \\
&= \int_0^1 -x^4\,\mathrm{d}x \\
&= -1
\end{aligned}
$$

Looking at things in aggregate then, we can write that

$$\Delta T = \int_1^2 \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}} = W(1 \to 2)$$

Remark 4.1

Be careful not to forget here that $\vec{\boldsymbol{F}}$ is your *net* force ($\dot{\vec{\boldsymbol{p}}}$ equal the net force by Newton's 2nd law). Or to put in otherwise, $W(1 \to 2)$ is the *net* work done on the system.

Therefore:

$$
\begin{aligned}
W(1 \to 2) &= \int \left(\sum_\alpha \vec{\boldsymbol{F}}_\alpha\right) \cdot \mathrm{d}\vec{\boldsymbol{r}} \\
&= \int \sum_\alpha \vec{\boldsymbol{F}}_\alpha \cdot \mathrm{d}\vec{\boldsymbol{r}} \\
&= \sum_\alpha \int \vec{\boldsymbol{F}}_\alpha \cdot \mathrm{d}\vec{\boldsymbol{r}} \\
&= \sum_\alpha W_\alpha(1 \to 2) \\
&= \text{ Sum of all the works from the individual forces}
\end{aligned}
$$

4.2 Conservative Forces

The impetus behind our interest in conservative forces are that these are forces which will allow us to write down corresponding potential energies. Since energy is usually easier to work with than forces, this is enticing to us.

Definition 4.2: Conservative Force

A conservative force must obey two rules:

1. It must depend only on the objects position (no $\vec{\boldsymbol{v}}$ or t dependencies)

$$\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})$$

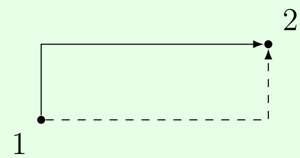
2. The work done by the force between any two points must be the same regardless of the path chosen

Example 4.2

Take gravity in two dimensions. The force due to gravity is given by:

$$\vec{\boldsymbol{F}} = (0, -mg)$$

Moving an object to a new height by two different routes:



Results in the same amount of work being done, since any movement not in the vertical direction yields no work:

$$\begin{aligned} W(1 \to 2) &= \int 0 - mg\,\mathrm{d}y \\ &= -mgy \quad \text{no matter what} \end{aligned}$$

Definition 4.3: Potential Energy

We'll now define an object's potential energy U as:

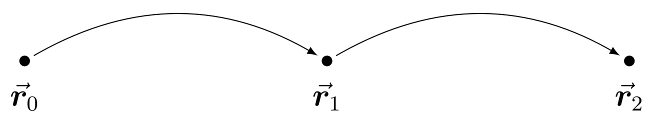
$$U(\vec{\boldsymbol{r}}) = -W(\vec{\boldsymbol{r}}_0 \to \vec{\boldsymbol{r}}) \equiv -\int_{\vec{\boldsymbol{r}}_0}^{\vec{\boldsymbol{r}}} \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}) \cdot \mathrm{d}\vec{\boldsymbol{r}}$$

where $\vec{\boldsymbol{r}}_0$ is the zero point of our potential energy.

Remark 4.2

Note the above definition wouldn't make sense if the work could depend on the path taken, else we'd have a non-unique definition of potential energy, which wouldn't be very helpful.

Say our conservative force is moving an object from the zero point to point $\vec{\boldsymbol{r}}_1$ and then onward to point $\vec{\boldsymbol{r}}_2$:



So:

$$\begin{aligned}
W(\vec{\boldsymbol{r}}_0 \to \vec{\boldsymbol{r}}_2) &= W(\vec{\boldsymbol{r}}_0 \to \vec{\boldsymbol{r}}_1) + W(\vec{\boldsymbol{r}}_1 \to \vec{\boldsymbol{r}}_2) \\
\Rightarrow W(\vec{\boldsymbol{r}}_1 \to \vec{\boldsymbol{r}}_2) &= \underbrace{\boxed{W(\vec{\boldsymbol{r}}_0 \to \vec{\boldsymbol{r}}_2)}}_{-U(\vec{\boldsymbol{r}}_2)} - \underbrace{\boxed{W(\vec{\boldsymbol{r}}_0 \to \vec{\boldsymbol{r}}_1)}}_{-U(\vec{\boldsymbol{r}}_1)} \\
&= -U(\vec{\boldsymbol{r}}_2) + U(\vec{\boldsymbol{r}}_1) \\
&= -\left(U(\vec{\boldsymbol{r}}_2) - U(\vec{\boldsymbol{r}}_1)\right) \\
&= -\Delta U
\end{aligned}$$

And then we have:

$$\begin{aligned}
\Delta T &= W(\vec{\boldsymbol{r}}_1 \to \vec{\boldsymbol{r}}_2) \\
&= -\Delta U \\
\Rightarrow \Delta T + \Delta U &= 0 \\
\Delta(T + U) &= 0
\end{aligned}$$

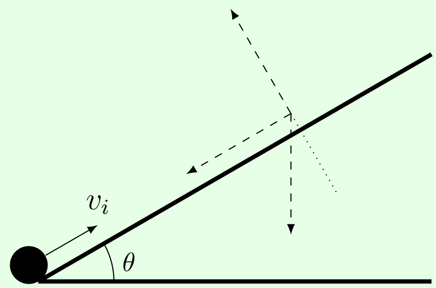
And thus mechanical energy is conserved! What if we have a non-conservative force (one that fails either of our two rules)? Well in that case:

$$\begin{aligned}
\Delta T &= W_{net} \\
&= \sum_{\alpha} W_{\alpha} \\
&= W_{cons} + W_{NC} \\
&= -\Delta U + W_{NC} \\
\Rightarrow \Delta(T + U) &= W_{NC}
\end{aligned}$$

and so we can still deal with non-conservative forces is this fashion!

Example 4.3

Say we have a object of mass m traveling up a slope with initial speed v_i. The slope has a coefficient of kinetic friction of μ. How high up the slope does the object travel?



Thus

$$\begin{aligned}\Delta T + \Delta U &= W_{NC}\\ \left(0 - \frac{1}{2}mv_i^2\right) + (mgh - 0) &= \mu mg\cos(\theta)\underbrace{\Delta s}_{\frac{h}{\sin(\theta)}}\\ mgh - \mu mg\cos(\theta)\frac{h}{\sin(\theta)} &= \frac{1}{2}mv_i^2\\ \Rightarrow h &= \frac{v_i^2}{2g\left(1 - \frac{\mu}{\tan(\theta)}\right)}\end{aligned}$$

4.3 Moving to 3 dimensions!

Moving to 3 dimensions will force us to start using some vector calculus machinery. I'll try to remind you and teach you the needed bits as we come to them.

So if we have a particle acted on by a conservative force from the point $\vec{\boldsymbol{r}}$ to the point $\vec{\boldsymbol{r}} + \mathrm{d}\vec{\boldsymbol{r}}$, then the work done:

$$W\left(\vec{\boldsymbol{r}} \to \vec{\boldsymbol{r}} + \mathrm{d}\vec{\boldsymbol{r}}\right) = \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}) \cdot \mathrm{d}\vec{\boldsymbol{r}}$$

where $\mathrm{d}\vec{\boldsymbol{r}}$ is a small displacement so we can assume it is linear and constant,

$$= F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z$$

Now, by a different tact:

$$
\begin{aligned}
W\left(\vec{\boldsymbol{r}} \rightarrow \vec{\boldsymbol{r}}+\mathrm{d}\vec{\boldsymbol{r}}\right) &= -dU \\
&= -\left(U\left(\vec{\boldsymbol{r}}+\mathrm{d}\vec{\boldsymbol{r}}\right)-U\left(\vec{\boldsymbol{r}}\right)\right) \\
&= -\left[U\left(x+dx, y+dy, z+dz\right)-U\left(x, y, z\right)\right] \\
&= -\left[U\left(x+dx\right)-U(x)+U(y+dy)-U(y)+\underbrace{U(z+dz)-U(z)}_{\frac{\partial U}{\partial z}=\frac{U(z+dz)-U(z)}{dz}}\right] \\
&= -\left[\frac{\partial U}{\partial x}\,\mathrm{d}x+\frac{\partial U}{\partial y}\,\mathrm{d}y+\frac{\partial U}{\partial z}\,\mathrm{d}z\right]
\end{aligned}
$$

By comparing the two different solutions, we arrive at the fact that:

$$
F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z}
$$

or

$$
\vec{\mathbf{F}} = -\frac{\partial U}{\partial x}\,\hat{\boldsymbol{x}} - \frac{\partial U}{\partial y}\,\hat{\boldsymbol{y}} - \frac{\partial U}{\partial z}\,\hat{\boldsymbol{z}}
$$

Definition 4.4: Gradient

The gradient is defined as:

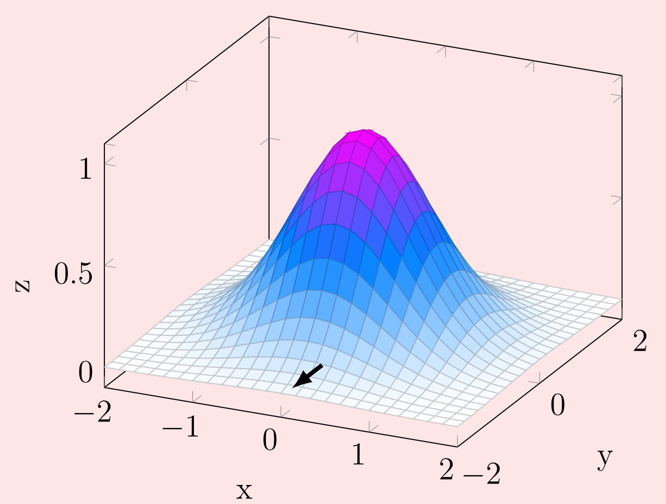
$$
\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\,\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\,\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\,\hat{\boldsymbol{z}}
$$

Thus we have that

$$
\vec{\mathbf{F}} = -\boldsymbol{\nabla} U
$$

Remark 4.3

Just like the derivative, the gradient gives us the tangent *vectors* to a surface!



Example 4.4

A potential energy curve is given by:

$$U(\vec{\mathbf{r}}) = xyz + 2y^2 + 3\sin(x)$$

Therefore

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = -\left(yz + 3\cos(x)\right)\,\hat{\boldsymbol{x}} - (xz + 4y)\,\hat{\boldsymbol{y}} - (xy)\,\hat{\boldsymbol{z}}$$

If we were interested in the force at the point (0,0,1):

$$\vec{\mathbf{F}}(0,0,1) = -3\,\hat{\boldsymbol{x}}$$

Often times we may be interested in the small amount of change in say the potential energy due to a small displacement. Based on our earlier equation for dU, we can generalize this to:

$$\mathrm{d}f = \boldsymbol{\nabla} f \cdot \mathrm{d}\vec{\mathbf{r}} = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z$$

4.4 Path Independence

So in addition to our first rule for conservative forces, that $\vec{\mathbf{F}}$ could only be a function of $\vec{\mathbf{r}}$, we had our second rule that the work done by $\vec{\mathbf{F}}$ must be the same along any path.

But that is a lot of paths to check...

Definition 4.5: Stoke's Theorem

W will be path independent if and only if

$$\boldsymbol{\nabla} \times \vec{\mathbf{F}} = 0$$

where $\boldsymbol{\nabla} \times \vec{\mathbf{F}}$ is call the "curl of F."

This requires that

$$\begin{aligned}
\boldsymbol{\nabla} \times \vec{\mathbf{F}} &= \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\
&= \hat{\boldsymbol{x}} \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) - \hat{\boldsymbol{y}} \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) + \hat{\boldsymbol{z}} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)
\end{aligned}$$

Remark 4.4

ALL of the components must equal zero in order for the curl of F to equal zero!

Example 4.5

Is Coulomb's Law a conservative force?

$$\vec{\mathbf{F}} = \frac{kQq}{r^2} \hat{\boldsymbol{r}} = \frac{kQq}{r^2} \hat{\boldsymbol{r}} \frac{r}{r} = \frac{kQq}{r^3} \vec{\mathbf{r}} = \frac{kQq}{r^3} \left(x\, \hat{\boldsymbol{x}} + y\, \hat{\boldsymbol{y}} + z\, \hat{\boldsymbol{z}} \right)$$

Looking at just the x-component of the curl, since the others will follow the same way:

$$\begin{aligned}
(\boldsymbol{\nabla} \times \vec{\mathbf{F}})_x &= \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\
&= \frac{\partial}{\partial y} \left(\frac{kQq}{r^3} z \right) - \frac{\partial}{\partial z} \left(\frac{kQq}{r^3} y \right) \\
&= kQq \left[z \frac{\partial}{\partial y} r^{-3} - y \frac{\partial}{\partial z} r^{-3} \right]
\end{aligned}$$

Now since

$$r = \left(x^2 + y^2 + z^2 \right)^{1/2}$$

then

$$\frac{\partial r}{\partial y} = \frac{1}{2} \left(x^2 + y^2 + z^2 \right)^{-1/2} (2y) = \frac{y}{r}$$

So

$$kQq\left[z\frac{\partial}{\partial y}r^{-3} - y\frac{\partial}{\partial z}r^{-3}\right] = kQq\left[z\left(-3r^{-4}\right)\frac{y}{r} - y\left(-3r^{-4}\right)\frac{z}{r}\right]$$
$$= \frac{-3kQq}{r^5}\left[-zy + yz\right]$$
$$= 0$$

All the other components will fall out in an identical fashion.

4.5 Time Dependence

We know that

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}, t)$$

is not a conservative force, as it fails the first rule, but we can still define a unique potential:

$$\vec{\mathbf{F}} = -\boldsymbol{\nabla} U$$

However, we can no longer guarantee that mechanical energy will be conserved in this situation! Recalling that

$$\mathrm{d}T = \frac{\mathrm{d}T}{\mathrm{d}t}\,\mathrm{d}t = \left(m\dot{\vec{\mathbf{v}}} \cdot \vec{\mathbf{v}}\right)\mathrm{d}t = \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$$

and

$$\mathrm{d}U = \underbrace{\frac{\partial U}{\partial x}\,\mathrm{d}x + \frac{\partial U}{\partial y}\,\mathrm{d}y + \frac{\partial U}{\partial z}\,\mathrm{d}z}_{\boldsymbol{\nabla} U \cdot \mathrm{d}\vec{\mathbf{r}} = -\vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}} + \frac{\partial U}{\partial t}\,\mathrm{d}t$$

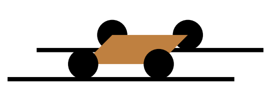
Means that

$$\mathrm{d}T + \mathrm{d}U = \mathrm{d}(T + U) = \frac{\partial U}{\partial t}\,\mathrm{d}t$$

4.6 The Power of Constraints

We've worked out how to treat an object free to move in all three dimensions, but in many cases our life is a bit easier. <u>Constraints</u> reduce the number of available dimensions we are forced to consider. To get us started, we'll constrain objects to a single dimension.

4.6.1 1D Systems

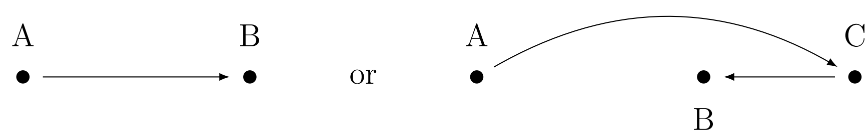


Let some force $\vec{\mathbf{F}}$ do work on our object over some interval:

$$W(x_1 \to x_2) = \int_{x_1}^{x_2} F_x(x)\, dx$$

Is $\vec{\mathbf{F}}$ conservative?

- Well, it does only depend on x
- For 1D systems, Rule 1 actually always implies Rule 2, as we don't exactly have large choice of paths...



$$W(AB) \qquad\qquad W(ACB) = W(AB) + \underbrace{W(BC) + W(CB)}_{\text{equal and opposite}}$$

Since our force is conservative, we have that

$$U(x) = -\int_{x_0}^{x} F_x(x)\,\mathrm{d}x$$

Example 4.6

If

$$F_x = -kx \quad \text{(Hooke's Law)}$$

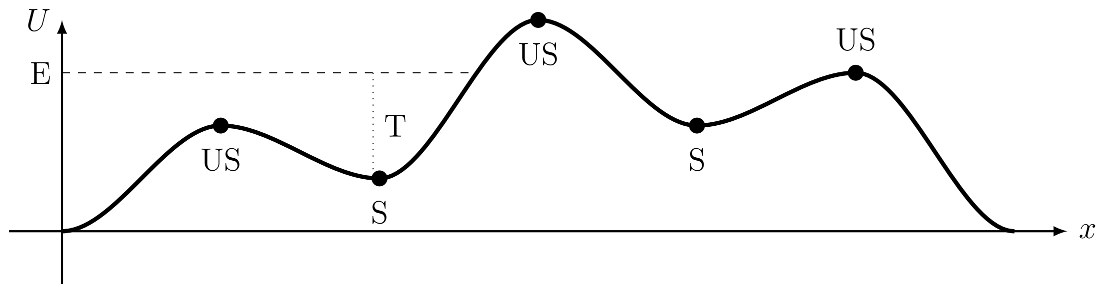
then

$$\begin{aligned} U(x) &= -\int_{x_0}^{x} -kx\,\mathrm{d}x \\ &= k\int_{0}^{x} x\,\mathrm{d}x \\ &= k\left.\frac{x^2}{2}\right|_0^x \\ &= \frac{1}{2}kx^2 \end{aligned}$$

Also, since we are purely in 1D atm:

$$F = -\boldsymbol{\nabla} U \quad \Rightarrow \quad F_x = \frac{\partial U}{\partial x}$$

which gives us a good chance to remind everyone about potential energy curves for a moment:



$$\frac{\mathrm{d}U}{\mathrm{d}x} = 0 = F \quad \Rightarrow \quad \text{points of equilibrium}$$

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} > 0 \quad \Rightarrow \quad \text{stable equilibrium (happy faces)}$$

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} < 0 \quad \Rightarrow \quad \text{unstable equilibrium (sad faces)}$$

Also note that since

$$E = T + U$$

then

$$T = E - U$$

Remark 4.5

U is a function of x here, so this could give us spacial information about how T (or v) varies with distance, but has no information on time dependence.

But if we write it as:

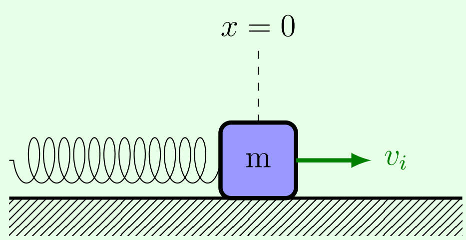
$$\begin{aligned} T &= E - U \\ \frac{1}{2}mv^2 &= E - U(x) \\ v &= \pm\sqrt{\frac{2(E-U(x))}{m}} \\ \Rightarrow \dot{x} &= \pm\sqrt{\frac{2}{m}}\sqrt{E-U(x)} \end{aligned}$$

Now $\dot{x}$ does have some time dependence hidden in it. We can uncover it with some separation of variables:

$$\begin{aligned}
\dot{x} &= \frac{\mathrm{d}x}{\mathrm{d}t} \\
\mathrm{d}t &= \frac{\mathrm{d}x}{\dot{x}} \\
t_f - t_i &= \int_{x_i}^{x_f} \frac{\mathrm{d}x}{\dot{x}} \\
t &= \int_{x_i}^{x} \frac{\mathrm{d}x}{\dot{x}(x)} \\
t &= \sqrt{\frac{m}{2}} \int_{x^i}^{x} \left(E - U(x)\right)^{-1/2} \mathrm{d}x
\end{aligned}$$

Example 4.7

Take for example a spring starting at its relaxed length with speed v_i:



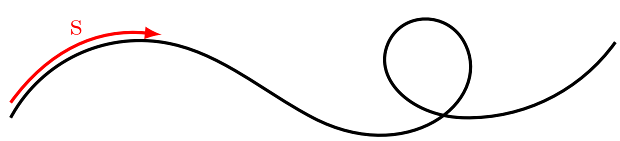
Here

$$\begin{aligned}
\dot{x}(x) &= \sqrt{\frac{2}{m}}\sqrt{E - U(x)} \\
&= \sqrt{\frac{2}{m}}\sqrt{\frac{1}{2}mv_i^2 - \frac{1}{2}kx^2} \\
&= \sqrt{v_i^2 - \frac{k}{m}x^2} \\
&= v_i\sqrt{1 - \underbrace{\frac{k}{v_i^2 m}}_{\alpha^2} x^2} \\
&= v_i\sqrt{1 - (\alpha x)^2}
\end{aligned}$$

And thus

$$\begin{aligned}
t &= \int_0^x \frac{\mathrm{d}x}{\dot{x}(x)} \\
&= \int_0^x \frac{\mathrm{d}x}{v_i\sqrt{1-(\alpha x)^2}} \\
&= \frac{1}{v_i}\int_0^x \frac{\mathrm{d}x}{\sqrt{1-(\alpha x)^2}} \\
&= \frac{1}{v_i}\left[\frac{\sin^{-1}(\alpha x)}{\alpha}\right|_0^x \\
&= \frac{\sin^{-1}(\alpha x)}{v_i\alpha} \\
&= \frac{1}{v_i}\sqrt{\frac{v_i^2 m}{k}}\sin^{-1}(\alpha x) \\
\sqrt{\frac{k}{m}}t &= \sin^{-1}(\alpha x) \\
\omega t &= \sin^{-1}(\alpha x) \\
\Rightarrow x(t) &= \frac{v_i}{\omega}\sin(\omega t)
\end{aligned}$$

So that was all on nice flat surfaces. What happens when we add curves?



So here we know that

$$T = \frac{1}{2}m\dot{s}^2$$

We'd like to relate this change in kinetic energy with the net work though, which means we'll need to look at what forces we have acting on the system. If everything is frictionless, then we just have:

- Gravity, this is constant
- Normal, this is *not* constant!
  - It is what keeps us on the track, so its magnitude will vary (constraining force)
  - *BUT IT DOES NO WORK*! ($N \perp s$)

And so we realize that only the forces tangential to the path will matter:

$$F_{tangential} = m\ddot{s}$$

If all the tangential forces are conservative, then we can write:

$$f_{tangential} = -\frac{\mathrm{d}U}{\mathrm{d}s}$$

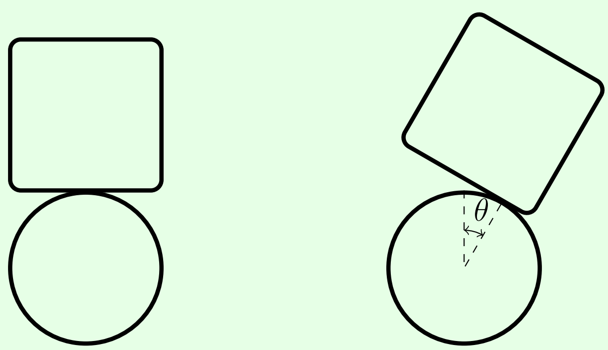
and return to

$$E = T + U(s)$$

being a conserved quantity.

Example 4.8

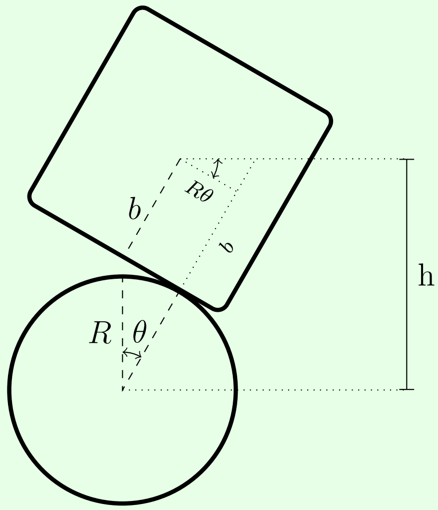
Say we have a situation where we are balancing a box on top of a ball. Static friction keeps the box from slipping, but it is free to "rock" back and forth. Presumably the location on the top of the box is an equilibrium point, but we want to know whether it is stable or non-stable!



Note that we can define the orientation of your system then purely through the variable θ! Which is nice. What forces do we have acting on the system?

- Gravity
- Friction $\leftarrow$ Constraining, no work done
- Normal $\leftarrow$ Constraining, no work done

If our ball then has a radius of R and the box a sidelength of $2b$, we have the following picture:



So

$$U = mgh = mg\left[(b+R)\cos(\theta) + r\theta\sin(\theta)\right]$$

We want to look at stability, which means taking a derivative:

$$\begin{aligned}\frac{\mathrm{d}U}{\mathrm{d}\theta} &= mg\left[(b+r)(-\sin(\theta)) + r\sin(\theta) + r\theta\cos(\theta)\right]\\ 0 &= mg\left[r\theta\cos(\theta) - b\sin(\theta)\right]\end{aligned}$$

which implies that

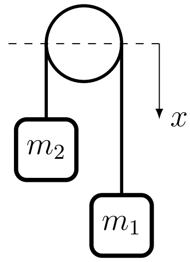
$$\begin{aligned}r\theta\cos(\theta) - b\sin(\theta) = 0\\ r\theta = b\tan(\theta)\\ \Rightarrow \theta = 0\end{aligned}$$

And so the top is an equilibrium point as expected. Is it stable?

$$\begin{aligned}\frac{\mathrm{d}^2U}{\mathrm{d}\theta^2} &= mg\left[r\cos\theta + r\theta(-\sin(\theta)) - b\cos(\theta)\right|_{\theta=0}\\ &= mg\left[r-b\right]\end{aligned}$$

So if $r > b$, then the second derivative is positive and thus the point is stable. If $r < b$, then the second derivative is negative and the point is unstable.

We can do a similar energy evaluation of attwood machines:



Total string length $= L$

So here we'd have

$$U = m_1 g x + m_2 g (L - x)$$

where we are taking advantage of the fact that gravity is conservative to write it as a potential energy. Tension is *not* conservative though, so we'd have:

$$\Delta T + \Delta U = W_{tension}$$

for each tension for on a mass. Thus we'd have:

$$\Delta T_1 + \Delta U_1 = (W_{ten})_1$$
$$\Delta T_2 + \Delta U_2 = (W_{ten})_2$$

But for perfect massless strings, the tension is the same everywhere on the string. And additionally, $\Delta x_1 = -\Delta x_2$, so that

$$(W_{ten})_1 = -(W_{ten})_2$$

Therefore, if we add both equations, we get:

$$\Delta T_1 + \Delta T_2 + \Delta U_1 + \Delta U_2 = 0$$
$$\Delta \underbrace{(T_1 + T_2 + U_1 + U_2)}_{\text{total mech energy conserved!}} = 0$$

So we see that, while constraining forces may dictate how the system moves, they do no work on the system and thus don't show up in the energies. Thus if all the forces *actually doing work* are conservative:

$$E = \sum_\alpha (T_\alpha + U_\alpha) \qquad \text{is conserved!}$$

4.7 Central Forces

The principle concept behind "central forces" is that they are everywhere directed relative to a "force center." If this "force center" is at the origin, then

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = f(\vec{\mathbf{r}})\hat{\mathbf{r}}$$

where $f(\vec{\mathbf{r}})$ gives the magnitude of the force, with positive values pushing away from the center and negative values pushing toward the center. For example:

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = \frac{kQq}{r^2}\hat{\mathbf{r}} \quad \text{or} \quad \vec{\mathbf{F}}(\vec{\mathbf{r}}) = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

Remark 4.6

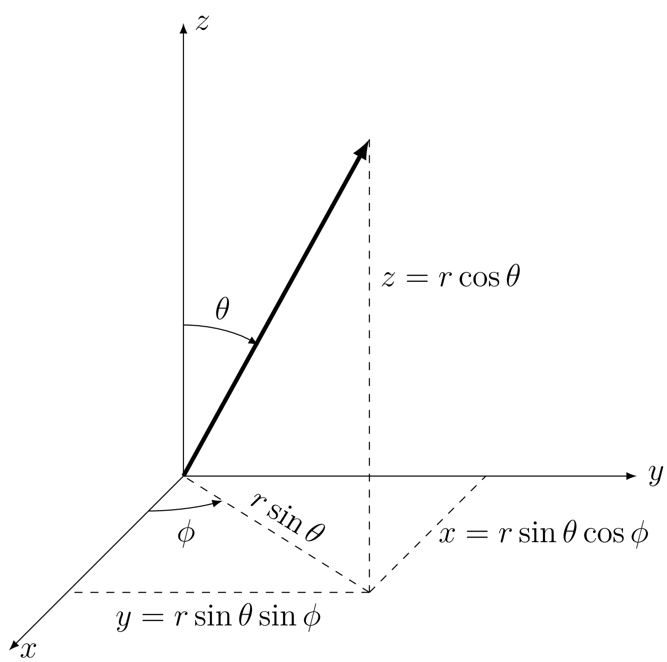
Note that in both of these common examples the force depends only on the *magnitude* of $\vec{\mathbf{r}} = |\vec{\mathbf{r}}|$. Functions that do not depend on the precise position, only the length of the position, are spherically symmetric.

Remark 4.7

A force that is central is conservative *if and only if* it is spherically symmetric!!

4.7.1 A Review of Spherical Coordinates

Recall that we define our spherical coordinates as:



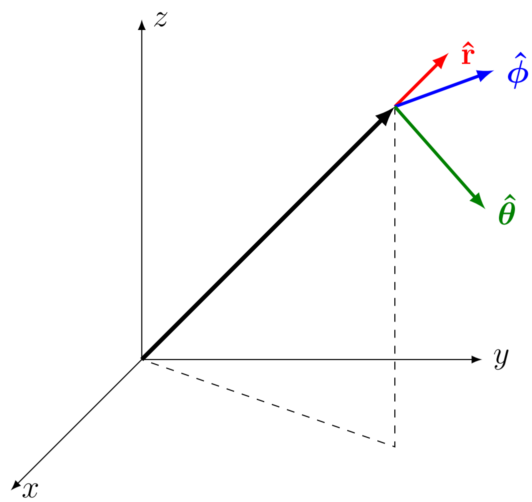
Remark 4.8

Note here that the way we've defined our spherical angles:

$$\begin{aligned} \theta &\Rightarrow \quad \text{latitudes} \\ \phi &\Rightarrow \quad \text{longitudes} \end{aligned}$$

Be careful, as many math books flip these!!!

And from this picture the unit vectors point:



Since all three unit vectors are still orthogonal to one another, dot products (and cross products) still function the same:

$$\vec{\mathbf{A}} = a\hat{\mathbf{r}} + b\hat{\boldsymbol{\theta}} + c\hat{\boldsymbol{\phi}}$$
$$\vec{\mathbf{B}} = d\hat{\mathbf{r}} + e\hat{\boldsymbol{\theta}} + f\hat{\boldsymbol{\phi}}$$
$$\Rightarrow \vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = ad + be + cf$$

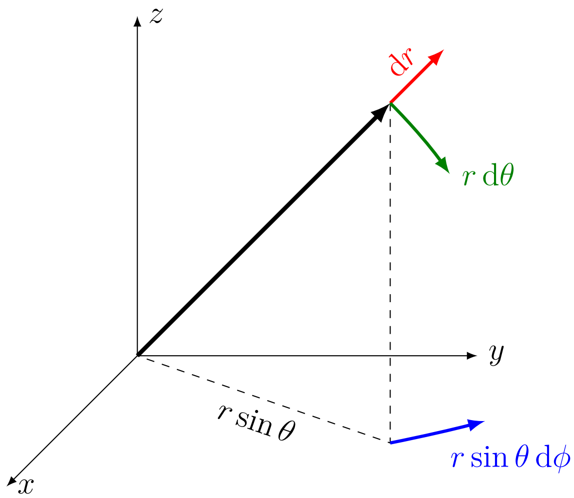
However, they *do* vary with position, so our calculus will be a bit more complicated!

4.7.2 Spherical Gradient

So a small change in $f(\vec{\mathbf{r}})$ is given by:

$$\mathrm{d}f = \boldsymbol{\nabla} f \cdot \mathrm{d}\vec{\mathbf{r}}$$

Our primary question is what is $\mathrm{d}\vec{\mathbf{r}}$ is spherical coordinates!



$$\hat{\mathbf{r}} \to \mathrm{d}r$$
$$\hat{\boldsymbol{\theta}} \to r\,\mathrm{d}\theta$$
$$\hat{\boldsymbol{\phi}} \to r\sin\theta\,\mathrm{d}\phi$$

Therefore, if we were to write out the above displacement:

$$\Rightarrow \mathrm{d}f = (\boldsymbol{\nabla} f)_r\,\mathrm{d}r + (\boldsymbol{\nabla} f)_\theta\, r\,\mathrm{d}\theta + (\boldsymbol{\nabla} f)_\phi\, r\sin\theta\,\mathrm{d}\phi$$

Now by chain rule, we also know that:

$$\mathrm{d}f = \frac{\partial f}{\partial r}\,\mathrm{d}r + \frac{\partial f}{\partial \theta}\,\mathrm{d}\theta + \frac{\partial f}{\partial \phi}\,\mathrm{d}\phi$$

By comparison, we can determine then that:

$$\begin{aligned}
(\boldsymbol{\nabla} f)_r &= \frac{\partial f}{\partial r} \\
(\boldsymbol{\nabla} f)_\theta &= \frac{1}{r}\frac{\partial f}{\partial \theta} \\
(\boldsymbol{\nabla} f)_\phi &= \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}
\end{aligned}$$

And therefore that, in spherical coordinates,

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Example 4.9

Previously we used cartesian coordinates to determine that Coulomb's law was conservative. We'll use spherical coordinates now to test Newton's law of gravity:

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

It only depends on the position, so the first rule is satisfied, so we need only check its path independence.

$$\begin{aligned}
\boldsymbol{\nabla} \times \vec{\mathbf{F}} &= \begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \\ -\frac{GMm}{r^2} & 0 & 0 \end{vmatrix} \\
&= 0\hat{\mathbf{r}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[-\frac{GMm}{r^2}\right](-\hat{\boldsymbol{\theta}}) + \frac{1}{r}\frac{\partial}{\partial\theta}\left[-\frac{GMm}{r^2}\right]\hat{\boldsymbol{\phi}} \\
&= 0
\end{aligned}$$

And thus it is indeed a conservative force, as we'd have expected since it is central and spherically symmetric.

4.7.3 Two Particle Systems

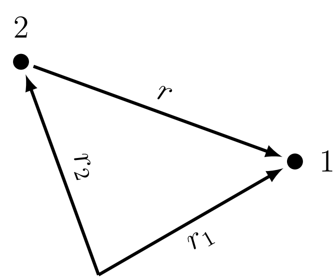
Say we have two particles interacting via $\vec{\mathbf{F}}_{12}$ and $\vec{\mathbf{F}}_{21}$, where the notation $\vec{\mathbf{F}}_{12}$ implies the forces acting *on* particle 1 *by* particle 2. We'll also let the force depend purely on the position of the two particles:

$$\vec{\mathbf{F}}_{12}(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)$$

So imagine a binary star system, where the two particles are two stars. Then:

$$\vec{\mathbf{F}}_{12} = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} \quad \text{or} \quad \vec{\mathbf{F}}_{12} = -\frac{Gm_1m_2}{r^3}\vec{\mathbf{r}}$$

Pictorally, we have:



And thus we can write:

$$\vec{\mathbf{F}}_{12} = -\frac{Gm_1m_2}{|r_1 - r_2|^3}(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2)$$

The lovely thing about this is that it depends *only* on the *difference* in the positions. Therefore it is translationally invarient: we could freely move the particles wherever we wanted and the force would stay the same as long as we kept them the same distance apart. We can make our life easier by moving one of the points (say point 2) to the origin. Then we just have that:

$$\vec{\mathbf{F}}_{12} = \vec{\mathbf{F}}_{12}(\vec{\mathbf{r}}_1)$$

As with any force, we can check to see if it is conservative by checking:

$$\nabla_1 \times \vec{\mathbf{F}}_{12} = 0$$

where ∇_1 signifies the derivatives with respect to $\vec{\mathbf{r}}_1$:

$$\nabla_1 = \hat{\boldsymbol{x}}\frac{\partial}{\partial x_1} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y_1} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z_1}$$

So, assuming it comes out to be conservative (like it did for the gravitational force), we can write:

$$\vec{\mathbf{F}}_{12} = -\nabla_1 U(\vec{\mathbf{r}}_1)$$

Then we can move back to the original position:

$$\vec{\mathbf{F}}_{12} = -\nabla_1 U\left(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\right)$$

Remark 4.9

We can do this without worrying about the derivative because $\vec{\mathbf{r}}_2$ is a constant with respect to ∇_1. So this is basically the same thing as saying

$$\frac{\mathrm{d}}{\mathrm{d}x}[5x] = \frac{\mathrm{d}}{\mathrm{d}x}[5x+1]$$

We are also interested in the reaction force, wherein we'll take advantage of two facts:

- $\vec{\mathbf{F}}_{12} = \vec{\mathbf{F}}_{21}$
- For each component:

$$a = x_1 - x_2$$

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial a}\frac{\partial a}{\partial x_1} = \frac{\partial f}{\partial a}$$
$$\frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial a}\frac{\partial a}{\partial x_2} = -\frac{\partial f}{\partial a}$$

$$\Rightarrow \frac{\partial f}{\partial x_1} = -\frac{\partial f}{\partial x_2}$$
$$\Rightarrow \nabla_1 U\left(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\right) = -\nabla_2 U\left(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\right)$$

So plugging both in gives us:

$$\begin{aligned}
\vec{\mathbf{F}}_{12} &= -\nabla_1 U\left(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\right)\\
-\vec{\mathbf{F}}_{21} &= -\nabla_1 U\left(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\right)\\
-\vec{\mathbf{F}}_{21} &= \nabla_2 U\left(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\right)\\
\vec{\mathbf{F}}_{21} &= \nabla_2 U\left(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\right)
\end{aligned}$$

And thus we see that

$$\begin{aligned}\text{force on particle 1 } &= -\nabla_1 U \\ \text{force on particle 2 } &= -\nabla_2 U\end{aligned}$$

The neat thing about this is that *both forces can be determined from the same potential energy!*

So now we want to work out our total energy equation. Starting with the kinetic energies, we have that:

$$\begin{aligned}\mathrm{d}T_1 &= \vec{\mathbf{F}}_{12} \cdot \mathrm{d}\vec{\mathbf{r}}_1 \\ \mathrm{d}T_2 &= \vec{\mathbf{F}}_{21} \cdot \mathrm{d}\vec{\mathbf{r}}_2\end{aligned}$$

$$\begin{aligned}\mathrm{d}T_{tot} &= \mathrm{d}T_1 + \mathrm{d}T_2 \\ &= \vec{\mathbf{F}}_{12} \cdot \mathrm{d}\vec{\mathbf{r}}_1 + \vec{\mathbf{F}}_{21} \cdot \mathrm{d}\vec{\mathbf{r}}_2 = W_{tot} \\ &= \vec{\mathbf{F}}_{12} \cdot \mathrm{d}\vec{\mathbf{r}}_1 - \vec{\mathbf{F}}_{12} \cdot \mathrm{d}\vec{\mathbf{r}}_2 \\ &= \vec{\mathbf{F}}_{12} \cdot (\mathrm{d}\vec{\mathbf{r}}_1 - \mathrm{d}\vec{\mathbf{r}}_2) \\ &= -\nabla_1 U(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2) \cdot (\mathrm{d}\vec{\mathbf{r}}_1 - \mathrm{d}\vec{\mathbf{r}}_2) \\ &= -\nabla_1 U(\vec{\mathbf{r}}) \cdot \mathrm{d}\vec{\mathbf{r}} \\ &= -\boldsymbol{\nabla} U(\vec{\mathbf{r}}) \cdot \mathrm{d}\vec{\mathbf{r}}\end{aligned}$$

because $\frac{\mathrm{d}}{\mathrm{d}x_1} U(x_1 - x_2) = \frac{\mathrm{d}}{\mathrm{d}x} U(x)$

$$= -\,\mathrm{d}U$$

$$\Rightarrow \mathrm{d}(T_{tot} + U) = 0$$

Or rather, the total energy

$$T_1 + T_2 + U$$

is conserved.

4.7.4 Multiparticle Systems

So the above was for two particles. We can build on that process to look at multiparticle systems. So

$$T_{tot} = T_1 + T_2 + T_3 + \cdots$$

We'll pair up all the particles as before so we are essentially looking at a lot of two particle interactions. Looking at the internal forces, we'll have our action and reaction forces:

$$\vec{\mathbf{F}}_{\alpha\beta} \quad \text{and} \quad \vec{\mathbf{F}}_{\beta\alpha}$$

where

$$\begin{aligned} U_{\alpha\beta} &= U_{\beta\alpha}\left(\vec{\mathbf{r}}_\alpha - \vec{\mathbf{r}}_\beta\right) \\ \vec{\mathbf{F}}_{\alpha\beta} &= -\nabla_\alpha U_{\alpha\beta} \\ \vec{\mathbf{F}}_{\beta\alpha} &= -\nabla_\beta U_{\beta\alpha} \end{aligned}$$

Any external forces that are acting on the system are going to depend on the particle's position. Assuming they are conservative we can write:

$$\vec{\mathbf{F}}_\alpha^{ext} = -\nabla_\alpha U_\alpha^{ext}(\vec{\mathbf{r}}_\alpha)$$

So altogether, we have that

$$U_{tot} = U^{int} + U^{ext} = \sum_\alpha \sum_{\alpha>\beta} U_{\alpha\beta} + \sum_\alpha U_\alpha^{ext}$$

Taking the negative gradient:

$$\begin{aligned} -\nabla_1 U &= -\nabla_1 \sum_{\beta>1} U_{1\beta} + -\nabla_1 U_1^{ext} \\ &= \sum_{\beta>1} (\text{forces on } 1) + \vec{\mathbf{F}}_1^{ext} \\ &= \text{ net force on particle 1} \end{aligned}$$

And thus

$$-\nabla_\alpha U = \text{net force on particle } \alpha$$

In an identical fashion to how we showed it for the two particle system, we could get that

$$\mathrm{d}E = \mathrm{d}T + \mathrm{d}U = 0$$

provided everything is conservative.

4.7.5 Rigid Bodies

A nice application of this concept is looking at rigid bodies. So we know the kinetic energy of a rigid body looks something like:

$$T_{tot} = T_{CM} + T_{rot}$$

Our potential energies are still

$$U_{tot} = U^{int} + U^{ext}$$

but

$$U^{int} = \sum_\alpha \sum_{\beta>\alpha} U_{\alpha\beta}\left(\underbrace{\vec{\mathbf{r}}_\alpha - \vec{\mathbf{r}}_\beta}_{\left|\vec{\mathbf{r}}_\alpha - \vec{\mathbf{r}}_\beta\right| \text{ if central}}\right)$$

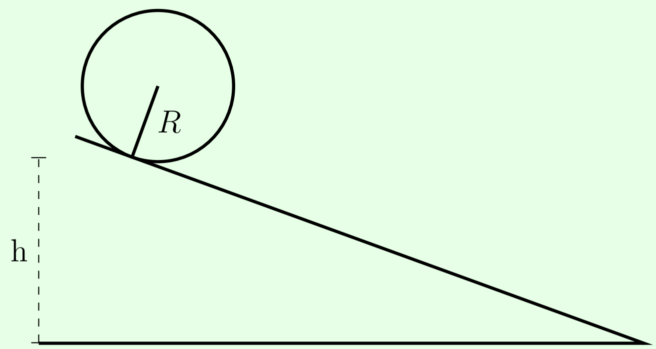
But for a rigid body the positions between particles on the interior of the object *are not changing*! (After all, it is *rigid*!) And thus

$$U^{int} = \text{ constant}$$

And thus we only need to consider U^{ext}, which is very nice!

Example 4.10

Consider a disk of size R rolling down a slope from an initial height of h.



And so we have that

$$U^{exit} = Mgh$$

and

$$\begin{aligned} T &= \frac{1}{2}Mv^2 + \frac{1}{2}\underbrace{I}_{\frac{1}{2}MR^2}\underbrace{\omega}_{\frac{v}{R}}{}^2 \\ &= \frac{3}{4}Mv^2 \end{aligned}$$

Thus:

$$\begin{aligned} \Delta T + \Delta U &= 0 \\ \left(0 - \frac{3}{4}Mv^2\right) + (Mgh - 0) &= 0 \\ \Rightarrow v &= \sqrt{\frac{4gh}{3}} \end{aligned}$$