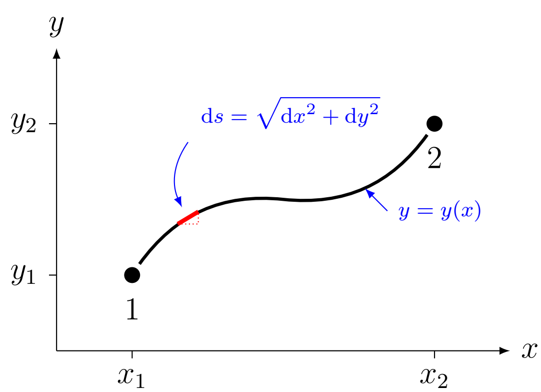
Chapter 6

Calculus of Variations

The Goal: To find functions that give the maximum or minimum values of a particular integral

Examples: Finding the shortest path along a curved surface, finding the fastest path, finding the path of least work, etc

6.1 Distance along a Path



We know that we are looking for a function where $y = y(x)$, so

$$\mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = y'(x)\,\mathrm{d}x$$

Remark 6.1

Note that we are back to using $y'(x)$ notation here to indicate the derivative. We are not dotting them because they are *not derivatives with respect to time!*

Plugging that into our tiny distance gives us

$$\begin{aligned} \mathrm{d}s &= \sqrt{\mathrm{d}x^2 + (y'(x))^2\,\mathrm{d}x^2} \\ &= \sqrt{1 + y'(x)^2}\,\mathrm{d}x \end{aligned}$$

Therefore the length of our path would be given by:

$$L = \int_1^2 \mathrm{d}s = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\,\mathrm{d}x$$

6.2 Time along a Path

Often times we'd like to know how long it takes something to travel along a particular path. A classic example is that light will always follow the path through a material that results in the fastest travel time (and the result in Snell's law!). So to determine the travel time, realize that:

$$\frac{\mathrm{d}s}{v} = \quad \text{time to travel a short distance } \mathrm{d}s$$

For light, it's velocity is defined as

$$v = \frac{c}{n} \quad \text{in a medium with a refractive index of } n$$

Thus, the time to travel between two points A and B:

$$t(A \to B) = \int_A^B \mathrm{d}t = \frac{1}{c}\int_A^B n\,\mathrm{d}s = \begin{cases} \frac{n}{c}\int_A^B \mathrm{d}s & \text{if n constant} \\ \frac{1}{c}\int_A^B n(x,y)\,\mathrm{d}s & \text{if not} \end{cases}$$

Remark 6.2

Let's compare what is going on here to standard calculus:

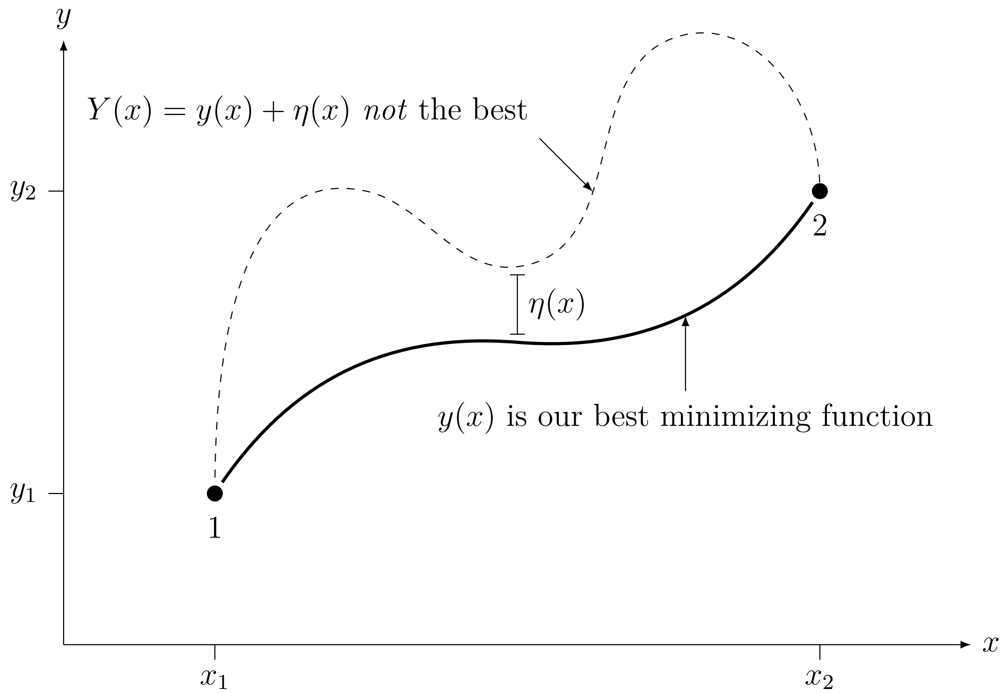
Normal Calc	Calc of Variations
Find min/max of function	Find min/max of integral
Returns a point	Returns a function
$\frac{\mathrm{d}f}{\mathrm{d}x} = 0$ is an equilibrium point	**What is our equivalent?**

6.3 The Calculus

So in general we have:

$$S = \int_{x_1}^{x_2} f(y, y', x)\,\mathrm{d}x \quad \text{using} \quad y(x_1) = y_1 \quad \text{and} \quad y(x_2) = y_2$$

Our goal is to find the equilibrium points of S! Say we have a solution $y(x)$ that minimizes the function:



Since $y(x)$ is our solution, then the nearby $Y(x)$ is decidedly *not* a solution that minimizes the integral.

Remark 6.3

Note that

$$\eta(x_1) = \eta(x_2) = 0$$

so that we are still ending at our endpoints!

Since the integral (and thus S) must always be greater along $Y(x)$ than $y(x)$, we will write:

$$Y(x) = y(x) + \alpha\eta(x)$$

This means that $S(\alpha)$ (and S) will be at an equilibrium point when $\alpha = 0$! This brings us back to standard calculus! So

$$\begin{aligned} S(\alpha) &= \int_{x_1}^{x_2} f(Y, Y', x)\,\mathrm{d}x \\ &= \int_{x_1}^{x_2} f(y + \alpha\eta, y' + \alpha\eta', x)\,\mathrm{d}x \end{aligned}$$

We want to find a min, so we'll take the derivative with respect to α:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} S(\alpha) = \frac{\mathrm{d}}{\mathrm{d}\alpha} \int_{x_1}^{x_2} f(y + \alpha\eta, y' + \alpha\eta', x)\,\mathrm{d}x$$

We'll need some chain rule to find the derivatives, so note that

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}\alpha} &= \frac{\partial f}{\partial Y}\frac{\mathrm{d}Y}{\mathrm{d}\alpha} + \frac{\partial f}{\partial Y'}\frac{\mathrm{d}Y'}{\mathrm{d}\alpha} \\ &= \frac{\partial f}{\partial Y}\eta + \frac{\partial f}{\partial Y'}\eta'\end{aligned}$$

Also note that

$$\frac{\partial f}{\partial Y} = \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}Y} = \frac{\partial f}{\partial y}$$

and identically for the primed term. Setting the derivative equal to zero then:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}S(\alpha) = \int_{x_1}^{x_2}\left(\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}\right)\mathrm{d}x = 0$$

We'll look at the integral of the second term a bit closer. Integrating it by parts gives us:

$$\int_{x^1}^{x_2}\eta'(x)\frac{\partial f}{\partial y'}\,\mathrm{d}x = \underbrace{\left[\eta(x)\frac{\partial f}{\partial y'}\right]_{x_1}^{x_2}}_{0 \text{ since } \eta=0 \text{ at ends}} - \int_{x_1}^{x_2}\eta(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\mathrm{d}x$$

Plugging this back in now we have that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}S(\alpha) = \int_{x_1}^{x_2}\eta(x)\left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\mathrm{d}x = 0$$

The important thing here is that this equation *must* be true for *any* value of $\eta(x)$. The only way for this to be true is if:

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} = 0$$

This is known as the Euler-Lagrange Equation!

Example 6.1

Say we want to find the shortest distance between two points in cartesian space (we already should know the answer to this, so it'll provide a nice check.) We wrote earlier that

$$L = \int_{x_1}^{x_2}\sqrt{1+y'^2}\,\mathrm{d}x \quad \text{then} \quad f(y,y',x) = (1+y'^2)^{1/2}$$

Thus:

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= 0 \\
\frac{\partial f}{\partial y'} &= \frac{1}{2}\left(1 + y'^2\right)^{-1/2} \cdot 2y' \\
\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} &= 0
\end{aligned}
$$

And thus

$$\frac{\partial f}{\partial y'} = \text{constant}$$

Calling this constant C, we have:

$$
\begin{aligned}
C &= \frac{y'}{(1 + y'^2)^{1/2}} \\
C^2(1 + y'^2) &= y'^2 \\
C^2 + C^2 y'^2 &= y'^2 \\
\Rightarrow y'^2 &= \frac{C^2}{1 - C^2} = \text{constant} \\
\Rightarrow y' &= \text{constant} \quad = m \\
\Rightarrow y &= mx + b
\end{aligned}
$$

Which is the equation for a straight line, as we'd have suspected! Victory!

Remark 6.4

Nothing we've done thus far technically tells us whether our equilibrium point is a maximum or a minimum! In reality it can be extremely difficult to determine which we are dealing with (though in the case of the shortest line it is fairly obvious). Fortunately, for our priority cases, all we are going to care about is that it is stationary, not whether or not it is max or min.

Remark 6.5

Multiple dimensions or parametrized functions are separable, each yielding their own set of Euler-Lagrange equations! Which is really nice!