Chapter 1

Newton's Laws of Motion

1.1 Go briefly through syllabus

- Mention homework, tests, project
- Mention helping each other, specially upper/lower classmen

1.2 Overview: Why do we care?

- Classical Mechanics: the study of how things move
  - planets around the sun
  - a car rolling down a hill
  - an electron circling in a magnetic field
  - etc
- You are already familiar with Newton's formulation, devised in the 17th century:
  1. An object in motion tends to stay in motion unless an outside force acts upon it
  2. $F = ma$ or $F = \frac{dp}{dt}$
  3. For every action there is an equal and opposite reaction
- In the 18th and 19th centuries, two new formulations were devised:
  1. Lagrange
  2. Hamilton

  Both are equivalent to Newton's formulation, but generally provide much simpler answers to more complicated problems.
- Failings of Classical Mechanics

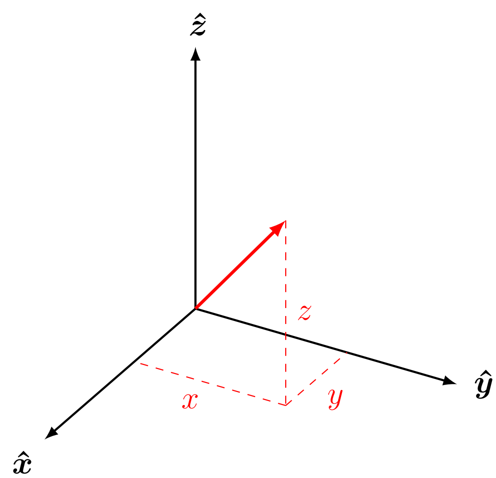
  - Struggles with super fast objects $\Rightarrow$ relativistic mechanics
  - Struggles with super tiny objects $\Rightarrow$ quantum mechanics

- There is still a *wide* swathe of physics where classical mechanics is king
  - In particular, recent work in chaos theory has helped bring work in classical mechanics back into popularity

1.3 The Basics of Vectors

- When explaining motion, we are primarily concerned with describing a point in space
- In 3 dimensions:

$$\vec{\boldsymbol{r}} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}}$$
$$\vec{\boldsymbol{r}} = (x, y, z)$$



- For many vectors, we'll use subscripts, eg:

$$\vec{\boldsymbol{v}} = v_x\,\hat{\boldsymbol{x}} + v_y\,\hat{\boldsymbol{y}} + v_z\,\hat{\boldsymbol{z}}$$

  or

$$\vec{\boldsymbol{a}} = (a_x, a_y, a_z)$$

- This is fine for 3-vectors, but when we move onto greater dimensions, sums do not lend themselves well to this shorthand. Thus let

$$r_x, r_y, r_z \rightarrow r_1, r_2, r_3$$
$$\hat{\boldsymbol{x}},\ \hat{\boldsymbol{y}},\ \hat{\boldsymbol{z}} \rightarrow \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3$$

Thus making our vector:

$$\vec{\boldsymbol{r}} = \sum_i^3 r_i \vec{\boldsymbol{e}}_i$$

Remark 1.1

This form is not as useful to us in the beginning. It will become more useful as things get more complicated!

1.4 Vector Operations

Adding

$$\begin{aligned}\vec{\boldsymbol{r}} + \vec{\boldsymbol{s}} &= (r_1 + s_1, r_2 + s_2, r_3 + s_3) \\ &= \sum_i^3 (r_i + s_i)\vec{\boldsymbol{e}}_i\end{aligned}$$

Adding vectors is very common probably arises most often in adding forces.

Scaling

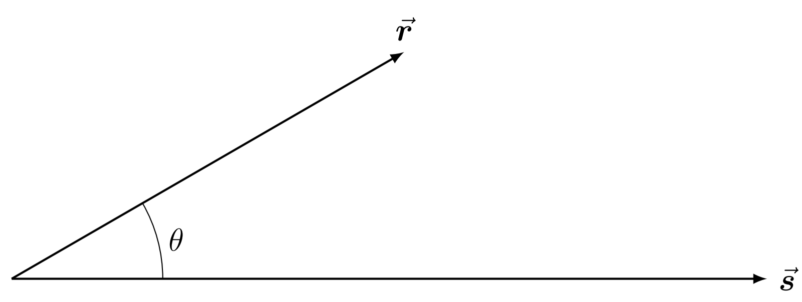
$$c\vec{\boldsymbol{r}} = (cr_1, cr_2, cr_3) = \sum_i^3 cr_i \vec{\boldsymbol{e}}_i$$

And thus scaling keeps the vector pointing in the same (or opposite) direction.

Scalar/Dot Product

$$\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{s}} = rs\cos\theta = r_1 s_1 + r_2 s_2 + r_3 s_3 = \sum_i^3 r_i s_i$$

This essentially gives you the length of s times the length of r in the direction of $\vec{\boldsymbol{s}}$.



This also defines our magnitude:

Definition 1.1: Magnitude

$$r = |\vec{\boldsymbol{r}}| = \sqrt{\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{r}}} = \sqrt{r_1^2 + r_2^2 + r_3^2}$$

Remark 1.2

This book will sometimes use

$$\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}^2$$

which I don't particularly like. (One of the very few things I don't like about this book!)

Vector/Cross Product

$$\vec{\boldsymbol{r}} \times \vec{\boldsymbol{s}} = (r_y s_z - r_z s_y)\, \hat{\boldsymbol{x}} + (r_z s_x - r_x s_z)\, \hat{\boldsymbol{y}} + (r_x s_y - r_y s_x)\, \hat{\boldsymbol{z}}$$

Or, generally more usefully:

$$\vec{\boldsymbol{r}} \times \vec{\boldsymbol{s}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{vmatrix}$$

Writing out the vector product in summation form isn't as nice:

$$\vec{\boldsymbol{r}} \times \vec{\boldsymbol{s}} = \sum_i \sum_j \sum_k \varepsilon_{ijk} r_i s_j \vec{\boldsymbol{e}}_k$$

Here, ε_{ijk} is the levi-civita symbol which takes on the values of:

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{if any index (i,j,k) is the same} \\ 1 & \text{if index permutations even or cyclic (123,231,312)} \\ -1 & \text{if index permutations odd (132, 213, 321)} \end{cases}$$

1.5 Vector Calculus

- Vector derivatives are defined in the same way the classic derivative of a function is defined

$$\frac{d\vec{\boldsymbol{r}}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\boldsymbol{r}}}{\Delta t}$$

- This means that normal sums of derivatives apply:

$$\frac{d}{dt}\left(\vec{\boldsymbol{r}} + \vec{\boldsymbol{s}}\right) = \frac{d\vec{\boldsymbol{r}}}{dt} + \frac{d\vec{\boldsymbol{s}}}{dt}$$

which can be proven easily using the summation definition of a vector if you desire.

- If a vector is multiplied by a function that is also a function of t, then the product rule applies:
$$\frac{d}{dt}\left(f(t)\vec{\boldsymbol{r}}(t)\right) = \frac{df}{dt}\vec{\boldsymbol{r}} + f\frac{d\vec{\boldsymbol{r}}}{dt}$$
- If we want to calculate the derivative of our vector in Cartesian coordinates then:
$$\begin{aligned}\frac{d\vec{\boldsymbol{r}}}{dt} &= \frac{d}{dt}\left[x(t)\,\hat{\boldsymbol{x}} + y(t)\,\hat{\boldsymbol{y}} + z(t)\,\hat{\boldsymbol{z}}\right] \\ &= \frac{dx}{dt}\,\hat{\boldsymbol{x}} + x\frac{d\,\hat{\boldsymbol{x}}}{dt} + \frac{dy}{dt}\,\hat{\boldsymbol{y}} + y\frac{d\,\hat{\boldsymbol{y}}}{dt} + \frac{dz}{dt}\,\hat{\boldsymbol{z}} + z\frac{d\,\hat{\boldsymbol{z}}}{dt}\end{aligned}$$
Since $\hat{x}, \hat{y}, \hat{z}$ don't depend on t (they always point the same direction), then their derivatives vanish.
$$\begin{aligned}&= \frac{dx}{dt}\,\hat{\boldsymbol{x}} + \frac{dy}{dt}\,\hat{\boldsymbol{y}} + \frac{dz}{dt}\,\hat{\boldsymbol{z}} \\ &= v_x\,\hat{\boldsymbol{x}} + v_y\,\hat{\boldsymbol{y}} + v_z\,\hat{\boldsymbol{z}} \\ &= \vec{\boldsymbol{v}}\end{aligned}$$
And acceleration falls out just as nicely:
$$\begin{aligned}\frac{d\vec{\boldsymbol{v}}}{dt} = \frac{d^2\vec{\boldsymbol{r}}}{dt^2} = \vec{\boldsymbol{a}} &= \frac{d^2x}{dt^2}\,\hat{\boldsymbol{x}} + \frac{d^2y}{dt^2}\,\hat{\boldsymbol{y}} + \frac{d^2z}{dt^2}\,\hat{\boldsymbol{z}} \\ &= a_x\,\hat{\boldsymbol{x}} + a_y\,\hat{\boldsymbol{y}} + a_z\,\hat{\boldsymbol{z}}\end{aligned}$$

Example 1.1

If
$$\begin{aligned}\vec{\boldsymbol{r}}(t) &= (3t)\,\hat{\boldsymbol{x}} + (t^2-1)\,\hat{\boldsymbol{y}} + 10\,\hat{\boldsymbol{z}} \\ \vec{\boldsymbol{s}}(t) &= \sum_{i=1}^{3}\left(it^i - 1\right)\vec{\boldsymbol{e}}_i\end{aligned}$$
Then answer the following:

1.
$$\vec{\boldsymbol{s}}\cdot\frac{d\vec{\boldsymbol{r}}}{dt} = ?$$
$$\begin{aligned}\vec{\boldsymbol{s}}\cdot\frac{d\vec{\boldsymbol{r}}}{dt} &= 3(t-1) + (2t)(2t^2-1) + (0)(3t^3-1) \\ &= 4t^3 + t - 3\end{aligned}$$

2.
$$\frac{d\vec{\boldsymbol{s}}}{dt}\times\vec{\boldsymbol{r}} = ?$$

$$\frac{d\vec{\boldsymbol{s}}}{dt} \times \vec{\boldsymbol{r}} = \left(-9t^4 + 9t^2 + 40t, 27t^3 - 10, -11t^2 - 1\right)$$

1.6 Newton's Laws in Inertial Frames

- Recall that, according to Newton's first two laws:
  1. In the absence of forces, a particle moves at a constant speed.
  2. $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$
- We will use the notation then that

$$\dot{\vec{\boldsymbol{r}}} = \frac{d\vec{\boldsymbol{r}}}{dt} = \vec{\boldsymbol{v}}$$
$$\ddot{\vec{\boldsymbol{r}}} = \frac{d^2\vec{\boldsymbol{r}}}{dt^2} = \vec{\boldsymbol{a}}$$

- We define:

Definition 1.2: Momentum

$$\vec{\boldsymbol{p}} = m\vec{\boldsymbol{v}}$$

And therefore can write that

$$\dot{\vec{\boldsymbol{p}}} = m\dot{\vec{\boldsymbol{v}}} + \dot{m}\vec{\boldsymbol{v}}$$

Since m is usually constant,

$$= m\dot{\vec{\boldsymbol{v}}}$$
$$= m\ddot{\vec{\boldsymbol{r}}}$$
$$= \vec{\boldsymbol{F}}$$

1.7 Differential Equations

- Equations like

$$\vec{\boldsymbol{F}} = m\ddot{\vec{\boldsymbol{r}}}$$

are differential equations. We will be seeing a lot of them throughout this course

- In general, this course boils down to:
  - Finding a differential equation that corresponds to the physics
  - Solving that differential equation using a variety of techniques
  - or, failing that, solving the equation numerically using a computer

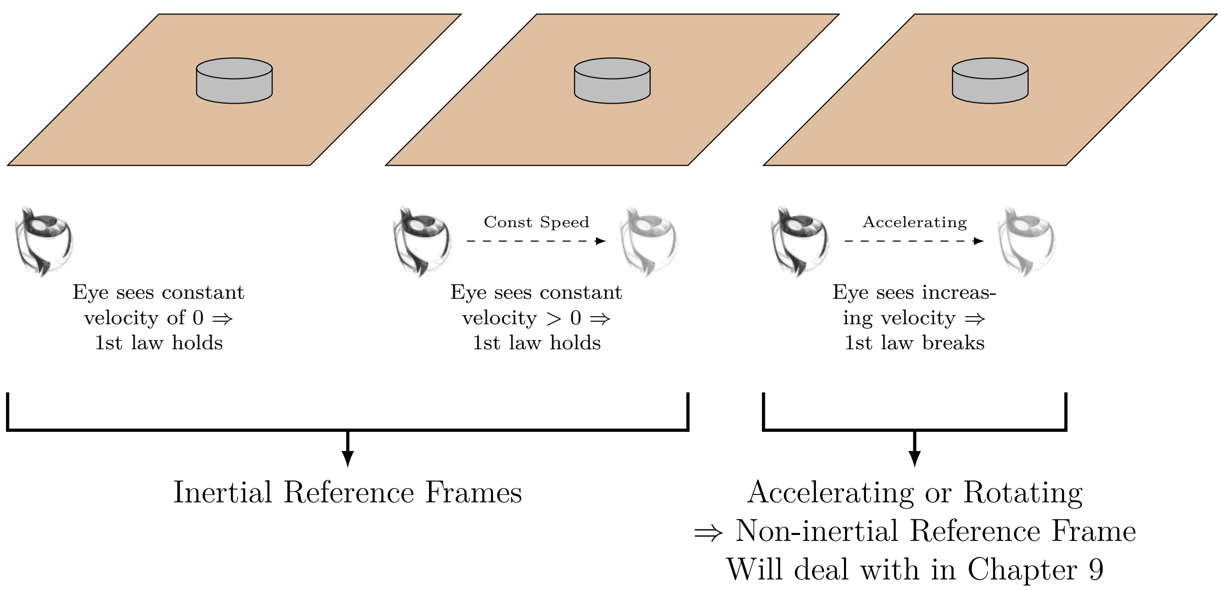
Example 1.2

Take the case where a particle is undergoing a constant force in one dimension.

$$\begin{aligned} F_0 &= m\ddot{x} \\ \Rightarrow \ddot{x}(t) &= \frac{F_0}{m} \\ \dot{x}(t) &= \frac{F_0}{m}t + v_0 \\ x(t) &= \frac{F_0}{2m}t^2 + v_0 t + x_0 \end{aligned}$$

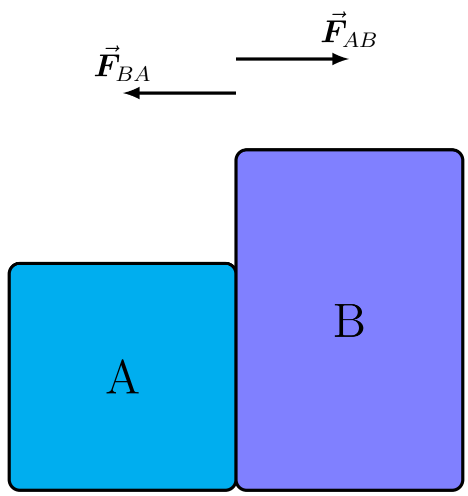
1.8 Reference Frames

- Newton's 1st law helps establish the idea of inertial reference frames



1.9 Newton's Third and Conservation of Momentum

- Recall that, for two objects:



And Newton's Third law states that:

$$\vec{\boldsymbol{F}}_{AB} = -\vec{\boldsymbol{F}}_{BA}$$

- Also recall that the conservation of momentum states that the total momentum is conserved only if the sum of all external forces equals zero. But mathematically:

$$\Rightarrow \dot{\vec{\boldsymbol{P}}} = \vec{\boldsymbol{F}}_{ext}$$

- Thus, in our two particle system, Newton's second law implies that

$$\begin{aligned}\dot{\vec{\boldsymbol{p}}}_A &= \vec{\boldsymbol{F}}_{A,net} = \vec{\boldsymbol{F}}_{AB} + \vec{\boldsymbol{F}}_{A,ext} \\ \dot{\vec{\boldsymbol{p}}}_B &= \vec{\boldsymbol{F}}_{B,net} = \vec{\boldsymbol{F}}_{BA} + \vec{\boldsymbol{F}}_{B,ext}\end{aligned}$$

And if the total momentum is equal to:

$$\begin{aligned}\vec{\boldsymbol{P}} &= \vec{\boldsymbol{p}}_A + \vec{\boldsymbol{p}}_B \\ \Rightarrow \dot{\vec{\boldsymbol{P}}} &= \dot{\vec{\boldsymbol{p}}}_A + \dot{\vec{\boldsymbol{p}}}_B \\ &= \vec{\boldsymbol{F}}_{AB} + \vec{\boldsymbol{F}}_{A,ext} + \vec{\boldsymbol{F}}_{BA} + \vec{\boldsymbol{F}}_{B,ext} \\ &= \vec{\boldsymbol{F}}_{A,ext} + \vec{\boldsymbol{F}}_{B,ext} \\ &= \vec{\boldsymbol{F}}_{ext}\end{aligned}$$

And thus momentum is conserved!

- This works for a many-particle system as well. If A is a particular particle and B is another particle creating a force on particle A, then by second law:

$$\dot{\vec{\boldsymbol{p}}}_A = \vec{\boldsymbol{F}}_{A,net} = \sum_{B\neq A} \vec{\boldsymbol{F}}_{AB} + \vec{\boldsymbol{F}}_{A,ext}$$

since every other particle B will act on A except A itself. Now the total momentum in the system is

$$\dot{\vec{\boldsymbol{P}}} = \sum_A \dot{\vec{\boldsymbol{p}}}_A$$

and thus

$$\begin{aligned}
\dot{\vec{\boldsymbol{P}}} &= \sum_A \dot{\vec{\boldsymbol{p}}}_A \\
&= \sum_A \left[\sum_{B\neq A} \vec{\boldsymbol{F}}_{AB} + \vec{\boldsymbol{F}}_{A,ext}\right] \\
&= \underbrace{\sum_A \sum_{B\neq A} \vec{\boldsymbol{F}}_{AB}}_{\text{third law pairs}} + \sum_A \vec{\boldsymbol{F}}_{A,ext} \\
&= \sum_A \vec{\boldsymbol{F}}_{A,ext} \\
&= \vec{\boldsymbol{F}}_{ext}
\end{aligned}$$

And thus CoM and N3 are linked, which is pretty neat!

- This breaks down when moving to relativistic speeds, as we need to be able to measure the momentum at exactly the same time

- Also breaks for magnetic forces on moving charges:

  Here it is clear that $\vec{\boldsymbol{F}}_{12} \neq -\vec{\boldsymbol{F}}_{21}$! *Mechanical* energy is therefore not conserved (EM fields also have momentum to consider!).

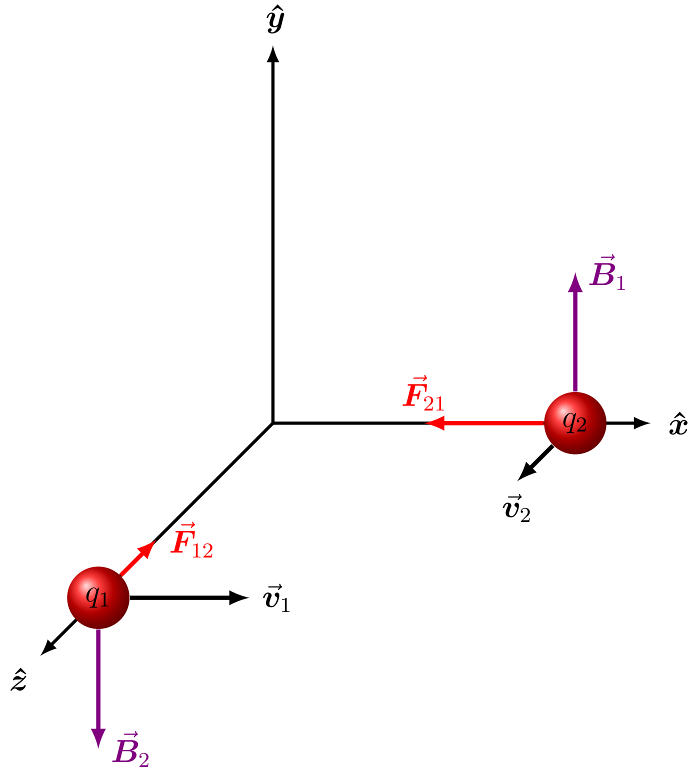
Remark 1.3

Fortunately the force due to the magnetic fields only approaches that due the electric fields when $v \to c$, so it's effects are largely negligible in day to day use.

1.10 Newton in 3D Cartesian Coordinates

- Recall that

$$\vec{\boldsymbol{r}} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}}$$



- In Cartesian coordinates then, Newton's second law can be written as:

$$\begin{aligned}\vec{\boldsymbol{F}} &= m\ddot{\vec{\boldsymbol{r}}}\\ &= m\left(\ddot{x}\,\hat{\boldsymbol{x}} + \ddot{y}\,\hat{\boldsymbol{y}} + \ddot{z}\,\hat{\boldsymbol{z}}\right)\\ &= m\ddot{x}\,\hat{\boldsymbol{x}} + m\ddot{y}\,\hat{\boldsymbol{y}} + m\ddot{z}\,\hat{\boldsymbol{z}}\end{aligned}$$

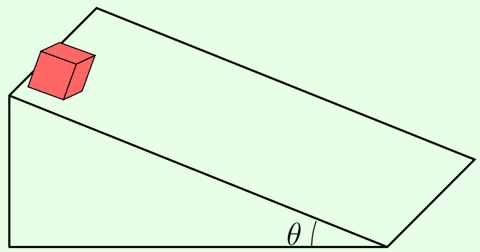
Or, written differently:

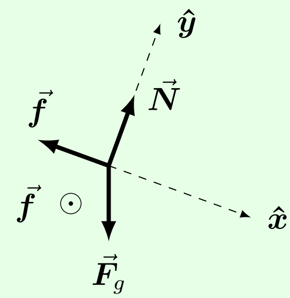
$$\begin{aligned}F_x &= m\ddot{x}\\ F_y &= m\ddot{y}\\ F_z &= m\ddot{z}\end{aligned}$$

Example 1.3

A block is sliding down an incline that makes an angle θ with the horizontal. Given the initial conditions below, describe it's motion over time with respect to the x, y, and z directions.

$$\vec{\boldsymbol{r}}_0 = (0,0,0) \qquad \dot{\vec{\boldsymbol{r}}}_0 = (0,0,-5)$$





Thus we have that

$$
\begin{aligned}
m\ddot{y} &= N - mg\cos\theta = 0 \\
\Rightarrow N &= mg\cos\theta \\
m\ddot{x} &= mg\sin\theta - \mu N \\
&= mg\sin\theta - \mu mg\cos\theta \\
m\ddot{z} &= \mu N \\
&= \mu mg\cos\theta
\end{aligned}
$$

Going from our differential equations to our solutions:

$$
\begin{aligned}
\ddot{x} &= g\left(\sin\theta - \mu\cos\theta\right) \\
\dot{x} &= g\left(\sin\theta - \mu\cos\theta\right)t + 0 \\
x &= \frac{1}{2}g\left(\sin\theta - \mu\cos\theta\right)t^2 + 0
\end{aligned}
$$

and

$$
\begin{aligned}
\ddot{z} &= \mu g\cos\theta \\
\dot{z} &= \mu g\cos\theta t - 5 \\
z &= \frac{1}{2}\mu g\cos\theta t^2 - 5t + 0
\end{aligned}
$$

Note that this solution for z only holds until $\dot{z}$ goes to zero, at which point the force in the z direction would vanish. Our final solution is thus:

$$
\vec{\boldsymbol{r}}(t) = \left(\frac{1}{2}g\left(\sin\theta - \mu\cos\theta\right)t^2, 0, \frac{1}{2}\mu g\cos\theta\, t^2 - 5t\right)
$$

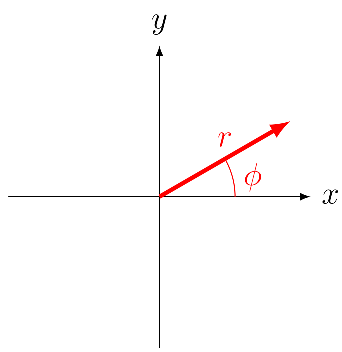
1.11 Polar Coordinates

Polar coordinates are defined in terms of Cartesian coordinates by:

$$r = \sqrt{x^2 + y^2}$$
$$\phi = \arctan\left(\frac{y}{x}\right)$$

or

$$x = r\cos\phi$$
$$y = r\sin\phi$$

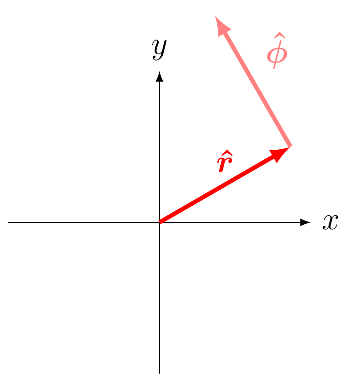


Definition 1.3: Unit Vectors

We can define unit vectors in the direction of any vector using

$$\hat{\boldsymbol{r}} = \frac{\vec{\boldsymbol{r}}}{|\vec{\boldsymbol{r}}|} = \frac{\vec{\boldsymbol{r}}}{\sqrt{\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{r}}}}$$

As such, in polar we have:

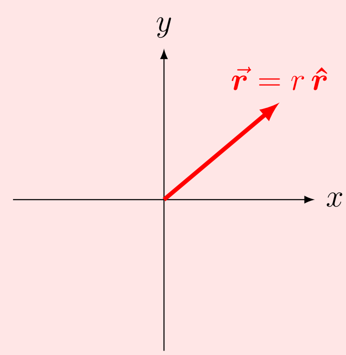


These two unit vectors are orthogonal, and thus any vector in two space can be described as a combination of the two. Eg:

$$\vec{\boldsymbol{F}} = F_r\,\hat{\boldsymbol{r}} + F_\phi\,\hat{\boldsymbol{\phi}}$$

Remark 1.4

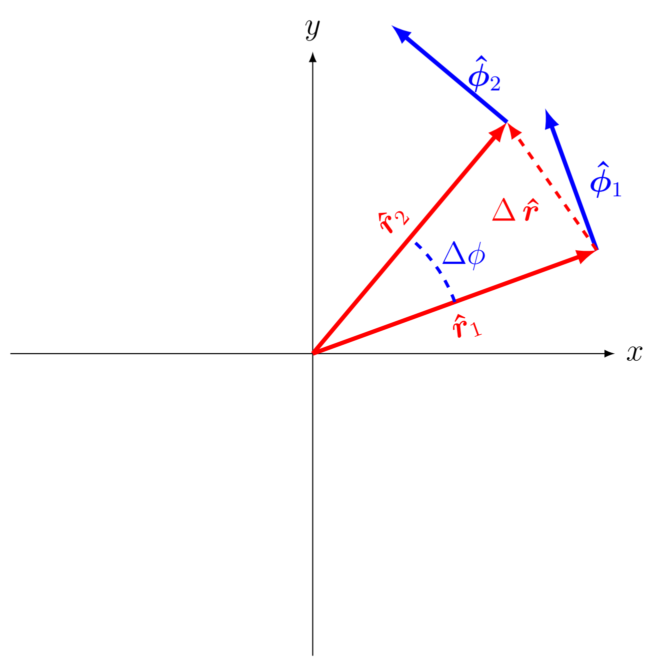
To describe any position, our vector has purely a component in the radial direction:



So now, if we want Newton's 2nd law in polar form, we need to find $\ddot{\vec{\boldsymbol{r}}}$.

$$\dot{\vec{\boldsymbol{r}}} = \dot{r}\,\hat{\boldsymbol{r}} + r\,\dot{\hat{\boldsymbol{r}}}$$

But $\hat{\boldsymbol{r}}$ is not constant this time! It's direction can change! So what is $\dot{\hat{\boldsymbol{r}}}$?



By small angle approximation:

$$\Delta \hat{\boldsymbol{r}} \approx \Delta\phi(1)\,\hat{\boldsymbol{\phi}}$$
$$\Rightarrow \frac{\Delta \hat{\boldsymbol{r}}}{\Delta t} \approx \frac{\Delta\phi}{\Delta t}\,\hat{\boldsymbol{\phi}}$$
$$\Rightarrow \dot{\hat{\boldsymbol{r}}} = \dot{\phi}\,\hat{\boldsymbol{\phi}}$$

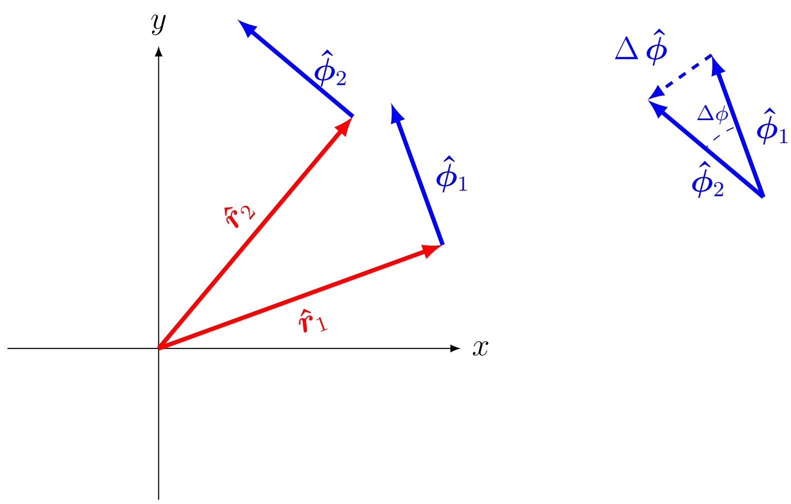
And thus:

$$\dot{\vec{\boldsymbol{r}}} = \dot{r}\,\hat{\boldsymbol{r}} + r\dot{\phi}\,\hat{\boldsymbol{\phi}}$$

Now we play the same game again!

$$\begin{aligned}\ddot{\vec{\boldsymbol{r}}} &= \ddot{r}\,\hat{\boldsymbol{r}} + \dot{r}\,\dot{\hat{\boldsymbol{r}}} + r\frac{d}{dt}\left[\dot{\phi}\,\hat{\boldsymbol{\phi}}\right] + \dot{r}\dot{\phi}\,\hat{\boldsymbol{\phi}} \\ &= \ddot{r}\,\hat{\boldsymbol{r}} + \dot{r}(\dot{\phi}\,\hat{\boldsymbol{\phi}}) + \dot{r}\dot{\phi}\,\hat{\boldsymbol{\phi}} + r\left(\ddot{\phi}\,\hat{\boldsymbol{\phi}} + \dot{\phi}\,\dot{\hat{\boldsymbol{\phi}}}\right)\end{aligned}$$

But now what is $\dot{\hat{\boldsymbol{\phi}}}$?! Again, by small angle:



$$\frac{\Delta\,\hat{\boldsymbol{\phi}}}{\Delta t} = \frac{\Delta\phi(1)(-\,\hat{\boldsymbol{r}})}{\Delta t}$$
$$\dot{\hat{\boldsymbol{\phi}}} = -\dot{\phi}\,\hat{\boldsymbol{r}}$$

And thus:

$$\begin{aligned}\ddot{\vec{\boldsymbol{r}}} &= \ddot{r}\,\hat{\boldsymbol{r}} + \dot{r}(\dot{\phi}\,\hat{\boldsymbol{\phi}}) + \dot{r}\dot{\phi}\,\hat{\boldsymbol{\phi}} + r\left(\ddot{\phi}\,\hat{\boldsymbol{\phi}} + \dot{\phi}(-\dot{\phi}\,\hat{\boldsymbol{r}})\right) \\ &= \left(\ddot{r} - r\dot{\phi}^2\right)\,\hat{\boldsymbol{r}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\,\hat{\boldsymbol{\phi}}\end{aligned}$$

So, putting it all together, in polar coordinates we have that

$$\begin{aligned}
\vec{\boldsymbol{r}} &= r\,\hat{\boldsymbol{r}} \\
\vec{\boldsymbol{v}} &= \dot{r}\,\hat{\boldsymbol{r}} + r\dot{\phi}\,\hat{\boldsymbol{\phi}} \\
&= \dot{r}\,\hat{\boldsymbol{r}} + r\omega\,\hat{\boldsymbol{\phi}} \\
\vec{\boldsymbol{a}} &= \left(\ddot{r} - r\dot{\phi}^2\right)\,\hat{\boldsymbol{r}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\,\hat{\boldsymbol{\phi}} \\
&= \left(\ddot{r} - r\omega^2\right)\,\hat{\boldsymbol{r}} + \left(r\alpha + 2\dot{r}\omega\right)\,\hat{\boldsymbol{\phi}}
\end{aligned}$$

Remark 1.5

In polar coordinates Newton's 2nd law looks like:

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} \Rightarrow \begin{cases} F_r = m\left(\ddot{r} - r\dot{\phi}^2\right) = m\left(\ddot{r} - r\omega^2\right) \\ F_\phi = m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) = m\left(r\alpha + 2\dot{r}\omega\right) \end{cases}$$

Example 1.4

Take a particle on a slope (so that ϕ is constant). Thus

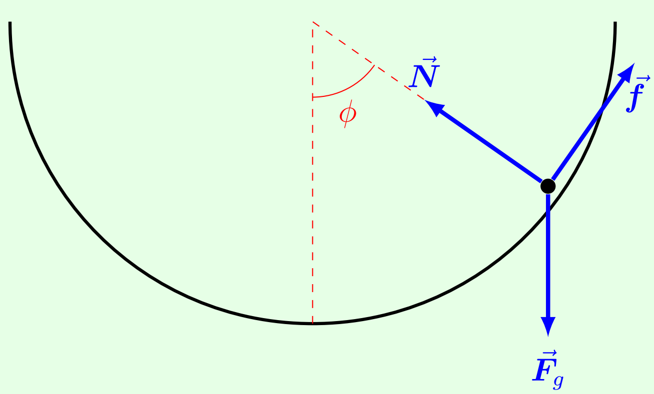
$$\vec{\boldsymbol{F}}_{net} = (m\ddot{r}, 0)$$

Which is exactly what we'd expect and identical to us rotating our axes.

Example 1.5

Take instead a particle on a rough circle, such that r is restricted to be constant. Our net force is thus:

$$\vec{\boldsymbol{F}}_{net} = \underbrace{\boxed{-mr\omega^2}}_{\text{Centripetal Force}}\,\hat{\boldsymbol{r}} + \underbrace{\boxed{mr\alpha}}_{\text{Tangential Force}}\,\hat{\boldsymbol{\phi}}$$



Thus:

$$\sum F_r = F_g \cos\phi - N = mg\cos\phi - N$$
$$\sum F_\phi = f - F_g \sin\phi = f - mg\sin\phi$$

Setting things equal to our expression for Newton's 2nd Law in polar:

$$mg\cos\phi - N = -mr\omega^2 \tag{1.1}$$
$$\mu N - mg\sin\phi = mr\alpha \tag{1.2}$$

Solving Eq 1.2 for N and plugging into Eq 1.1:

$$N = \frac{mr\ddot{\phi} + mg\sin\phi}{\mu}$$
$$\Rightarrow mg\cos\phi - \frac{mr\ddot{\phi}}{\mu} - \frac{mg\sin\phi}{\mu} = -mr\dot{\phi}^2$$
$$\Rightarrow r\ddot{\phi} = r\mu\dot{\phi}^2 + g\left(\mu\cos\phi - \sin\phi\right)$$
$$\ddot{\phi} = \mu\dot{\phi}^2 + \frac{g}{r}(\mu\cos\phi - \sin\phi)$$

This is non-trivial to solve! It looks a bit like a damped oscillator but with a squared term and some extra trig terms floating around. So how do we solve it? Numerically!

Remark 1.6

Note that the sign of the friction force switches when the object is traveling the other direction. In these cases the signs of the $\dot{\phi}^2$ term and the cosine term become negative.