Chapter 5

Oscillations

Our general gameplan for this chapter will be:

Start simple: multiple ways to solve a SHO

→ Add Damping

→ Add drivers and forcing

5.1 The Simple Case

Hooke's law tells us that

$$\vec{\mathbf{F}}_x(x) = -kx \quad \text{or} \quad U(x) = \frac{1}{2}kx^2$$

The potential tells us that

$$\frac{\mathrm{d}U}{\mathrm{d}x} = kx = 0 \quad \text{so equilbrium point at 0}$$

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} = k \quad \text{so long as } k \text{ is positive, we are stable}$$

Remark 5.1

Note that if k wasn't positive and we weren't stable, then we certainly aren't going to get oscillations near that point!

So what about *any* potential that happens to have a stable equilibrium point? Will they all exhibit simple harmonic motion near that point?

We'll shift the equilibrium point to 0 to keep things simple. Taking a Taylor series near that point gives us:

$$U(x) = \underbrace{\boxed{U(0)}}_{\text{A constant (include in our zero)}} + \underbrace{\boxed{\dot{U}(0)}}_{\text{0 b/c our equil point at 0}} x + \frac{1}{2}\ddot{U}(0)x^2 + \underbrace{\boxed{\cdots}}_{\text{just keeping 3 terms so long as x is small}}$$

And thus

$$U(x) \approx \frac{1}{2}\underbrace{\ddot{U}(0)}_{k}x^2$$

Remark 5.2

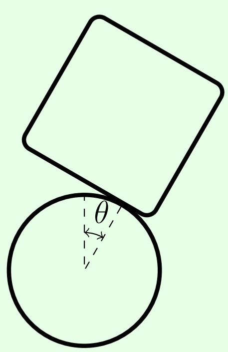
Thus *any* potential with a stable equilibrium point can be approximated with

$$U(x) = \frac{1}{2}kx^2$$

for small displacements about the equilibrium point.

Example 5.1

Recall our rocking cube, where we had



$$U(\theta) = mg\left[(r+b)\cos\theta + r\theta\sin\theta\right]$$

So when θ is small:

$$\sin\theta \approx \theta$$

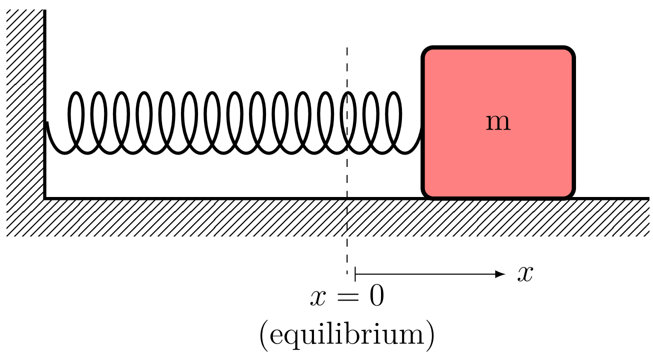
$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$

And thus we have

$$
\begin{aligned}
U(\theta) &= mg\left[(r+b)\left(1-\frac{\theta^2}{2}\right)+r\theta^2\right] \\
&= mg(r+b) - mg(r+b)\frac{\theta^2}{2} + r\theta^2 mg \\
&= \underbrace{\boxed{mg(r+b)}}_{\text{const}} + \underbrace{\boxed{mg(r-b)\frac{\theta^2}{2}}}_{\text{k}}
\end{aligned}
$$

And note that k is still positive and thus stable only when $r > b$.

Now to review our known solutions to the SHM. So at it's most basic we have that:



$$\ddot{x} = -\frac{k}{m}x = -\omega^2 x \tag{5.1}$$

where

$$\omega = \sqrt{\frac{k}{m}}$$

So now we'll discuss the various ways to solve Eq 5.1.

5.1.1 Solution Form 1: Exponential Solutions

Note that

$$x(t) = e^{i\omega t} \quad \text{or} \quad x(t) = e^{-i\omega t}$$

both satisfy Eq 5.1. Thus we have a general solution of the form:

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \tag{5.2}$$

Remark 5.3

Note here that $x(t)$ needs to be real, so the constants C_1 and C_2 would need to be chosen accordingly.

5.1.2 Solution Form 2: Trig Solutions

Recall that we can use Euler's equation to relate complex exponentials and trig terms:

$$e^{\pm i\omega t} = \cos(\omega t) \pm i\sin(\omega t)$$

We can rewrite Eq 5.2 thus as:

$$x(t) = \underbrace{\boxed{(C_1 + C_2)}}_{B_1}\cos(\omega t) + \underbrace{\boxed{i\,(C_1 - C_2)}}_{B_2}\sin(\omega t)$$

And so we have

$$x(t) = B_1\cos(\omega t) + B_2\sin(\omega t) \tag{5.3}$$

If the object starts at some position x_0 with some speed v_0, then plugging in $t = 0$ gives us that:

$$B_1 = x_0$$
$$\omega B_2 = v_0$$

$$\Rightarrow x(t) = x_0\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t)$$

which has a period of oscillation equal to

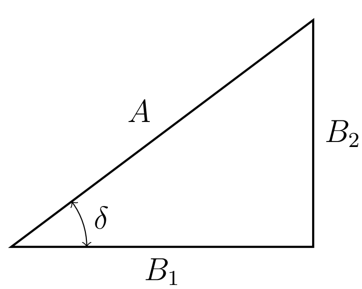
$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

5.1.3 Solution Form 3: Phase Shifted Solutions

Let

$$A = \sqrt{B_1^2 + B_2^2}$$

so that it describes the triangle:



Then, multiplying Eq 5.3 by $\frac{A}{A}$ gives:

$$\begin{aligned}x(t) &= \frac{A}{A}\left[B_1\cos(\omega t) + B_2\sin(\omega t)\right] \\ &= A\left[\frac{B_1}{A}\cos(\omega t) + \frac{B_2}{A}\sin(\omega t)\right] \\ &= A\left[\cos(\delta)\cos(\omega t) + \sin(\delta)\sin(\omega t)\right]\end{aligned}$$

Recalling that $\cos(u \pm v) = \cos(u)\cos(v) \mp \sin(u)\sin(v)$:

$$= A\cos(\omega t - \delta)$$

So we also have that

$$x(t) = A\cos(\omega t - \delta) \tag{5.4}$$

5.1.4 Solution Form 4: Real Part of Complex Exponential

Recalling that

$$\begin{aligned}C_1 + C_2 &= B_1 \\ i(C_1 - C_2) &= B_2\end{aligned}$$

Then we can rearrange and solve in terms of the C's to find that

$$\begin{aligned}C_1 &= \frac{1}{2}\left(B_1 - iB_2\right) \\ C_2 &= \frac{1}{2}\left(B_1 + iB_2\right)\end{aligned}$$

Or in other words, that

$$C_1 = C_2^*$$

This means that we can simplify Eq 5.2 to:

$$x(t) = C_1 e^{i\omega t} + \underbrace{\boxed{C_1^* e^{-i\omega t}}}_{\text{complex conjugate of 1st term!}}$$

Recall that

$$z + z^* = (x + iy) + (x - iy) = 2x = 2\,\mathrm{Re}[z]$$

Thus we can write that

$$x(t) = 2\,\mathrm{Re}\left[C_1 e^{i\omega t}\right]$$

If we let $C = 2C_1$:

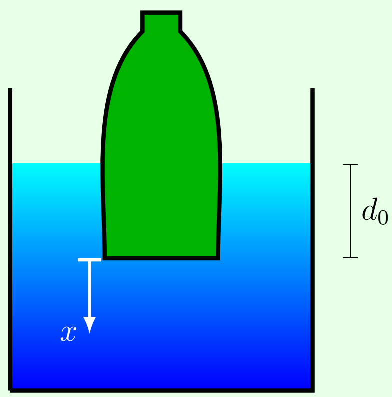
$$\begin{aligned} C &= B_1 - iB_2 \\ &= A\cos(\delta) - Ai\sin(\delta) \\ &= A\left(\cos(\delta) - i\sin(\delta)\right) \\ &= Ae^{-i\delta} \end{aligned}$$

And hence

$$x(t) = \operatorname{Re}\left[Ae^{-i\delta}e^{i\omega t}\right] = \operatorname{Re}\left[Ae^{i(\omega t-\delta)}\right]$$

Example 5.2

Suppose we have a bottle floating in a barrel of water, currently at it's equilibrium position. We push the bottle slightly downward and are interested in finding the period of its resulting oscillations.



The bottle is experiencing two forces:

$$\begin{aligned} F_{buoy} &= \rho g V \\ F_{grav} &= mg \end{aligned}$$

And thus

$$\begin{aligned} mg &= \rho g A d_0 \\ \Rightarrow d_0 &= \frac{m}{\rho A} \end{aligned}$$

So if we push the bottle down some distance x:

$$\begin{aligned} \sum \vec{\mathbf{F}} : m\ddot{x} &= mg - \rho g A\left(d_0 + x\right) \\ &= \rho A d_0 g - \rho g A\left(d_0 + x\right) \\ &= -\rho g A x \\ \rho A d_0 \ddot{x} &= -\rho g A x \\ \ddot{x} &= -\frac{g}{d_0}x \end{aligned}$$

Which exhibits simple harmonic motion! We know that

$$\omega = \sqrt{\frac{g}{d_0}}$$

and the period is thus

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{d_0}{g}}$$

5.1.5 A note on energies

Using the 3rd form (Eq 5.4):

$$x(t) = A\cos(\omega t - \delta)$$

Then the potential energy can be written as

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t - \delta)$$

Taking a derivative to find the velocity:

$$\dot{x} = v(t) = -\omega A\sin(\omega t - \delta)$$

And thus we can find the kinetic energy:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\sin^2(\omega t - \delta) = \frac{1}{2}\cancel{m}\frac{k}{\cancel{m}}A^2\sin^2(\omega t - \delta)$$

And so putting it all together yields:

$$T + U = \frac{1}{2}kA^2\left(\cos^2(\omega t - \delta) + \sin^2(\omega t - \delta)\right) = \frac{1}{2}kA^2$$

5.2 Two (or Three) Dimensional Oscillators

5.2.1 Isotropic Harmonic Oscillators

Isotropic harmonic oscillators are those where all dimensions are oscillating with the same spring constant. Thus:

$$\vec{\mathbf{F}} = -k\vec{\mathbf{r}}$$

where

$$F_x = -kx$$
$$F_y = -ky$$
$$F_z = -kz$$

Remark 5.4

These are all completely unlinked equations, and thus each will have its own independent solutions

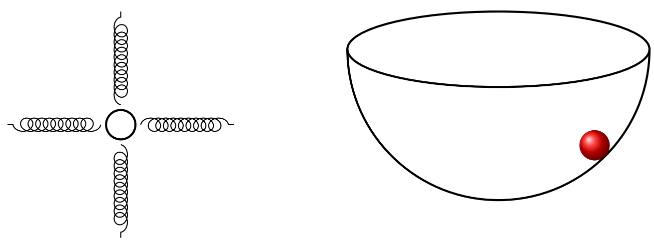
We therefore get the solutions:

$$\ddot{x} = -\omega^2 x$$
$$\ddot{y} = -\omega^2 y$$
$$\ddot{z} = -\omega^2 z$$

where

$$\omega = \sqrt{\frac{k}{m}}$$

as per usual. You can picture these as something like:



We therefore get solutions of the form:

$$x(t) = A_x \cos\left(\omega t - \delta_x\right)$$
$$y(t) = A_y \cos(\omega t - \delta_y)$$

If we shift our stopwatch so that we start time at $t = t + \delta_x$, then:

$$x(t) = A_x \cos(\omega t)$$
$$y(t) = A_y \cos(\omega t - \delta)$$

where

$$\delta = \delta_y - \delta_x$$

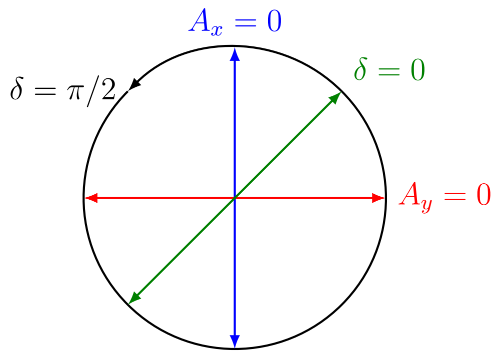
The motion then ends up depending on the three parameters A_x, A_y, δ

$$A_x \quad \text{or} \quad A_y = 0 \quad \Rightarrow \quad \text{1d oscillations}$$

$$\delta = 0 \quad \Rightarrow \quad \text{Oscillating together}$$

Remark 5.5

Note that a $\delta = \pi/2$ will given perfectly out of phase, or circular motion!



Other values of δ will give results between!

5.2.2 Anisotropic Harmonic Oscillators

Anisotropic oscillators are those which have different spring constants in each dimension. Eg.

$$F_x = -k_x x$$
$$F_y = -k_y y$$
$$F_z = -k_z z$$

And thus

$$\ddot{x} = -\omega_x^2 x$$
$$\ddot{y} = -\omega_y^2 y$$
$$\ddot{z} = -\omega_z^2 z$$

Again, in two dimensions we could write this as:

$$x(t) = A_x \cos(\omega_x t)$$
$$y(t) = A_y \cos(\omega_y t - \delta)$$

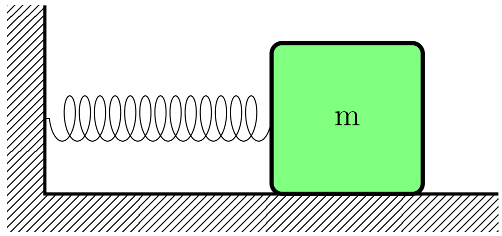
where we define δ the same as earlier. This means the exact resulting orbits will depend on the ratio between ω_x and ω_y!

- If $\dfrac{\omega_x}{\omega_y}$ is rational, then we see periodic motion
- If $\dfrac{\omega_x}{\omega_y}$ is irrational, then we call the system quasiperiodic and the paths *never* repeat exactly!

5.3 Damped Oscillators

We'll now add situations where the oscillator has some resistive force acting against its motion. Technically this force could be velocity independent, linear, quadratic, etc. **We'll be looking at the special case for *linear* damping!**

Let us return to looking in only 1-dimension:



$$\Rightarrow \vec{\mathbf{F}} = -kx - b\dot{x}$$

$$\Rightarrow m\ddot{x} + b\dot{x} + kx = 0$$

Remark 5.6

This is actually identical to the equation governing the voltage through a LCR circuit!

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0$$

Dividing everything through by m:

$$\ddot{x} + \underbrace{\boxed{\frac{b}{m}}}_{2\beta}\dot{x} + \underbrace{\boxed{\frac{k}{m}}}_{\omega_0^2}x = 0$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

This is a 2nd order differential equation, so to solve it generally we need to find two independent solutions. Let's try

$$x(t) = e^{rt} \quad \Rightarrow \dot{x} = re^{rt} \quad \Rightarrow \ddot{x} = r^2 e^{rt}$$

So

$$\begin{aligned} \ddot{x} + 2\beta\dot{x} + \omega_0^2 x &= 0 \\ r^2 e^{rt} + 2\beta r e^{rt} + \omega_0^2 e^{rt} &= 0 \\ r^2 + 2\beta r + \omega_0^2 &= 0 \end{aligned}$$

Solving the quadratic equation gives us the two solutions we need!

$$r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}$$

$$r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$$

So so we can form our general solution of

$$\begin{aligned} x(t) &= C_1 e^{r_1 t} + C_2 e^{r_2 t} \\ &= C_1 \exp\left((-\beta + \sqrt{\beta^2 - \omega_0^2})t\right) + C_2 \exp\left((-\beta - \sqrt{\beta^2 - \omega_0^2})t\right) \\ &= e^{-\beta t}\left(C_1 \exp\left(t\sqrt{\beta^2 - \omega_0^2}\right) + C_2 \exp\left(-t\sqrt{\beta^2 - \omega_0^2}\right)\right) \end{aligned}$$

So now we'll look at the results in a variety of situations.

- When $\beta = 0$

  $$x(t) = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$$

  which is, of course, the non-damped solution and boring

- When β is small (weak damping):

  In the instance where β is smaller than ω_0, we can write:

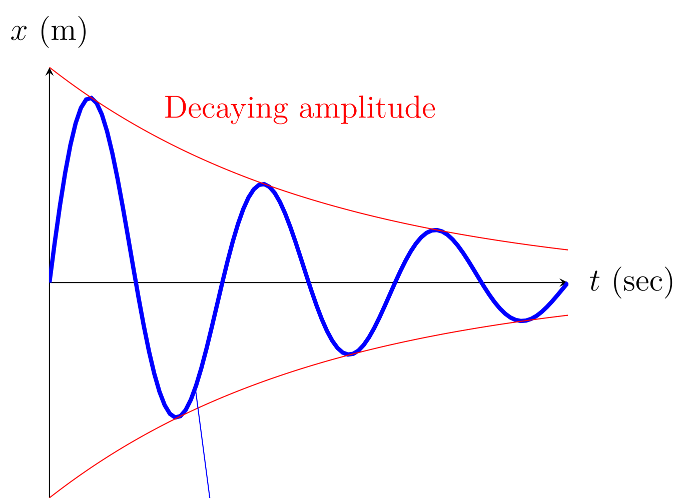
  $$\sqrt{\beta^2 - \omega_0^2} = i\sqrt{\underbrace{\omega_0^2 - \beta^2}_{\omega_1^2}} = i\omega_1$$

  which means that

  $$x(t) = e^{-\beta t}\left(C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right)$$

  or

  $$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$$



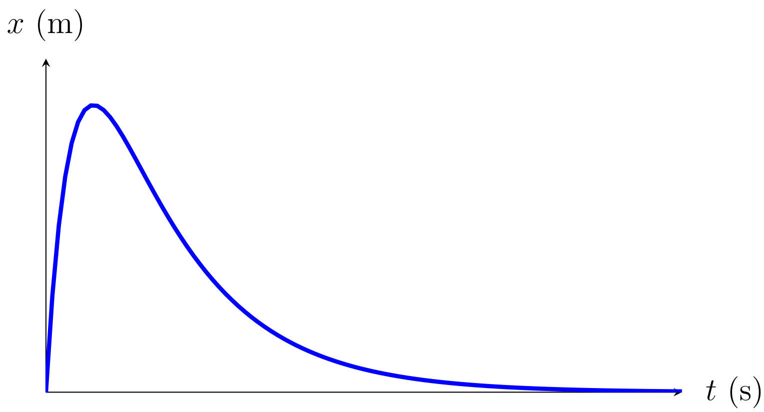
- When $\beta > \omega_0$ (strong damping)

$$\Rightarrow \sqrt{\beta^2 - \omega_0^2} \quad \text{is real}$$

Thus the solution is simply the combination of two decaying exponentials:

$$x(t) = C_1 \exp\left((-\beta + \sqrt{\beta^2 - \omega_0^2})t\right) + C_2 \exp\left((-\beta - \sqrt{\beta^2 - \omega_0^2})t\right)$$

The first term is a smaller (negative) number. Thus it decays slower and will eventually dominate the long term motion.



- When $\beta = \omega_0$ (critical damping)

In this case, we would only get a single root from our characteristic equation:

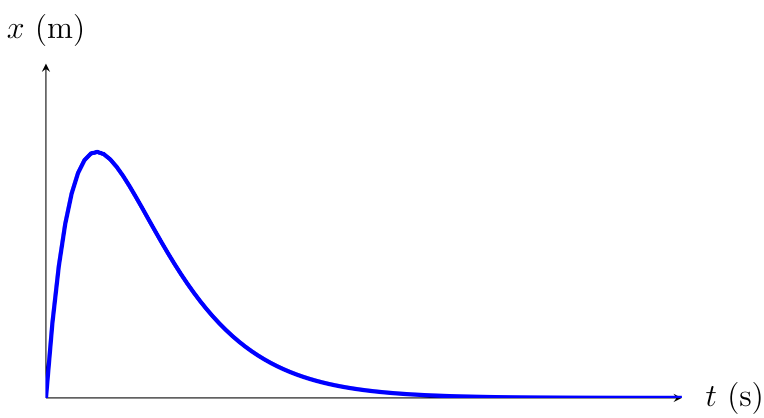
$$r = -\beta$$

This only gives us one solution and thus we don't have enough to form a complete general solution. It turns out though for these type problems that multiplying one solution by t will still yield another solution:

$$x(t) = C_2 t e^{-\beta t}$$

You can check that this still works in the DE. Putting the two solutions together:

$$x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t}$$



5.4 Driven Oscillators

If we have an external driving force (say a push on a swing each oscillation) then

$$m\ddot{x} = -b\dot{x} - kx + \underbrace{\boxed{F(t)}}_{\text{only depends on time!}}$$

and thus

$$\begin{aligned} m\ddot{x} + b\dot{x} + kx &= F(t) \\ \ddot{x} + 2\beta\dot{x} + \omega_0^2 x &= f(t) \end{aligned}$$

where

$$f(t) = \frac{F(t)}{m}$$

Remark 5.7

Note that plugging in $x = e^{rt}$ here will no longer give us a nice quadratic equation in r, so we need a new plan of attack to solve this beast.

Say we write the above equation as

$$Dx = f$$

where

$$D = \frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\beta\frac{\mathrm{d}}{\mathrm{d}t} + \omega_0^2$$

Remark 5.8

This is just another form of differential operator, similar to how

$$\nabla = \frac{\mathrm{d}}{\mathrm{d}x}\,\hat{\boldsymbol{x}} + \frac{\mathrm{d}}{\mathrm{d}y}\,\hat{\boldsymbol{y}} + \frac{\mathrm{d}}{\mathrm{d}z}\,\hat{\boldsymbol{z}}$$

Remark 5.9

Also note that this is *linear* in x! (It looks similary to your linear algebra $Ax = b$ type equations.)

This means that your normal properties for linear equations hold:

$$\begin{aligned} D(ax) &= aD(x) \\ D(x_1 + x_2) &= D(x_1) + D(x_2) \\ D(ax_1 + bx_2) &= aD(x_1) + bD(x_2) \end{aligned}$$

So our old damped case could be written as

$$\begin{aligned} Dx &= 0 \\ D(C_1x_1 + C_2x_2) &= C_1Dx_1 + C_2Dx_2 \\ &= 0 + 0 = 0 \end{aligned}$$

Definition 5.1: Homogeneous Equation

This form, when the right-hand side is equal to zero, is known as the **homogeneous equation**.

$$Dx = 0$$

By contrast,

Definition 5.2: Inhomogeneous Equation

When the right-hand side is *not* equal to zero, then we have an **inhomogeneous equation**:

$$Dx = f$$

5.4.1 Solving the Inhomogenous Equation

Say we can find one specific solution to

$$Dx_p = f$$

We already know the solution to

$$Dx_n = 0$$

which gives us

$$x_n = C_1e^{r_1t} + C_2e^{r_2t}$$

Importantly, then:

$$\begin{aligned} D(x_p + x_n) &= Dx_p + Dx_n \\ &= f + 0 \\ &= f \end{aligned}$$

And so we see that $x_p + x_n$ is also a solution to our inhomogeneous equation. Moreover, it is a *general solution* to our equation because it already has two arbitrary constants (C_1 and C_2) in it. So our equation then becomes: how do we find x_p?

5.4.2 Finding x_p: The Particular Solution

The exact particular solution is going to depend on the form of f. For our purposes, we'll assume the oscillator is being driven in a periodic fashion. Thus we can write:

$$f(t) = f_0 \cos(\omega t)$$

where

$$\begin{aligned} f_0 = &\quad \text{the driving amplitude} \\ \omega = &\quad \text{the driving frequency} \end{aligned}$$

Thus we have that

$$Dx = f_0 \cos(\omega t)$$

Note that simply by shifting our time axis, we could also write a different solution of the form

$$Dy = f_0 \sin(\omega t)$$

If we combine these into a single complex number:

$$z = x(t) + iy(t)$$

Then

$$\begin{aligned} Dz &= D(x + iy) \\ &= Dx + iDy \\ &= f_0 \left(\cos(\omega t) + i\sin(\omega t)\right) \\ &= f_0 e^{i\omega t} \end{aligned}$$

While the complex values might alarm you, this is actually an improvement! Exponentials are generally much easier to work with (and have far fewer identities to remember!), and we just need to remember to only take the real part of the solution afterwards.

So now if we try a solution of the form

$$z(t) = Ce^{i\omega t}$$

then

$$Dz = C\left(-\omega^2 e^{i\omega t} + i2\beta\omega e^{i\omega t} + \omega_0^2 e^{i\omega t}\right) = f_0 e^{i\omega t}$$

$$\Rightarrow C = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega} \quad \text{works!!}$$

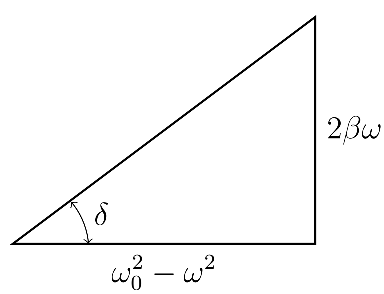
C here is clearly complex. To make our life a bit nicer, we'll write C in the form $Ae^{-i\delta}$:

$$\begin{aligned}
C &= Ae^{-i\delta} \\
\Rightarrow CC^* &= A^2 \\
\Rightarrow A^2 &= \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega} \cdot \frac{f_0}{\omega_0^2 - \omega^2 - 2i\beta\omega} \\
A^2 &= \frac{f_0^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2} \\
A &= \sqrt{\frac{f_0^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}}
\end{aligned}$$

So that gets us A. It might look a bit ugly but it is really just a matter of plugging in all the known constants and crunching the calculation. To get δ:

$$\begin{aligned}
Ae^{-i\delta} &= C \\
&= \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega} \\
A\left(\omega_0^2 - \omega^2 + 2i\beta\omega\right) &= f_0 e^{i\delta}
\end{aligned}$$

We can then make a triangle out of the real and imaginary parts:



And thus

$$\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

With a method of calculating those constants, we can write out:

$$z(t) = Ae^{i(\omega t - \delta)}$$

and then, recalling that we only want the real part:

$$x(t) = A\cos(\omega t - \delta)$$

So our full solution, with both particular and becomes:

$$x(t) = A\cos(\omega t - \delta) + \underbrace{\boxed{C_1 e^{r_1 t} + C_2 e^{r_2 t}}}_{\substack{\text{vanish in time,} \\ \text{called transients}}}$$

Thus if things are weakly damped:

$$x(t) = A\cos(\omega t - \delta) + A_{tr}e^{-\beta t}\cos(\omega_1 t - \delta_{tr})$$

Example 5.3

Suppose we are driving an oscillator at an angular frequency $\omega = 3\pi$, whose natural frequency is $\omega_0 = 6\omega$. The damping coefficient is $\beta = \omega_0/60$ and the amplitude of the driving oscillation is $f_0 = 1000$.
$\beta^2 < \omega_0^2$ so we are in the weakly damping case. So this is largely a question of crunching the constants that we need:

$$A = \sqrt{\frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta\omega^2}} = \sqrt{\frac{1000^2}{((18\pi)^2 - (3\pi)^2)^2 + 4\left(\frac{18\pi}{60}\right)^2(3\pi)^2}} = 0.321649$$

$$\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) = \tan^{-1}\left(\frac{2\left(\frac{18\pi}{60}\right)(3\pi)}{(18\pi)^2 - (3\pi)^2}\right) = 0.005714$$

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2} = \sqrt{(18\pi)^2 - \left(\frac{18\pi}{60}\right)^2} = 56.5408$$

So thus far we have that:

$$0.32\cos(3\pi t - 0.005714) + e^{-\beta t}\left(B_1\cos(\omega_1 t) + B_2\sin(\omega_1 t)\right)$$

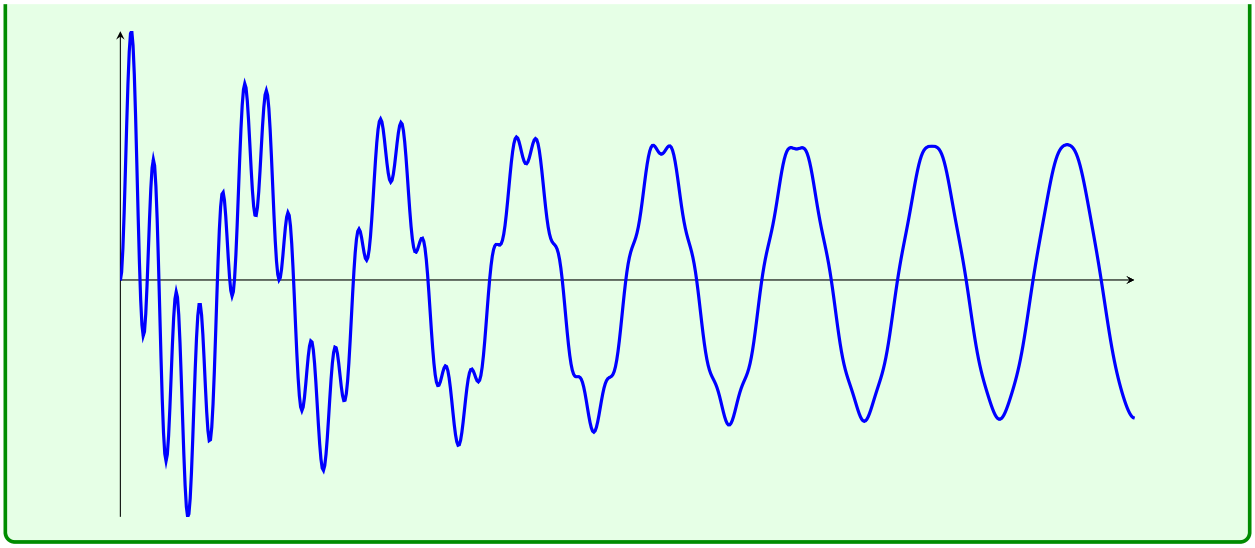
So if we start from rest at the origin:

$$\begin{aligned} x(0) &= A\cos(-\delta) + B_1 \\ &= 0 \\ \Rightarrow B_1 &= -0.321 \end{aligned}$$

And for B_2:

$$\begin{aligned} \dot{x}(0) &= -\omega A\sin(-\delta) - \beta B_1 + \omega_1 B_2 \\ &= 0 \\ \Rightarrow B_2 &= \frac{\beta B_1 + \omega A\sin(-\delta)}{\omega_1} \\ &= -0.00504 \end{aligned}$$

Plotting this madness gives us:

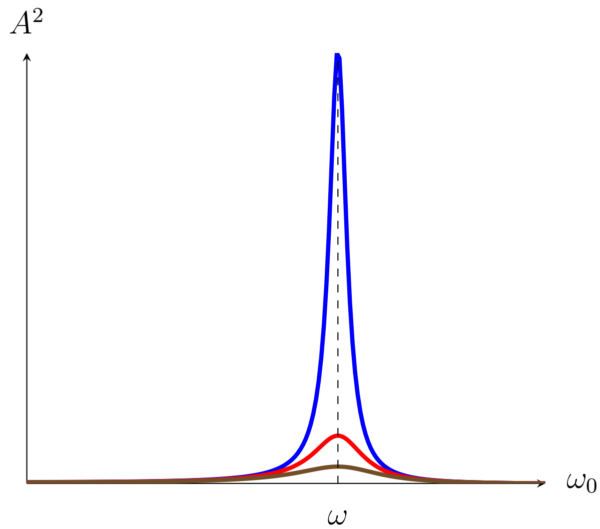


5.4.3 Resonance

Recall that we found that

$$A^2 = \frac{f_0^2}{\left(\omega_0^2 - \omega^2\right)^2 - 4\beta^2\omega^2}$$

So when β is small, then as $\omega_0 \to \omega$, the bottom goes tiny and amplitude explodes.



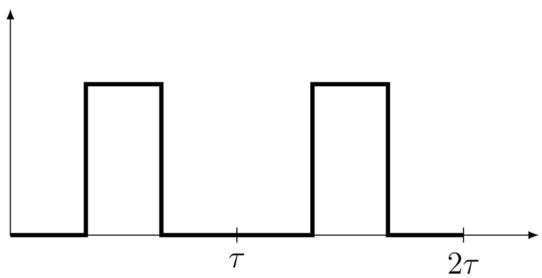
This large response when the system is driven at near the rest frequency is called resonance. This is what causes your radio to work as your vary the natural frequency of your oscillator until it syncs with the radio station frequency, amplifying the signal and making your station audible.

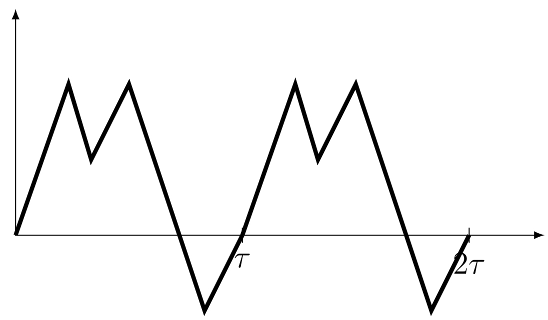
Remark 5.10

Note that the sharpness of the peak can be measured in terms of the full-width, half max value. This turns out to be equal to 2β. Thus very slight damping results in a very narrow range where resonance happens, but a very strong response!

5.5 Fourier Series

Consider any periodic function with a period of τ ($f(t+\tau) = f(t)$)





A whole bunch of cosines and sines also meet this criteria:

$$\cos\left(\frac{2\pi}{\tau}t\right), \quad \cos\left(\frac{4\pi}{\tau}t\right), \quad \cos\left(\frac{6\pi}{\tau}t\right), \ldots$$

Accounting for the fact that $\frac{2\pi}{\tau} = \omega$, these can be written as

$$\cos(n\omega t) \quad \text{or} \quad \sin(n\omega t) \quad \text{for} \quad n = 0, 1, 2, 3, \ldots$$

Definition 5.3: Fourier's Theorem

Fourier's Theorem states that we can replicate *any* periodic function with a combination of cosine and sine terms via:

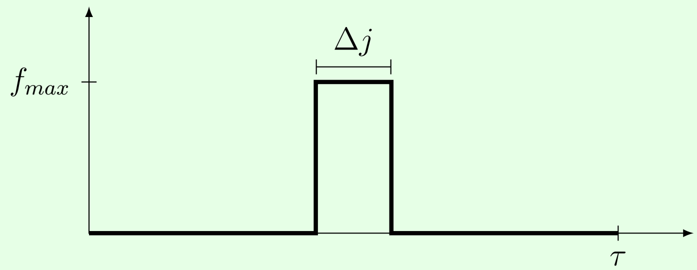
$$f(t) = \sum_{n=0}^{\infty} \left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

The coefficients are given by:

$$\begin{aligned}
a_0 &= \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\, \mathrm{d}t \\
b_0 &= 0 \\
a_n &= \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\cos(n\omega t)\, \mathrm{d}t \quad \text{for} \quad n \geq 1 \\
b_n &= \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\sin(n\omega t)\, \mathrm{d}t \quad \text{for} \quad n \geq 1
\end{aligned}$$

Example 5.4

Take, for instance, a square wave:



So, calculating a_0:

$$a_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\,\mathrm{d}t = \frac{1}{\tau}\int_{-\Delta j/2}^{\Delta j/2} f_{max}\,\mathrm{d}t = \frac{f_{max}\Delta j}{\tau}$$

b_0 we get for free, and then

$$\begin{aligned} a_n &= \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)cos(n\omega t)\,\mathrm{d}t \\ &= \frac{2f_{max}}{\tau}\int_{-\Delta j/2}^{\Delta j/2} cos(n\omega t)\,\mathrm{d}t \\ &= \frac{4f_{max}}{\tau}\int_{0}^{\Delta j/2} cos(n\omega t)\,\mathrm{d}t \\ &= \frac{2f_{max}}{\pi n}\sin\left(\frac{\pi n\Delta j}{\tau}\right) \end{aligned}$$

Note that all the b_n terms will vanish because sin is an odd function being integrated over an even interval.

The amazing part of this all is that we already know that the sums of solutions are solutions in and of themselves. Eg if

$$Dx_1 = f_1 \quad \text{and} \quad Dx_2 = f_2$$

then

$$D(x) = D(x_1 + x_2) = Dx_1 + Dx_2 = f_1 + f_2 = f$$

So if we can write our driving force as a sum of functions:

$$f(t) = \sum_n f_n(t)$$

then we get that our solution is a sum of the solutions to those individual driving froces:

$$x(t) = \sum_n x_n(t)$$

where

$$Dx_n = f_n$$

Remark 5.11

This is why us solving the driven oscillator for cosine functions is so nice! In reality it sets us up to be able to easily solve the system for ANY periodic function!