Chapter 7

Lagrange's Equations

7.1 Motivations

We will start with unconstrained motion. Say we have a single particle moving in three dimensions, subject to a conservative force. Then

$$\begin{aligned}
T &= \frac{1}{2}m\dot{\vec{\mathbf{r}}}^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \\
U &= U(\vec{\mathbf{r}}) = U(x, y, z)
\end{aligned}$$

Definition 7.1: The Lagrangian

We define the **Lagrangian** as

$$\mathcal{L} = T - U$$

Remark 7.1

Yes, the Lagrangian is defined with a **minus**. Fair enough that this seems backward, but as we'll see it makes sense with the math!

Now let's look at two derivatives:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial}{\partial x}(T - U) = -\frac{\partial U}{\partial x} = F_x \\
\frac{\partial \mathcal{L}}{\partial \dot{x}} &= \frac{\partial}{\partial \dot{x}}(T - U) \\
&= \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)\right) \\
&= m\dot{x} = p_x
\end{aligned}$$

But recall by Newton's 2nd law that

$$\dot{p} = \frac{\mathrm{d}}{\mathrm{d}t}p = F$$

And thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{\partial\mathcal{L}}{\partial x}$$

So from Newton's Second law then we get three equations:

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}}, \quad \frac{\partial\mathcal{L}}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{y}}, \quad \frac{\partial\mathcal{L}}{\partial z} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{z}}$$

and the methods are equivalent.

Notice that we can rearrange things to get back our Euler-Lagrange Equations from Chapter 6:

$$\frac{\partial\mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}} = 0$$

Thus, the path that the particle travels is such that:

$$S = \int_{t_1}^{t_2} \mathcal{L}\,\mathrm{d}t$$

is stationary along the path. This is called Hamilton's Principle. The integral itself is called the "action integral". The the path any object follows is the path that minimizes the action!

Remark 7.2

Hamilton's Principle here is completely unrelated to Hamilton's equations and his formulation of Classical Mechanics. Same guy though!

And so we find ourselves with 3 equivalencies:

Path as given by Newton's 2nd law $\longleftrightarrow$ Path as given by the three Lagrange Equations $\longleftrightarrow$ Path as give by Hamilton's Principle

7.2 Independent Variables

Recall that our calculus of variations was coordinate system independent. It didn't matter what the variables were, the calculus still held the same.

Definition 7.2: Generalized Coordinates

Let us define **Generalized Coordinates** q_1, q_2, q_3 where at every point $\vec{\mathbf{r}}$ can be uniquely represented by

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}(q_1, q_2, q_3)$$

Remark 7.3

This means we could have

$$(q_1, q_2, q_3) = \begin{cases} (r, \theta, \phi) & \text{for spherical} \\ (\rho, \phi, z) & \text{for cylindrical} \\ (x, y, z) & \text{for cartesian} \end{cases}$$

So the Lagrangian would thus be

$$\mathcal{L} = \mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)$$

Implying that

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_1}, \quad \frac{\partial \mathcal{L}}{\partial q_2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_2}, \quad \frac{\partial \mathcal{L}}{\partial q_3} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_3}$$

since our action integral is unaffected by the change in variables. This means we can use *any* coordinate system and Lagranges Equations will have the same form. **This is awesome!**. (Compare to our awful formulations of Newton's 2nd law in other forms...)

Remark 7.4

Warning: We do have to be careful of one thing. We used the fact that $F_x = \dot{p}_x$, which is only true in inertial reference frames. So when we first write down $\mathcal{L}$, we must make sure we do so in an inertial frame.

Since

$$\frac{\partial \mathcal{L}}{\partial x} = F_x \quad \text{then} \quad \frac{\partial \mathcal{L}}{\partial q_i} \quad \text{is the } i^{th} \text{ component of the generalize force}$$
$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = p_x \quad \text{then} \quad \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad \text{is the } i^{th} \text{ component of the generalized momentum}$$

Example 7.1

Consider one particle in two-dimensional polar coordinates. I can write the velocity in polar coordinates as:

$$v = (\dot{r}, r\dot{\phi})$$

Therefore I could express the Lagrangian in any potential as

$$\mathcal{L} = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r, \phi)$$

Looking at the Euler-Lagrange equations for the r coordinate gives me:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial r} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{r}} \\
mr\dot{\phi}^2 - \frac{\partial U}{\partial r} &= \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) \\
mr\dot{\phi}^2 - \frac{\partial U}{\partial r} &= m\ddot{r} \\
mr\dot{\phi}^2 + F_r &= m\ddot{r} \\
\Rightarrow F_r &= m\ddot{r} - mr\dot{\phi}^2 \\
&= m\left(\ddot{r} - r\dot{\phi}^2\right)
\end{aligned}$$

which is the same thing we got back in chapter 1! But much easier this time!

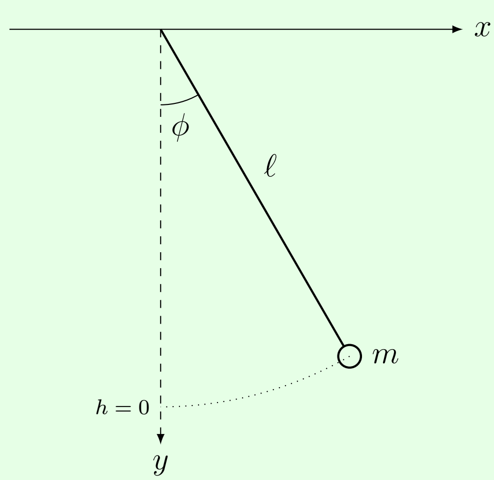
7.3 Constraints

Similar to our discussion of constraints from before, we are going to focus on situations when the motions is limited or constrained to move in a particular way. Example include:

- Beads on a wire
- Rigid bodies
- Things on the ground
- etc

Example 7.2

Take for instance a pendulum:



Clearly the position of the pendulum varies in both the x and the y directions. But because

$$\ell^2 = x^2 + y^2$$

the coordinates are not independent.
We could account for this either by writing

$$y = \sqrt{\ell^2 - x^2}$$

OR

$$x = \ell \sin \phi$$
$$y = \ell \cos \phi$$

where ϕ is our one independent variable now. Therefore

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(\ell^2\cos^2(\phi)\dot{\phi}^2 + \ell^2\left(-\sin^2(\phi)\right)\dot{\phi}^2\right) = \frac{1}{2}m\ell^2\dot{\phi}^2$$

And

$$U = mgh = mg(\ell - \ell\cos(\phi)) = mg\ell(1 - \cos\phi)$$

Thus the Lagrangian looks like

$$\mathcal{L} = T - U = \frac{1}{2}m\ell^2\dot{\phi}^2 - mg\ell(1 - \cos\phi)$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} = mgl(-\sin\phi)$$
$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m\ell^2\dot{\phi}$$

So

$$\underbrace{-mg\ell\sin\phi}_{\Gamma} = \frac{\mathrm{d}}{\mathrm{d}t}\left[m\ell^2\dot{\phi}\right] = \underbrace{m\ell^2\ddot{\phi}}_{\dot{L}\ \text{ or }\ I\alpha}$$

Thus we get back the expected Newton's 2nd law for rotations!

7.3.1 Back to Generalized Coordinates

Going back to our generalized coordinates, we can express the state of the system uniquely in terms of $q_1, \ldots, q_n$ and potentially t:

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}(q_1, \ldots, q_n, t)$$

We do require though that n should be the smallest number needed to parameterize the system (eg: no redundancies).

Therefore, a system in 3-dimensions with N particles:

$$n_{max} = 3N$$

and likely much less with constraints.

Example 7.3

For a solid body, say you have a mole of particles, so

$$N \approx 10^{23}$$

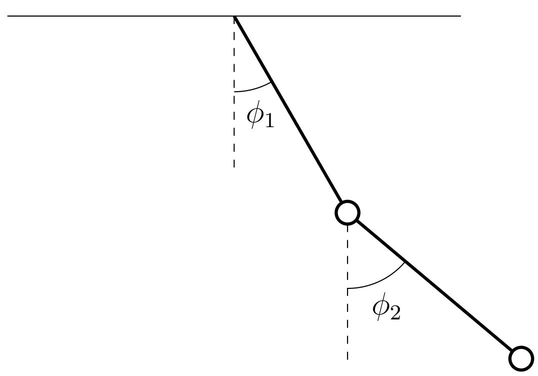
But if the body is rigid, you only ever need to worry about it's position and rotation, thus

$$n \approx 6$$

That is some lovely and *sizeable* simplification!!

Therefore:

- For our pendulum:
  - 1 object in 2-dimensions $\Rightarrow n_{max} = 2$
  - But $\vec{\mathbf{r}} = (x, y) = (\ell \sin \phi, \ell \cos \phi)$, which only depends on ϕ and thus $n = 1$
- For a double pendulum:

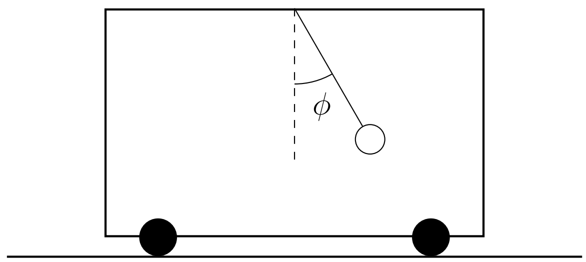


  - 2 objects in two dimensions $\Rightarrow n_{max} = 4$
  - But

$$r_1 = (x_1, y_1) = (\ell_1 \sin \phi_1, \ell_1 \cos \phi_1)$$
$$r_2 = (x_1 + x_2, y_1 + y_2) = (\ell_1 \sin \phi_1 + \ell_2 \sin \phi_2, \ell_1 \cos \phi_1 + \ell_2 \cos \phi_2)$$

  only depends on ϕ_1, ϕ_2 and thus $n = 2$

- For a moving, accelerating pendulum:



  - 1 object in 2 dimensions $\Rightarrow n_{max} = 2$
  - We need the postition relative to the non-accelerating ground:

$$\begin{aligned} r &= (x, y) \\ &= \left(x + \frac{1}{2}at^2, y\right) \\ &= \left(\ell \sin\phi + \frac{1}{2}at^2, \ell\cos\phi\right) \end{aligned}$$

  - Depends on both ϕ and t

Remark 7.5

We'll sometimes call coordinates "natural" if they do *not* depend on time.

Definition 7.3: Degrees of Freedom

We define the number of degrees of freedom (DoF) of a system to be the number of independent "directions" in which the system can move. Eg:

- Pendulum = 1
- Double Pendulum = 2
- Free particle = 3
- etc

Definition 7.4: Constrained

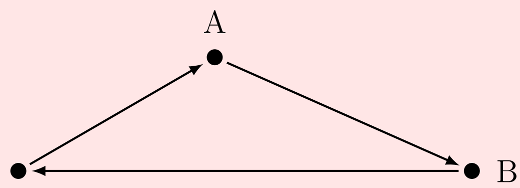
If the number of degrees of freedom is less than the n_{max}, then we say the system is constrained.

Definition 7.5: Holonomic

When the number of degrees of freedom equals the number of generalized coordinates, then the system is said to be **holonomic**.

Remark 7.6

Most normal systems we'll look at will indeed be holonomic, but there are some odd exceptions. For example, consider a ball rolling on a table without slipping to a point A, then to a point B, then back to where it started.



The ball can only move in two directions, so it has two degrees of freedom. But if we put a dot on top of the ball and follow it around the path, it most likely will not end up with the dot on top again at the end. Therefore, we'd need more coordinates to also describe the ball's orientation!

7.3.2 Proof the Lagrange's Equations work with Constraints

See the book!

The basic requirements though are:

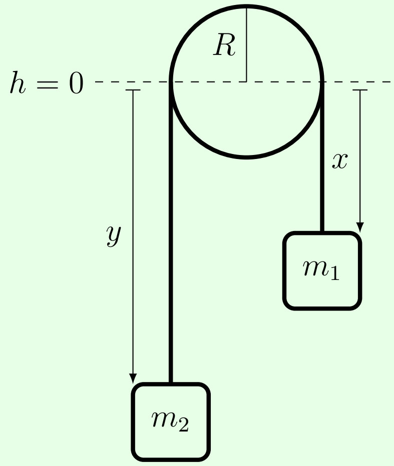
1. Non-constraing forces must be conservative and able to be written in terms of potential energy

2. Constraining forces must do no work

7.4 Examples!

Example 7.4

Consider an Atwood machine with a pulley of some radius:



Where the string has a total length ℓ. Therefore we have 2 objects in a 1-dimensional problem, so

$$n_{max} = 2$$

The string gives us one constraint since

$$\begin{aligned} x + y + \pi R &= \ell \\ \Rightarrow y &= -x + \ell - \pi R \\ \Rightarrow \dot{y} &= -\dot{x} \end{aligned}$$

Let's let x be our generalized coordinate:

$$\begin{aligned} T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 &= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(-\dot{x}\right)^2 \\ &= \frac{1}{2}\left(m_1 + m_2\right)\dot{x}^2 \\ U = -m_1 g x - m_2 g y &= -m_1 g x - m_2 g\left(-x + \ell - \pi R\right) \\ &= -xg(m_1 - m_2) \underbrace{\boxed{-m_2 g\left(\ell - \pi R\right)}}_{\text{constant so shift U zero point so this vanishes}} \end{aligned}$$

Thus

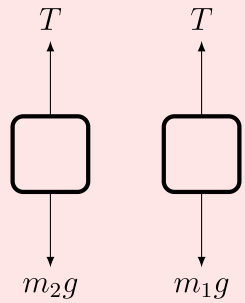
$$\mathcal{L} = T - U = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx$$

$$\frac{\partial \mathcal{L}}{\partial x} = (m_1 - m_2)g, \quad \frac{\partial \mathcal{L}}{\partial \dot{x}} = (m_1 + m_2)\dot{x}$$

$$\begin{aligned} \Rightarrow (m_1 - m_2)g &= \frac{\mathrm{d}}{\mathrm{d}t}\left[(m_1 + m_2)\dot{x}\right] = (m_1 + m_2)\ddot{x} \\ \Rightarrow \ddot{x} &= \frac{m_1 - m_2}{m_1 + m_2}g \end{aligned}$$

Remark 7.7

Had we done things by our class Intro Mechanics methods:



and

$$\sum F_y = T - m_2 g = m_2 a_y$$
$$\sum F_x = T - m_1 g = m_1 a_x$$

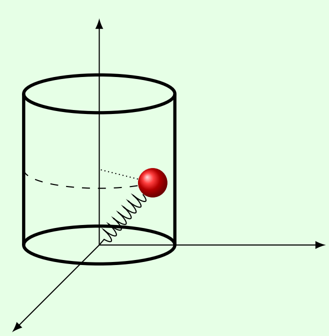
and then $a_y = -a_x$, so

$$\begin{aligned} \sum F_y = T - m_2 g &= -m_2 a_x \\ \Rightarrow -m_1 a_x + m_1 g &= m_2 g + m_2 a_x \\ (m_1 - m_2) g &= (m_1 + m_2) a_x \\ \Rightarrow a_x = \ddot{x} &= \frac{m_1 - m_2}{m_1 + m_2} g \end{aligned}$$

So we get the same thing! Yay!

Example 7.5

Consider a particle constrained to move on the surface of a cylinder attached to the origin via a spring:



It probably makes the most sense to work in cylindrical coordinates here

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}(\rho, \phi, z)$$

But ρ will be fixed, so only ϕ and z will vary.

$$\Rightarrow \quad \text{two degrees of freedom}$$

Our particles velocity will then be given by:

$$\vec{\mathbf{v}} = (0, R\dot{\phi}, \dot{z})$$

$$\Rightarrow T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(R^2\dot{\phi}^2 + \dot{z}^2\right)$$

For the potential, we are assuming the springs equilibrium point is at the origin itself. Therefore:

$$\begin{aligned} \vec{\mathbf{F}} &= -k\vec{\mathbf{r}} \\ &= -k\sqrt{R^2 + z^2} \end{aligned}$$

and thus

$$U = \frac{1}{2}kr^2 = \frac{1}{2}k\left(R^2 + z^2\right)$$

The Lagrangian is thus

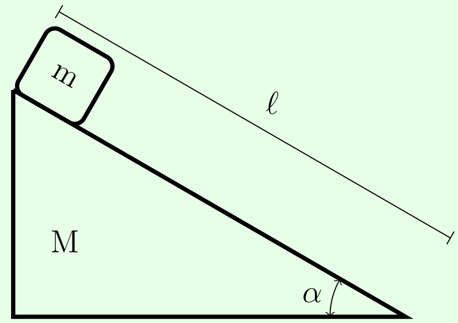
$$\mathcal{L} = \frac{1}{2}m\left(R^2\dot{\phi}^2 + \dot{z}^2\right) - \frac{1}{2}k\left(R^2 + z^2\right)$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \phi} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}, & \frac{\partial \mathcal{L}}{\partial z} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{z}} \\ 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left[mR^2\dot{\phi}\right], & -kz &= \frac{\mathrm{d}}{\mathrm{d}t}\left[m\dot{z}\right] = m\ddot{z} \\ 0 &= \frac{\mathrm{d}}{\mathrm{d}t}[L], & \ddot{z} &= -\frac{k}{m}z \end{aligned}$$

And so we see that our angular momentum is conserved and that we get simple harmonic motion up and down in the z-direction!

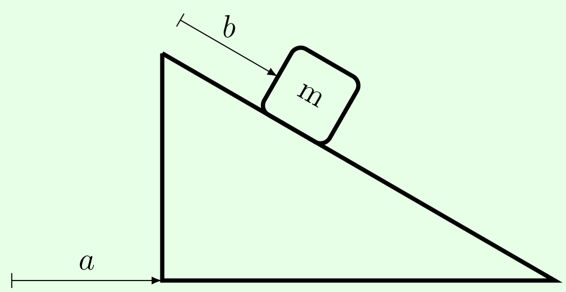
Example 7.6

Suppose I have a block atop a wedge. The block is free to slide down the wedge, and the wedge is free to move left or right on the table surface. How long will it take the block to slide from the top to the bottom of the wedge?



How many degrees of freedom do we have?

- 2 blocks in 2 space $\Rightarrow n_{max} = 4$
- 2 constraining forces: N_{wedge}, N_{block}
- Therefore 2 degrees of freedom



Remark 7.8

It is fine for us to determine our coordinates in this way, but we need to realize that we need the blocks velocity *relative to the table* not to the wedge!

Writing out the position of the block relative to the stationary ground:

$$\vec{\mathbf{r}}_b = (a + b\cos\alpha, b\sin\alpha)$$
$$\dot{\vec{\mathbf{r}}}_b = (\dot{a} + \dot{b}\cos\alpha, \dot{b}\sin\alpha)$$

Therefore:

$$\begin{aligned}
T &= \frac{1}{2}M\dot{a}^2 + \frac{1}{2}m\dot{\vec{\mathbf{r}}}_b^2 \\
&= \frac{1}{2}M\dot{a}^2 + \frac{1}{2}m\left(v_x^2 + v_y^2\right) \\
&= \frac{1}{2}M\dot{a}^2 + \frac{1}{2}m\left(\dot{a}^2 + \dot{b}^2\cos^2(\alpha) + 2\dot{a}\dot{b}\cos(\alpha) + \dot{b}^2\sin^2(\alpha)\right) \\
&= \frac{1}{2}M\dot{a}^2 + \frac{1}{2}m\left(\dot{a}^2 + \dot{b}^2 + 2\dot{a}\dot{b}\cos(\alpha)\right)
\end{aligned}$$

For our potential energy, we'll initially call the top of the wedge the zero point. Thus:

$$\begin{aligned}
U &= U_W + U_b \\
&= \text{ constant } - mgb\sin\alpha
\end{aligned}$$

Adjusting the zero point to let our constant be zero:

$$= -mgb\sin\alpha$$

And thus our Lagrangian is

$$\mathcal{L} = \frac{1}{2}M\dot{a}^2 + \frac{1}{2}m\left(\dot{a}^2 + \dot{b}^2 + 2\dot{a}\dot{b}\cos\alpha\right) + mgb\sin\alpha$$

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial a} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{a}}, & \frac{\partial\mathcal{L}}{\partial b} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{b}} \\
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left[M\dot{a} + m\dot{a} + m\dot{b}\cos\alpha\right], & mg\sin\alpha &= \frac{\mathrm{d}}{\mathrm{d}t}\left[m\dot{b} + m\dot{a}\cos\alpha\right] \\
k &= M\dot{a} + m\underbrace{\left(\dot{a} + \dot{b}\cos\alpha\right)}_{\dot{r}_{bx}}, & g\sin\alpha &= \ddot{b} + \ddot{a}\cos\alpha
\end{aligned}$$

The first is just telling us that momentum is conserved in the x-direction! To advance further we need to solve for one of the accelerations and plug it into the other.

$$\begin{aligned}
0 &= M\ddot{a} + m\left(\ddot{a} + \ddot{b}\cos\alpha\right) \\
-m\ddot{b}\cos\alpha &= (M+m)\ddot{a} \\
\Rightarrow \ddot{a} &= \frac{-m\cos\alpha}{M+m}\ddot{b}
\end{aligned}$$

Then plugging that back into the second equation:

$$\begin{aligned}
g\sin\alpha &= \ddot{b} + \left(\frac{-m\cos\alpha}{M+m}\right)\cos\alpha\ddot{b} \\
\Rightarrow \ddot{b} &= \frac{g\sin\alpha}{1 - \frac{m\cos^2(\alpha)}{M+m}}
\end{aligned}$$

Which is just a constant value! We therefore have that:

$$\ell = \frac{1}{2}\ddot{b}t^2$$
$$\Rightarrow t = \sqrt{\frac{2\ell}{\ddot{b}}}$$

Remark 7.9

If $\alpha = 90°$, then we see that

$$\ddot{b} = g$$

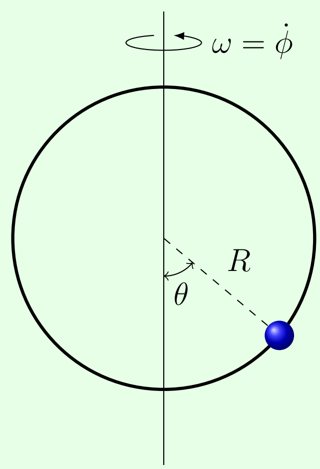
as we'd expect. Similarly, if $M \to \infty$, then the large wedge will no longer move and we see that

$$\ddot{b} = \frac{g \sin\alpha}{1 - 0}$$

also as we'd expect!

Example 7.7

Suppose we have a bead on a circular loop and is free to move about the loop. The loop is mounted vertically and spinning with angular velocity ω.



Degrees of Freedom?

- 1 obj, 3 space $\Rightarrow n_{max} = 3$
- Wire constraint
- Angular constraint
- $\Rightarrow$ 1 degree of freedom

– Use θ as coordinate!

We will have two components of velocity:

$$v_\theta = R\dot{\theta}, \quad v_\phi = R\sin\theta\dot{\phi} = R\sin\theta\omega$$

Thus our kinetic energy looks like:

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(v_\theta^2 + v_\phi^2\right) \\ &= \frac{1}{2}m\left(R^2\dot{\theta}^2 + R^2\sin^2(\theta)\omega^2\right) \\ &= \frac{1}{2}mR^2\left(\dot{\theta}^2 + \sin^2(\theta)\omega^2\right) \end{aligned}$$

We'll take the zero point of the potential energy to be at the bottom of the loop, so

$$U = mgh = mg\left(R - R\cos\theta\right) = mgR(1-\cos\theta)$$

Our Lagrangian is thus:

$$\mathcal{L} = T - U = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \omega^2\sin^2(\theta)\right) - mgR(1-\cos\theta)$$

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\theta} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} \\ m\omega^2R^2\sin\theta\cos\theta - mgR\sin\theta &= \frac{\mathrm{d}}{\mathrm{d}t}\left[mR^2\dot{\theta}\right] \\ mR^2\left(\omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta\right) &= mR^2\ddot{\theta} \\ \left(\omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta\right) &= \ddot{\theta} \\ \Rightarrow \ddot{\theta} &= \left(\omega^2\cos\theta - \frac{g}{R}\right)\sin\theta \end{aligned}$$

This equation of motion is distinctly non-linear, so it solve it exactly would require numerical solutions. But we can get a feel for its properties nonetheless! Consider that at points where the beads acceleration and velocity are both zero it will not move. Or rather, it will be in equilibrium around these points. We can determine where these points lie by breaking it into first-order equations:

$$\begin{aligned} \dot{\theta} &= \Omega \\ \dot{\Omega} &= \left(\omega^2\cos\theta - \frac{g}{R}\right)\sin\theta \end{aligned}$$

For the particle to be stable we require that both the above equations equal zero! So

$$\Omega = 0$$
$$\left(\omega^2 \cos\theta - \frac{g}{R}\right)\sin\theta = 0$$

So either $\sin\theta = 0$, in which case

$$\theta = 0^\circ \quad \text{or} \quad 180^\circ$$

or $\omega^2 \cos\theta = g/R$ and thus

$$\theta = \arccos\left(\frac{g}{R\omega^2}\right)$$

Note that the second arccos term does not exist when $\omega^2 < \frac{g}{R}$. So when the loop is spinning slowly there are only equilibrium points at the top or the bottom. But when the loop spins past a certain speed, extra equilibrium points form on either side of the bottom.
So those are our equilibrium points, but what does their stability look like?

- Looking at the $\theta = 0^\circ$ point first, realize that small perturbations about this point have $\cos\theta \approx 1$ and $\sin\theta \approx \theta$:

  $$\Rightarrow \ddot{\theta} = \left(\omega^2 - \frac{g}{R}\right)\theta$$

  So when $\omega^2 < \frac{g}{R}$:
  $$\Rightarrow \ddot{\theta} = -(\text{number})\theta$$
  which is the equation for a restoring force (think spring)! So 0 is stable. But when $\omega^2 > \frac{g}{R}$:
  $$\ddot{\theta} = +(\text{number})\theta$$
  and so the bead accelerates away and the 0° point is unstable.

- Looking at when $\theta = 180^\circ$, if we take the Taylor series about the point π we get

  $$\ddot{\theta} = \left(\omega^2 + \frac{g}{R}\right)(t - \pi)$$

  Note in this case the $\omega^2 + \frac{g}{R}$ term will always be positive, so it will be unstable at all times.

- We already know $\theta = \cos^{-1}\left(\frac{g}{R\omega^2}\right)$ is the equilibrium point, so imagine we poked it and make θ slightly larger.

$$\ddot{\theta} = \left(\omega^2 \underbrace{\boxed{\cos\theta}}_{\substack{\text{goes down}\\ \text{a bit}}} - \frac{g}{R}\right)\underbrace{\boxed{\sin\theta}}_{\substack{\text{goes up}\\ \text{a bit}}}$$

$$\Rightarrow \ddot{\theta} = -(\text{number})\sin\theta$$

which is a restoring force so it is stable!

7.5 Lagrangian Multipliers

One of the strengths of Lagrangian mechanics is that it can allow us to bypass constraint forces. Usually this is useful, since we are more concerned about the motion than the constraints, but sometimes we actually want to know what these constraints are:

- Knowing the tension to ensure that your cord doesn't break
- Knowing the normal force to ensure that your object doesn't crumple
- etc

To this end, we can use the idea of Lagrangian Multipliers.

Key Idea: With the normal Lagrangian, we want the fewest number of independent coordinates. With Lagrangian multipliers, we will want the full number of independent coordinates and then we'll use constraint equations to limit things.

Our constraint equations will take the form of:

$$f(x, y) = \text{ constant}$$

$$\text{Pendulum:} \quad \sqrt{x^2 + y^2} = \ell, \quad \text{Attwood:} \quad x + y = \ell$$

Definition 7.6: Lagrangian Multiplier Equations

In two dimensions, the Lagrangian Multiplier equations will take on the form

$$\frac{\partial \mathcal{L}}{\partial x} + \lambda \frac{\partial f}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}}, \quad \frac{\partial \mathcal{L}}{\partial y} + \lambda \frac{\partial f}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{y}}$$

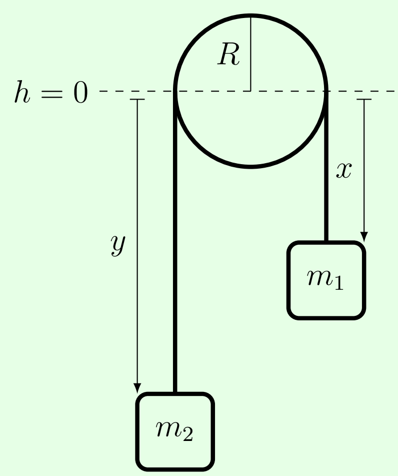
plus the constaint equation:

$$f(x, y) = \quad \text{constant}$$

All told, that is three equations for three unknowns (x, y, λ).

Example 7.8

Let us return to the Atwood machine, with the same configuration as before:



Our constraint equation is

$$x + y = \ell \Rightarrow x + y - \ell = 0 \tag{†}$$

And our unconstrained Lagrangian is:

$$\mathcal{L} = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + m_1 g x + m_2 g y$$

Applying our first LME:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} + \lambda \frac{\partial f}{\partial x} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} \\ m_1 g + \lambda(1) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[m_1\dot{x}\right] \\ m_1 g + \lambda &= m_1 \ddot{x} \end{aligned} \tag{‡}$$

From the second LME:

$$\frac{\partial \mathcal{L}}{\partial y} + \lambda \frac{\partial f}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{y}}$$
$$m_2 g + \lambda(1) = \frac{\mathrm{d}}{\mathrm{d}t}\left[m_2 \dot{y}\right]$$
$$m_2 g + \lambda = m_2 \ddot{y} \tag{$\star$}$$

If we realize from our constraint equation that $\ddot{y} = -\ddot{x}$, then subtracting (‡) and ($\star$) yields:

$$(\ddagger) - (\star) = g\left(m_1 - m_2\right) = \ddot{x}\left(m_1 + m_2\right)$$
$$\Rightarrow \ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g$$

which is the same result we got before. But further note that if we plug this back into (‡) we get:

$$m_1 g + \lambda = m_1 \left(\frac{(m_1 - m_2)}{(m_1 + m_2)} g\right)$$
$$(m_1 + m_2)(m_1 g + \lambda) = m_1(m_1 - m_2) g$$
$$\cancel{m_1^2} g + m_1 \lambda + m_2 m_1 g + m_2 \lambda = \cancel{m_1^2} g - m_1 m_2 g$$
$$\lambda(m_1 + m_2) = -2 m_1 m_2 g$$
$$\Rightarrow \lambda = \frac{-2 m_1 m_2}{(m_1 + m_2)} g$$

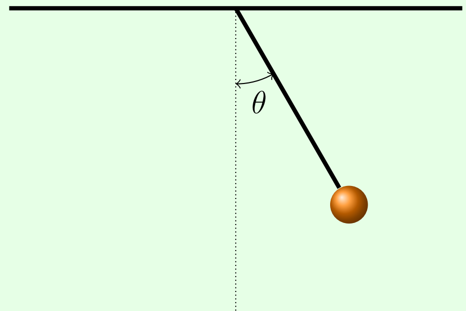
Which coincides with the T force we'd get if we'd done things the old way! So the λ tells us the magnitude of that constraining force!

Remark 7.10

The extra negative sign here is the result of us defining both x and y downwards, where as the tension forces are always pointing up.

Example 7.9

Suppose we take another simple example of a pendulum:



Our Lagrangian can be written as:

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + mgy$$

with our constraint:

$$x^2 + y^2 = \ell^2$$
$$\Rightarrow \sqrt{x^2 + y^2} = \ell$$

Therefore:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} + \lambda\frac{\partial f}{\partial x} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}}\\ 0 + \lambda\left(\frac{x}{\sqrt{x^2+y^2}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}[m\dot{x}]\\ \frac{\lambda x}{\sqrt{x^2+y^2}} &= m\ddot{x}\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial y} + \lambda\frac{\partial f}{\partial y} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{y}}\\ mg + \lambda\left(\frac{y}{\sqrt{x^2+y^2}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}[m\dot{y}]\\ mg + \frac{\lambda y}{\sqrt{x^2+y^2}} &= m\ddot{y}\end{aligned}$$

We could write these in terms of θ now to see that

$$\begin{aligned}m\ddot{x} &= \lambda\sin\theta\\ m\ddot{y} &= mg + \lambda\cos\theta\end{aligned}$$

Which should make it clear now that λ is indeed our tension force.