Chapter 3

Momentum and Angular Momentum

3.1 Momentum Conservation

Recall that for an n-dimensional system:

$$\vec{\boldsymbol{P}} = \vec{\boldsymbol{p}}_1 + \vec{\boldsymbol{p}}_2 + \cdots + \vec{\boldsymbol{p}}_N = \sum_{\alpha=1}^{N} \vec{\boldsymbol{p}}_\alpha$$

and

$$\dot{\vec{\boldsymbol{P}}} = \vec{\boldsymbol{F}}_{ext}$$

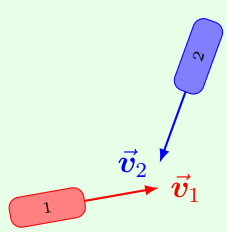
since all the internal forces cancel out by Newton's 3rd law. Thus, when

$$\vec{\boldsymbol{F}}_{ext} = 0 \quad \Rightarrow \quad \dot{\vec{\boldsymbol{P}}} = 0 \quad \Rightarrow \quad \vec{\boldsymbol{P}} = \text{ constant}$$

or, put differently, *mechanical* momentum (mv) is conserved. When N=1 this just gives us the first law.

Example 3.1

Two cars collide and stick together. What is their final velocity?



By conservation of momentum:

$$\vec{\boldsymbol{P}} = m_1\vec{\boldsymbol{v}}_1 + m_2\vec{\boldsymbol{v}}_2 = \text{ constant } = (m_1 + m_2)\vec{\boldsymbol{v}}_f$$

$$\Rightarrow \vec{\boldsymbol{v}}_f = \frac{m_1\vec{\boldsymbol{v}}_1 + m_2\vec{\boldsymbol{v}}_2}{m_1 + m_2}$$

If car 2 happens to be stationary at first:

$$\vec{\boldsymbol{v}}_f = \frac{m_1}{m_1 + m_2}\vec{\boldsymbol{v}}_1$$

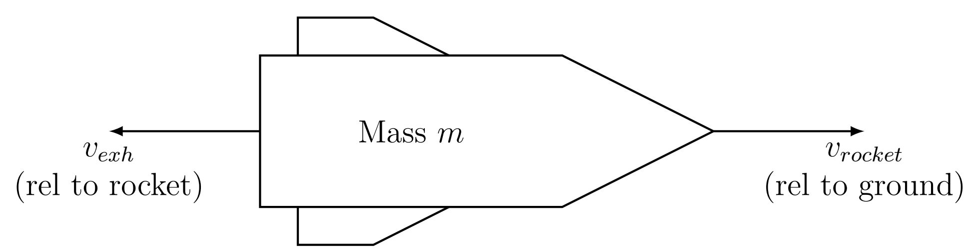
3.1.1 Rockets!

If we want to propel a rocket through the void of space, we have no external forces available. So how do we propel ourselves?

- Same as asking how you'd propel yourself if you were stranded on a frictionless lake
    - Throw your boots! ($\vec{\boldsymbol{P}}_{you} = -\vec{\boldsymbol{P}}_{boots}$)

We do the same thing with rockets, just with spent fuel instead of boots...

Imagine a rocket:



At time t, we have

$$\vec{\boldsymbol{p}} = m\vec{\boldsymbol{v}}_r$$

A short time later, at time $t + \Delta t$, we burned an amount of fuel Δm:

$$\vec{\boldsymbol{p}}_{rocket} = \left(m + \underbrace{\boxed{\Delta m}}_{\text{Negative}}\right)\left(v_r + \underbrace{\boxed{\Delta v}}_{\text{positive}}\right)$$

$$\vec{\boldsymbol{p}}_{fuel} = -\Delta m \underbrace{\boxed{(v_r - v_{exh})}}_{\text{To make rel to ground}}$$

The total momentum is thus:

$$\begin{aligned}
P &= p_r + p_{fuel} \\
&= (m + \Delta m)(v_r + \Delta v) - \Delta m (v_r - v_{exh}) \\
&= mv_r + m\Delta v + v_r \Delta m - \Delta m v_r + \Delta m v_{exh} + \underbrace{\boxed{\Delta m \Delta v}}_{\text{tiny}} \\
&= mv_r + m\Delta v + v_{exh}\Delta m
\end{aligned}$$

Now consider ΔP:

$$\Delta P = P(t + \Delta t) - P(t) = m\Delta v + v_{exh}\Delta m$$

So when we divide by Δt and take the limit as $t \to 0$, we get:

$$\dot{P} = m\dot{v} + v_{exh}\dot{m} = F_{exh} = 0$$

And thus we get that

$$m\dot{v} = -\dot{m}v_{exh}$$

Definition 3.1: Thrust

The rhs of the equation is indeed a force, and one which we will generally call the thrust!

$$\text{Thrust } = -\dot{m}v_{exh}$$

We want to get rid of the time dependence and just look at how the velocity of the rocket depends on the velocity of the exhaust. If we multiply both sides by dt:

$$\begin{aligned}
mdv &= -dmv_{exh} \\
dv &= -v_{exh}\frac{dm}{m} \\
\int_{v_0}^{v} dv &= -v_{exh}\int_{m_0}^{m} m^{-1}\, dm \\
v - v_0 &= -v_{exh} \ln(m)\big|_{m_0}^{m} \\
&= -v_{exh}(\ln(m) - \ln(m_0)) \\
&= v_{exh}(\ln(m_0) - \ln(m)) \\
&= v_{exh}\ln\left(\frac{m_0}{m}\right) \\
\Rightarrow v(m) &= v_{exh}\ln\left(\frac{m_0}{m}\right) + v_0
\end{aligned}$$

The entire mass of the rocket however is not fuel. Rather, we can say that the initial mass of the rocket is some combination:

$$m_0 = m_r + m_f$$

and the rocket will be at its maximum speed when it is out of fuel:

$$m = m_r$$

Thus the rocket is at maximum speed when:

$$\frac{m_r + m_f}{m_r} = 1 + \frac{m_f}{m_r}$$

Example 3.2

Imagine we have 900 kg of fuel and our rocket is 100 kg when empty.

$$\Rightarrow 1 + \frac{900}{100} = 10$$

$$\ln(10) \approx 2.3$$

And so our rocket would only gain 2.3x our exhaust speed even after burning all its fuel! The moral of the story:

- Make good nozzles to increase exhaust speed
- or jettison old mass as you go to keep things light

3.2 Center of Mass

Definition 3.2: Center of Mass

For N particles, their center of mass is given by:

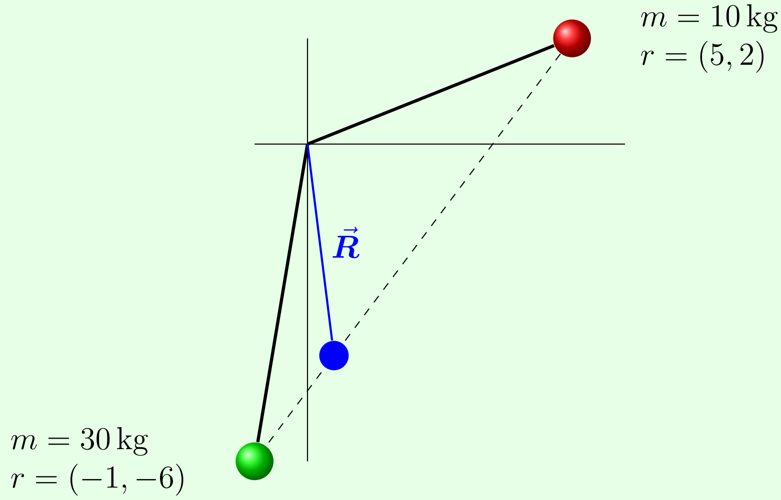
$$\vec{\boldsymbol{R}} = \frac{\sum_{\alpha=1}^{N} m_\alpha \vec{\boldsymbol{r}}_\alpha}{\sum_{\alpha=1}^{N} m_\alpha} = \frac{m_1\vec{\boldsymbol{r}}_1 + m_2\vec{\boldsymbol{r}}_2 + \cdots + m_N\vec{\boldsymbol{r}}_N}{M} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \vec{\boldsymbol{r}}_\alpha$$

Remark 3.1

Note that we are still sticking with the convention that capital letters denote totals over a larger distribution. $(\vec{\boldsymbol{P}}, \vec{\boldsymbol{R}}, M)$

Example 3.3

Consider two particles:



$$\vec{\boldsymbol{R}} = \frac{m_1\vec{\boldsymbol{r}}_1 + m_2\vec{\boldsymbol{r}}_2}{m_1 + m_2}$$
$$X = \frac{10 \cdot 5 + 30(-1)}{40} = 0.5$$
$$Y = \frac{10 \cdot 2 + 30(-6)}{40} = -4$$

Note that

$$\dot{\vec{\boldsymbol{R}}} = \frac{1}{M}\sum_\alpha m_\alpha \dot{\vec{\boldsymbol{r}}}_\alpha$$

Therefore, when we look at the total momentum:

$$\begin{aligned}\vec{\boldsymbol{P}} &= \sum_\alpha \vec{\boldsymbol{p}}_\alpha \\ &= \sum_\alpha m_\alpha \dot{\vec{\boldsymbol{r}}}_\alpha \\ &= M\dot{\vec{\boldsymbol{R}}}\end{aligned}$$

So the total momentum of the system is the same as that of an object located at the center of mass with the total mass of the system! This also means that

$$\dot{\vec{\boldsymbol{P}}} = M\ddot{\vec{\boldsymbol{R}}} = \vec{\boldsymbol{F}}_{ext}$$

And so the system moves as a single body with total mass M located at the center of mass.

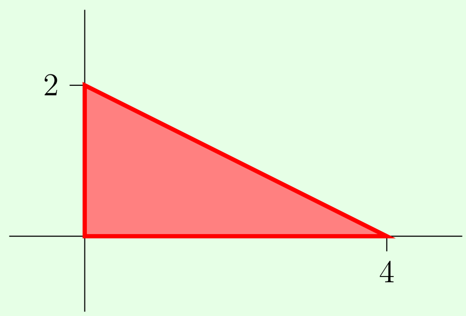
3.2.1 Continuous Distributions

Frequently, we'll need or want to look at more continuous distributions of mass to determine where the center of mass is. In these cases:

$$\vec{\boldsymbol{R}} = \frac{1}{M}\int \vec{\boldsymbol{r}}\,dm = \frac{1}{M}\int \vec{\boldsymbol{r}}\rho\,dV$$

Example 3.4

Suppose we wanted to find the center of mass of the uniformly distributed triangle below:



We thus have that:

$$\vec{\boldsymbol{R}} = \frac{1}{M}\int \vec{\boldsymbol{r}}\rho\,dV$$

$$\Rightarrow X = \frac{1}{M}\int x\rho\,dx\,dy \qquad Y = \frac{1}{M}\int y\,dx\,dy$$

$$\begin{aligned} X &= \frac{\rho}{M}\int_0^4\int_0^{2-x/2} x\,dydx \\ &= \frac{\rho}{M}\int_0^4 x\,y\big|_0^{2-x/2}\,dx \\ &= \frac{\rho}{M}\int_0^4 x\left(2-x/2\right)\,dx \\ &= \frac{\rho}{M}\int 2x - \frac{x^2}{2}\,dx \\ &= \frac{\rho}{M}\left[x^2 - \frac{x^3}{6}\right]_0^4 \\ &= \frac{16}{3}\frac{\rho}{M} \end{aligned}$$

$$\begin{aligned} Y &= \frac{\rho}{M}\int_0^4\int_0^{2-x/2} y\,dydx \\ &= \frac{\rho}{M}\int_0^4 \frac{y^2}{2}\bigg|_0^{2-x/2}\,dx \\ &= \frac{\rho}{2M}\int_0^4 \left(2-x/2\right)^2\,dx \\ &= \frac{\rho}{2M}\left[\frac{1}{3}\left(2-x/2\right)^3(-2)\right]_0^4 \\ &= \frac{\rho}{2M}\left[-\frac{2}{3}(2-4/2)^3 + \frac{2}{3}(2)^3\right] \\ &= \frac{16}{6}\frac{\rho}{M} \end{aligned}$$

Finding the mass:

$$M = \int_0^4\int_0^{2-x/2}\rho\,dxdy = 4\rho$$

And thus

$$X = \frac{4}{3} \qquad Y = \frac{2}{3}$$

3.3 Angular Momentum

Definition 3.3: Angular Momentum

Recall that angular momentum is defined as:

$$\vec{\boldsymbol{\ell}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}} = \vec{\boldsymbol{r}} \times m\vec{\boldsymbol{v}}$$

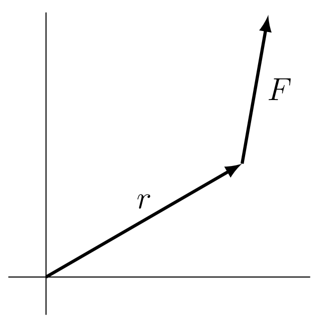
Remark 3.2

Note that we are using lowercase ℓ for the angular momentum, not the classic L. This is to help keep with our capital letter convention.

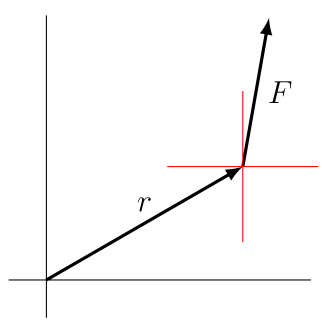
In our quest for conservation laws, we usually want to look at time derivatives:

$$\begin{aligned}
\dot{\vec{\boldsymbol{\ell}}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}\right] \\
&= \left(\dot{\vec{\boldsymbol{r}}} \times \vec{\boldsymbol{p}}\right) + \left(\vec{\boldsymbol{r}} \times \dot{\vec{\boldsymbol{p}}}\right) \\
&= \underbrace{\boxed{\left(\dot{\vec{\boldsymbol{r}}} \times m\dot{\vec{\boldsymbol{r}}}\right)}}_{\vec{r} \times \vec{r} = 0} + \underbrace{\boxed{\left(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}\right)}}_{\text{Torque!} \Rightarrow \Gamma} \\
&= \vec{\boldsymbol{\Gamma}}
\end{aligned}$$

Note that this is torque about the origin (and thus analogous to $\dot{\vec{\boldsymbol{p}}} = \vec{\boldsymbol{F}}$).

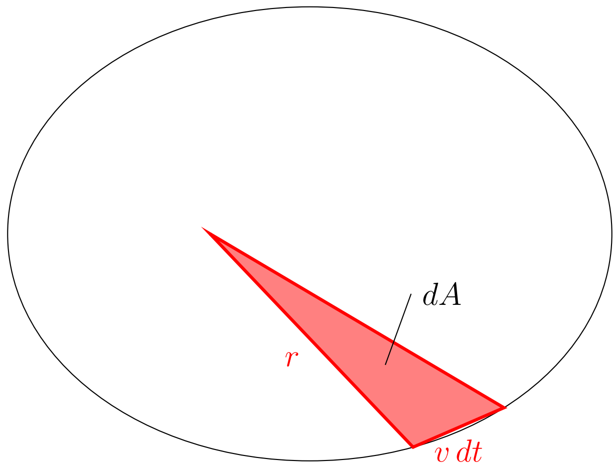


For many one particle or particularly simple systems, we can move the origin without loss of information:



This results in a torque equal to zero, and hence lets us consider conservation of angular momentum!

3.3.1 Kepler Aside



As seen in the homework, the area of the triangle is half the magnitude of the cross product:

$$\begin{aligned}
\mathrm{d}A &= \frac{1}{2}\|\vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}\,\mathrm{d}t\| \\
&= \frac{1}{2}\left\|\vec{\boldsymbol{r}} \times \frac{\vec{\boldsymbol{p}}}{m}\,\mathrm{d}t\right\| \\
\Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{1}{2m}\|\vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}\| \\
&= \frac{1}{2m}\ell \\
&= \text{ constant}
\end{aligned}$$

3.3.2 Angular Momentum for Several Bodies

Following our capital letter conventions:

$$\begin{aligned}
\vec{\boldsymbol{L}} &= \sum_\alpha \vec{\boldsymbol{\ell}}_\alpha = \sum_\alpha \vec{\boldsymbol{r}}_\alpha \times \vec{\boldsymbol{p}}_\alpha \\
\dot{\vec{\boldsymbol{L}}} &= \sum_\alpha \dot{\vec{\boldsymbol{\ell}}}_\alpha = \boxed{\sum_\alpha \vec{\boldsymbol{r}}_\alpha \times \vec{\boldsymbol{F}}_\alpha}
\end{aligned}$$

Can follow a similar path to Ch 1 momentum discussion to separate internal and external forces. The internal cancel by 3rd law *and assuming central forces!*

$$\Rightarrow \dot{\vec{\boldsymbol{L}}} = \vec{\boldsymbol{\Gamma}}_{ext}$$

And thus if there is no net external torque, then the total angular momentum is conserved.

Remark 3.3

Recall the Moment of Inertia

- Measure of an object's "rotational mass"
- Can look up for various solids
  - Disk – $\frac{1}{2}MR^2$
  - Hoop – MR^2
  - Solid Sphere – $\frac{2}{5}MR^2$
  - Hollow Sphere – $\frac{2}{3}MR^2$

So the angular momentum component along the axis of rotation can be written as:

$$L_z = I\omega$$

Example 3.5

Suppose a bird feeder is comprised of a solid disk of radius R and mass M and is initially stationary. A bird of mass m with initial speed v arrives tangentially to the rim of the bird feeder and comes to a stop perched on the rim. What is the angular velocity of the bird feeder + bird afterwards?
The bird + feeder are our system, so there are no external forces. Therefore angular momentum is conserved.

$$\begin{aligned}
L_{zi} &= \|\vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}\| = rp\sin(\theta) = Rmv \\
L_{zf} &= (I_{feeder} + I_{bird})\,\omega \\
&= \left(\frac{1}{2}MR^2 + mR^2\right)\omega \\
&= \omega R^2\left(m + \frac{M}{2}\right)
\end{aligned}$$

And thus:

$$\begin{aligned}
mRv &= \omega R^2\left(m + \frac{M}{2}\right) \\
\Rightarrow \omega &= \frac{m}{m + \frac{M}{2}}\frac{v}{R}
\end{aligned}$$

3.3.3 Angular Momentum about Center of Mass

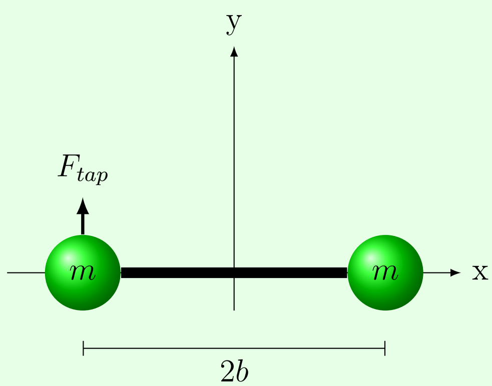
Originally, we assumed that the origin was in a fixed, inertial frame (as we usually do). It turns out that

$$\dot{\vec{\boldsymbol{L}}} = \vec{\boldsymbol{\Gamma}}_{ext}$$

still holds true about the center of mass, *even for accelerating frames!* We'll prove this later (Ch 9-10), but for now it is useful!

Example 3.6

Take the situation below:



where we have a connected barbell system that we are giving a brief "tap." Based on momentum conservation:

$$\begin{aligned}
\dot{\vec{\boldsymbol{P}}} &= \vec{\boldsymbol{F}}_{ext} \\
\mathrm{d}\vec{\boldsymbol{P}} &= \vec{\boldsymbol{F}}_{ext}\,\mathrm{d}t \\
\int_0^P \mathrm{d}\vec{\boldsymbol{P}} &= \vec{\boldsymbol{F}}_{ext}\int_0^t \mathrm{d}t \\
\vec{\boldsymbol{P}} &= \vec{\boldsymbol{F}}_{ext}t \\
&= M\dot{\vec{\boldsymbol{R}}} \\
\Rightarrow \dot{R} &= \frac{F_{tap}}{M}t = v_{cm}
\end{aligned}$$

We can do a similar thing via angular momentum conservation:

$$\begin{aligned}
\dot{\vec{\boldsymbol{L}}} &= \vec{\boldsymbol{\Gamma}}_{ext} \\
\mathrm{d}\vec{\boldsymbol{L}} &= \vec{\boldsymbol{\Gamma}}_{ext}\,\mathrm{d}t \\
\vec{\boldsymbol{L}} &= \vec{\boldsymbol{\Gamma}}_{ext} t \\
&= \left(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}\right) t \\
&= bFt \\
&= I\omega \\
&= Mb^2\omega \\
\Rightarrow \omega &= \frac{F_{tap} bt}{Mb^2} \\
\omega &= \frac{F_{tap} t}{Mb}
\end{aligned}$$

We can look at how both masses are moving immediately after the poke:

$$\begin{aligned}
v_{left,i} = v_{cm} + v_{rot} &= \frac{F_{tap} t}{M} + b\frac{F_{tap} t}{Mb} = \frac{2F_{tap} t}{M} \\
v_{right,i} = v_{cm} - v_{rot} &= \frac{F_{tap} t}{M} - b\frac{F_{tap} t}{Mb} = 0
\end{aligned}$$

After the poke, note that $\vec{\boldsymbol{F}}_{ext} = 0$ and thus $\vec{\boldsymbol{\Gamma}}_{ext} = 0$. Therefore $\vec{\boldsymbol{P}}$ and $\vec{\boldsymbol{L}}$ are constant and $\dot{R}$ and ω will continue at their present rates.