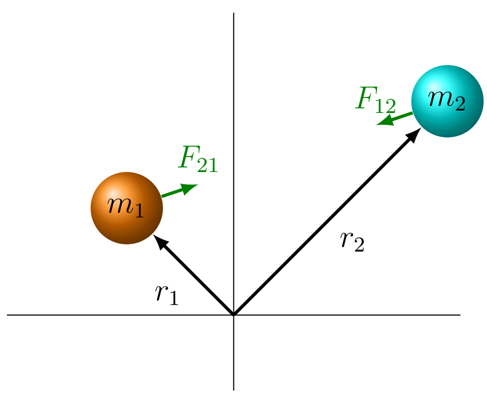
Chapter 8

Orbital Motion

In this chapter we are primarily going to be concerned with looking at two body, central force type problems most common when looking at orbits (of either gravity or electrostatic types).

8.1 Setup, the Lagrangian, and Simplifications

Take an instance like the image below:



where

$$F_{12} = -F_{21}$$

and both are conservative and central. So generally, our potential energies for these situations looks like:

$$U(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2) = -\frac{Gm_1m_2}{|\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2|} \quad \text{for gravity}$$

$$U(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2) = -\frac{ke^2}{|\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2|} \quad \text{for Coulomb}$$

Since U only depends on the difference in positions, then we can let

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2,$$

where $\vec{\mathbf{r}}$ will be the relative position. Whereupon we can write our Lagrangian thus far as:

$$\mathcal{L} = \frac{1}{2}m_1\dot{\vec{\mathbf{r}}}_1^2 + \frac{1}{2}m_2\dot{\vec{\mathbf{r}}}_2^2 - U(\vec{\mathbf{r}})$$

We want to choose the best generalized coordinates for the job, to make our lives as easy as possible. The relative position $\vec{\mathbf{r}}$ is already looking like a strong contender, as it makes the potential energy simple. For the other we'll use the center of mass $\vec{\mathbf{R}}$!

$$\vec{\mathbf{R}} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{m_1 + m_2} = \frac{m_1\vec{\mathbf{r}}_1}{M} + \frac{m_2\vec{\mathbf{r}}_2}{M}$$

where M is the total mass of the system. Since $\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$, we can write:

$$\begin{aligned}
\vec{\mathbf{R}} &= \frac{m_1\vec{\mathbf{r}}_1}{M} + \frac{m_2\vec{\mathbf{r}}_2}{M} \\
&= \frac{m_1\left(\vec{\mathbf{r}} + \vec{\mathbf{r}}_2\right)}{M} + \frac{m_2\vec{\mathbf{r}}_2}{M} \\
M\vec{\mathbf{R}} &= m_1\vec{\mathbf{r}} + (m_1 + m_2)\,\vec{\mathbf{r}}_2 \\
\frac{M\vec{\mathbf{R}} - m_1\vec{\mathbf{r}}}{M} &= \vec{\mathbf{r}}_2 \\
\Rightarrow \vec{\mathbf{r}}_2 &= \vec{\mathbf{R}} - \frac{m_1\vec{\mathbf{r}}}{M} \\
\Rightarrow \vec{\mathbf{r}}_1 &= \vec{\mathbf{R}} + \frac{m_2\vec{\mathbf{r}}}{M}
\end{aligned}$$

So our kinetic energy thus becomes:

$$\begin{aligned}
T &= \frac{1}{2}m_1\dot{\vec{\mathbf{r}}}_1^2 + \frac{1}{2}m_2\dot{\vec{\mathbf{r}}}_2^2 \\
&= \frac{1}{2}m_1\left(\dot{\vec{\mathbf{R}}} + \frac{m_2}{M}\dot{\vec{\mathbf{r}}}\right)^2 + \frac{1}{2}m_2\left(\dot{\vec{\mathbf{R}}} - \frac{m_1}{M}\dot{\vec{\mathbf{r}}}\right)^2 \\
&= \frac{1}{2}m_1\left(\dot{\vec{\mathbf{R}}}^2 + 2\dot{\vec{\mathbf{R}}}\dot{\vec{\mathbf{r}}}\frac{m_2}{M} + \left(\frac{m_2}{M}\right)^2\dot{\vec{\mathbf{r}}}^2\right) + \frac{1}{2}m_2\left(\dot{\vec{\mathbf{R}}}^2 - 2\dot{\vec{\mathbf{R}}}\dot{\vec{\mathbf{r}}}\frac{m_1}{M} + \left(\frac{m_1}{M}\right)^2\dot{\vec{\mathbf{r}}}^2\right) \\
&= \frac{1}{2}\left(M\dot{\vec{\mathbf{R}}}^2 + \frac{m_1m_2^2}{M^2}\dot{\vec{\mathbf{r}}}^2 + \frac{m_2m_1^2}{M^2}\dot{\vec{\mathbf{r}}}^2\right) \\
&= \frac{1}{2}\left(M\dot{\vec{\mathbf{R}}}^2 + m_1m_2\dot{\vec{\mathbf{r}}}^2\left(\frac{m_2 + m_1}{M^2}\right)\right) \\
&= \frac{1}{2}\left(M\dot{\vec{\mathbf{R}}}^2 + \underbrace{\boxed{\frac{m_1m_2}{M}}}_{\substack{\text{reduced mass} \\ \mu}}\dot{\vec{\mathbf{r}}}^2\right) \\
&= \frac{1}{2}\left(M\dot{\vec{\mathbf{R}}}^2 + \mu\dot{\vec{\mathbf{r}}}^2\right)
\end{aligned}$$

So, in the coordinates $\vec{\mathbf{R}}$ and $\vec{\mathbf{r}}$, we have that

$$\mathcal{L} = T - U = \frac{1}{2}M\dot{\vec{\mathbf{R}}}^2 + \left(\frac{1}{2}\mu\dot{\vec{\mathbf{r}}}^2 - U(\vec{\mathbf{r}})\right)$$

or, in other words, we could say:

$$\mathcal{L} = \mathcal{L}_{CM} + \mathcal{L}_{rel}$$

Note that

$$\mathcal{L}_{CM} = \frac{1}{2}M\dot{\vec{\mathbf{R}}}^2 \quad \Rightarrow \quad \frac{\partial \mathcal{L}_{CM}}{\partial \vec{\mathbf{R}}} = 0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}_{CM}}{\partial \dot{\vec{\mathbf{R}}}} = M\dot{\vec{\mathbf{R}}}\right] = M\ddot{\vec{\mathbf{R}}}$$

$$\Rightarrow \dot{\vec{\mathbf{R}}} = \quad \text{constant!}$$

and similarly:

$$\mathcal{L}_{rel} = \frac{1}{2}\mu\dot{\vec{\mathbf{r}}}^2 - U(\vec{\mathbf{r}}) \quad \Rightarrow \quad \frac{\partial \mathcal{L}_{rel}}{\partial \vec{\mathbf{r}}} = -\boldsymbol{\nabla}U(\vec{\mathbf{r}}) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}_{rel}}{\partial \dot{\vec{\mathbf{r}}}} = \mu\dot{\vec{\mathbf{r}}}\right] = \mu\ddot{\vec{\mathbf{r}}}$$

Remark 8.1

This gives us important information about how we can further simplify this problem. Since $\dot{\vec{\mathbf{R}}}$ is a constant, then our center of mass is not accelerating, and is thus an inertial reference frame. We can therefore look at things from some frame that would cause $\dot{\vec{\mathbf{R}}}$ to be equal to 0 (generally called the center of mass frame...).

$$\Rightarrow \mathcal{L}_{CM} = 0$$

$$\Rightarrow \mathcal{L} = \mathcal{L}_{rel} = \frac{1}{2}\mu\dot{\vec{\mathbf{r}}}^2 - U(\vec{\mathbf{r}})$$

So we have a nicer, one (vector) coordinate problem now! But we can actually simplify this even further if we look at the total angular momentum!

$$\begin{aligned}\vec{\mathbf{L}} &= \vec{\mathbf{r}}_1 \times \vec{\mathbf{p}}_1 + \vec{\mathbf{r}}_2 \times \vec{\mathbf{p}}_2 \\ &= m_1\vec{\mathbf{r}}_1 \times \dot{\vec{\mathbf{r}}}_1 + m_2\vec{\mathbf{r}}_2 \times \dot{\vec{\mathbf{r}}}_2\end{aligned}$$

Now since

$$\vec{\mathbf{r}}_1 = \cancelto{0}{\vec{\mathbf{R}}} + \frac{m_2}{M}\vec{\mathbf{r}} \quad \text{and} \quad \vec{\mathbf{r}}_2 = \cancelto{0}{\vec{\mathbf{R}}} - \frac{m_1}{M}\vec{\mathbf{r}}$$

So we have

$$\begin{aligned}\mathcal{L} &= \frac{m_1 m_2^2}{M^2}(\vec{\mathbf{r}} \times \dot{\vec{\mathbf{r}}}) + \frac{m_2 m_1^2}{M^2}(\vec{\mathbf{r}} \times \dot{\vec{\mathbf{r}}}) \\ &= \frac{m_1 m_2(m_1 + m_2)}{M^2}(\vec{\mathbf{r}} \times \dot{\vec{\mathbf{r}}}) \\ &= \mu(\vec{\mathbf{r}} \times \dot{\vec{\mathbf{r}}})\end{aligned}$$

Since we have no external torques acting on the system, we know that the total angular momentum is conserved, so $\vec{\mathbf{L}}$ is a constant. This means that $\vec{\mathbf{r}} \times \dot{\vec{\mathbf{r}}}$ always must point in the same direction. And this means that $\vec{\mathbf{r}}$ and $\dot{\vec{\mathbf{r}}}$ must lie in a plane!

Thus we can reduce the entire problem into a one-particle, two dimensional system! Sweet! If we use polar coordinates to write out $\vec{\mathbf{r}}$:

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

The Lagrangian equation wrt ϕ gives us:

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2 \dot{\phi}\right] \quad \Rightarrow \mu r^2 \dot{\phi} = \text{ constant } = \ell$$

which is just our conservation of angular momentum back. The Lagrangian equation wrt r yields:

$$\frac{\partial \mathcal{L}}{\partial r} = \mu r \dot{\phi}^2 - \frac{\mathrm{d}U}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{r}} = \mu \dot{r}\right] = \mu \ddot{r}$$

or

$$\mu \ddot{r} = \mu r \dot{\phi}^2 - \frac{\mathrm{d}U}{\mathrm{d}r}$$

Notice that the first term on the right looks an awful lot like the classic centripetal force $m\omega^2 r$, except it is pointing *outwards*. Thus it is playing the role of the fabled "fictional" centifugal force! If we write

$$\mu r^2 \dot{\phi} = \ell \quad \text{as} \quad \dot{\phi} = \frac{\ell}{\mu r^2}$$

then

we could say that

$$F_{cf} = \mu r \dot{\phi}^2 = \mu r \left(\frac{\ell}{\mu r^2}\right)^2 = \frac{\ell^2}{\mu r^3}$$

We could integrate this to work out the corresponding potential energy for this centrifugal force:

$$U_{cf} = \frac{\ell^2}{2\mu r^2}$$

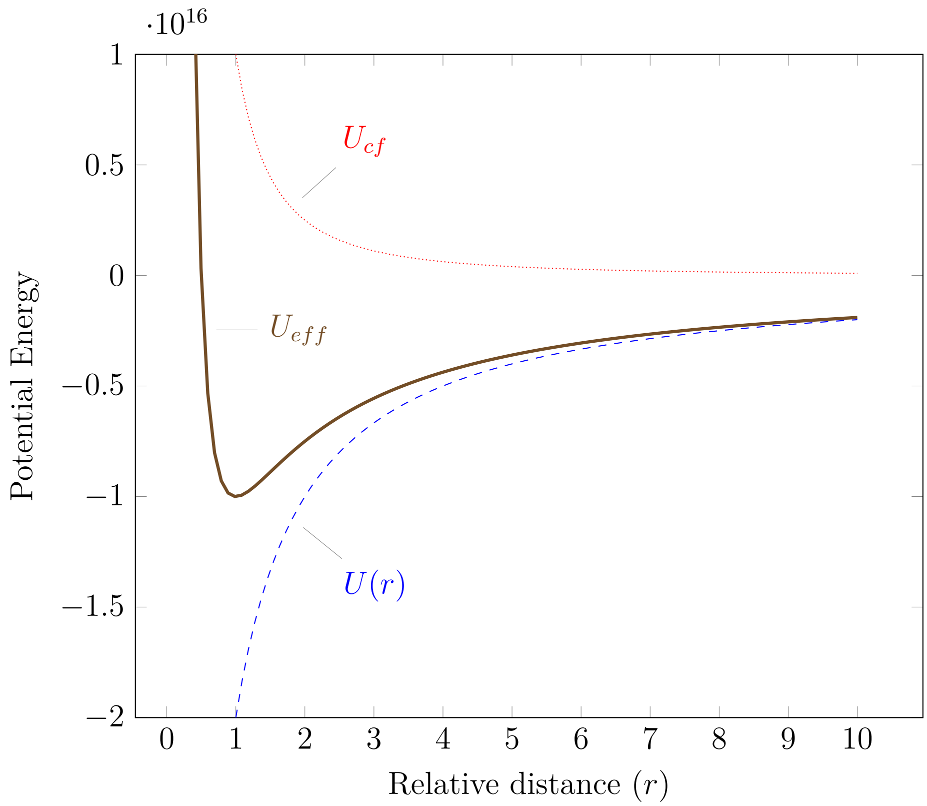
And thus we can write

$$\mu \ddot{r} = -\frac{\mathrm{d}}{\mathrm{d}r}\underbrace{\boxed{[U + U_{cf}]}}_{U_{eff} = \text{effective potential}}$$

So commonly we'll talk about the effective potential:

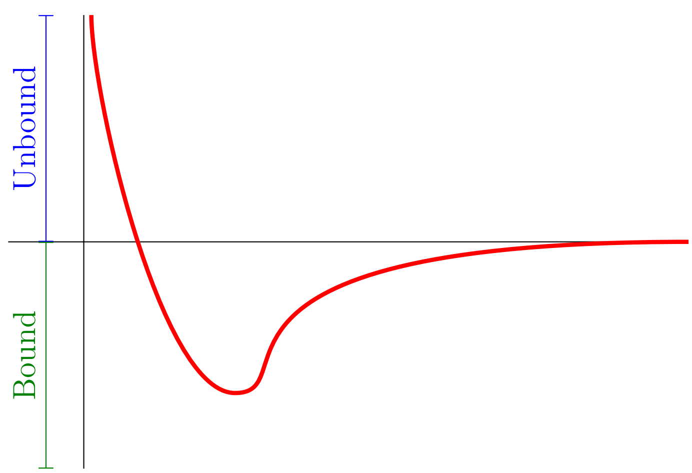
$$U_{eff} = U(r) + \frac{\ell^2}{2\mu r^2}$$

For some gravitational potential, the commonly looks like this:



8.2 Shape of Orbits

So we have determined that our potential energy curve looks like:



With bound orbits corresponding to negative effective potentials and unbound orbits corresponding to positive effective potentials! Often times, though, we'd rather know the *shape* of the orbit $r(\phi)$ instead of how the radius varies with time ($r(t)$). So our goal now will be to transition our equation of state to being in terms of ϕ! In terms of our forces:

$$\mu\ddot{r} = \underbrace{\boxed{F(r)}}_{\text{central force}} + \frac{\ell^2}{\mu r^3}$$

To transition to writing our equation in terms of ϕ, we are going to do two things. First, not the excess of r's hanging out in the denominators in the terms above. So that we don't have to deal with this as much, we'll make the substitution:

$$u = \frac{1}{r}$$

We also need to rewrite our time derivatives into ϕ derivatives. Note that:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\phi} = \dot{\phi}\frac{\mathrm{d}}{\mathrm{d}\phi} = \frac{\ell}{\mu r^2}\frac{\mathrm{d}}{\mathrm{d}\phi} = \frac{\ell u^2}{\mu}\frac{\mathrm{d}}{\mathrm{d}\phi}$$

We can then work out our time derivatives:

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\ell u^2}{\mu}\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{1}{u}\right) = \frac{\ell u^2}{\mu}\left(-u^{-2}\frac{\mathrm{d}u}{\mathrm{d}\phi}\right) = -\frac{\ell}{\mu}\frac{\mathrm{d}u}{\mathrm{d}\phi}$$
$$\ddot{r} = \frac{\mathrm{d}}{\mathrm{d}t}\dot{r} = \frac{\ell u^2}{\mu}\frac{\mathrm{d}}{\mathrm{d}\phi}\left(-\frac{\ell}{\mu}\frac{\mathrm{d}u}{\mathrm{d}\phi}\right) = -\frac{\ell^2 u^2}{\mu}\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2}$$

Therefore, plugging those in, we have

$$\mu\left(-\frac{\ell^2 u^2}{\mu^2}\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2}\right) = F + \frac{\ell^2 u^3}{\mu}$$

or

$$u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2}F \tag{8.1}$$

Example 8.1

Let's look at this in a very simple case with ZERO central force (so a free particle). In that case, we have:

$$u''(\phi) = -u(\phi)$$

But this is just an equation for simple harmonic motion! So

$$u(\phi) = A\cos(\phi - \delta)$$

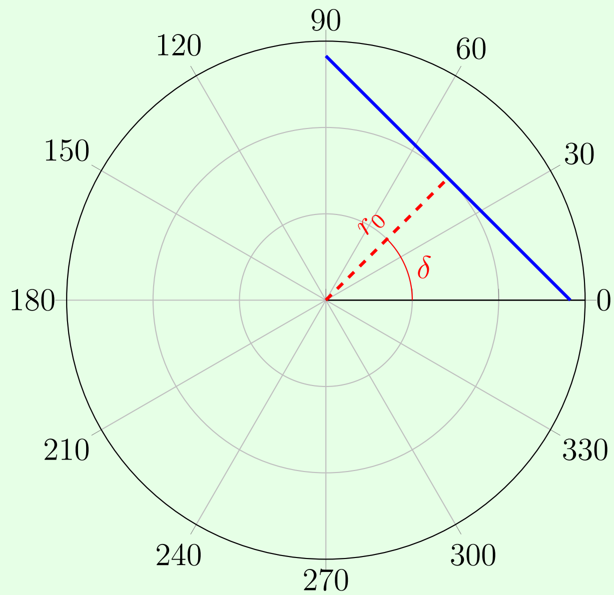
and

$$r(\phi) = \frac{1}{u(\phi)} = \frac{1}{A\cos(\phi - \delta)} = \frac{r_0}{\cos(\phi - \delta)}$$

where

$$r_0 = \frac{1}{A}$$

If your polar plotting is not well visualized, you can always plot this to see what is happening:



So we get a straight line as expected!

So now we'll go ahead and add our central force:

$$F = \frac{-Gm_1m_2}{r^2} \quad \text{or} \quad \frac{-kq_1q_2}{r^2} = -\frac{\gamma}{r^2} = -\gamma u^2$$

Therefore:

$$\begin{aligned} u''(\phi) &= -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2}\left(-\gamma u(\phi)^2\right) \\ &= -u(\phi) + \frac{\gamma\mu}{\ell^2} \end{aligned} \tag{†}$$

Note that that last term is a constant!! So we can shift things (since derivatives of constants are zero). Let

$$w(\phi) = u(\phi) - \frac{\gamma\mu}{\ell^2}$$

So

$$u'(\phi) = w'(\phi) \quad \text{and} \quad u''(\phi) = w''(\phi)$$

Thus (†) simplifies to:

$$w''(\phi) = -w(\phi)$$

which we can solve!

$$w(\phi) = A\cos(\phi - \delta)$$

initially, we'll choose $\phi = 0$ such that $\delta = 0$

$$= A\cos(\phi)$$

Writing things back in terms of $u(\phi)$:

$$\begin{aligned} u(\phi) &= w(\phi) + \frac{\gamma\mu}{\ell^2} \\ &= \frac{\gamma\mu}{\ell^2} + A\cos(\phi) \\ &= \frac{\gamma\mu}{\ell^2}(1 + \varepsilon\cos(\phi)) \end{aligned}$$

where $\varepsilon = \frac{A\ell^2}{\gamma\mu} = Ac$

$$= \frac{1}{c}(1 + \varepsilon\cos(\phi))$$

where $c = \frac{\ell^2}{\gamma\mu}$

Thus we finally can write:

$$r(\phi) = \frac{1}{u(\phi)} = \frac{c}{1 + \varepsilon\cos\phi} \tag{8.2}$$

Yay! So what does this look like?

Remark 8.2

Note that if $\varepsilon \geq 1$, then $r(\phi)$ blows up at some ϕ. But if $\varepsilon < 1$, then $r(\phi)$ will stay bounded. Thus $\varepsilon < 1$ corresponds to bound orbits!

Remark 8.3

The maximum cosine can be is 1 and the minimum it can be is -1. Thus we can find the maximum and minimum radial distances:

$$r_{min} = \frac{c}{1+\varepsilon} \quad \text{and} \quad r_{max} = \frac{c}{1-\varepsilon}$$

For some terminology,r_{min} is called perihelion and r_{max} is called aphelion

Let's take a look at the different shapes of the orbits!

8.2.1 Circles ($\varepsilon = 0$)

The most obvious case, but if $\varepsilon = 0$ then we just have that

$$r(\phi) = c$$

And we have a perfect circle!

8.2.2 Parabola ($\varepsilon = 1$)

Here we have that:

$$r(\phi) = \frac{c}{1 + \cos\phi}$$

Rewriting that it terms of cartesian coordinates, we know that:

$$x = r\cos\phi, \quad y = r\sin\phi \quad \text{then} \quad \cos\phi = \frac{x}{r}, \quad r^2 = x^2 + y^2$$

and so

$$\begin{aligned} r &= \frac{c}{1 + \frac{x}{r}} \\ \Rightarrow (1 + \frac{x}{r})r &= c \\ (r + x) &= c \\ r^2 &= (c - x)^2 \\ x^2 + y^2 &= c^2 - 2cx + x^2 \\ y^2 = c^2 - 2cx \end{aligned}$$

which is the equation for a parabola! (Sideways the way we have done things)

8.2.3 Hyperbola ($\varepsilon > 1$)

We can make similar substitutions (though the algebra is a good deal more messy) to find that, when $\varepsilon > 1$, we get

$$\frac{(x - \delta)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$$

which you hopefully recognize as the equation for a hyperbola

8.2.4 Ellipse ($\varepsilon < 1$)

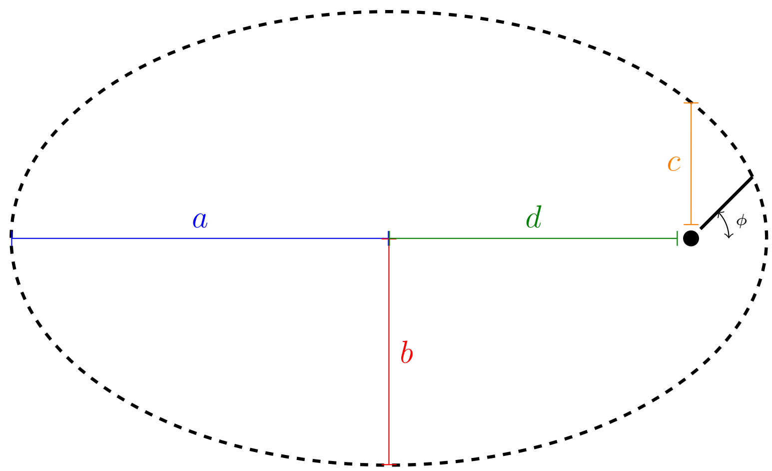
And finally, we can also make the same substitutions to determine was is happening in the case of our bound orbits, which yields:

$$\frac{(x + d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where

$$
\begin{aligned}
a &= \frac{c}{1-\varepsilon^2} \\
b &= \frac{c}{\sqrt{1-\varepsilon^2}} \\
d &= \varepsilon a
\end{aligned}
$$

Hopefully you recognize this as the equation of an ellipse!



Remark 8.4

Note that

$$\frac{b}{a} = \sqrt{1-\varepsilon^2}$$

So ε is really giving us information about how circular the orbit is!

8.3 Energetics

So we are working with the fact that

$$r(\phi) = \frac{c}{1+\varepsilon\cos\phi} \quad \text{and} \quad U_{eff} = -\frac{\gamma}{r} + \frac{\ell^2}{2\mu r^2}$$

At closest approach then:

$$\begin{aligned} E &= T_r + U_{eff} \\ &= 0 + U_{eff} \\ &= -\frac{\gamma}{r_{min}} + \frac{\ell^2}{2\mu r_{min}^2} \\ &= \frac{1}{2r_{min}}\left(\frac{\ell^2}{\mu r_{min}} - 2\gamma\right) \end{aligned}$$

Recall that we previously had that

$$r_{min} = \frac{c}{1+\varepsilon} \quad \text{and} \quad c = \frac{\ell^2}{\gamma\mu}$$

or

$$r_{min} = \frac{\ell^2}{\gamma\mu(1+\varepsilon)}$$

Therefore, plugging that into our energy equation:

$$\begin{aligned} E &= \frac{1}{2}\frac{\gamma\mu(1+\varepsilon)}{\ell^2}\left[\frac{\ell^2}{\mu}\frac{\gamma\mu(1+\varepsilon)}{\ell^2} - 2\gamma\right] \\ &= \frac{\gamma\mu(1+\varepsilon)}{2\ell^2}\left[\gamma(1+\varepsilon) - 2\gamma\right] \\ &= \frac{\gamma^2\mu}{2\ell^2}\left[(1+\varepsilon)^2 - 2(1+\varepsilon)\right] \\ &= \frac{\gamma^2\mu}{2\ell^2}\left(1 + 2\varepsilon + \varepsilon^2 - 2 - 2\varepsilon\right) \\ &= \frac{\gamma^2\mu}{2\ell^2}(\varepsilon^2 - 1) \end{aligned}$$

And thus we see that ε is directly related to the energy of the system!

8.4 Changing Orbits

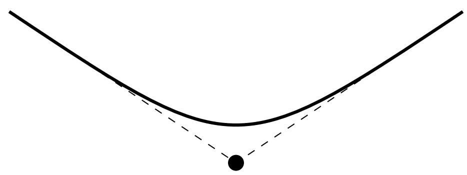
Remark 8.5

Recall that, for a single orbit, we can always rotate everything such that our $\phi = 0$ point is at $\delta = 0$. Unfortunately, for multiple orbits we can not do this for both orbits simultaneously, so we'll need to leave the rotation angle in our expression:

$$r(\phi) = \frac{c}{1 + \varepsilon\cos(\phi - \delta)}$$

This leaves us with three orbital parameters: c, ε, δ.

Often times we transition orbits with a rocket firing over some time interval. We can approximate this as a brief impulse at some angle ϕ_0.



Moment of Impulse

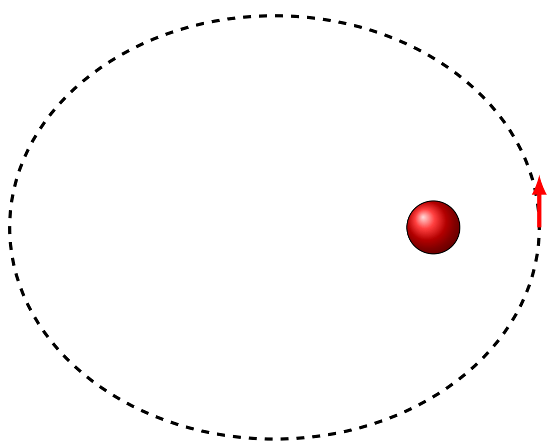
The result of this impulse is a change in velocity, which then proceeds to imply a new energy, and thus a new angular momentum, and thus a new c and ε value. Finally, we also need the new orbits to overlap at the point of impulse and the spacecraft transitions from one to the other:

$$r_1(\phi_0) = r_2(\phi_0)\frac{c_1}{1+\varepsilon\cos(\phi_0-\delta_1)} \qquad = \frac{c_2}{1+\varepsilon\cos(\phi_0-\delta_2)}$$

which, in general, tends to be ugly.

8.4.1 A Special Case

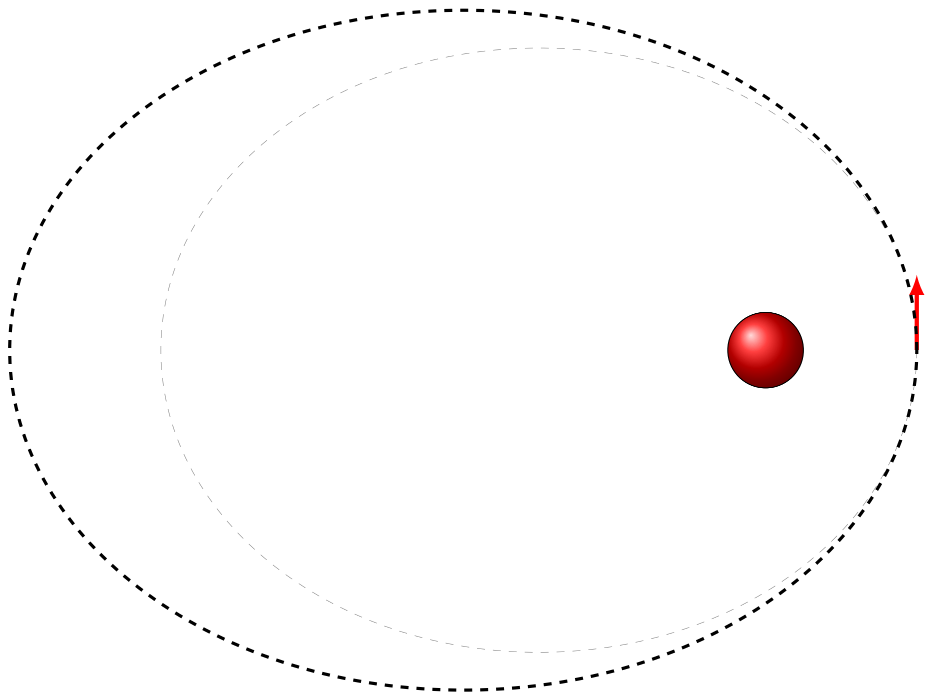
Let's look at the case then where we fire the rocket's thrusters forward tangentially when at perigee (closest approach to the central orbiting body).



Remark 8.6

Note that we have adjusted our picture again so that $\phi_0 = 0$ and $\delta_1 = 0$!

After the impulse, our velocity is *still in the tangential direction*! Which means that we are still at perigee for the *new* orbit! And thus $\delta_2 = 0$ as well!



And so we have

$$\frac{c_1}{1+\varepsilon_1} = \frac{c_2}{1+\varepsilon_2}$$

Now let us define a <u>thrust factor</u> λ such that

$$\lambda = \frac{v_2}{v_1}$$

Out at perigee our angular momentum is given by:

$$\ell = \mu r v$$

and thus our new angular momentum could be written as:

$$\ell_2 = \lambda \ell_1$$

Thus, since

$$c = \frac{\ell^2}{\gamma\mu}$$

we have that

$$c_2 = \lambda^2 c_1$$

Altogether then, cross multiplying:

$$\begin{aligned} \cancel{c_1}(1+\varepsilon_2) &= \lambda^2 \cancel{c_1}(1+\varepsilon_1) \\ \varepsilon_2 &= \lambda^2 \varepsilon_1 + \lambda^2 - 1 \end{aligned}$$

So if

- $\varepsilon_2 > \varepsilon_1$ then we've gotten more eccentric
- $\varepsilon_2 < \varepsilon_1$ we've gotten less eccentric
- $\varepsilon_2 = 0$ then we've gone circular
- $\varepsilon_2 < 0$ then we've flipped r_{min} and r_{max}
- $\varepsilon_2 > 1$ we've escaped the gravity well!