Consider the system of equations

$$\frac{dI}{dt} = \beta S \frac{I}{N} - \gamma I \tag{1}$$

$$\frac{dC}{dt} = \beta S \frac{I}{N} \tag{2}$$

where $\frac{S}{N} = 1$ is relatively unchanging. Note $\beta = R_0 \gamma$. The equations are linear and decoupled. Let,

$$R_0 = \begin{cases} R_1 > 1 & \text{if } 0 < t < t_e \\ R_2 < 1 & \text{if } t \geq t_e \end{cases} \tag{3}$$

$$I(t) = \begin{cases} I(0)\exp(\gamma(R_1 - 1)t) & \text{if } 0 < t < t_e \\ I(t_e)\exp(\gamma(R_2 - 1)(t - t_e)) & \text{if } t \geq t_e \end{cases} \tag{4}$$

where

$$I(t_e) = I(0)\exp(\gamma(R_1 - 1)t_e)$$

Note $C(\infty) = C^- + C^+$ where C^- is the cumulative number of infections up until $t = t_e$ and C^+ is the cumulative infections from $t = t_e$ to ∞. Note C(0)=0.

$$\int_0^{t_e} \frac{dC}{dt} = \gamma R_1 \int_0^{t_e} I(t)dt \tag{5}$$

$$C^- - C(0) = \gamma R_1 \int_0^{t_e} I(0)\exp(\gamma(R_1 - 1)t)dt \tag{6}$$

$$C^- = \frac{R_1 I(0)}{R_1 - 1} \exp(\gamma(R_1 - 1)t)\Big|_0^{t_e} \tag{7}$$

$$C^- = \frac{R_1 I(0)}{R_1 - 1}(\exp(\gamma(R_1 - 1)t_e) - 1) \tag{8}$$

$$\tag{9}$$

$$\int_{t_e}^{\infty} \frac{dC}{dt} = \gamma R_2 \int_{t_e}^{\infty} I(t)dt \tag{10}$$

$$C^+ - 0 = \gamma R_2 \int_{t_e}^{\infty} I(t_e)\exp(\gamma(R_2 - 1)(t - t_e))dt \tag{11}$$

$$C^+ = \frac{R_2 I(t_e)}{R_2 - 1} \exp(\gamma(R_2 - 1)(t - t_e))\Big|_{t_e}^{\infty} \tag{12}$$

$$C^+ = \frac{R_2 I(t_e)}{R_2 - 1}(0 - 1) \tag{13}$$

$$C^+ = \frac{R_2 I(t_e)}{(1 - R_2)} \tag{14}$$

Final size is $C(\infty) = C^- + C^+$.

$$C^- + C^+ = \frac{R_1 I(0)}{R_1 - 1}(\exp(\gamma(R_1 - 1)t_e) - 1) + \frac{R_2 I(t_e)}{1 - R_2} \tag{15}$$

$$= \frac{R_1 I(0)}{R_1 - 1}(\exp(\gamma(R_1 - 1)t_e) - 1) + \frac{R_2 I(0)\exp(\gamma(R_1 - 1)t_e)}{1 - R_2} \tag{16}$$

$$= I(0)\exp(\gamma(R_1 - 1)t_e)\left(\frac{R_1}{R_1 - 1} + \frac{R_2}{1 - R_2}\right) \tag{17}$$

$$C^- + C^+ = C^- + \frac{R_2 I(t_e)}{1 - R_2} \quad (18)$$

$$= C^- + \frac{R_2 I(t_e)}{1 - R_2} \quad (19)$$

Note that,

$$C^- = \frac{R_1}{R_1 - 1}(I(t_e) - 1) \quad (20)$$

$$I(t_e) = \left(C^- + \frac{R_1}{R_1 - 1}\right)\frac{R_1 - 1}{R_1} \quad (21)$$

$$= C^- \frac{R_1 - 1}{R_1} + 1 \quad (22)$$

Then,

$$C^- + C^+ = C^- + \frac{R_2}{1 - R_2}\left(C^- \frac{R_1 - 1}{R_1} + 1\right) \quad (23)$$

$$= C^- \left(1 + \frac{R_2}{1 - R_2}\frac{R_1 - 1}{R_1}\right) + \frac{R_2}{1 - R_2} \quad (24)$$

The above states that the cumulative number of infections for the outbreak is a linear function of the cumulative number of cases when escalation begins.