— Supplementary Material —

Fiscal Fatigue, Fiscal Limits, and Sovereign Credit Spreads

The supplementary material is organized as follows:

- Section I gathers the proofs of the propositions of Section 2 of the paper.
- Section II presents the formulas we use to compute bond prices, CDS spreads, and probabilities of default. They are presented in the context of a framework that is more general than the one presented in the main text. The last subsection of this appendix (Subsection II.4) illustrates the quality of these approximations. For this, we exploit the stylized model of Subsection 2.3 of the paper. (Indeed, we can employ compute prices in the latter context (using numerical solutions); this allows us to verify our analytical approximate formulas.)
- Section III presents results of panel regression-based analyses.
- Section IV provides details on the data.
- Section V contains additional tables and figures (baseline estimation).
- Section VI presents robustness analyses.

I. Proofs of Section 2

I.1. *Proof of Proposition 1*

Let us denote by I_t the proceeds of date-t issuances and by X_t the resulting first payments (settled on date $t+1$). By definition of q_t, the yield-to-maturity associated with the perpetuity, we have:

$$I_t = \sum_{j=1}^{\infty} \frac{\chi^{j-1} X_t}{(1+q_t)^j} = \frac{X_t}{1+q_t-\chi}.$$

Consider the date-t (residual) face value of the issuances that took place on date $t-h$. According to the concept of nominal valuation of debt securities (see International Monetary Fund, Bank for International Settlements and European Central Bank, 2015), this face value is computed as the sum of future associated payoffs $\chi^{h+1} X_{t-h}, \chi^{h+2} X_{t-h}, \dots$, discounted using the issuance yield-to-maturity that materialized on date $t-h$, that is q_{t-h}. This is equal to $\chi^h I_{t-h}$. As a consequence, and because current debt D_t is the sum of the (residual) face values of all past issuances (for $h \geq 0$), we obtain:

$$D_t \equiv I_t + \chi I_{t-1} + \chi^2 I_{t-2} + \dots = I_t + \chi D_{t-1}. \tag{I.1}$$

Using $X_t = (1+q_t-\chi) I_t = (1+q_t-\chi)(D_t - \chi D_{t-1})$, past debt issuances give rise to the following debt payments on date $t+1$:

$$\begin{aligned} & CF_{t+1} = X_t + \chi X_{t-1} + \chi^2 X_{t-2} + \dots \\ = \quad & (1+q_t-\chi)(D_t - \chi D_{t-1}) + \\ & \chi(1+q_{t-1}-\chi)(D_{t-1} - \chi D_{t-2}) + \chi^2(1+q_{t-2}-\chi)(D_{t-2} - \chi D_{t-3}) + \dots \\ = \quad & D_t - \chi D_t + q_t(D_t - \chi D_{t-1}) + \chi q_{t-1}(D_{t-1} - \chi D_{t-2}) + \chi^2 q_{t-2}(D_{t-2} - \chi D_{t-3}) + \dots \end{aligned} \tag{I.2}$$

Let us now take a cash-flow perspective. On date t, the sum of the issuance proceeds (I_t) and of the primary budget surplus (S_t) has to equate date-t payments associated with previous issuances (CF_t). That is: $I_t = CF_t - S_t$. Using Eq. (I.1), we get:

$$D_{t+1} - \chi D_t = CF_{t+1} - S_{t+1}. \tag{I.3}$$

Substituting for CF_t (Eq. I.2) into Eq. (I.3), we have:

$$\begin{aligned} D_{t+1} \quad = \quad & D_t - S_{t+1} + \\ & \underbrace{q_t(D_t - \chi D_{t-1}) + \chi q_{t-1}(D_{t-1} - \chi D_{t-2}) + \chi^2 q_{t-2}(D_{t-2} - \chi D_{t-3}) + \dots}_{\text{interest payments on date } t+1 \,\equiv\, R_{t+1}}, \end{aligned} \tag{I.4}$$

which proves Proposition 1.

I.2. *Proof of Proposition 2*

Let us determine how $\mathcal{P}_t$ depends on q_{t+1}. On date $t+1$, the payoff of the perpetuity is:

$$\begin{cases} 1+\chi\mathcal{P}_{t+1} & \text{if } \mathcal{D}_{t+1}=0, \\ RR+\mathbb{E}_{t+1}\left(\sum_{h=2}^{\infty}\mathcal{M}_{t+1,t+h}\chi^{h-1}RR\right) & \text{if } \mathcal{D}_{t+1}=1. \end{cases} \tag{I.5}$$

In the stylized model (described in Subsections 2.1 to 2.3), the s.d.f. is given by

$$\mathcal{M}_{t,t+1}=\delta\exp(\gamma b_y(\mathcal{D}_{t+1}-\mathcal{D}_t)-\mu). \tag{I.6}$$

Therefore, after a default on date $t+1$ (which implies $\mathcal{D}_{t+k}=1$ for all $k>0$), the s.d.f. becomes deterministic:

$$\mathcal{M}_{t+1,t+1+h}=\mathcal{M}_{t+1,t+2}\times\cdots\times\mathcal{M}_{t+h-1,t+h}=\exp(\log(\delta)-\mu)^h. \tag{I.7}$$

Using Eqs. (I.5) and (I.7), we have:

$$\begin{aligned} \mathcal{P}_t &= \mathbb{E}_t\left(\mathcal{M}_{t,t+1}\left[\mathcal{D}_{t+1}RR\left(1+\sum_{h=1}^{\infty}\exp(\log(\delta)-\mu)^h\chi^h\right)+(1-\mathcal{D}_{t+1})(1+\chi\mathcal{P}_{t+1})\right]\right) \\ &= \mathbb{E}_t\left(\mathcal{M}_{t,t+1}\left[\mathcal{D}_{t+1}\frac{RR}{1-\chi\exp(\log(\delta)-\mu)}+(1-\mathcal{D}_{t+1})(1+\chi\mathcal{P}_{t+1})\right]\right). \end{aligned}$$

Eq. (8) is obtained by rearranging the terms of the previous equation, using Eq. (I.6), together with $\mathcal{P}_t=1/(1+q_t-\chi)$, and $\mathcal{P}_{t+1}=1/(1+q_{t+1}-\chi)$.

I.3. *Proof of Proposition 3*

We have:

$$\begin{aligned} \mathcal{B}_{t,h} &= \mathbb{E}_t\big(\exp(h\log(\delta)-h\mu+\gamma b_y\mathcal{D}_{t+h})(1-[1-RR]\mathcal{D}_{t+h})|\mathcal{D}_t=0\big) \\ &= \mathbb{E}_t\big(\exp(h\log(\delta)-h\mu)\{1-[1-RR\exp(b_y\gamma)]\mathcal{D}_{t+h}\}|\mathcal{D}_t=0\big), \end{aligned}$$

which gives Eq. (10).

Turning to the probabilities of default, we have:

$$\begin{aligned} p_h(d_t,d_{t-1},r_t) &= \mathbb{E}_t\big(\mathbb{E}_{t+1}(\mathcal{D}_{t+h}|\mathcal{D}_t=0)|\mathcal{D}_t=0\big) \\ &= \mathbb{E}_t\big(\mathcal{D}_{t+1}+(1-\mathcal{D}_{t+1})p_{h-1}(d_{t+1},d_t,r_{t+1})|\mathcal{D}_t=0\big) \\ &= \mathbb{E}_t\big(\mathcal{D}_{t+1}[1-p_{h-1}(d_{t+1},d_t,r_{t+1})]+p_{h-1}(d_{t+1},d_t,r_{t+1})|\mathcal{D}_t=0\big), \end{aligned}$$

which proves Eq. (11).

I.4. *Proof of Corollary 1*

According to Eq. (10), when $RR\exp(b_y\gamma) = 1$, we have $\mathcal{B}_{t,h} = \exp(h\log(\delta) - h\mu)$. Since $\mathcal{P}_t = \sum_{h=1}^{\infty}\chi^{h-1}\mathcal{B}_{t,h}$, this gives:

$$
\begin{aligned}
\mathcal{P}_t &= \sum_{h=1}^{\infty}\chi^{h-1}(\exp(\log(\delta)-\mu))^h = \exp(\log(\delta)-\mu)\sum_{h=0}^{\infty}(\chi\exp(\log(\delta)-\mu))^h \\
&= \frac{\delta\exp(-\mu)}{1-\chi\delta\exp(-\mu)}.
\end{aligned}
$$

Using $\mathcal{P}_t = 1/(1+q_t-\chi)$ leads to the expression of q_t given in Corollary 1.

I.5. *Short-term risk-free rate*

The following proposition gives an explicit formula for the short-term risk-free real rate in the context of our stylized model.

Proposition 4. *In the context of the model described in Subsections 2.2 to 2.1, and if $\mathcal{D}_t = 0$, the one-period risk-free real rate is given by:*

$$
\begin{aligned}
r_t &= \mu - \log(\delta) \\
&\quad - \log\left(\exp(\gamma b_y) + (1-\exp(\gamma b_y))\mathbb{E}_t\left[\exp(-\underline{\lambda}_{t+1})\right]\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{E}_t\left[\exp(-\underline{\lambda}_{t+1})\right] &= \Phi\left(\frac{-\beta\times(d_{t-1}-d^*)+s^*}{\sigma_s}\right) + \\
&\quad \Phi\left(\frac{\beta\times(d_{t-1}-d^*)-s^*-\sigma_s^2}{\sigma_s}\right)e^{-\beta\times(d_{t-1}-d^*)+s^*+\sigma_s^2/2}.
\end{aligned}
$$

(And $r_t = \mu - \log(\delta)$ if $\mathcal{D}_t = 1$.)

Proof. The short-term risk-free real rate is given by $r_t = -\log(\mathbb{E}_t(\mathcal{M}_{t,t+1}))$. Using Eq. (6)—i.e., $\mathcal{M}_{t,t+1} = \exp(\log(\delta) - \mu + \gamma b_y(\mathcal{D}_{t+1} - \mathcal{D}_t))$—, we have:

$$
r_t = \mu - \log(\delta) - \log\mathbb{E}_t\left[\exp(\gamma b_y(\mathcal{D}_{t+1}-\mathcal{D}_t))\right] = \mu - \log(\delta) - \psi_t(\gamma b_y),
$$

where ψ_t is the log-Laplace transform of $\mathcal{D}_{t+1} - \mathcal{D}_t$, that is: $\psi_t(u) = \log\mathbb{E}_t[\exp(u(\mathcal{D}_{t+1}-\mathcal{D}_t))]$. It is easily seen that $\psi_t(u) = 0$ when $\mathcal{D}_t = 1$. Let us consider the case where $\mathcal{D}_t = 0$:

$$
\begin{aligned}
\mathbb{E}_t[\exp(u(\mathcal{D}_{t+1}-\mathcal{D}_t))|\mathcal{D}_t=0] &= \mathbb{E}_t[\mathbb{E}_t[\exp(u\mathcal{D}_{t+1})|\eta_{t+1},\mathcal{D}_t=0]|\mathcal{D}_t=0] \\
&= \mathbb{E}_t\left[\exp(-\underline{\lambda}_{t+1}) + \exp(u)(1-\exp(-\underline{\lambda}_{t+1}))\right],
\end{aligned}
$$

where $\underline{\lambda}_{t+1}$ is the default intensity, given by $\alpha \max(0, s_{t+1} - s^*) = \alpha \max(0, \beta \times (d_{t-1} - d^*) + \eta_{t+1} - s^*)$. Using standard results on the truncated normal distribution, it comes that:

$$\begin{aligned} & \mathbb{E}_t \left[\exp(-\max(0, \beta \times (d_{t-1} - d^*) + \eta_{t+1} - s^*))\right] \\ = \; & \Phi\left(\frac{-\beta \times (d_{t-1} - d^*) + s^*}{\sigma_s}\right) + \Phi\left(\frac{\beta \times (d_{t-1} - d^*) - s^* - \sigma_s^2}{\sigma_s}\right) e^{-\beta \times (d_{t-1} - d^*) + s^* + \sigma_s^2/2}, \end{aligned}$$

which proves the proposition. □

II. Pricing bonds and CDS in the extended framework

This section presents the formulas used to value bonds, CDSs, and perpetuities in the context of our extended model. Subsection II.1 starts by presenting three assumptions that describe a generic econometric framework of which our model is a specific case. Subsection II.2 presents propositions and lemmas that underlie our pricing formulas. The latter are presented in Subsection II.3. Finally, Subsection II.4 illustrates the quality of these approximations. For this, we exploit the stylized model of Subsection 2.3 of the paper. Indeed, we can employ compute prices in the latter context (using numerical solutions); this allows us to verify our analytical approximate formulas.

II.1. Assumptions

Assumption 1. *w_t follows an exogenous Gaussian VAR process, that is:*

$$w_t = \Phi_w w_{t-1} + \Sigma_w \varepsilon_t, \tag{II.1}$$

with $\varepsilon_t \sim$ i.i.d. $\mathcal{N}(0, Id)$.

The full state vector x_t is of the form $x_t = [w_t', \bullet']'$, where $\bullet$ denotes a vector of additional variables (that may correlate to w_t). As long as $\mathcal{D}_t = 0$, x_t's dynamics also takes the form of a Gaussian VAR(1) process:

$$x_t = \mu_x + \Phi_x x_{t-1} + \Sigma_x \varepsilon_t. \tag{II.2}$$

Because w_t coincides with the first entries of x_t, we necessarily have:

$$\mu_x = \left[\begin{array}{c} \mathbf{0}_{\{n_w \times 1\}} \\ \bullet \end{array}\right], \quad \Phi_x = \left[\begin{array}{cc} \Phi_w & \bullet \\ \bullet & \bullet \end{array}\right], \quad and \quad \Sigma_x = \left[\begin{array}{c} \Sigma_w \\ \bullet \end{array}\right].$$

Finally, we denote by $\mathbf{x}_t$ the process that follows (II.2) whatever the default status of the government. That is, $\mathbf{x}_t = x_t$ as long as $\mathcal{D}_t = 0$. Since the ε_t's are exogenous, it comes that $\mathcal{D}_t$ does not Granger-cause $\mathbf{x}_t$ (while it may Granger-cause x_t).

Assumption 2. *The nominal stochastic discount factor is given by:*

$$\mathcal{M}^n_{t,t+1} = \exp\left[\varphi_0 + \varphi_1' w_{t+1} + \varphi_2(\mathcal{D}_{t+1} - \mathcal{D}_t)\right]. \tag{II.3}$$

Specifically, in the framework described in Section 2, we have (starting from Eq. 6):

$$\begin{aligned}
\mathcal{M}^n_{t,t+1} &= \exp(\log(\delta) - \gamma(\Delta y_{t+1} - \mu) - \mu - \pi_{t+1}) \\
&= \exp(\log(\delta) - \mu - \mu_\pi - (\gamma\sigma_y + \sigma_\pi)' w_{t+1} + (\gamma b_y + b_\pi)(\mathcal{D}_{t+1} - \mathcal{D}_t)) \\
&= \exp(\varphi_0 + \varphi_1' w_{t+1} + \varphi_2(\mathcal{D}_{t+1} - \mathcal{D}_t)),
\end{aligned} \tag{II.4}$$

which corresponds to (II.3), with $\varphi_0 = \log(\delta) - \mu - \mu_\pi$, $\varphi_1 = -(\gamma\sigma_y + \sigma_\pi)$, and $\varphi_2 = \gamma b_y + b_\pi$.

Assumption 3. *Denoting by $\mathcal{I}_t$ the information available on date t (i.e., $\mathcal{I}_t = \{x_t, x_{t-1}, \dots\}$), the probability of default is given by:*

$$\mathbb{P}(\mathcal{D}_{t+1} = 1|\mathcal{D}_t = 0, w_{t+1}, \mathcal{I}_t) = 1 - \exp(-\underbrace{\max(0, \lambda(x_{t+1}))}_{=\underline{\lambda}(x_{t+1})}),$$

with $\lambda(x_{t+1}) = a + b'x_{t+1}$, where x_t is of the form $x_t = [w_t', \bullet']'$, where $\bullet$ denotes a vector of additional variables (that may correlate to w_t).

[Note that since $\underline{\lambda}(x_{t+1})$ is assumed to be a function of w_{t+1} and of $\mathcal{I}_t$, vector b can only load on those entries of x_{t+1} that correspond to w_{t+1}, as well as on those components of x_{t+1} that were determined before date t, for instance d_t.]

II.2. Auxiliary propositions and lemmas

This subsection presents propositions and lemmas that underlie our pricing formulas.

Proposition 5. *Under Assumptions 1 and 3, we have:*

$$\begin{aligned} & \mathbb{E}(f(x_{t+1}, \dots, x_{t+h})(1 - \mathcal{D}_{t+h})|\mathcal{D}_t = 0, \mathcal{I}_t) \\ = \; & \mathbb{E}(f(\mathbf{x}_{t+1}, \dots, \mathbf{x}_{t+h}) \exp(-\underline{\lambda}(\mathbf{x}_{t+1}) - \dots - \underline{\lambda}(\mathbf{x}_{t+h}))|\mathcal{D}_t = 0, \mathcal{I}_t), \end{aligned}$$

where $\mathcal{I}_t$ denotes the information available on date t, i.e., $\mathcal{I}_t = \{x_t, x_{t-1}, \dots\}$.

Proof. For any variable ω_t, let us use the following notation: $\underline{\omega_t} = \{\omega_t, \omega_{t-1}, \dots\}$. We have:

$$\begin{aligned} & \mathbb{E}(f(x_{t+1}, \dots, x_{t+h})(1 - \mathcal{D}_{t+h})|\mathcal{D}_t = 0, \mathcal{I}_t) \\ = \; & \mathbb{E}(f(\mathbf{x}_{t+1}, \dots, \mathbf{x}_{t+h})(1 - \mathcal{D}_{t+h})|\mathcal{D}_t = 0, \mathcal{I}_t) \\ = \; & \mathbb{E}(\mathbb{E}(f(\mathbf{x}_{t+1}, \dots, \mathbf{x}_{t+h})(1 - \mathcal{D}_{t+h})|\mathcal{D}_{t+h-1}, \mathcal{D}_t = 0, \mathcal{I}_{t+h-1})|\mathcal{D}_t = 0, \mathcal{I}_t) \\ = \; & \mathbb{E}(\mathbb{E}(f(\mathbf{x}_{t+1}, \dots, \mathbf{x}_{t+h})(1 - \mathcal{D}_{t+h})|\mathcal{D}_{t+h-1} = 0, \mathcal{D}_t = 0, \mathcal{I}_{t+h-1})|\mathcal{D}_t = 0, \mathcal{I}_t) \\ = \; & \mathbb{E}(\mathbb{E}(f(\mathbf{x}_{t+1}, \dots, \mathbf{x}_{t+h})(1 - \mathcal{D}_{t+h})|\mathcal{D}_{t+h-1} = 0, \mathbf{x}_{t+h}, \mathcal{D}_t = 0, \mathcal{I}_{t+h-1})|\mathcal{D}_t = 0, \mathcal{I}_t) \\ = \; & \mathbb{E}(f(\mathbf{x}_{t+1}, \dots, \mathbf{x}_{t+h})(1 - \mathcal{D}_{t+h-1}) \exp(-\underline{\lambda}(\mathbf{x}_{t+h}))|\mathcal{D}_t = 0, \mathcal{I}_t) \\ = \; & \mathbb{E}(\mathbb{E}(f(\mathbf{x}_{t+1}, \dots, \mathbf{x}_{t+h})(1 - \mathcal{D}_{t+h-1}) \exp(-\underline{\lambda}(\mathbf{x}_{t+h}))|\mathcal{D}_{t+h-2}, \mathcal{D}_t = 0, \underline{\mathbf{x}_{t+h}}, \mathcal{I}_t)|\mathcal{D}_t = 0, \mathcal{I}_t) \\ = \; & \mathbb{E}(\mathbb{E}(f(\mathbf{x}_{t+1}, \dots, \mathbf{x}_{t+h})(1 - \mathcal{D}_{t+h-1}) \exp(-\underline{\lambda}(\mathbf{x}_{t+h}))|\mathcal{D}_{t+h-2} = 0, \mathcal{D}_t = 0, \underline{\mathbf{x}_{t+h}}, \mathcal{I}_t)|\mathcal{D}_t = 0, \mathcal{I}_t) \\ = \; & \mathbb{E}(f(\mathbf{x}_{t+1}, \dots, \mathbf{x}_{t+h})(1 - \mathcal{D}_{t+h-2}) \exp(-\underline{\lambda}(\mathbf{x}_{t+h-1}) - \underline{\lambda}(\mathbf{x}_{t+h}))|\mathcal{D}_t = 0, \mathcal{I}_t), \end{aligned}$$

where the last equality results from the fact that, since $\mathcal{D}_t$ does not Granger-cause $\mathbf{x}_t$, the distribution of $\mathcal{D}_t$ conditional on $\mathcal{D}_{t-1}$ and $\underline{\mathbf{x}_{t+h}}$ is the same as that of $\mathcal{D}_t$ conditional on $\mathcal{D}_{t-1}$ and $\underline{\mathbf{x}_t}$ (due to the equivalence between Sims' and Granger's causalities).

Using the same type of conditioning in a backward fashion (progressively replacing $1-\mathcal{D}_{t+k}$ by $\exp(-\underline{\lambda}(\mathbf{x}_{t+k})))$ leads to the result. $\square$

Lemma 1. *Consider the following Gaussian VAR:*

$$\mathbf{x}_t = \mu_x + \Phi_x \mathbf{x}_{t-1} + \Sigma_x \varepsilon_t,$$

where $\varepsilon_t \sim$ i.i.d. $\mathcal{N}(0, Id)$. The conditional expectation:

$$K_{t,n} \quad \equiv \quad \mathbb{E}_t\left[\exp(-\dot{b}'(\mathbf{x}_{t+1}+\cdots+\mathbf{x}_{t+n}) - (\underline{\lambda}_{t+1}+\cdots+\underline{\lambda}_{t+n})\right], \tag{II.5}$$

with $\underline{\lambda}_{t+1} = \max(0, \lambda_t)$ and $\lambda_t = a + b'\mathbf{x}_t$, can be approximated as follows:

$$\mathcal{K}_{t,n}(a, b, \dot{b}) \approx \exp(F_{0,1} + \cdots + F_{n-1,n}),$$

where

$$\begin{aligned} F_{n-1,n,t} \quad = \quad & \dot{b}'\mu_{t,n} + \Phi\left(\mu_{\lambda,t,n}/\sigma_{\lambda,n}\right)\mu_{\lambda,t,n} + \phi(-\mu_{\lambda,t,n}/\sigma_{\lambda,n})\sigma_{\lambda,n} \\ & -\frac{1}{2}\left(p_{t,n}\left[\dot{b}+b\right]'\Gamma_{n,0}\left[\dot{b}+b\right] + \left[1-p_{t,n}\right]\dot{b}'\Gamma_{n,0}\dot{b}\right) \\ & -\sum_{j=1}^{n-1}\left\{p_{t,n-j}\left[\dot{b}+b\right]'\Gamma_{n,j}\left[\dot{b}+b\right] + \left[1-p_{t,n-j}\right]\dot{b}'\Gamma_{n,j}\dot{b}\right\}, \end{aligned} \tag{II.6}$$

where:

- *the $\mu_{t,n}$'s and $\Gamma_{n,j}$'s are given by:*

$$\left\{\begin{array}{lllll} \mu_{t,n} & = & \mathbb{E}_t(\mathbf{x}_{t+n}) & = & (I-\Phi_x)^{-1}(I-\Phi_x^n)\mu_x + \Phi_x^n \mathbf{x}_t, \\ \Gamma_{n,0} & = & \mathbb{V}ar_t(\mathbf{x}_{t+n}) & = & \Sigma_x\Sigma_x' + \Phi_x\Gamma_{n-1,0}\Phi_x', \quad with \quad \Gamma_{1,0} = \Sigma_x\Sigma_x' \\ & & & = & \Sigma_x\Sigma_x' + \Phi_x\Sigma_x\Sigma_x'\Phi_x' + \cdots + \Phi_x^{n-1}\Sigma_x\Sigma_x'\Phi_x^{n-1\prime}, \\ \Gamma_{n,j} & = & \mathbb{C}ov_t(\mathbf{x}_{t+n}, \mathbf{x}_{t+n-j}) & = & \Phi_x^j\Gamma_{n-j,0} \quad if \quad n-j>0, \end{array}\right.$$

- *and*

$$\begin{aligned} \mu_{\lambda,t,n} &= \mathbb{E}_t(\lambda_{t+n}) = a + b'\mu_{t,n} \\ \sigma_{\lambda,n} &= \sqrt{\mathbb{V}ar_t(\lambda_{t+n})} = \sqrt{b'\Gamma_{n,0}b} \\ p_{t,n} &= \mathbb{P}_t(\lambda_{t+n} > 0) = \Phi\left(\frac{\mu_{\lambda,t,n}}{\sigma_{\lambda,n}}\right). \end{aligned}$$

Proof. Using the notation:

$$f_{n-1,n} \quad = \quad -\log K_{t,n} + \log K_{t,n-1}, \tag{II.7}$$

we have:

$$K_{t,n} = \exp(f_{0,1} + \cdots + f_{n-1,n}). \tag{II.8}$$

Following Wu and Xia (2016), we will approximate $K_{t,n}$ by, first, determining approximations to the $f_{h-1,h}$'s (that will be denoted by $F_{h-1,h}$), and, second, substituting for the $f_{h-1,h}$s' into (II.8).

Using, in (II.5), that $\log \mathbb{E}[\exp(Z)] \approx \mathbb{E}(Z) + 1/2\mathbb{V}ar(Z)$ for any random variable Z (the approximation being exact in the Gaussian case), and substituting for $K_{t,n}$ and $K_{t,n-1}$ in (II.7) yields:

$$\begin{aligned} f_{n-1,n} &= -\log K_{t,n} + \log K_{t,n-1} \\ &\approx \mathbb{E}_t(-\dot{b}'\mathbf{x}_{t+n} + \underline{\lambda}_{t+n}) \\ &\quad -\frac{1}{2}\mathbb{V}ar_t\left(-\dot{b}'\mathbf{x}_{t+n} + \underline{\lambda}_{t+n}\right) - \mathbb{C}ov_t\left(-\dot{b}'\mathbf{x}_{t+n} + \underline{\lambda}_{t+n}, \sum_{i=1}^{n-1}(-\dot{b}'\mathbf{x}_{t+i} + \underline{\lambda}_{t+i})\right). \end{aligned} \tag{II.9}$$

As in Wu and Xia (2016), we use, for $0 < n$ and $0 \leq j \leq n$:

$$\mathbb{C}ov_t(-\dot{b}'\mathbf{x}_{t+n}, \underline{\lambda}_{t+n-j}) \approx p_{t,n-j}\mathbb{C}ov_t\left[-\dot{b}'\mathbf{x}_{t+n}, \lambda_{t+n-j}\right], \tag{II.10}$$

$$\mathbb{C}ov_t(\underline{\lambda}_{t+n}, \underline{\lambda}_{t+n-j}) \approx p_{t,n-j}\mathbb{C}ov_t\left[\lambda_{t+n}, \lambda_{t+n-j}\right]. \tag{II.11}$$

Using the last two equations, we obtain an approximation to (II.9):

$$\begin{aligned} f_{n-1,n,t} \approx\ & \mathbb{E}_t\left[-\dot{b}'\mathbf{x}_{t+n} + \underline{\lambda}_{t+n}\right] \\ & -\frac{1}{2}\left(p_{t,n}\mathbb{V}ar_t\left[-\dot{b}'\mathbf{x}_{t+n} + \lambda_{t+n}\right] + (1-p_{t,n})\mathbb{V}ar_t(-\dot{b}'\mathbf{x}_{t+n})\right) \\ & -\sum_{j=1}^{n-1}\left\{p_{t,j}\mathbb{C}ov_t\left[-\dot{b}'\mathbf{x}_{t+n} + \lambda_{t+n}, -\dot{b}'\mathbf{x}_{t+j} + \lambda_{t+j}\right] + (1-p_{t,j})\mathbb{C}ov_t(-\dot{b}'\mathbf{x}_{t+n}, -\dot{b}'\mathbf{x}_{t+j})\right\}, \end{aligned} \tag{II.12}$$

which leads to the result (denoting by $F_{n-1,n,t}$ the right-hand-side term of the previous equation). □

Lemma 1 in practice. The estimation of our model involves a large number of computations of the $\Gamma_{n,j}$'s. In order to speed up the computation, one can employ the following approach.

Consider a vector κ of dimension n_x, that is the dimension of $\mathbf{x}_t$, and let us denote by ξ_i^κ the vector defined by $\xi_i^\kappa = (\Phi_x^i)'\kappa$ (κ will typically be b, or $(b + \dot{b})$, see Eq. II.6).

Because we have $\Gamma_{n,j} = \Phi_x^j\Omega + \Phi_x^{j+1}\Omega\Phi_x' + \cdots + \Phi_x^{n-1}\Omega\Phi_x^{n-1-j'}$, it comes that:

$$\kappa'\Gamma_{n,j}\kappa = \xi_j^{\kappa\prime}\Omega\xi_0^\kappa + \xi_{j+1}^{\kappa}{}'\Omega\xi_1^\kappa + \cdots + \xi_{n-1}^{\kappa}{}'\Omega\xi_{n-1-j}^\kappa. \tag{II.13}$$

Let us consider a maximal value for n, say H, and let us denote by Ξ_κ the $n_x \times (H+1)$ matrix whose i^{th} column is ξ_{i-1}^κ. It can then be seen that the (j,k) entry of $\Psi^\kappa := \Xi_\kappa{}'\Omega\Xi_\kappa$—which is a matrix of dimension $(H+1)\times(H+1)$—is equal to $\xi_{j-1}^{\kappa}{}'\Omega\xi_{k-1}^\kappa$. The sum of the entries $(j+1,1)$, $(j+2,2)$, ...,

$(j+k,k)$ of Ψ^κ therefore is

$$\xi_j^{\kappa\prime}\Omega\xi_0^\kappa + \xi_{j+1}^{\kappa}{}'\Omega\xi_1^\kappa + \dots + \xi_{j+k-1}^{\kappa}{}'\Omega\xi_{k-1}^\kappa,$$

which is equal to $\kappa'\Gamma_{j+k,j}\kappa$ according to (II.13). Equivalently, $\kappa'\Gamma_{n,j}\kappa$ is the sum of the entries $(j+1,1)$, $(j+2,2),\dots,(n,n-j)$ of Ψ^κ.

In particular, the entry $(1,1)$ of Ψ^κ is equal to $\kappa'\Gamma_{1,0}\kappa$, the sum of the entries $(1,1)$ and $(2,2)$ is equal to $\kappa'\Omega\kappa + \kappa'\Phi_x\Omega\Phi_x'\kappa = \kappa'\Gamma_{2,0}\kappa$, and, more generally, the sum of the entries $(1,1),\dots,(n-1,n-1)$ of Ψ^κ is equal to $\kappa'\Gamma_{n,0}\kappa$.

Lemma 2. *If w_t follows the following Gaussian VAR model:*

$$w_t = \Phi_w w_{t-1} + \Sigma_w \varepsilon_t,$$

where $\varepsilon_t \sim$ i.i.d. $\mathcal{N}(0, Id)$, we have:

$$\mathcal{L}_{t,h}(u) := \mathbb{E}_t\left[\exp\{u'(w_{t+1}+\dots+w_{t+h})\}\right] = \exp(\mathcal{A}_h(u) + \mathcal{B}_h(u)'w_t), \tag{II.14}$$

where functions $\mathcal{A}_h(\bullet)$ and $\mathcal{B}_h(\bullet)$ satisfy the following recursive equations:

$$\left\{\begin{array}{lcl} \mathcal{B}_h(u) &=& \Phi'(\mathcal{B}_{h-1}(u)+u) \\ \mathcal{A}_h(u) &=& \mathcal{A}_{h-1}(u) + \frac{1}{2}(\mathcal{B}_{h-1}(u)+u)'\Sigma_w\Sigma_w(\mathcal{B}_{h-1}(u)+u), \end{array}\right.$$

with $\mathcal{A}_0(u)=0$ and $\mathcal{B}_0(u)=0$.

Proof. If $\mathbb{E}_t\left[\exp\{u'(w_{t+1}+\dots+w_{t+h-1})\}\right] = \exp(\mathcal{A}_{h-1}(u) + \mathcal{B}_{h-1}(u)'w_t)$ holds for any vector u, then:

$$\begin{array}{cl} & \mathbb{E}_t\left[\exp\{u'(w_{t+1}+\dots+w_{t+h})\}\right] \\ = & \mathbb{E}_t\left[\exp\{u'w_{t+1}\}\mathbb{E}_{t+1}\left[\exp\{u'(w_{t+2}+\dots+w_{t+h})\}\right]\right] \\ = & \mathbb{E}_t[\exp\{u'w_{t+1} + \mathcal{A}_{h-1}(u) + \mathcal{B}_{h-1}(u)'w_{t+1}\}] \quad \text{(using the recursive assumption)} \\ = & \mathbb{E}_t[\exp\{(\mathcal{B}_{h-1}(u)+u)'w_{t+1} + \mathcal{A}_{h-1}(u)\}] \\ = & \mathbb{E}_t\left[\exp\left\{\mathcal{A}_{h-1}(u) + [\Phi'(\mathcal{B}_{h-1}(u)+u)]'w_t + \dfrac{1}{2}(\mathcal{B}_{h-1}(u)+u)'\Sigma_w\Sigma_w(\mathcal{B}_{h-1}(u)+u)\right\}\right], \end{array}$$

where the last equality results from w_t's law of motion. □

II.3. Pricing formulas

This subsection provides formulas to price zero-coupon bonds issued by the government (Proposition 6), perpetuities (Proposition 7), CDSs (Propositions 8 for the general formula, and 9 for its numeric application), and risk-free bonds (Proposition 10).

Proposition 6. *Under Assumptions 1 and 2, and if* $RR = \exp(-\varphi_2)$*, then the date-t price of a generic zero-coupon bond providing the nominal payoff* $1-(1-RR)\mathcal{D}_{t+h}$ *on date* $t+h$ *is:*

$$\mathcal{B}_{t,h} = \exp[A_h(\varphi_0,\varphi_1) + B_h(\varphi_0,\varphi_1)'w_t],$$

where functions $A_h(\bullet)$ *and* $B_h(\bullet)$ *satisfy the following recursive equations:*

$$\left\{\begin{array}{lcl} B_h(v,u) & = & \Phi'(B_{h-1}(v,u)+u) \\ A_h(v,u) & = & v + A_{h-1}(u) + \frac{1}{2}(B_{h-1}(u)+u)'\Sigma_w\Sigma_w'(B_{h-1}(u)+u), \end{array}\right. \tag{II.15}$$

with $A_0(u) = 0$ *and* $B_0(u) = 0$.

Proof. We have:

$$\begin{array}{rcl} \mathcal{B}_{t,h} & = & \mathbb{E}_t\Big\{\mathcal{M}^n_{t,t+1}\times\cdots\times\mathcal{M}^n_{t+h-1,t+h}[(1-(1-RR)\mathcal{D}_{t+h}]\Big\} \\ & = & \mathbb{E}_t\Big\{\exp[h\varphi_0+\varphi_1'(w_{t+1}+\cdots+w_{t+h})+\varphi_2\mathcal{D}_{t+h}][(1-(1-RR)\mathcal{D}_{t+h}]\Big\} \\ & = & \mathbb{E}_t\Big\{\exp[h\varphi_0+\varphi_1'(w_{t+1}+\cdots+w_{t+h})]\{(1-[1-RR\exp(\varphi_2)]\mathcal{D}_{t+h}\}\Big\}. \end{array}$$

If $RR = \exp(-\varphi_2)$, we therefore obtain $\mathcal{B}_{t,h} = \mathbb{E}_t\Big\{\exp[h\varphi_0+\varphi_1'(w_{t+1}+\cdots+w_{t+h})]\Big\}$. The recursive equations (II.15) then result from Lemma 2. □

Definition 1. *We define a defaultable decaying-coupon perpetuity as an infinitely-lived asset providing the following payoff on date* $t+h$*:*

$$\chi^{h-1}(1-(1-RR)\mathcal{D}_{t+h}).$$

The date-t price of this perpetuity is, therefore, $\mathcal{P}_t := \sum_{h=1}^{\infty}\chi^{h-1}\mathcal{B}_{t,h}$*, where* $\mathcal{B}_{t,h}$ *is the date-t price of a generic zero-coupon bond providing the nominal payoff* $1-(1-RR)\mathcal{D}_{t+h}$ *on date* $t+h$.

By definition, the yield-to-maturity of the perpetuity, denoted by q_t*, satisfies:*

$$\mathcal{P}_t = \sum_{h=1}^{\infty}\frac{\chi^{h-1}}{(1+q_t)^h} = \frac{1}{1+q_t-\chi}. \tag{II.16}$$

Proposition 7. *Under Assumptions 1 and 2, and if* $RR = \exp(-\varphi_2)$*, the yield-to-maturity* q_t *of the defaultable decaying-coupon perpetuity (see Definition 1) can be approximated as follows:*

$$q_t \approx a_H(\varphi_0,\varphi_1) + b_H(\varphi_0,\varphi_1)'w_t, \tag{II.17}$$

with $a_H(\bullet) = -\frac{1}{H}A_H(\bullet)$ *and* $b_H(\bullet) = -\frac{1}{H}B_H(\bullet)$*, where functions* A_H *and* B_H *are defined in Proposition 6, and where the pair* (χ, H) *satisfies:*

$$H \approx \frac{1}{1+\bar{q}-\chi} + \frac{\mathbb{V}ar(q)}{(1+\bar{q}-\chi)^3} + \frac{3\mathbb{V}ar(q)^2}{(1+\bar{q}-\chi)^5}, \tag{II.18}$$

with

$$\mathbb{V}ar(q) = b_H(\varphi)'\mathbb{V}ar(w)b_H(\varphi) \quad \textit{and} \quad vec[\mathbb{V}ar(w)] = (I - \Phi_w \otimes \Phi_w)vec(\Sigma_w \Sigma_w'). \tag{II.19}$$

Proof. Because the perpetuity is a collection of zero-coupons of price $\mathcal{B}_{t,h}$ (with geometrically-decaying weights, see Definition 1), the yield-to-maturity of the perpetuity is expected to be close to the yield of an "average" zero-coupon, that is to one of the $r_{t,h}$'s, where $r_{t,h} = -1/h \log \mathcal{B}_{t,h}$. A natural candidate for h is the average debt maturity (i.e., the average duration of the perpetuities), which we denote by H. According to Proposition 6, under Assumptions 1 and 2, we have (for any h, but in particular for $h = H$):

$$r_{t,h} = -\frac{1}{h}A_h(\varphi_1) - \frac{1}{h}B_h(\varphi_1)'w_t,$$

which gives (II.17).

Since the duration of the perpetuity is equal to its price, and if we want H to be, on average, equal to the duration of the perpetuity, we should have:

$$H \approx \mathbb{E}\left(\frac{1}{1-\chi+q_t}\right). \tag{II.20}$$

Using a fourth-order Taylor expansion of q_t around its mean $\bar{q}$ leads to:

$$H \approx \frac{1}{1+\bar{q}-\chi} + \frac{\mathbb{V}ar(q)}{(1+\bar{q}-\chi)^3} + \frac{Skew(q)\mathbb{V}ar(q)^{3/2}}{(1+\bar{q}-\chi)^4} + \frac{Kurt(q)\mathbb{V}ar(q)^2}{(1+\bar{q}-\chi)^5},$$

where $\bar{q} = \mathbb{E}(q_t)$. Since, under Assumption 1, w_t follows a Gaussian VAR, and since q_t approximately linearly depends on w_t, it comes that $Skew(q) \approx 0$ and $Kurt(q) \approx 3$, which leads to (II.18). The variances given in Eq. (II.19) directly result from (II.17) and (II.1). □

Definition 2. *In a CDS contract, a protection buyer pays a regular premium to a protection seller. These payments end either after a given period of time—the maturity of the CDS, that we denote by h—or upon default of the reference entity. Upon the default of the debtor (a third party), the protection seller compensates the protection buyer for the loss incurred, assuming the latter was holding defaulted bonds.*

Following the "Recovery of Treasury" (RT) convention of Duffie and Singleton (1999), we assume that the bond recovery payment, upon default, is a fraction RR of the price of a risk-free zero-coupon bond of equivalent residual maturity. Accordingly, if t is the inception date of a maturity-h CDS:

- *The amount paid on date $t+k$ (with $0 < k \le h$) by the protection seller to the protection buyer is:*

  $$(\mathcal{D}_{t+k} - \mathcal{D}_{t+k-1})(1-RR)\mathbb{E}_{t+k}(\mathcal{M}^n_{t+k,h-k}),$$

  where $\mathbb{E}_{t+k}(\mathcal{M}^n_{t+k,h-k})$ is the price, as of date $t+k$, of a nominal risk-free bond of residual maturity $h-k$.

- *On date $t+k$, the protection buyer pays $S^{cds}_{t,h}(1-\mathcal{D}_{t+k})$ to the protection seller, where $S^{cds}_{t,h}$ denotes the CDS premium—as negotiated on date t—expressed in percentage of the notional.*

At inception of the CDS contract (date t), there is no cash-flow exchanged between both parties; that is, the CDS spread $S^{cds}_{t,h}$ is determined so as to equalize the present discounted values of the payments promised by each of them. Therefore:

$$\underbrace{\mathbb{E}_t\left\{\sum_{k=1}^{h}\mathcal{M}^n_{t,t+k}(\mathcal{D}_{t+k}-\mathcal{D}_{t+k-1})(1-RR)\mathbb{E}_{t+k}(\mathcal{M}^n_{t+k,h-k})\right\}}_{\textit{Protection leg}} = \underbrace{S^{cds}_{t,h}\mathbb{E}_t\left\{\sum_{k=1}^{h}\mathcal{M}^n_{t,t+k}(1-\mathcal{D}_{t+k})\right\}}_{\textit{Premium leg}}. \quad \text{(II.21)}$$

Proposition 8. *Consider the CDS presented in Definition 2. If $\mathcal{D}_t = 0$ (i.e., the reference entity has not defaulted before date t), then the CDS premium ($S^{cds}_{t,h}$) satisfies:*

$$S^{cds}_{t,h} = (1-RR)\frac{\mathbb{E}_t\left\{\mathcal{M}^n_{t,t+h}\mathcal{D}_{t+h}\right\}}{\mathbb{E}_t\left\{\sum_{k=1}^{h}\mathcal{M}^n_{t,t+k}(1-\mathcal{D}_{t+k})\right\}}. \quad \text{(II.22)}$$

Proof. The date-t value of the protection leg is:

$$\begin{aligned}
& \mathbb{E}_t\left\{\sum_{k=1}^{h}\mathcal{M}^n_{t,t+k}(\mathcal{D}_{t+k}-\mathcal{D}_{t+k-1})(1-RR)\mathbb{E}_{t+k}(\mathcal{M}^n_{t+k,t+h})\right\}\\
= & \mathbb{E}_t\left\{\sum_{k=1}^{h}\mathcal{M}^n_{t,t+h}(\mathcal{D}_{t+k}-\mathcal{D}_{t+k-1})(1-RR)\right\}\\
= & (1-RR)\mathbb{E}_t\left\{\mathcal{M}^n_{t,t+h}(\mathcal{D}_{t+h}-\mathcal{D}_t)\right\},
\end{aligned}$$

where we have used $\mathcal{M}^n_{t,t+h} = \mathcal{M}^n_{t,t+k}\mathcal{M}^n_{t+k,t+h}$, as well as the law of iterated expectations.

Using the previous expression in (II.21) leads to the result. □

A consequence of Proposition 8 is that the computation of the CDS spread $S^{cds}_{t,h}$ necessitates the knowledge of the following two conditional expectations: $\mathbb{E}_t[\mathcal{M}^n_{t,t+h}\mathcal{D}_{t+h-1}]$ and $\mathbb{E}_t[\mathcal{M}^n_{t,t+h}(1-\mathcal{D}_{t+h})]$, which can be seen as "binary CDSs," in the sense that they correspond to date-t prices of instruments providing a binary payoff (0 or 1) depending on the default status of the government on date $t+h$.

The following proposition explains how to approximate these conditional expectations.

Proposition 9. *We suppose that Assumptions 1, 2, and 3 hold. We introduce the vector $\tilde{\varphi}_1$ that is such that $\varphi_1' w_t = \tilde{\varphi}_1' x_t$. (That is, $\tilde{\varphi}_1$ is of the form $[\varphi_1', \mathbf{0}']'$.)*

If the reference entity has not defaulted before date t (i.e., $\mathcal{D}_t = 0$), the CDS premium ($S^{cds}_{t,h}$, as defined in Definition 2) can be approximated as follows:

$$S^{cds}_{t,h} \approx (1-RR)\frac{\exp(h\varphi_0+\varphi_2)[\mathcal{K}_{t,h}(0,0,-\tilde{\varphi}_1)-\mathcal{K}_{t,h}(a,b,-\tilde{\varphi}_1)]}{\sum_{k=1}^{h}\exp(k\varphi_0)\mathcal{K}_{t,k}(a,b,-\tilde{\varphi}_1)},$$

where function $\mathcal{K}_{t,h}$ is defined in Lemma 1.

Proof. According to Proposition 8, the computation of the CDS spread $S^{cds}_{t,h}$ necessitates the knowledge of the following two conditional expectations: $\mathbb{E}_t[\mathcal{M}^n_{t,t+h}\mathcal{D}_{t+h-1}]$ and $\mathbb{E}_t[\mathcal{M}^n_{t,t+h}(1-\mathcal{D}_{t+h})]$. We start with the computation of $\mathbb{E}_t[\mathcal{M}^n_{t,t+h}(1-\mathcal{D}_{t+h})]$. Since $\mathcal{D}_t = 0$, we have:[1]

$$\begin{aligned}
& \mathbb{E}_t\left[\mathcal{M}^n_{t,t+1}\times\cdots\times\mathcal{M}^n_{t+h-1,t+h}(1-\mathcal{D}_{t+h})\right] \\
= \ & \exp\left(h\varphi_0\right)\mathbb{E}_t\left[\exp\{\varphi_1'(w_{t+1}+\cdots+w_{t+h})+\varphi_2\mathcal{D}_{t+h}\}(1-\mathcal{D}_{t+h})\right] \\
= \ & \exp(h\varphi_0)\mathbb{E}_t\left[\exp\{\varphi_1'(w_{t+1}+\cdots+w_{t+h})\}\mathbb{1}_{\{\mathcal{D}_{t+h}=0\}}\right] \\
= \ & \exp(h\varphi_0)\mathbb{E}_t\left[\mathbb{E}_t\left(\exp\{\varphi_1'(w_{t+1}+\cdots+w_{t+h})\}\mathbb{1}_{\{\mathcal{D}_{t+h}=0\}}|\underline{x_{t+h}}\right)\right] \\
= \ & \exp(h\varphi_0)\mathbb{E}_t\left[\exp\{\varphi_1'(w_{t+1}+\cdots+w_{t+h})-\underline{\lambda}_{t+1}-\cdots-\underline{\lambda}_{t+h}\}\right],
\end{aligned} \tag{II.23}$$

where the last equality results from Proposition 5. Lemma 1 gives:

$$\mathbb{E}_t[\mathcal{M}^n_{t,t+h}(1-\mathcal{D}_{t+h})] \approx \exp(h\varphi_0)\mathcal{K}_{t,h}(a,b,-\tilde{\varphi}_1). \tag{II.24}$$

We then turn to the computation of $\mathbb{E}_t\left[\mathcal{M}^n_{t,t+h}\mathcal{D}_{t+h}\right]$. We have:

$$\begin{aligned}
& \mathbb{E}_t\left[\mathcal{M}^n_{t,t+1}\times\cdots\times\mathcal{M}^n_{t+h-1,t+h}\mathcal{D}_{t+h}\right] \\
= \ & \exp(h\varphi_0)\mathbb{E}_t\left[\exp\{\varphi_1'(w_{t+1}+\cdots+w_{t+h})+\varphi_2\mathcal{D}_{t+h}\}\mathcal{D}_{t+h}\right] \\
= \ & \exp(h\varphi_0+\varphi_2)\mathbb{E}_t\left[\exp\{\varphi_1'(w_{t+1}+\cdots+w_{t+h})\}\mathbb{1}_{\{\mathcal{D}_{t+h}=1\}}\right] \\
= \ & \exp(h\varphi_0+\varphi_2)\mathbb{E}_t\left[\exp\{\varphi_1'(w_{t+1}+\cdots+w_{t+h})\}\left(1-\mathbb{1}_{\{\mathcal{D}_{t+h}=0\}}\right)\right] \\
= \ & \exp(h\varphi_0+\varphi_2)\mathbb{E}_t\left[\exp\{\varphi_1'(w_{t+1}+\cdots+w_{t+h})\}\right] \\
& -\exp(h\varphi_0+\varphi_2)\mathbb{E}_t\left[\exp\{\varphi_1'(w_{t+1}+\cdots+w_{t+h})-\underline{\lambda}_{t+1}-\cdots-\underline{\lambda}_{t+h}\}\right],
\end{aligned} \tag{II.25}$$

where we have made use of Proposition 5. Lemma 1 gives:

$$\mathbb{E}_t\left[\mathcal{M}^n_{t,t+h}\mathcal{D}_{t+h}\right] = \exp(h\varphi_0+\varphi_2)[\mathcal{K}_{t,h}(0,0,-\tilde{\varphi}_1)-\mathcal{K}_{t,h}(a,b,-\tilde{\varphi}_1)]. \tag{II.26}$$

Using Eqs. (II.24) and (II.26) in Eq. (II.22) leads to the result. □

Proposition 10. *We suppose that Assumptions 1, 2, and 3 hold. We introduce the vector $\tilde{\varphi}_1$ that is such that $\varphi_1'w_t = \tilde{\varphi}_1'x_t$. (That is, $\tilde{\varphi}_1$ is of the form $[\varphi_1', \mathbf{0}']'$.)*

The price of a risk-free zero-coupon bond of maturity h is given by:

$$\mathbb{E}_t\left[\mathcal{M}^n_{t,t+1}\times\cdots\times\mathcal{M}^n_{t+h-1,t+h}\right] = \exp(h\varphi_0)(1-\exp(\varphi_2))\mathcal{K}_{t,n}(a,b,-\tilde{\varphi}_1), \tag{II.27}$$

[1]Let us recall the following notation: $\underline{x_t} = \{x_t, x_{t-1},\ldots\}$.

where function $\mathcal{K}_{t,n}$ is defined in Lemma 1.

Proof. We have:

$$
\begin{aligned}
& \mathbb{E}_t\left[\mathcal{M}^n_{t,t+1}\times\cdots\times\mathcal{M}^n_{t+h-1,t+h}\right] \\
= \; & \exp(h\varphi_0)\mathbb{E}_t\left[\exp\{\tilde{\varphi}_1'(x_{t+1}+\cdots+x_{t+h})+\varphi_2\mathcal{D}_{t+h}\}\right] \\
= \; & \exp(h\varphi_0)\mathbb{E}_t\left[\exp\{\tilde{\varphi}_1'(x_{t+1}+\cdots+x_{t+h})\}\left(\exp(\varphi_2)+\mathbb{1}_{\{\mathcal{D}_{t+h}=0\}}(1-\exp(\varphi_2))\right)\right] \\
= \; & \exp(h\varphi_0+\varphi_2)\mathbb{E}_t\left[\exp\{\tilde{\varphi}_1'(x_{t+1}+\cdots+x_{t+h})\}\right]+ \\
& \exp(h\varphi_0)(1-\exp(\varphi_2))\mathbb{E}_t\left[\exp\{\tilde{\varphi}_1'(x_{t+1}+\cdots+x_{t+h})-\underline{\lambda}_{t+1}-\cdots-\underline{\lambda}_{t+h}\}\right],
\end{aligned}
\tag{II.28}
$$

where the last equality results from Proposition 5. □

Proposition 11. *Under Assumptions 2 and 3, and if $\mathcal{D}_{t-1}=0$, the risk-neutral default intensity is given by:*

$$
\underline{\lambda}_t^{\mathbb{Q}} = \underline{\lambda}_t + \log(\exp(\varphi_2)\{1-\exp(-\underline{\lambda}_t)\}+\exp(-\underline{\lambda}_t)). \tag{II.29}
$$

Note: the physical and risk-neutral default probabilities, $\underline{\lambda}_t$ and $\underline{\lambda}_t^{\mathbb{Q}}$, are respectively defined by

$$
\begin{aligned}
\exp(-\underline{\lambda}_t) &= \mathbb{P}(\mathcal{D}_t=0|\mathcal{D}_{t-1}=0,\mathcal{I}_{t-1},w_t) \\
\exp(-\underline{\lambda}_t^{\mathbb{Q}}) &= \mathbb{Q}(\mathcal{D}_t=0|\mathcal{D}_{t-1}=0,\mathcal{I}_{t-1},w_t),
\end{aligned}
$$

where the risk-neutral measure $\mathbb{Q}$ (from date $t-1$ to date t) is defined with respect to the physical one through the Radon-Nikodym derivative $\mathcal{M}^n_{t-1,t}/\mathbb{E}(\mathcal{M}^n_{t-1,t}|\mathcal{I}_{t-1})$.

Proof. On each date t, the representative agent observes the new information $X_t=\{\mathcal{D}_t,w_t\}$; the total agent's information then is $\mathcal{I}_t=\{X_t,X_{t-1},\dots\}$. By Bayes, we have:

$$
f^{\mathbb{Q}}(\mathcal{D}_t|w_t,\mathcal{I}_{t-1}) = \frac{f^{\mathbb{Q}}(\mathcal{D}_t,w_t|\mathcal{I}_{t-1})}{f^{\mathbb{Q}}(w_t|\mathcal{I}_{t-1})}. \tag{II.30}
$$

Under Assumption 2, we have $\mathcal{M}^n_{t,t+1}=\exp\left(\varphi_0+\varphi_1'w_{t+1}+\varphi_2(\mathcal{D}_{t+1}\right)$. Assume $\mathcal{D}_{t-1}=0$. We have:

$$
\begin{aligned}
f^{\mathbb{Q}}(\mathcal{D}_t,w_t|\mathcal{I}_{t-1}) &= \frac{\mathcal{M}^n_{t-1,t}}{\mathbb{E}(\mathcal{M}^n_{t-1,t}|\mathcal{I}_{t-1})}f^{\mathbb{P}}(\mathcal{D}_t,w_t|\mathcal{I}_{t-1}) \\
&= \frac{\exp(\varphi_1'w_t+\varphi_2\mathcal{D}_t)}{\mathbb{E}[\exp(\varphi_1'w_t+\varphi_2\mathcal{D}_t)|\mathcal{I}_{t-1}]}f^{\mathbb{P}}(\mathcal{D}_t,w_t|\mathcal{I}_{t-1}) \\
&= \frac{\exp(\varphi_1'w_t+\varphi_2\mathcal{D}_t)}{\mathbb{E}[\exp(\varphi_1'w_t+\varphi_2\mathcal{D}_t)|\mathcal{I}_{t-1}]}f^{\mathbb{P}}(\mathcal{D}_t|w_t,\mathcal{I}_{t-1})f^{\mathbb{P}}(w_t|w_{t-1}) \\
&= \frac{\exp(\varphi_1'w_t+\varphi_2\mathcal{D}_t)}{\mathbb{E}[\exp(\varphi_1'w_t+\varphi_2\mathcal{D}_t)|\mathcal{I}_{t-1}]}\times \\
&\quad \left(\mathcal{D}_t\{1-\exp(-\underline{\lambda}_t)\}+(1-\mathcal{D}_t)\{\exp(-\underline{\lambda}_t)\}\right)f^{\mathbb{P}}(w_t|w_{t-1}).
\end{aligned}
\tag{II.31}
$$

Integrating both sides w.r.t. $\mathcal{D}_t$, we obtain:

$$f^{\mathbb{Q}}(w_t|\mathcal{I}_{t-1}) = \exp\left(\varphi_1' w_t\right) \frac{\exp(\varphi_2)\{1-\exp(-\underline{\lambda}_t)\} + \exp(-\underline{\lambda}_t)}{\mathbb{E}[\exp(\varphi_1' w_t + \varphi_2 \mathcal{D}_t)|\mathcal{I}_{t-1}]} f^{\mathbb{P}}(w_t|w_{t-1}). \tag{II.32}$$

Using (II.31) and (II.32)in (II.30) leads to:

$$f^{\mathbb{Q}}(\mathcal{D}_t|w_t, \mathcal{I}_{t-1}) = \frac{\exp(\varphi_2\mathcal{D}_t)\left(\mathcal{D}_t\{1-\exp(-\underline{\lambda}_t)\} + (1-\mathcal{D}_t)\{\exp(-\underline{\lambda}_t)\}\right)}{\exp(\varphi_2)\{1-\exp(-\underline{\lambda}_t)\} + \exp(-\underline{\lambda}_t)},$$

which implies:

$$\exp(-\underline{\lambda}_t^{\mathbb{Q}}) \equiv \mathbb{Q}(\mathcal{D}_t = 0|\mathcal{D}_{t-1} = 0, w_t, \mathcal{I}_{t-1}) = \frac{\exp(-\underline{\lambda}_t)}{\exp(\varphi_2)\{1-\exp(-\underline{\lambda}_t)\} + \exp(-\underline{\lambda}_t)},$$

which gives (II.29). □

II.4. Performance of the approximate pricing formulas in the stylized model

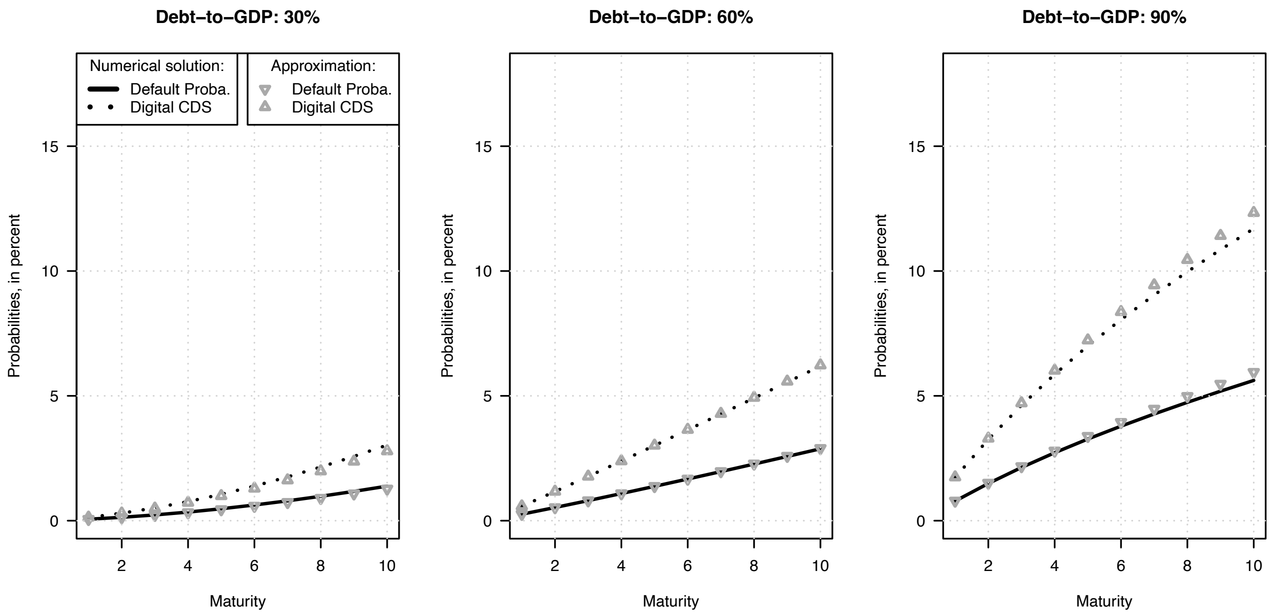
This appendix illustrates the quality of the approximate formulas presented in Section II. For that, it compares physical and risk-neutral probabilities of default (that can be obtained numerically in the context of the stylized model described in Subsections 2.1 to 2.3 of the paper) with those resulting from the approximate formulas (that are valid when $RR = \exp(-\gamma b_y)$).

The risk-neutral probabilities of default can be interpreted as prices of digital Credit Default Swaps, defined as a forward contract providing $\mathcal{D}_{t+h}$ on date $t+h$, with payment deferred to date $t+h$. The price of such a contract is given by:[2] $\mathbb{E}_t^{\mathbb{Q}^h}(\mathcal{D}_{t+h}) = \mathbb{E}_t\left(\mathcal{M}_{t,t+h}\mathcal{D}_{t+h}\right)/\mathbb{E}_t\left(\mathcal{M}_{t,t+h}\right)$. (It is easily checked that $\mathbb{E}_t^{\mathbb{Q}^h}(\mathcal{D}_{t+h}) = \mathbb{E}_t(\mathcal{D}_{t+h})$ when $\gamma = 0$.)

Figure B.8 presents the term structures of default probabilities for three different values of the debt-to-GDP ratio d_t, namely 30%, 60%, and 90%. We take $d_{t-1} = d_t$, $r_t = 2\%$, and we use the same calibration as the one underlying Figures 1 and 2.

[2] $\mathbb{Q}^h$ denotes the h-forward risk-neutral measure, that is, the measure whose Radon-Nikodym derivative with respect to the physical distribution is $\mathcal{M}_{t,t+h}/\mathbb{E}_t\left(\mathcal{M}_{t,t+h}\right)$.

Figure B.8: Probabilities of default and digital CDSs



Note: This figure shows term structures of physical and risk-neutral default probabilities in the context of the stylized model described in Subsection 2.3. Triangles are based on approximate formulas given in Section II. We use the same calibration as the one underlying Figures 1 and 2, that is: : $\gamma = 4$, $\mu = 2\%$, $\delta = 0.99$, $b = 0.2$, $\chi = 0.7$, $\beta = 0.1$, $d^* = 0.6$, $\sqrt{\mathbb{V}ar(\eta_t)} = 4\%$, $\alpha = 0.5$, and $s^* = 3\%$.

III. Panel regressions

Tables C.3 shows the results of panel regressions where fiscal space is accounted for by its first lag, the economic policy uncertainty (EPU) index (Baker, Bloom, and Davis, 2016), the assets held by the national central bank and the MOVE index to capture bond market volatility.[3,4]

Using two specifications for the regression model (with country fixed effects and in first difference), we obtain significant estimates for the three covariates, especially for the EPU index. As expected, fiscal space negatively comoves with economic policy uncertainty and bond market volatility, while it positively relates to the size of the central bank's balance sheet. Comparable findings arise if we run our panel regressions on subsets of countries (advanced and emerging economies).

In Table C.4, we report results from regressing the fourth latent factor $w_{4,t}$— that is the factor driving surplus threshold— on the same set of covariates. We reach similar conclusions, with the exception of advanced economies for which the results are less robust for the EPU index.

Data for the EPU index are drawn from the Economic Policy Uncertainty database at the monthly frequency and, then, aggregated at the quarterly frequency.[5] The Merrill Lynch MOVE index and the assets held by the central bank are taken from Refinitiv Eikon Datastream.

[3]The EPU index is constructed from three types of underlying components: (i) newspaper coverage of policy-related economic uncertainty, (ii) the number of tax code provisions set to expire in future years, (iii) disagreement among economic forecasters as a proxy for uncertainty. The Japanese and Chinese EPU indexes are respectively obtained from Arbatli Saxegaard, Davis, Ito, and Miake (2022) and Davis, Liu, and Sheng (2019).

[4]The MOVE index measures options-implied volatility in the US bond market. It represents the fixed-income counterparty of the VIX index (30-day expected volatility of the US stock market), which is usually taken as a proxy for global volatility in stock markets.

[5]The database can be publicly accessed at the following link: https://www.policyuncertainty.com/.

Table C.3: Fiscal space - Panel regression results

	(a) All countries		*(b) Advanced Economies*		*(c) Emerging Economies*	
	Country FE	FD	Country FE	FD	Country FE	FD
	FS_t	FS_t	FS_t	FS_t	FS_t	FS_t
FS_{t-1}	0.787^{***}		0.814^{***}		0.664^{***}	
	(0.046)		(0.043)		(0.048)	
EPU_t	-1.123^{***}	-1.008^{*}	-1.801^{***}	-2.627^{***}	-0.566^{***}	0.056
	(0.253)	(0.600)	(0.390)	(0.681)	(0.195)	(0.345)
$CBAssets_t$	3.660	10.961^{*}	6.440^{*}	11.443^{*}	5.294	13.576^{*}
	(4.598)	(5.657)	(3.710)	(6.058)	(8.475)	(7.744)
$MOVE_t$	-7.700^{***}	-5.746^{**}	-10.011^{***}	-5.544	-6.076^{***}	-6.161^{***}
	(2.077)	(2.526)	(2.655)	(3.563)	(1.790)	(1.980)

Note: This table reports the results of panel regressions of fiscal space (FS) estimates on the Economic Policy Uncertainty (EPU), Central Bank assets (Δ%) and the ICE BofAML MOVE Index (MOVE). The estimation sample goes from 2004Q1 to 2022Q3. FE stands for Fixed Effects, FD for first difference. We employ two-way clustering for the standard errors (country and quarter). See text for more details. *p<0.1; **p<0.05; ***p<0.01

Table C.4: Latent factor ($w_{4,t}$) - Panel regression results

	(a) All countries		*(b) Advanced Economies*		*(c) Emerging Economies*	
	Country FE	FD	Country FE	FD	Country FE	FD
	$w_{4,t}$	$w_{4,t}$	$w_{4,t}$	$w_{4,t}$	$w_{4,t}$	$w_{4,t}$
$w_{4,t-1}$	0.937^{***}		0.961^{***}		0.920^{***}	
	(0.025)		(0.010)		(0.036)	
EPU_t	-0.056^{**}	-0.059^{**}	-0.025	-0.051	-0.095^{**}	-0.037
	(0.027)	(0.029)	(0.039)	(0.069)	(0.041)	(0.025)
$CBAssets_t$	11.187^{*}	11.868^{**}	12.222^{**}	7.823^{***}	4.181	26.731^{**}
	(5.894)	(4.970)	(5.743)	(2.839)	(11.964)	(10.773)
$MOVE_t$	-46.479^{***}	-42.531^{**}	-42.517^{**}	-13.654	-50.638^{**}	-69.093^{***}
	(17.631)	(19.077)	(17.378)	(13.473)	(22.118)	(21.237)

Note: This table reports the results of panel regressions of the latent factor estimates $w_{4,t}$ on the Economic Policy Uncertainty (EPU), Central Bank assets (Δ%) and the ICE BofAML MOVE Index (MOVE). The estimation sample goes from 2004Q1 to 2022Q3. FE stands for Fixed Effects, FD for first difference. We employ two-way clustering for the standard errors (country and quarter). See text for more details. *p<0.1; **p<0.05; ***p<0.01

IV. Data

IV.1. Overview

We consider eight countries: Brazil, Canada, China, India, Japan, Russia, the United Kingdom, and the United States. According to the IMF sovereign debt investor bases for advanced economies and emerging markets, more than three-quarters of sovereign debts are held domestically in these eight countries (IMF sovereign debt investor bases build on Arslanalp and Tsuda, 2014). The fact that most bondholders are domestic helps to make the data consistent with our closed-economy framework.

Estimation samples vary across countries due to data availability (CDS prices being the main limiting factor); on average, they cover the last 13 years, at the quarterly frequency. We use government yields of three maturities (2, 5, and 10 years), and CDS spreads of 5 maturities (1, 2, 3, 5, and 10 years). CDS spreads and bond yields are extracted from CMA and Refinitiv Eikon Datastream, respectively.

The macroeconomic variables (GDP growth, inflation based on the GDP deflator, debt, budget surplus, and interest payments) are extracted from Refinitiv Eikon Datastream but come from different sources. Whenever possible we prefer data drawn from official national sources or international organization (e.g., OECD) datasets. Further country-specific details are provided below and in Tables D.6- D.13.

We augment the set of macroeconomic variables with forecasts extracted from past vintages of IMF World Economic Outlook forecasts. This is to ensure that our model is able to replicate, as much as possible, the trajectories of debt and growth as they were expected at different points in time. Forecasts from the IMF WEO are bi-annual, except for 2020, in which projections in the April round were limited. Details on the time span of forecasts at the country level are provided in Tables D.6- D.13.

IV.2. Country-specific details

US (Table D.6)

GDP at constant and current prices, and the GDP deflator are taken from the Bureau of Economic Analysis. The same goes for personal consumption expenditure for non-durables and services at constant prices. Series for the public debt outstanding and the budget balance are drawn from the Bureau of the Fiscal Service, while interest payments are taken from the Bureau of Economic Analysis. Due to the availability of CDS prices, the sample for the US goes from 2008Q1 to 2022Q3.

UK (Table D.7)

GDP at market constant prices and current prices, the GDP deflator, and final private consumption expenditure for services and non-durables are drawn from the Office for National Statistics. General government debt at nominal values is collected from the Bank for International Settlements. The series for general government interest payments is taken from the IMF - International Finance Statistics database, while the primary surplus/deficit is drawn from the Office for National Statistics. Due to the availability of CDS prices, the sample for the UK goes from 2008Q1 to 2022Q3.

Japan (Table D.8)

GDP at market constant prices and current prices, final private consumption expenditure for services and non-durables are drawn from the Cabinet Office database (Government of Japan). The GDP deflator series is provided by Refinitiv. National government debt for Japan is drawn from the Bank of Japan. Gross government interest payments and government primary balance are taken from the OECD. The sample for Japan goes from 2004Q1 to 2022Q3.

Canada (Table D.9)

GDP at market constant prices and current prices, the GDP deflator, final private consumption expenditure for services and non-durables are drawn from the Canadian Socio-Economic Information Management System (Statistics Canada). General government debt at nominal values is taken from the Bank for International Settlements, while the series for interest payments and budget balance are drawn from Canadian Socio-Economic Information Management System (Statistics Canada). Due to the availability of CDS prices, the sample for Canada goes from 2012Q4 to 2022Q3.

Brazil (Table D.10)

GDP at market constant prices and current prices, the GDP deflator, and final private consumption expenditure are drawn from OECD databases. Public finance statistics (general government debt, primary deficit/surplus and nominal interest payments) are collected from Banco Central do Brasil. The sample for Brazil goes from 2006Q3 to 2022Q3.

China (Table D.11)

GDP at constant market prices and the implicit price deflator are taken from the National Bureau of Statistics of China. GDP at current prices is drawn from the OECD Quarterly National Accounts. Private consumption, primary balance, and gross interest payments are taken from Oxford Economics. Last, general government debt at nominal values is drawn from the Bank for International Settlements. The sample for China goes from 2004Q1 to 2022Q3.

India (Table D.12)

GDP at constant market and current prices, GDP deflator and private final consumption expenditure are drawn from OECD Quarterly National Accounts. General government debt at nominal values is taken from the Bank for International Settlements. Series for central government interest payments and primary surplus/deficit are drawn from the Controller General of Accounts. Given that sovereign CDSs for India have been traded only for a few years, we proxy for these CDSs from

2004Q4 until 2013Q4 with those written on the State Bank of India, which is common practice (see, e.g., de Boyrie and Pavlova, 2016). The sample for India goes from 2004Q4 to 2022Q3.

Russia (Table D.13)

GDP at constant market prices is drawn from the Federal State Statistics Service. GDP at current prices is taken from the IMF International Finance Statistics database and, given that the latter was discontinued, from the Federal State Statistics Service. General government debt at nominal values is drawn from the Bank for International Settlements. Federal government interest payments and primary surplus/deficit series are taken from the Ministry of Finance of the Russian Federation. Final private consumption expenditure and the implicit GDP deflator are collected from the OECD. The sample for Russia goes from 2005Q3 to 2021Q3.

IV.3. Average debt maturities

The parametrization of the model involves H, the duration of the perpetuity (which, in turn, is used to determine χ, the decay rate of the coupons, using Proposition 7). Parameter H is calibrated so as to match the average debt maturities of the sovereign debts. Table D.5 reports the values used in the present study.

Table D.5: Average debt maturities

Country	**Avg debt maturity**	**Source**
Brazil	4	IMF Art. IV/country report (2021)[a]
Canada	6	IMF Art. IV/country report (2022)[b]
China	6	IMF Art. IV/country report (2022)[c]
India	11.7	BIS (2022)[d]
Japan	9.0	Financial Bureau, Ministry of Finance (2020)[e]
Russia	6.2	BIS (2013)[f]
United Kingdom	14.7	BIS (2022)[d]
United States of America	5.7	BIS (2022)[d]

[a]: Table p. 51 (between 3.5 and 4.8 years).
[b]: Annex I - Figure 3.
[c]: Appendix III - Figure 3.
[d]: BIS (2022): `https://www.bis.org/statistics/secstats.htm`.
[e]: Figure 1-13.
[f]: BIS (2013): `https://www.bis.org/publ/qtrpdf/r_qa1303_anx17c.pdf`.

Table D.6: Data Panel: United States of America

Variable	Horizon / Maturity	Source	Period	N. of Obs.
Nominal GDP Forecasts	1 Years	IMF WEO[a]	04/2008-10/2022	29
	2 Years	IMF WEO	04/2008-10/2022	29
	3 Years	IMF WEO	04/2008-10/2022	29
	5 Years	IMF WEO	04/2008-10/2022	29
	10 Years	IMF WEO	04/2008-10/2022	29
Inflation Forecasts (based on GDP deflator)	1 Years	IMF WEO	04/2008-10/2022	29
	2 Years	IMF WEO	04/2008-10/2022	29
	3 Years	IMF WEO	04/2008-10/2022	29
	5 Years	IMF WEO	04/2008-10/2022	29
	10 Years	IMF WEO	04/2008-10/2022	29
Government Debt	1 Years	IMF WEO	04/2008-10/2022	29
	2 Years	IMF WEO	10/2009-10/2022	26
	3 Years	IMF WEO	10/2009-10/2022	26
	5 Years	IMF WEO	10/2009-10/2022	26
	10 Years	IMF WEO	10/2009-10/2022	26
Senior CDS	1 Year	CMA	2008Q1-2021Q2	59
	2 Years	CMA	2008Q1-2022Q3	59
	3 Years	CMA	2008Q1-2022Q3	59
	5 Years	CMA	2008Q1-2022Q3	59
	10 Years	CMA	2008Q1-2022Q3	59
Yields	1 Year	Federal Reserve, US	2008Q1-2021Q2	54
	2 Years	Federal Reserve, US	2008Q1-2022Q3	59
	3 Years	Federal Reserve, US	2008Q1-2022Q3	59
	5 Years	Federal Reserve, US	2008Q1-2022Q3	59
	10 Years	Federal Reserve, US	2008Q1-2022Q3	59
GDP, market constant prices (CHND 2012)	-	Bureau of Economic Analysis	2008Q1-2022Q3	59
GDP, market current prices	-	Bureau of Economic Analysis	2008Q1-2022Q3	59
Final Consumption Expenditure, Services	-	Bureau of Economic Analysis	2008Q1-2022Q3	59
Final Consumption Expenditure, Non-Durables	-	Bureau of Economic Analysis	2008Q1-2022Q3	59
GDP Implicit Price Deflator (Index 2012=100)	-	Bureau of Economic Analysis	2008Q1-2022Q3	59
Gross Federal Government Debt, Current Prices	-	Bureau of the Fiscal Service	2008Q1-2022Q3	59
Government Interest Payments, Current Prices	-	Bureau of Economic Analysis	2008Q1-2022Q3	59
Government Budget Balance, Current Prices	-	Bureau of the Fiscal Service	2008Q1-2022Q3	59

[a] IMF WEO: International Monetary Fund World Economic Outlook.

Table D.7: Data Panel: United Kingdom

Variable	Horizon / Maturity	Source	Period	N. of Obs.
Nominal GDP Forecasts	1 Years	IMF WEO[a]	04/2008-10/2022	29
	2 Years	IMF WEO	04/2008-10/2022	29
	3 Years	IMF WEO	04/2008-10/2022	29
	5 Years	IMF WEO	04/2008-10/2022	29
	10 Years	IMF WEO	04/2008-10/2022	29
Inflation Forecasts (based on GDP deflator)	1 Years	IMF WEO	04/2008-10/2022	29
	2 Years	IMF WEO	04/2008-10/2022	29
	3 Years	IMF WEO	04/2008-10/2022	29
	5 Years	IMF WEO	04/2008-10/2022	29
	10 Years	IMF WEO	04/2008-10/2022	29
Government Debt	1 Years	IMF WEO	04/2008-10/2022	29
	2 Years	IMF WEO	10/2009-10/2022	26
	3 Years	IMF WEO	10/2009-10/2022	26
	5 Years	IMF WEO	10/2009-10/2022	26
	10 Years	IMF WEO	10/2009-10/2022	26
Senior CDS	1 Year	CMA	2008Q1-2022Q3	59
	2 Years	CMA	2008Q1-2022Q3	59
	3 Years	CMA	2008Q1-2022Q3	59
	5 Years	CMA	2008Q1-2022Q3	59
	10 Years	CMA	2008Q1-2022Q3	59
Yields	1 Year	ICAP - Refinitiv Eikon Datastream	2008Q1-2022Q3	59
	2 Years	ICAP - Refinitiv Eikon Datastream	2008Q1-2022Q3	59
	3 Years	ICAP - Refinitiv Eikon Datastream	2008Q1-2022Q3	59
	5 Years	ICAP - Refinitiv Eikon Datastream	2008Q1-2022Q3	59
	10 Years	ICAP - Refinitiv Eikon Datastream	2008Q1-2022Q3	59
GDP, market constant prices (2019 prices)	-	Office for National Statistics	2008Q1-2022Q3	59
GDP, market current prices	-	Office for National Statistics	2008Q1-2022Q3	59
Final Consumption Expenditure, Services (2019 prices)	-	Office for National Statistics	2008Q1-2022Q3	59
Final Consumption Expenditure, Non-Durables (2019 prices)	-	Office for National Statistics	2008Q1-2022Q3	59
GDP Implicit Price Deflator (Index 2019=100)	-	Office for National Statistics	2008Q1-2022Q3	59
General Government Debt, nominal value	-	Bank for International Settlements	2008Q1-2022Q3	59
General Government Interest Payments, Current Prices	-	IMF - International Financial Statistics	2008Q1-2022Q3	59
Government primary surplus/deficit, Current Prices	-	Office for National Statistics	2008Q1-2022Q3	59

[a] International Monetary Fund - World Economic Outlook.

Table D.8: Data Panel: Japan

Variable	**Horizon / Maturity**	**Source**	**Period**	**N. degree of Obs.**
	1 Years	IMF WEO[a]	04/2004-10/2022	37
	2 Years	IMF WEO	04/2008-10/2022	29
Nominal GDP Forecasts	3 Years	IMF WEO	04/2008-10/2022	29
	5 Years	IMF WEO	04/2008-10/2022	29
	10 Years	IMF WEO	04/2008-10/2022	29
	1 Years	IMF WEO	04/2004-10/2022	37
	2 Years	IMF WEO	04/2008-10/2022	29
Inflation Forecasts (based on GDP deflator)	3 Years	IMF WEO	04/2008-10/2022	29
	5 Years	IMF WEO	04/2008-10/2022	29
	10 Years	IMF WEO	04/2008-10/2022	29
	1 Years	IMF WEO	04/2004-10/2022	37
	2 Years	IMF WEO	10/2009-10/2022	26
Government Debt	3 Years	IMF WEO	10/2009-10/2022	26
	5 Years	IMF WEO	10/2009-10/2022	26
	10 Years	IMF WEO	10/2009-10/2022	26
	1 Year	CMA	2004Q1-2022Q3	75
	2 Years	CMA	2004Q1-2022Q3	75
Senior CDS	3 Years	CMA	2004Q1-2022Q3	75
	5 Years	CMA	2004Q1-2022Q3	75
	10 Years	CMA	2004Q1-2022Q3	75
	1 Year	ICAP - Refinitiv Eikon Datastream	2004Q1-2022Q3	75
	2 Years	ICAP - Refinitiv Eikon Datastream	2004Q1-2022Q3	75
Yields	3 Years	ICAP - Refinitiv Eikon Datastream	2004Q1-2022Q3	75
	5 Years	ICAP - Refinitiv Eikon Datastream	2004Q1-2022Q3	75
	10 Years	ICAP - Refinitiv Eikon Datastream	2004Q1-2022Q3	75
GDP, market constant prices (CHND 2015)	-	Cabinet Office (Gov. of Japan)	2004Q1-2022Q3	75
GDP, market current prices	-	Cabinet Office (Gov. of Japan)	2004Q1-2022Q3	75
Final Consumption Expenditure, Services (CHND 2015)	-	Cabinet Office (Gov. of Japan)	2004Q1-2022Q3	75
Final Consumption Expenditure, Non-Durables (CHND 2015)	-	Cabinet Office (Gov. of Japan)	2004Q1-2022Q3	75
GDP Implicit Price Deflator (Index 2015=100)	-	Refinitiv Eikon Datastream	2004Q1-2022Q3	75
National Government Debt, Total, Current Prices	-	Bank of Japan	2004Q1-2022Q3	75
Gross Government Interest Payments, Current Prices	-	OECD Economic Outlook	2004Q1-2022Q3	75
Government Primary Balance, Current Prices	-	OECD Economic Outlook	2004Q1-2022Q3	75

[a] International Monetary Fund - World Economic Outlook.

Table D.9: Data Panel: Canada

Variable	Horizon / Maturity	Source	Period	N. of Obs.
Nominal GDP Forecasts	1 Years	IMF WEO[a]	10/2012-10/2022	20
	2 Years	IMF WEO	10/2012-10/2022	20
	3 Years	IMF WEO	10/2012-10/2022	20
	5 Years	IMF WEO	10/2012-10/2022	20
	10 Years	IMF WEO	10/2012-10/2022	20
Inflation Forecasts (based on GDP deflator)	1 Years	IMF WEO	10/2012-10/2022	20
	2 Years	IMF WEO	10/2012-10/2022	20
	3 Years	IMF WEO	10/2012-10/2022	20
	5 Years	IMF WEO	10/2012-10/2022	20
	10 Years	IMF WEO	10/2012-10/2022	20
Government Debt	1 Years	IMF WEO	10/2012-10/2022	20
	2 Years	IMF WEO	10/2012-10/2022	20
	3 Years	IMF WEO	10/2012-10/2022	20
	5 Years	IMF WEO	10/2012-10/2022	20
	10 Years	IMF WEO	10/2012-10/2022	20
Senior CDS	1 Year	Refinitiv Eikon Datastream	2012Q4-2022Q3	40
	2 Years	Refinitiv Eikon Datastream	2012Q4-2022Q3	40
	3 Years	Refinitiv Eikon Datastream	2012Q4-2022Q3	40
	5 Years	Refinitiv Eikon Datastream	2012Q4-2022Q3	40
	10 Years	Refinitiv Eikon Datastream	2012Q4-2022Q3	40
Yields	1 Year	Refinitiv Eikon Datastream	2012Q4-2022Q3	40
	2 Years	Refinitiv Eikon Datastream	2012Q4-2022Q3	40
	3 Years	Refinitiv Eikon Datastream	2012Q4-2022Q3	40
	5 Years	Refinitiv Eikon Datastream	2012Q4-2022Q3	40
	10 Years	Refinitiv Eikon Datastream	2012Q4-2022Q3	40
GDP, market constant prices (CHND 2012)	-	CANSIM[a]	2012Q4-2022Q3	40
GDP, market current prices	-	CANSIM	2012Q4-2022Q3	40
Final Consumption Expenditure, Services (CHND 2012)	-	CANSIM	2012Q4-2022Q3	40
Final Consumption Expenditure, Non-Durables (CHND 2012)	-	CANSIM	2012Q4-2022Q3	40
GDP Implicit Price Deflator (Index 2012=100)	-	CANSIM	2012Q4-2022Q3	40
General Government Debt, Total, nominal values	-	Bank for International Settlements	2012Q4-2022Q3	40
Gross Government Interest Payments, Current Prices	-	CANSIM	2012Q4-2022Q3	40
Government Primary Surplus/Deficit, Current Prices	-	CANSIM	2012Q4-2022Q3	40

[a] International Monetary Fund - World Economic Outlook. [b] Canadian Socio-Economic Information Management System (Statistics Canada).

Table D.10: Data Panel: Brazil

Variable	Horizon / Maturity	Source	Period	N. of Obs.
Nominal GDP Forecasts	1 Years	IMF WEO[a]	09/2006-10/2022	32
	2 Years	IMF WEO	04/2008-10/2022	29
	3 Years	IMF WEO	04/2008-10/2022	29
	5 Years	IMF WEO	04/2008-10/2022	29
	10 Years	IMF WEO	04/2008-10/2022	29
Inflation Forecasts (based on GDP deflator)	1 Years	IMF WEO	09/2006-10/2022	32
	2 Years	IMF WEO	04/2008-10/2022	29
	3 Years	IMF WEO	04/2008-10/2022	29
	5 Years	IMF WEO	04/2008-10/2022	29
	10 Years	IMF WEO	04/2008-10/2022	29
Government Debt	1 Years	IMF WEO	10/2010-10/2022	24
	2 Years	IMF WEO	10/2012-10/2022	24
	3 Years	IMF WEO	10/2012-10/2022	24
	5 Years	IMF WEO	10/2012-10/2022	24
	10 Years	IMF WEO	10/2012-10/2022	24
Senior CDS	1 Year	CMA	2006Q3-2022Q3	65
	2 Years	CMA	2006Q3-2022Q3	65
	3 Years	CMA	2006Q3-2022Q3	65
	5 Years	CMA	2006Q3-2022Q3	65
	10 Years	CMA	2006Q3-2022Q3	65
Yields	1 Year	Refinitiv Eikon Datastream	2006Q3-2022Q3	65
	2 Years	Refinitiv Eikon Datastream	2006Q3-2022Q3	65
	3 Years	Refinitiv Eikon Datastream	2006Q3-2022Q3	65
	5 Years	Refinitiv Eikon Datastream	2006Q3-2022Q3	65
	10 Years	Refinitiv Eikon Datastream	2006Q3-2022Q3	65
GDP, market constant prices (1995 prices)	-	Quarterly National Accounts, OECD	2006Q3-2022Q3	65
GDP, market current prices	-	Quarterly National Accounts, OECD	2006Q3-2022Q3	65
Final Consumption Expenditure (1995 prices)	-	Quarterly National Accounts, OECD	2006Q3-2022Q3	65
GDP Implicit Price Deflator (Index 2015=100)	-	OECD Main Economic Indicators	2006Q3-2022Q3	65
Public Debt, General Government, Gross, Domestic, Current Prices	-	Banco Central do Brasil	2006Q3-2022Q3	65
Nominal Interest Payments, Current Prices	-	Banco Central do Brasil	2006Q3-2022Q3	65
Consolidated Public Sector Primary Deficit/Surplus, Current Prices	-	Banco Central do Brasil	2006Q3-2022Q3	65

[a] International Monetary Fund - International Financial Statistics.

Table D.11: Data Panel: China

Variable	Horizon / Maturity	Source	Period	N. of Obs.
	1 Years	IMF WEO[a]	04/2004-10/2022	37
	2 Years	IMF WEO	04/2008-10/2022	29
Nominal GDP Forecasts	3 Years	IMF WEO	04/2008-10/2022	29
	5 Years	IMF WEO	04/2008-10/2022	29
	10 Years	IMF WEO	04/2008-10/2022	29
	1 Years	IMF WEO	04/2004-10/2022	37
	2 Years	IMF WEO	04/2008-10/2022	29
Inflation Forecasts (based on GDP deflator)	3 Years	IMF WEO	04/2008-10/2022	29
	5 Years	IMF WEO	04/2008-10/2022	29
	10 Years	IMF WEO	04/2008-10/2022	29
	1 Years	IMF WEO	10/2010-10/2022	24
	2 Years	IMF WEO	10/2012-10/2022	24
Government Debt	3 Years	IMF WEO	10/2012-10/2022	24
	5 Years	IMF WEO	10/2012-10/2022	24
	10 Years	IMF WEO	10/2012-10/2022	24
	1 Year	CMA	2004Q1-2022Q3	75
	2 Years	CMA	2004Q1-2022Q3	75
Senior CDS	3 Years	CMA	2004Q1-2022Q3	75
	5 Years	CMA	2004Q1-2022Q3	75
	10 Years	CMA	2004Q1-2022Q3	75
	1 Year	Refinitiv Eikon Datastream	2004Q1-2022Q3	75
	2 Years	Refinitiv Eikon Datastream	2004Q1-2022Q3	75
Yields	3 Years	Refinitiv Eikon Datastream	2004Q1-2022Q3	75
	5 Years	Refinitiv Eikon Datastream	2004Q1-2022Q3	75
	10 Years	Refinitiv Eikon Datastream	2004Q1-2022Q3	75
GDP, market constant prices (2020 prices)	-	National Bureau of Statistics of China/Refinitiv	2004Q1-2022Q3	75
GDP, current prices	-	Quarterly National Accounts, OECD	2004Q1-2022Q3	75
Private Consumption (2020 prices)	-	Oxford Economics	2004Q1-2022Q3	75
GDP Implicit Price Deflator (Index 2010=100)	-	National Bureau of Statistics of China/Refinitiv	2004Q1-2022Q3	75
General Government Debt, nominal values	-	Bank for International Settlements	2004Q1-2022Q3	75
Gross Government Interest Payments, Current Prices	-	Oxford Economics	2004Q1-2022Q3	75
General Government Primary Balance, Current Prices	-	Oxford Economics	2004Q1-2022Q3	75

[a] International Monetary Fund - International Financial Statistics.

Table D.12: Data Panel: India

Variable	Horizon / Maturity	Source	Period	N. of Obs.
	1 Years	IMF WEO[a]	04/2005-10/2022	35
	2 Years	IMF WEO	04/2008-10/2022	29
Nominal GDP Forecasts	3 Years	IMF WEO	04/2008-10/2022	29
	5 Years	IMF WEO	04/2008-10/2022	29
	10 Years	IMF WEO	04/2008-10/2022	29
	1 Years	IMF WEO	04/2005-10/2022	35
	2 Years	IMF WEO	04/2008-10/2022	29
Inflation Forecasts (based on GDP deflator)	3 Years	IMF WEO	04/2008-10/2022	29
	5 Years	IMF WEO	04/2008-10/2022	29
	10 Years	IMF WEO	04/2008-10/2022	29
	1 Years	IMF WEO	10/2010-10/2022	24
	2 Years	IMF WEO	10/2012-10/2022	24
Government Debt	3 Years	IMF WEO	10/2012-10/2022	24
	5 Years	IMF WEO	10/2012-10/2022	24
	10 Years	IMF WEO	10/2012-10/2022	24
	1 Year	CMA	2004Q4-2013Q3	36
	2 Years	CMA	2004Q4-2013Q3	36
State Bank of India CDS	3 Years	CMA	2004Q4-2013Q3	36
	5 Years	CMA	2004Q4-2013Q3	36
	10 Years	CMA	2004Q4-2013Q3	36
	1 Year	CMA	2013Q4-2022Q3	36
	2 Years	CMA	2013Q4-2022Q3	36
Senior CDS	3 Years	CMA	2013Q4-2022Q3	36
	5 Years	CMA	2013Q4-2022Q3	36
	10 Years	CMA	2013Q4-2022Q3	36
	1 Year	Refinitiv Eikon Datastream	2004Q4-2022Q3	72
	2 Years	Refinitiv Eikon Datastream	2004Q4-2022Q3	72
Yields	3 Years	Refinitiv Eikon Datastream	2004Q4-2022Q3	72
	5 Years	Refinitiv Eikon Datastream	2004Q4-2022Q3	72
	10 Years	Refinitiv Eikon Datastream	2004Q4-2022Q3	72
GDP, market constant prices (2011-2012 prices)	-	Quarterly National Accounts, OECD	2004Q4-2022Q3	72
GDP, market current prices	-	Quarterly National Accounts, OECD	2004Q4-2022Q3	72
Private Final Consumption Expenditure (2011-2012 prices)	-	Quarterly National Accounts, OECD	2004Q4-2022Q3	72
GDP Implicit Price Deflator (Index 2015=100)	-	Quarterly National Accounts, OECD	2004Q4-2022Q3	72
General Government Debt, nominal values	-	Bank for International Settlements	2004Q4-2022Q3	72
Central Government, Interest Payments, Current Prices	-	Controller General of Accounts, India	2004Q4-2022Q3	72
Central Government, Deficit/Surplus, Primary	-	Controller General of Accounts, India	2004Q4-2022Q3	72

[a] International Monetary Fund - International Financial Statistics.

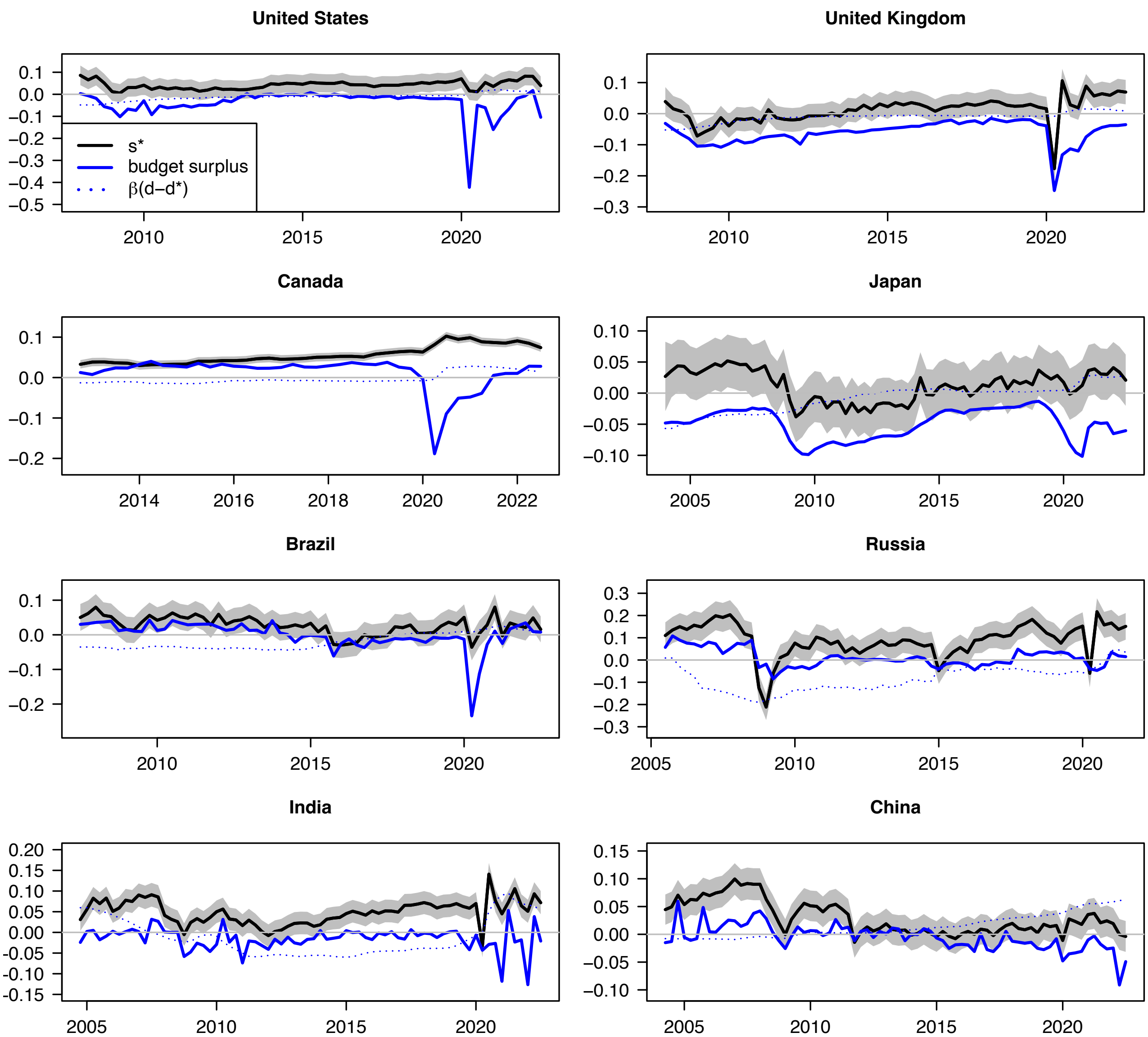
Table D.13: Data Panel: Russia

Variable	Horizon / Maturity	Source	Period	N. of Obs.
Nominal GDP Forecasts	1 Years	IMF WEO[a]	04/2005-10/2021	33
	2 Years	IMF WEO	04/2008-10/2021	27
	3 Years	IMF WEO	04/2008-10/2021	27
	5 Years	IMF WEO	04/2008-10/2021	27
	10 Years	IMF WEO	04/2008-10/2021	27
Inflation Forecasts (based on GDP deflator)	1 Years	IMF WEO	04/2005-10/2021	33
	2 Years	IMF WEO	04/2008-10/2021	27
	3 Years	IMF WEO	04/2008-10/2021	27
	5 Years	IMF WEO	04/2008-10/2021	27
	10 Years	IMF WEO	04/2008-10/2021	27
Government Debt	1 Years	IMF WEO	04/2008-10/2021	27
	2 Years	IMF WEO	04/2008-10/2021	27
	3 Years	IMF WEO	10/2012-10/2021	22
	5 Years	IMF WEO	10/2012-10/2021	22
	10 Years	IMF WEO	10/2012-10/2021	22
Senior CDS	1 Year	CMA	2005Q3-2021Q3	65
	2 Years	CMA	2005Q3-2021Q3	65
	3 Years	CMA	2005Q3-2021Q3	65
	5 Years	CMA	2005Q3-2021Q3	65
	10 Years	CMA	2005Q3-2021Q3	65
Yields	1 Year	Refinitiv Eikon Datastream	2005Q3-2021Q3	65
	2 Years	Refinitiv Eikon Datastream	2005Q3-2021Q3	65
	3 Years	Refinitiv Eikon Datastream	2005Q3-2021Q3	65
	5 Years	Refinitiv Eikon Datastream	2005Q3-2021Q3	65
	10 Years	Refinitiv Eikon Datastream	2005Q3-2021Q3	65
GDP, market constant prices (2016 prices)	-	Federal Statistics Service	2005Q3-2021Q3	65
GDP, current prices	-	IMF-IFS[b]/Federal Statistics Service	2005Q3-2021Q3	65
Final Consumption Expenditure (2016 prices)	-	Quarterly National Accounts, OECD	2005Q3-2021Q3	65
GDP Implicit Price Deflator (Index 2015=100)	-	OECD Main Economic Indicators	2005Q3-2021Q3	65
General Government Debt, nominal values	-	Bank for International Settlements	2005Q3-2021Q3	65
Federal Government Interest Payments, Current Prices	-	Ministry of Finance of the Russian Federation	2005Q3-2021Q3	65
Federal Government Primary Surplus/Deficit, Current Prices	-	Ministry of Finance of the Russian Federation	2005Q3-2021Q3	65

[b] International Monetary Fund World Economic Outlook; [b] International Monetary Fund - International Financial Statistics.

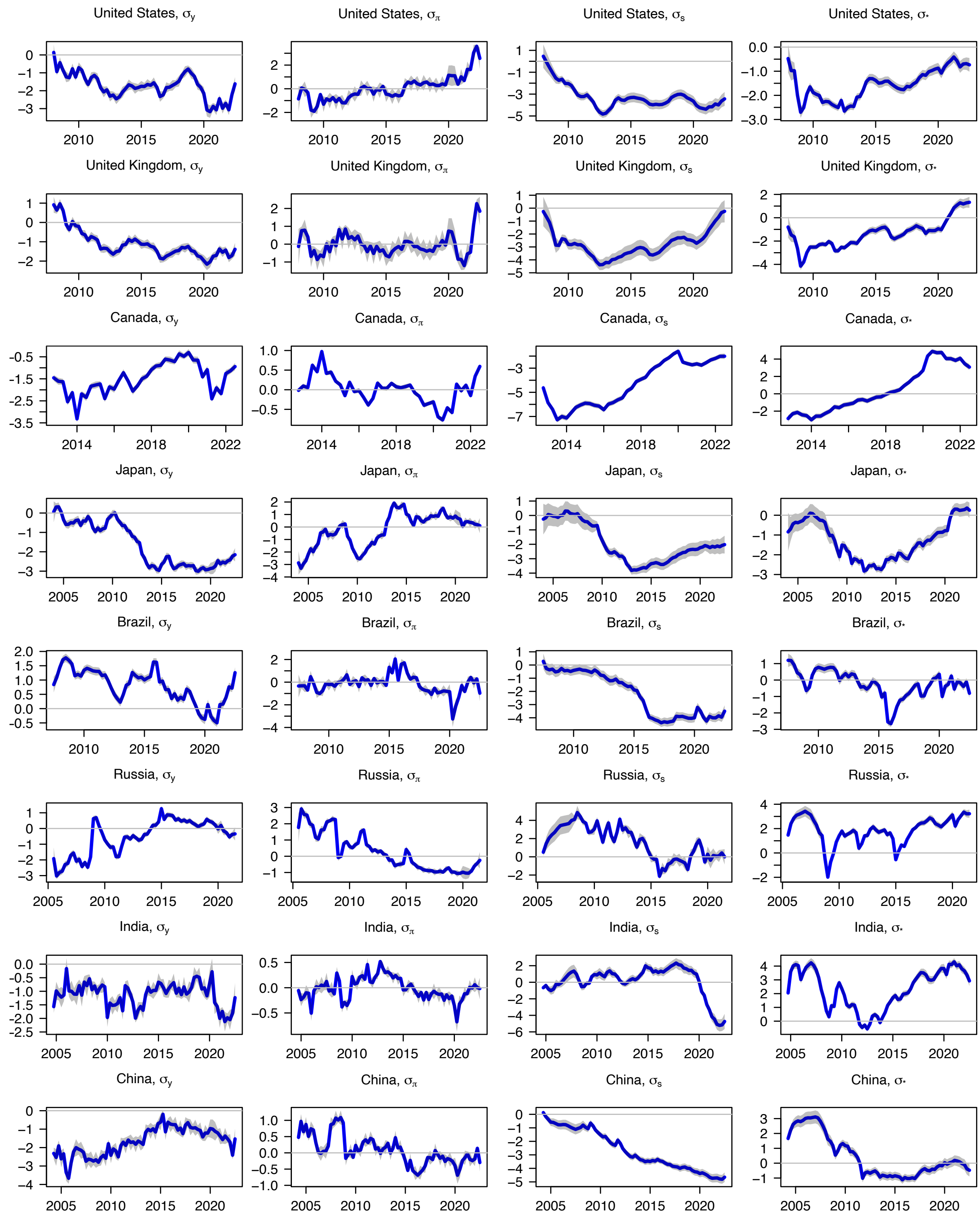
V. Additional tables and figures

Figure E.9: Surplus threshold estimates



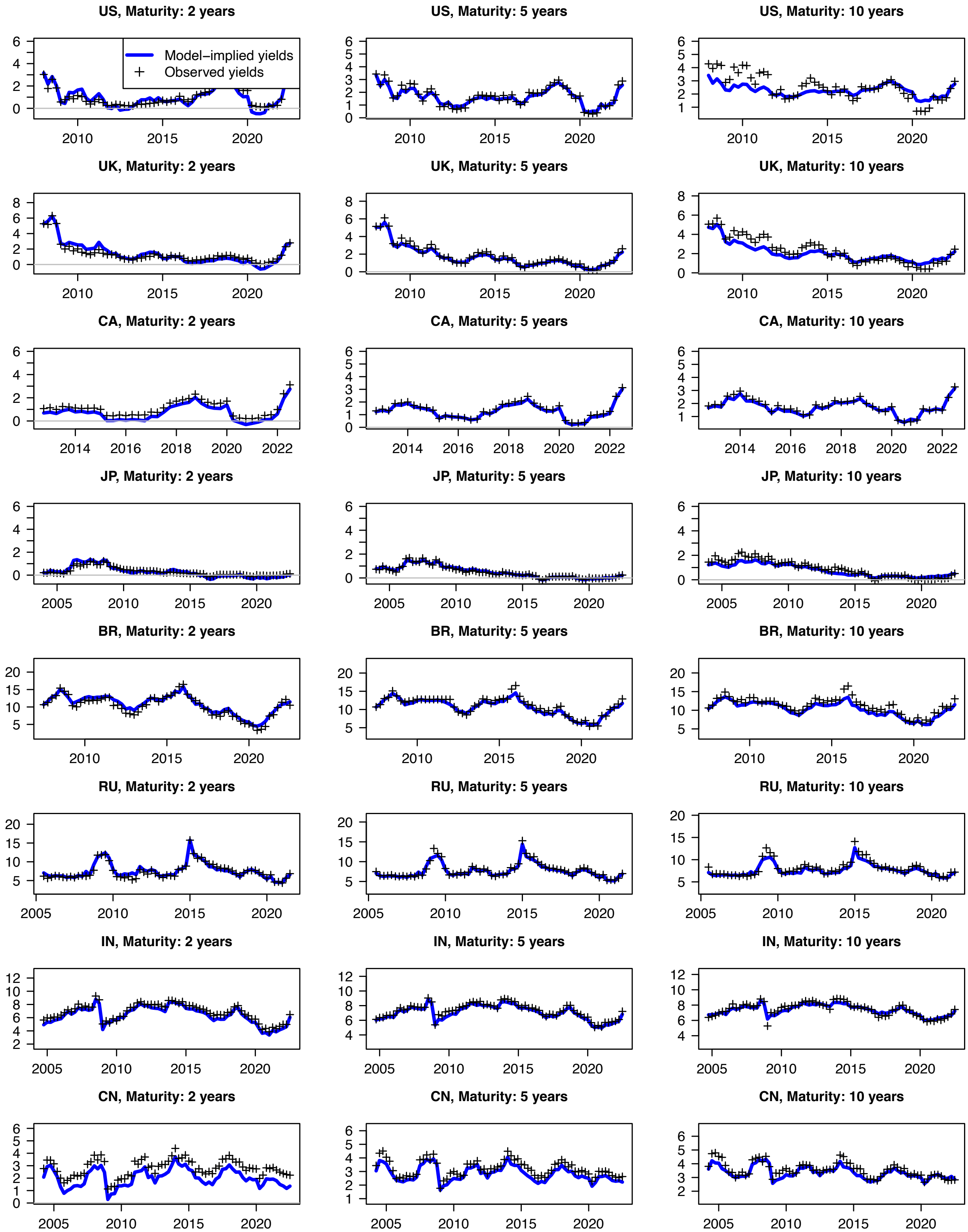
These plots show the estimates of the surplus threshold s_t^* (black solid lines), the actual budget surplus s_t (blue line), together with $\beta(d_t - d^*)$ (blue dotted lines). On each date t, the default intensity is equal to $\alpha \max(0, s_t - s_t^*)$ (see Eq. 17). The shaded area surrounding s_t^* indicates the 95% confidence interval (accounting for Kalman-smoothing uncertainty).

Figure E.10: Estimated factors



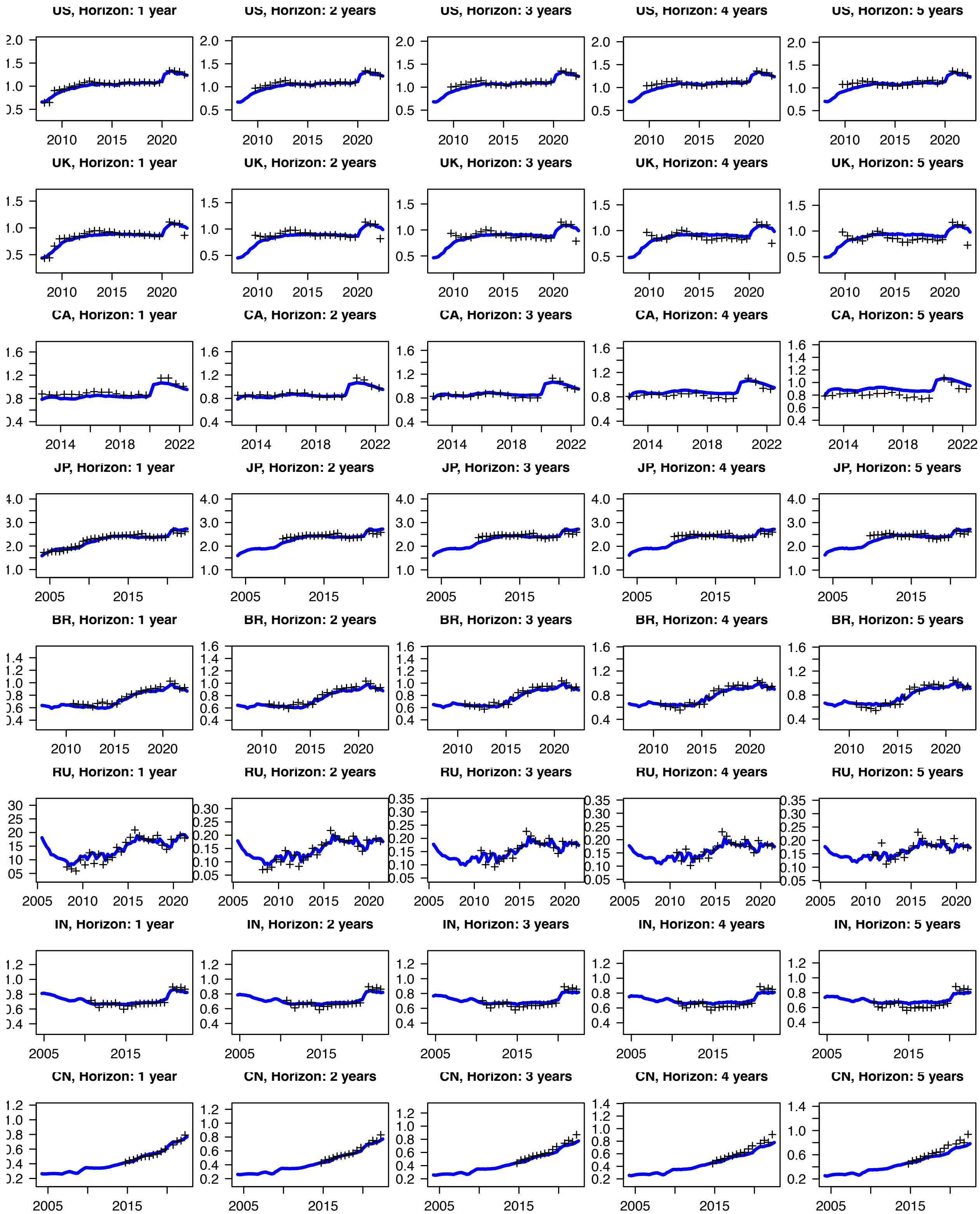
Note: This figure displays smoothed factors $w_{i,t}$, $i = 1, \ldots, 4$, for each country. The first, second, third, and fourth columns respectively show $w_{1,t}$ (the persistent component of Δy_t), $w_{2,t}$ (the persistent component of inflation), $w_{3,t}$ (the persistent component of budget surplus), and $w_{4,t}$ (the persistent component of s_t^*). These estimates result from the Extended Kalman Filter (see Section **??**). The shaded area indicates the 95% confidence interval (accounting for filtering uncertainty).

Figure E.11: Observed vs model-implied yields



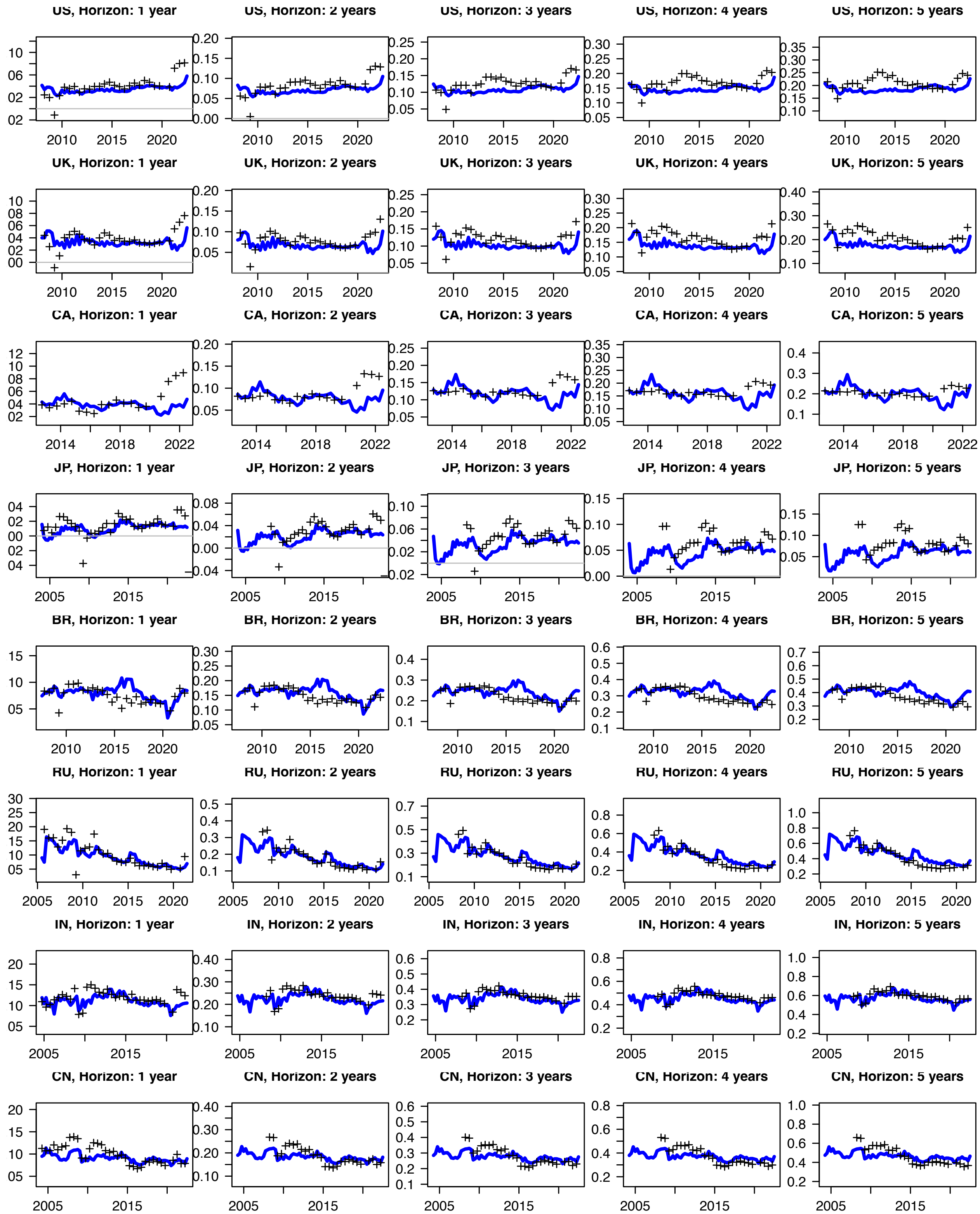
Note: This figure compares model-implied and observed quarterly yields of zero-coupon government yields. The computation of model-implied yields is based on Proposition 6. (The maturity-h yield is given by $-\frac{1}{h}\log \mathcal{B}_{t,h}$, where $\mathcal{B}_{t,h}$ is the date-t price of a zero-coupon bond of maturity h).

Figure E.12: Observed vs model-implied forecasts of the debt-to-GDP ratio



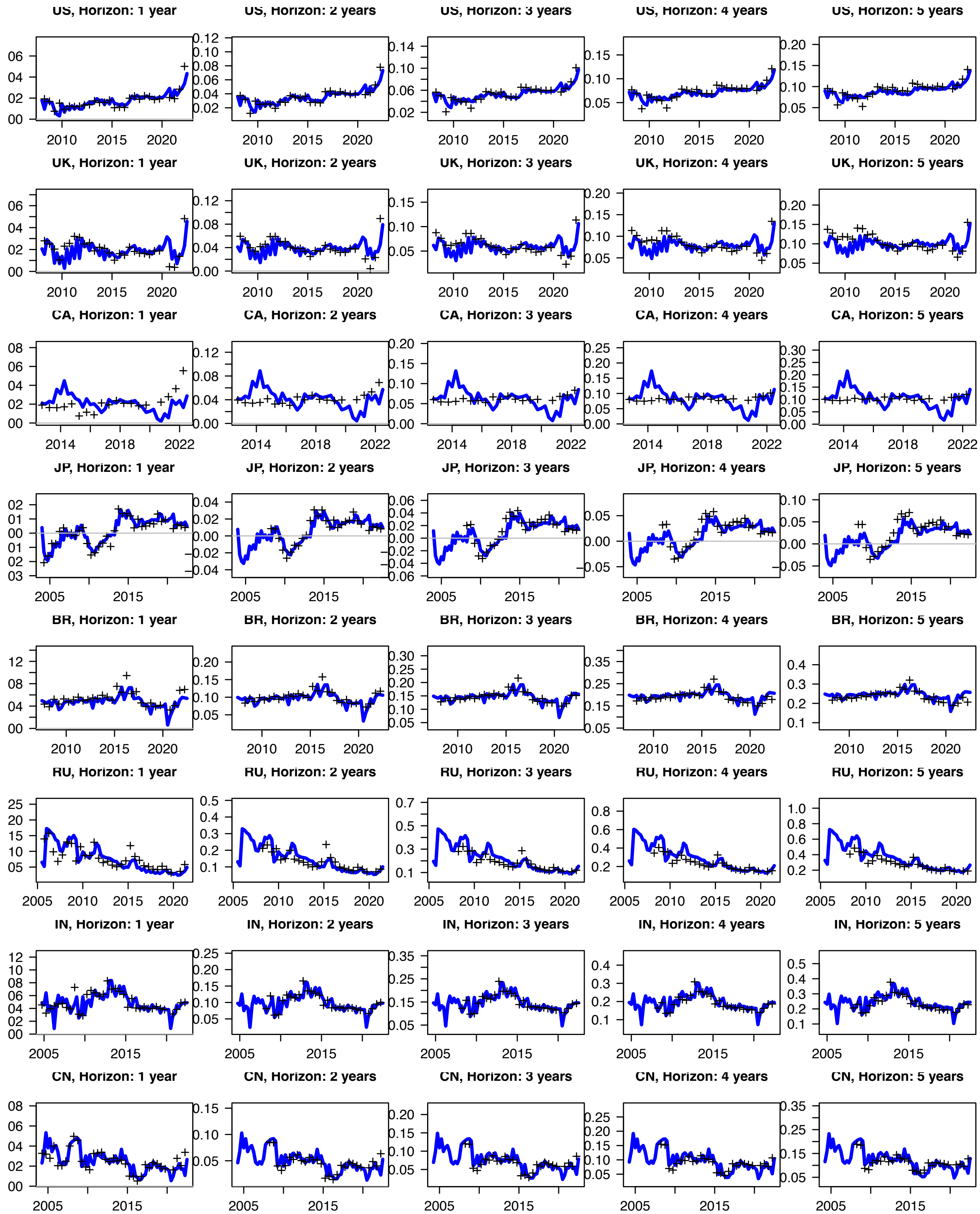
Note: This figure compares model-implied (blue line) and observed (crosses) forecasts of the debt-to-GDP ratio. Observed values are those from the IMF World Economic Outlook.

Figure E.13: Observed vs model-implied forecasts of nominal growth



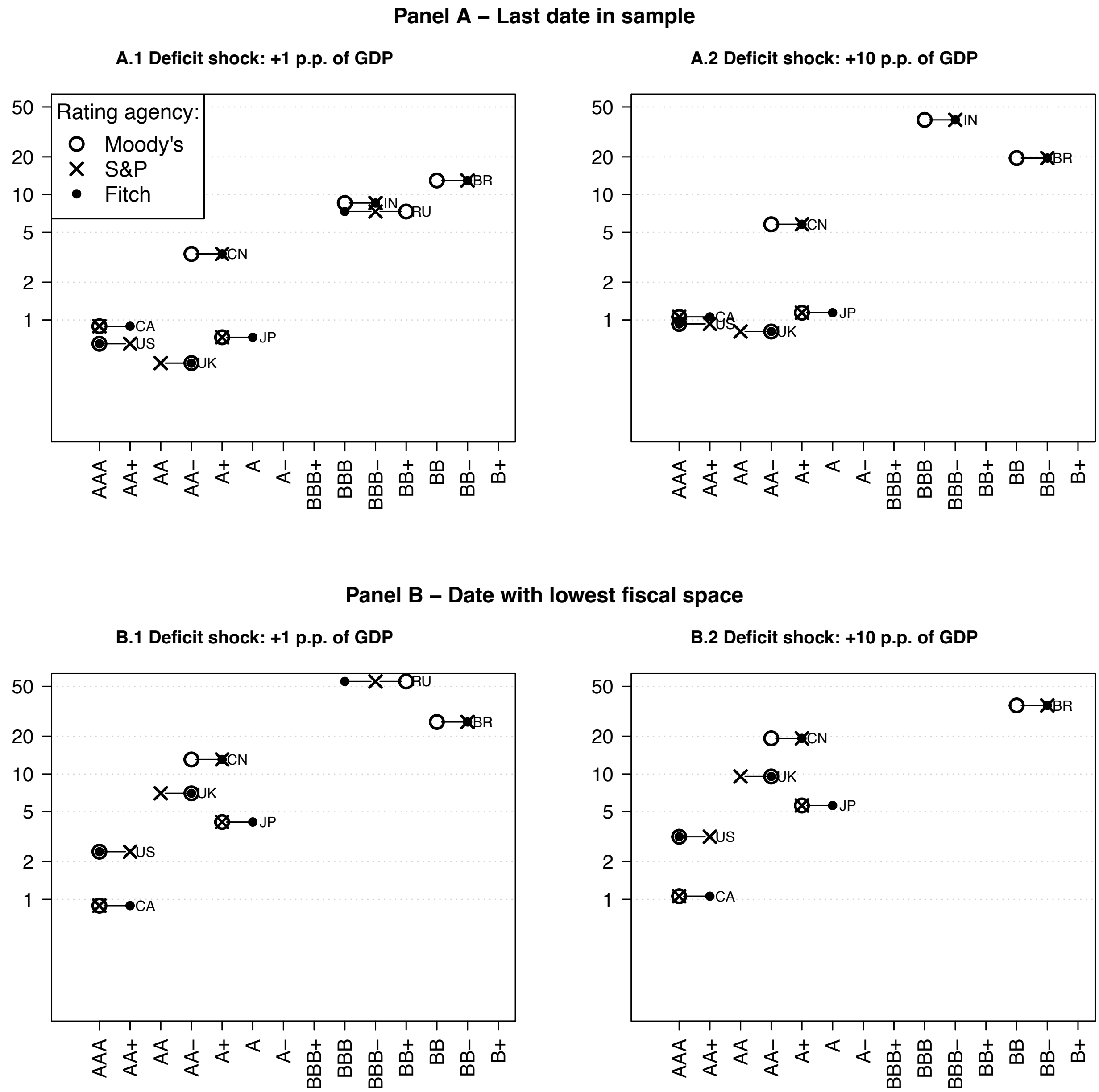
Note: This figure compares model-implied (blue line) and observed (crosses) forecasts of nominal GDP growth. Observed values are those from the IMF World Economic Outlook.

Figure E.14: Observed vs model-implied inflation forecasts



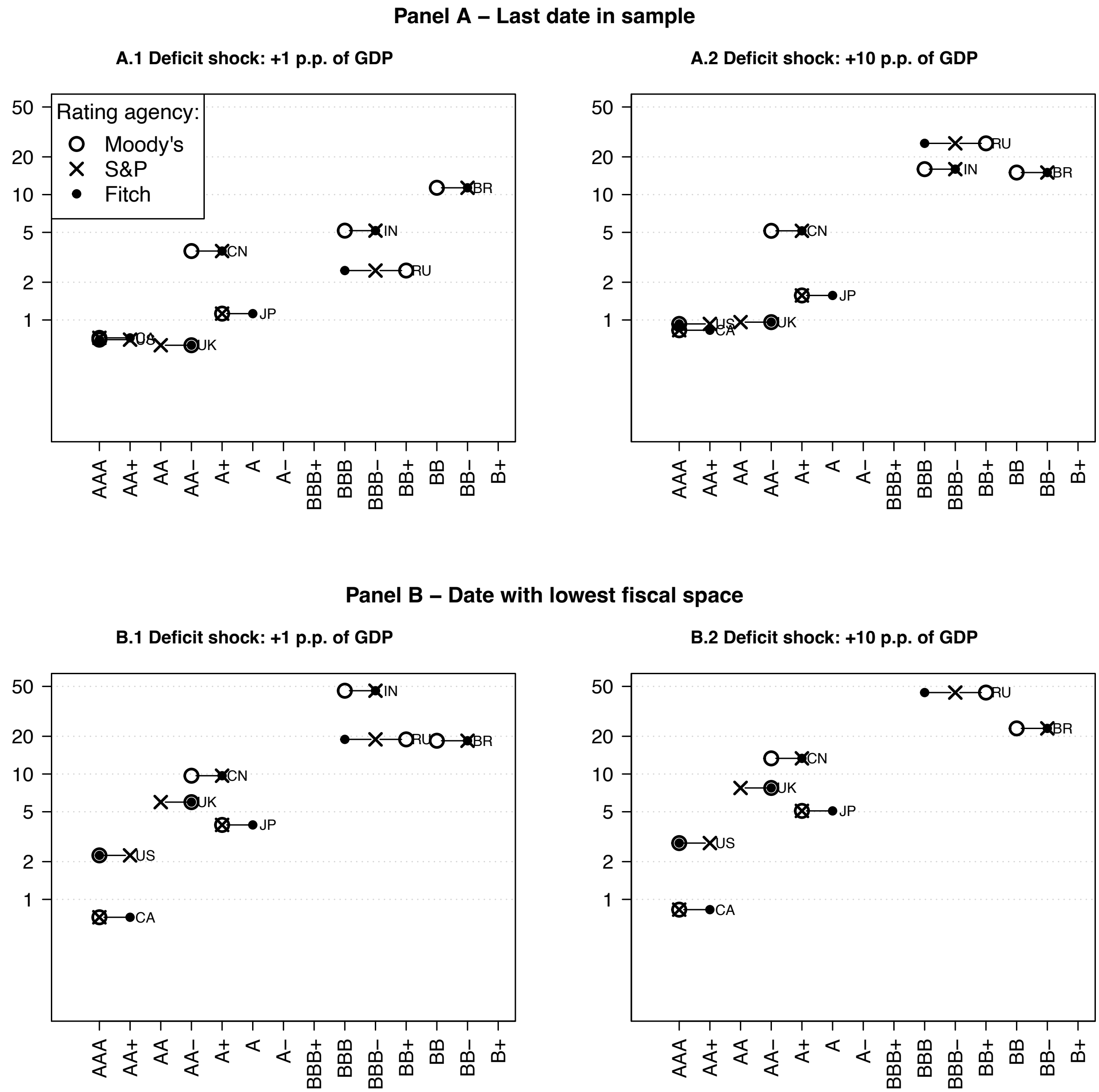
Note: This figure compares model-implied (blue line) and observed (crosses) forecasts of changes in the price index. Observed values are those from the IMF World Economic Outlook.

Figure E.15: Sensitivity of 2-year CDS spreads to increases in deficits



Note: These plots display the model-predicted sensitivities of 2-year CDS spreads to fiscal conditions. Effects are measured in basis points. We consider two magnitudes of fiscal shocks (increases in primary deficits by 1% and 10% of GDP). The benchmark conditions underlying Panel A (respectively Panel B) are those observed on the last date of the estimation sample (respectively on the date where the fiscal space is the tightest). The fiscal space is measured by the difference between the level of debt-to-GDP that would provide a one-year probability of default of 1% (everything else equal) and the observed debt-to-GDP; these fiscal spaces are shown in Figure 5. Reported figures correspond to the marginal influence of an additional unit increase in the debt-to-GDP (i.e., from "benchmark debt-to-GDP ratio" to "benchmark +1 p.p." for left-hand-side plots and from "benchmark +9 p.p." to "benchmark +10 p.p." for right-hand-side plots). The credit ratings are those observed in August 2021.

Figure E.16: Sensitivity of 5-year CDS spreads to increases in deficits



Note: These plots display the model-predicted sensitivities of 5-year CDS spreads to fiscal conditions. Effects are measured in basis points. We consider two magnitudes of fiscal shocks (increases in primary deficits by 1% and 10% of GDP). The benchmark conditions underlying Panel A (respectively Panel B) are those observed on the last date of the estimation sample (respectively on the date where the fiscal space is the tightest). The fiscal space is measured by the difference between the level of debt-to-GDP that would provide a one-year probability of default of 1% (everything else equal) and the observed debt-to-GDP; these fiscal spaces are shown in Figure 5. Reported figures correspond to the marginal influence of an additional unit increase in the debt-to-GDP (i.e., from "benchmark debt-to-GDP ratio" to "benchmark +1 p.p." for left-hand-side plots and from "benchmark +9 p.p." to "benchmark +10 p.p." for right-hand-side plots). The credit ratings are those observed in August 2021.

Table E.14: Models' parameterization (Σ_w)

	US	UK	CA	JP	BR	RU	IN	CN
$\Sigma_{w,1,1}$	0.340	0.226	0.381	0.206	0.141	0.259	0.381	0.434
$\Sigma_{w,2,1}$	−0.256	−0.246	−0.065	−0.175	−0.178	−0.129	−0.141	−0.118
$\Sigma_{w,2,2}$	0.097	0.088	0.102	0.080	0.083	−0.086	0.051	0.034
$\Sigma_{w,3,1}$	0.053	0.091	0.018	0.057	−0.066	−0.092	−0.052	0.021
$\Sigma_{w,3,2}$	0.491	0.473	0.125	0.335	0.485	0.248	0.271	0.227
$\Sigma_{w,3,3}$	0.017	0.034	−0.050	0.100	−0.074	−0.086	−0.178	−0.036
$\Sigma_{w,4,1}$	0.029	0.046	−0.071	0.044	0.024	−0.047	0.038	0.022
$\Sigma_{w,4,2}$	0.185	0.239	0.248	0.213	0.141	0.555	0.375	0.132
$\Sigma_{w,4,3}$	0.031	0.106	−0.023	0.060	0.026	0.043	0.084	0.012
$\Sigma_{w,4,4}$	0.153	0.203	0.170	0.173	0.175	0.255	0.221	0.169
$\Sigma_{w,5,5}$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\Sigma_{w,6,5}$	0.297	−0.462	0.010	−0.274	−0.305	0.325	0.284	0.096
$\Sigma_{w,6,6}$	−0.069	0.126	−0.461	−0.006	−0.117	−0.138	−0.249	−0.031
$\Sigma_{w,7,5}$	−0.461	0.462	0.462	−0.024	−0.147	0.462	0.462	0.462
$\Sigma_{w,7,6}$	0.955	0.887	1.000	0.962	0.952	0.946	0.959	0.995
$\Sigma_{w,7,7}$	0.118	−0.190	−0.365	0.091	0.206	0.099	−0.341	0.246
$\Sigma_{w,8,5}$	0.202	−0.410	0.410	0.349	0.457	0.410	−0.410	0.410
$\Sigma_{w,8,6}$	0.991	0.974	0.809	0.996	0.972	0.985	0.907	0.969
$\Sigma_{w,8,7}$	0.399	0.363	−0.363	0.433	0.405	0.363	0.363	0.363
$\Sigma_{w,8,8}$	0.766	0.698	0.698	0.831	0.778	0.698	0.698	0.698

Note: This table reports the estimated parameterization of Σ_w. Given Eq. (13), we have that $\Sigma_w \Sigma_w'$ is the conditional covariance matrix of w_{t+1} (as of date t). This matrix is block diagonal. The 4×4 upper-left block (respectively lower-right block) is lower triangular and corresponds to the persistent components (resp. volatile components) of w_t, its specification is given in the upper part of the table (resp. in the lower part of the table). The parameterization is such that, for the sake of identification, the unconditional variance of each of the $w_{i,t}$'s is equal to one.

VI. Robustness analysis

This section presents the results of different alternative estimations of the model. More precisely:

(a) **Parameter** α: We estimate models while imposing a small value (0.01) and a large value for α, that is the elasticity of the default intensity with respect to the surplus gap ($s_t - s_t^*$), see Eq. (1). (In the baseline case, this parameter is estimated, with a cap of 2.)

(b) **Output drop upon default** b_y: We estimate models with $b_y = 10\%$ (versus 20% in the baseline model).

(c) **Coefficient of relative risk aversion** γ: We estimate models with smaller and larger values for γ. Since our approach requires $RR = \exp(-\gamma b_y - b_\pi)$, modifying γ, everything else equal, results in a change in RR. Accordingly, we also consider cases where b_y is adjusted in order to keep the same recovery rate RR as in the baseline case. This is summarized in Table F.15.

(d) **No CDS in the estimation dataset**: We remove CDS data from the estimation sample. In other words, we remove the CDS measurement equations in the state-space model.

Table F.15: Robustness analysis (changes in γ)

Version	Description	γ	b_y	RR
	Baseline	4	0.20	46%
Case A.i	low γ, high RR	2	0.20	68%
Case A.ii	low γ	2	0.40	46%
Case B.i	high γ, low RR	6	0.20	31%
Case B.ii	high γ	6	0.13	46%

In all case, Condition $\star$ is satisfied, i.e., we have $RR = \exp(-\gamma b_y - b_\pi)$ (or, equivalently, $b_y = [-\log(RR) - b_\pi]/\gamma$). We use $b_\pi = -2.1\%$.

The results of these exercises can be summarized as follows:

(a) **Parameter** α: The resulting fiscal limit estimates are displayed on Figure F.17 (dotted and solid red lines). Except for China and Canada, imposing a large α has no strong effects on the estimated fiscal limits. The changes in estimated fiscal limits are larger when α is small. For all countries and the two cases (small or large α), likelihood ratio tests strongly reject all these models (against the baseline), at any significance level.

(b) **Output drop upon default** b_y: The resulting fiscal limit estimates are shown on Figure F.18 (pink line). For most countries the effects of this change on the fiscal limit estimates are mild. India is the exception: for $b_y = 10\%$, the estimation leads to a low estimate for β (see Table F.16, row "Low b_y"), which implies a volatile fiscal limit estimate. Indeed, when β is small, large debt fluctuations are needed to affect the default probability. (Remember that we define the fiscal limit as the level of debt that gives a 1% default probability.)

(c) **Coefficient of relative risk aversion** γ: The results are shown on Figure F.18 (see blue and green lines). In most cases, fiscal limits are higher for larger risk aversion (and vice versa). This results from the fact that, when γ is higher, a larger share of the credit spreads corresponds to risk premiums. Accordingly, estimated *physical* probabilities of default are lower. That is, in those models featuring higher γ, the physical probability of default is less sensitive to the debt level (in particular). Since we define our fiscal limits as the levels of debt resulting in a given *physical* probability of default, it comes that the estimated fiscal space is larger when γ is higher, hence the larger fiscal limits.

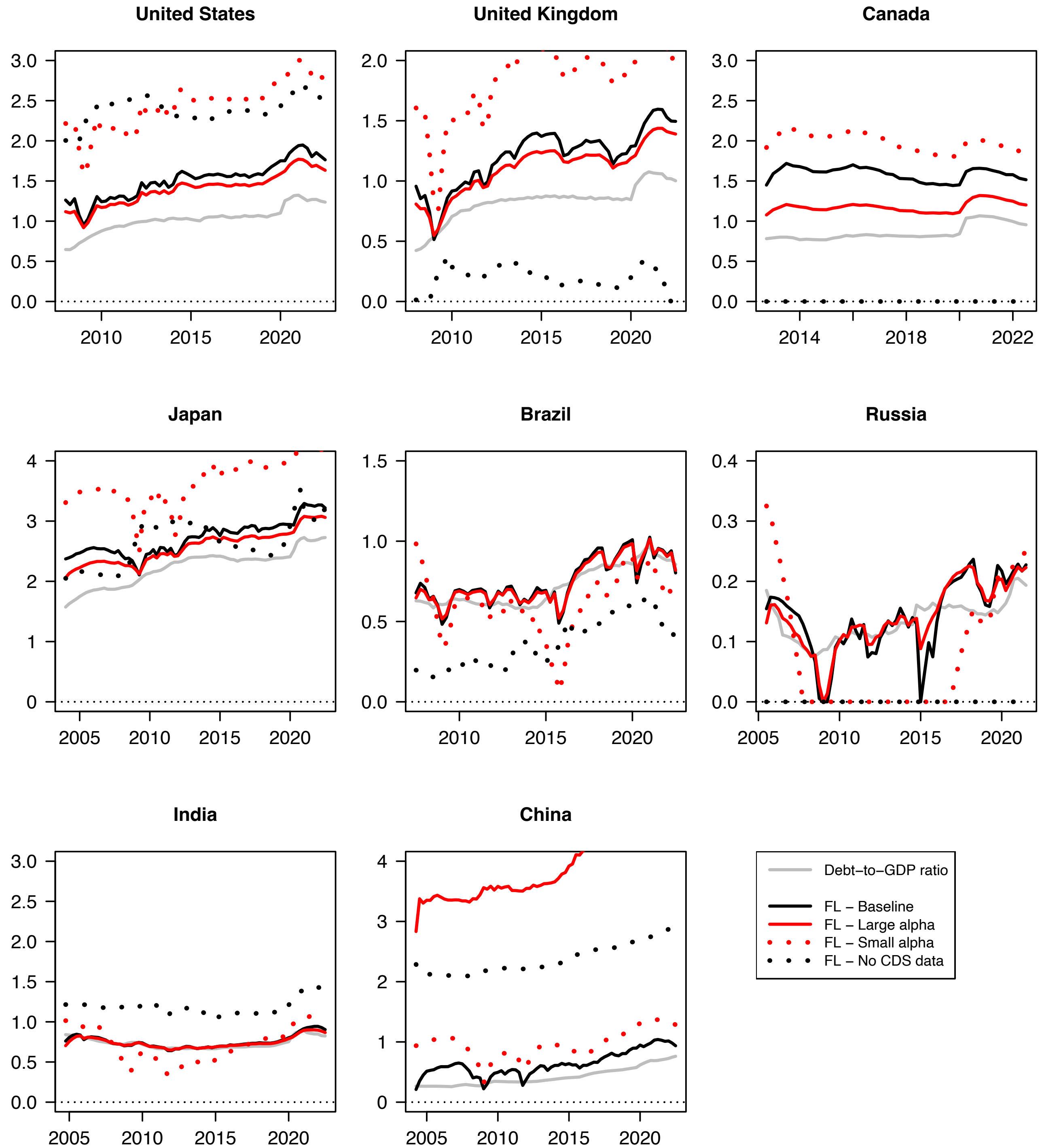
For some countries and variants, the changes in the fiscal limits resulting from these changes is substantial. First, the Canadian estimates appear sensitive to these changes. Second, in three instances, we obtain highly volatile fiscal limit estimates for India and China; again, this happens when the estimates of β is low (see numbers in red in Table F.16).

(d) **No CDS in the estimation dataset**: The resulting fiscal limits, displayed in Figure F.17 (black dotted lines), are very different from the baseline case and show implausible fluctuations. This highlights the importance of credit spreads to identify fiscal limits.

An additional exercise is the following: we take the baseline parametrization (Table 2 of the paper), but simply remove the CDS data from the set of observed variables. That is, we switch off the associated measurement equations. Figure F.19 compares the filtered fiscal limits when the measurement equations include (black lines) or do not include (red lines) the CDS data. The dotted lines indicate 99% confidence intervals, reflecting the filtering uncertainty. The results show that the estimates of the fiscal limits depend strongly on the inclusion, or not, of the CDS spreads in the state-space model—even when the parametrization is unchanged. Moreover, the confidence intervals show that the fiscal limit estimates are much less accurate when credit spreads are not included in the estimation.

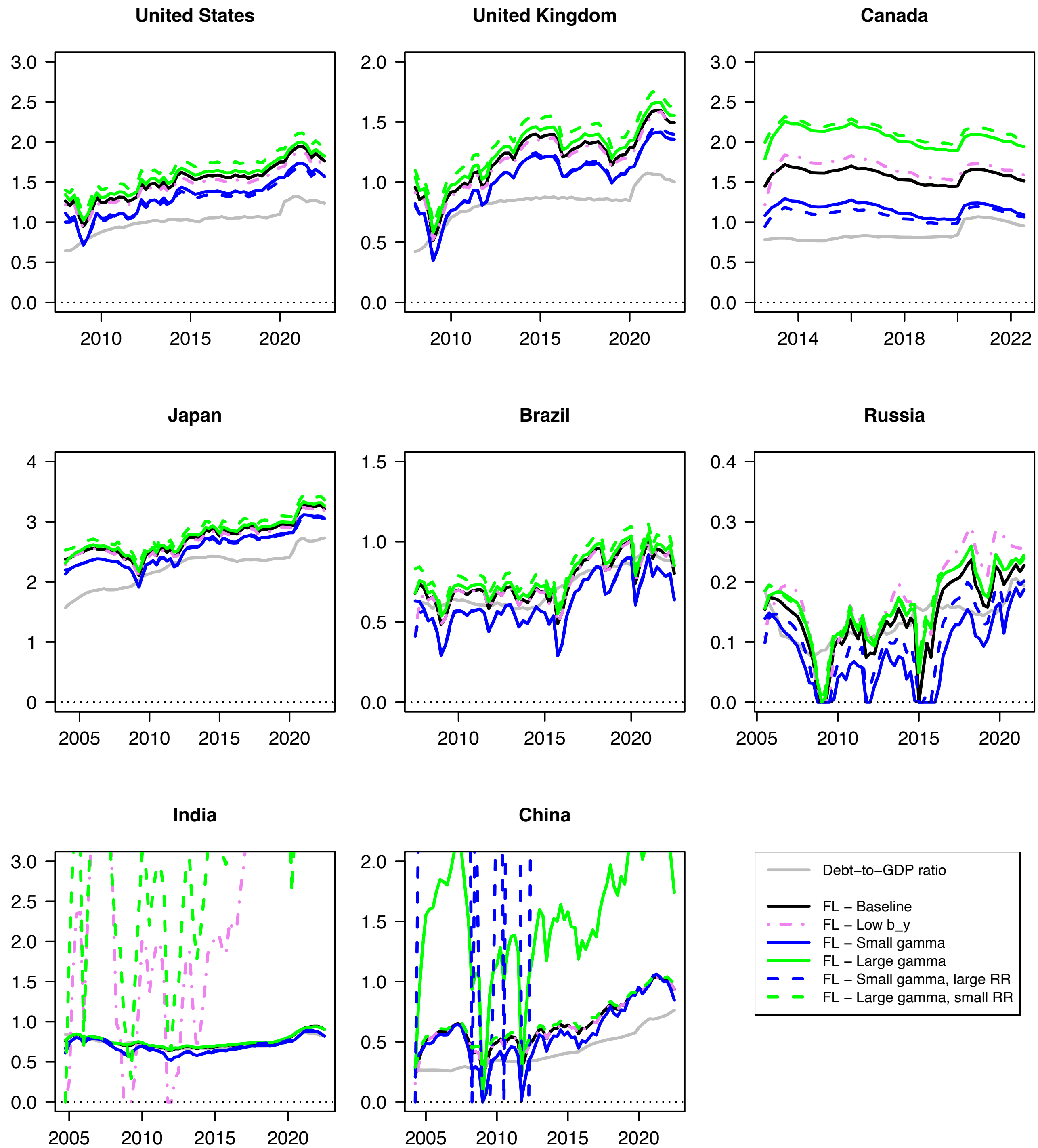
This may seem puzzling since, even when CDS are removed from the measurement equations, these equations still include bond yields, which are also forward-looking and feature a default-compensation component. However, the set of observed variables is then short of information allowing the filter to decompose these yields into its two components (risk-free yield and default compensation). This implies, in particular, that s_t^* is inaccurately estimated, which further translates into uncertain fiscal limit estimates.

Figure F.17: Fiscal limits – Robustness analysis: α and CDS data



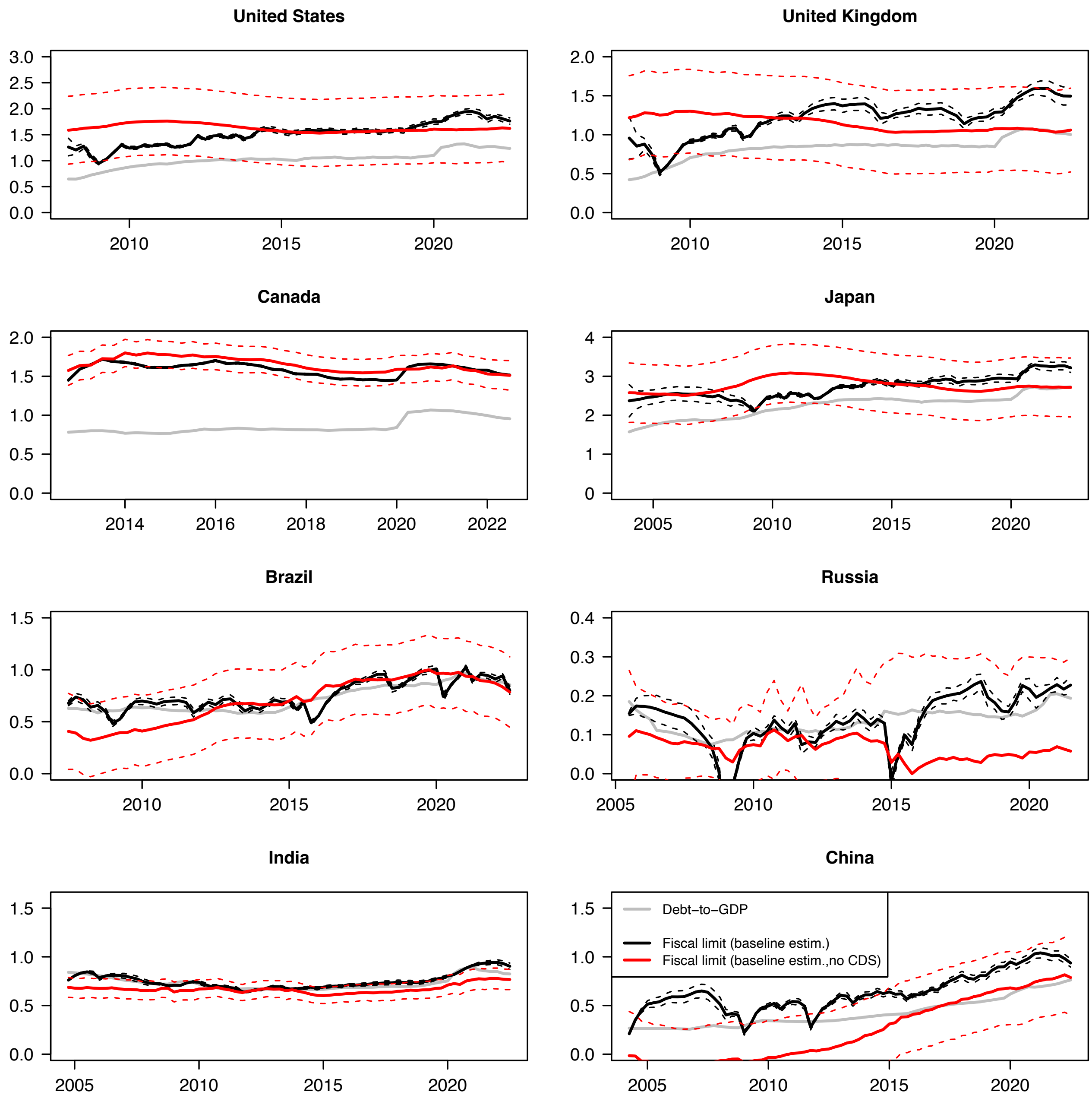
Note: These plots show the estimates of the fiscal limits (FL) obtained while imposing different types of restrictions. **Large alpha**: α is set to 10 (in the baseline case, it is estimated, but smaller than 2); **Small alpha**: α is set to 0.01; **No CDS data**: no CDS data are used in the estimation approach (i.e., there is no measurement equations involving CDS spreads).

Figure F.18: Fiscal limits – Robustness analysis: b_y and γ



Note: These plots show the estimates of the fiscal limits (FL) obtained while imposing different types of restrictions. **Low b_y**: b_y is set to 10% (versus 20% in the baseline case); **Small gamma**: γ is set to 3 (versus 4 in the baseline case); **Large gamma**: γ is set to 5 (versus 4 in the baseline case); **Small gamma, large RR**: γ is set to 3 (versus 4 in the baseline case) and b_y is adjusted to give the same RR as in the baseline case; **Large gamma, small RR**: γ is set to 5 (versus 3 in the baseline case) and b_y is adjusted to give the same RR as in the baseline case.

Figure F.19: Fiscal limits in the baseline model: with and without CDS data



Note: These plots compare the estimates of the fiscal limit when the measurement equations include (black lines) or do not include (red lines) the CDS data. The model setting is the baseline model (documented in Table 2 of the paper). The dotted lines indicate the 99% confidence intervals, reflecting the filtering uncertainty. The results show that the estimates of the fiscal limits depend strongly on the inclusion of the CDS spreads in the state-space model; they also show that the filtering uncertainty is strongly reduced when using the CDS data.

Table F.16: Estimates of β

	US	UK	CA	JP	BR	RU	IN	CN
Baseline	0.0253	0.0259	0.0361	0.0186	0.0460	0.4556	0.1698	0.0372
Low b_y	0.0252	0.0261	0.0187	0.0185	0.0343	0.4177	**0.0040**	0.0354
Large α	0.0162	0.0176	0.0181	0.0172	0.0272	0.0910	0.1455	**0.0094**
Small α	0.0456	0.0648	0.0763	0.0532	0.1349	0.4761	0.2571	0.1310
Small γ	0.0418	0.0410	0.0702	0.0393	0.0408	0.4474	0.2718	0.0211
Large γ	0.0245	0.0246	0.0196	0.0188	0.0496	0.4574	0.1549	**0.0058**
Small γ, large RR	0.0419	0.0349	0.0701	0.0388	0.0320	0.4011	0.3271	**0.0002**
Large γ, small RR	0.0239	0.0236	0.0235	0.0185	0.0487	0.4644	**0.0019**	0.0354
No CDS data	0.0908	0.1418	0.2949	0.0214	0.1343	0.4366	0.2453	0.0503

Note: This table reports the estimates of parameter β obtained while imposing different types of restrictions during the estimation. **Low b_y**: b_y is set to 10% (versus 20% in the baseline case); **Large alpha**: α is set to 10 (in the baseline case, it is estimated, but smaller than 2); **Small alpha**: α is set to 0.01; **Small gamma**: γ is set to 3 (versus 4 in the baseline case); **Large gamma**: γ is set to 5 (versus 4 in the baseline case); **Small gamma, large RR**: γ is set to 3 (versus 4 in the baseline case) and b_y is adjusted to give the same RR as in the baseline case; **Large gamma, small RR**: γ is set to 5 (versus 3 in the baseline case) and b_y is adjusted to give the same RR as in the baseline case; **No CDS data**: no CDS data are used in the estimation approach (i.e., there is no measurement equations involving CDS spreads).

References

Acharya, V., I. Drechsler, and P. Schnabl (2014). A Pyrrhic Victory? Bank Bailouts and Sovereign Credit Risk. *Journal of Finance 69*(6), 2689–2739.

Aguiar, M. and G. Gopinath (2006). Defaultable Debt, Interest Rates and the Current Account. *Journal of International Economics 69*(1), 64–83.

Alessi, L., P. Balduzzi, and R. Savona (2020). Anatomy of a Sovereign Debt Crisis: CDS Spreads and Real-Time Macroeconomic Data. Technical report, European Commission.

Ang, A. and F. A. Longstaff (2013). Systemic Sovereign Credit Risk: Lessons from the U.S. and Europe. *Journal of Monetary Economics 60*(5), 493–510.

Ang, A. and M. Piazzesi (2003). A No-Arbitrage Vector Autoregression of Term Structure Dynamics with Macroeconomic and Latent Variables. *Journal of Monetary Economics 50*(4), 745–787.

Arbatli Saxegaard, E. C., S. J. Davis, A. Ito, and N. Miake (2022). Policy Uncertainty in Japan. *Journal of the Japanese and International Economies 64*, 101192.

Arellano, C. (2008). Default Risk and Income Fluctuations in Emerging Economies. *American Economic Review 98*(3), 690–712.

Arellano, C. and A. Ramanarayanan (2012). Default and the Maturity Structure in Sovereign Bonds. *Journal of Political Economy 120*(2), 187 – 232.

Arslanalp, S. and T. Tsuda (2014). Tracking Global Demand for Advanced Economy Sovereign Debt. *IMF Economic Review 62*(3), 430–464.

Augustin, P., M. Chernov, L. Schmid, and D. Song (2021). Benchmark Interest Rates when the Government is Risky. *Journal of Financial Economics 140*(1), 74–100.

Augustin, P. and R. Tedongap (2016). Real Economic Shocks and Sovereign Credit Risk. *Journal of Financial and Quantitative Analysis 51*, 541–587.

Bai, J., P. Collin-Dufresne, R. S. Goldstein, and J. Helwege (2015). On Bounding Credit-Event Risk Premia. *Review of Financial Studies 28*(9), 2608–2642.

Baker, S. R., N. Bloom, and S. J. Davis (2016). Measuring Economic Policy Uncertainty. *Quarterly Journal of Economics 131*(4), 1593–1636.

Barro, R. J. (2006). Rare Disasters and Asset Markets in the Twentieth Century. *Quarterly Journal of Economics 121*(3), 823–866.

Barro, R. J. and T. Jin (2011). On the Size Distribution of Macroeconomic Disasters. *Econometrica 79*(5), 1567–1589.

Barro, R. J. and J. F. Ursúa (2012). Rare Macroeconomic Disasters. *Annual Review of Economics 4*(1), 83–109.

Bekaert, G., E. C. Engstrom, and N. R. Xu (2022). The Time Variation in Risk Appetite and Uncertainty. *Management Science 68*(6), 3975–4004.

Bi, H. (2012). Sovereign Default Risk Premia, Fiscal Limits, and Fiscal Policy. *European Economic Review 56*(3), 389–410.

Bi, H. and E. M. Leeper (2013). Analyzing Fiscal Sustainability. Staff working papers, Bank of Canada.

Bi, H. and N. Traum (2012). Estimating Sovereign Default Risk. *American Economic Review, Papers & Proceedings 102*(3), 161–166.

Black, F. (1995). Interest Rates as Options. *Journal of Finance 50*(5), 1371–76.

Bohn, H. (1998). The Behavior of U.S. Public Debt and Deficits. *Quarterly Journal of Economics 113*(3), 949–963.

Chen, L., P. Collin-Dufresne, and R. S. Goldstein (2009). On the Relation Between the Credit Spread Puzzle and the Equity Premium Puzzle. *Review of Financial Studies 22*(9), 3367–3409.

Chernov, M., L. Schmid, and A. Schneider (2020). A Macrofinance View of U.S. Sovereign CDS Premiums. *Journal of Finance 75*(5), 2809–2844.

Christensen, J. H. and G. D. Rudebusch (2013). Modeling Yields at the Zero Lower Bound: Are Shadow Rates the Solution? Working Paper Series 2013-39, Federal Reserve Bank of San Francisco.

Collard, F., M. Habib, and J.-C. Rochet (2015). Sovereign Debt Sustainability In Advanced Economies. *Journal of the European Economic Association 13*(3), 381–420.

Coroneo, L. and S. Pastorello (2020). European Spreads at the Interest Rate Lower Bound. *Journal of Economic Dynamics and Control 119*(C), 1–21.

Cox, J. C., J. E. Ingersoll, and S. A. Ross (1985). An Intertemporal General Equilibrium Model of Asset Prices. *Econometrica 53*(2), 363–84.

Davis, S. J., D. Liu, and X. S. Sheng (2019). Economic Policy Uncertainty in China Since 1949: The View from Mainland Newspapers. Technical report, Chicago Booth School of Business.

de Boyrie, M. E. and I. Pavlova (2016). Dynamic Interdependence of Sovereign Credit Default Swaps in BRICS and MIST Countries. *Applied Economics 48*(7), 563–575.

De Paoli, B., G. Hoggarth, and V. Saporta (2006). Output Costs of Sovereign Defaults: Some Empirical Estimates. Financial Stability Paper 1, Bank of England.

Doshi, H., J. Ericsson, K. Jacobs, and S. M. Turnbull (2013). Pricing Credit Default Swaps with Observable Covariates. *Review of Financial Studies 26*(8), 2049–2094.

Driessen, J. (2005). Is Default Event Risk Priced in Corporate Bonds? *Review of Financial Studies 18*(1), 165–195.

Duffie, D., L. H. Pedersen, and K. J. Singleton (2003). Modeling Sovereign Yield Spreads: A Case Study of Russian Debt. *Journal of Finance 58*(1), 119–159.

Duffie, D. and K. J. Singleton (1999). Modeling Term Structures of Defaultable Bonds. *Review of Financial Studies 12*(4), 687–720.

Eaton, J. and M. Gersovitz (1981). Debt with Potential Repudiation: Theoretical and Empirical Analysis. *Review of Economic Studies 48*(2), 289–309.

Elenev, V., T. Landvoigt, P. J. Shultz, and S. Van Nieuwerburgh (2021). Can Monetary Policy Create Fiscal Capacity? NBER Working Papers 29129, National Bureau of Economic Research, Inc.

Farah-Yacoub, J. P., C. M. H. Graf Von Luckner, R. Ramalho, and C. M. Reinhart (2022). The Social Costs of Sovereign Default. Policy Research Working Paper Series 10157, The World Bank.

Gabaix, X. (2012). Variable Rare Disasters: An Exactly Solved Framework for Ten Puzzles in Macro-Finance. *Quarterly Journal of Economics 127*(2), 645–700.

Garcia, R., E. Renault, and A. Semenov (2006). Disentangling Risk Aversion and Intertemporal Substitution Through a Reference Level. *Finance Research Letters 3*(3), 181–193.

Ghosh, A., J. I. Kim, E. Mendoza, J. Ostry, and M. Qureshi (2013). Fiscal Fatigue, Fiscal Space and Debt Sustainability in Advanced Economies. *The Economic Journal 123*(566), F4–F30.

Ghosh, A. R., J. D. Ostry, and M. S. Qureshi (2013). Fiscal Space and Sovereign Risk Pricing in a Currency Union. *Journal of International Money and Finance 34*, 131 – 163.

Gouriéroux, C., A. Monfort, and J. Renne (2014). Pricing Default Events: Surprise, Exogeneity and Contagion. *Journal of Econometrics 182*(2), 397–411.

Hamilton, J. D. (1986). A Standard Error for the Estimated State Vector of a State-space Model. *Journal of Econometrics 33*(3), 387–397.

Hatchondo, J. C. and L. Martinez (2009). Long-Duration Bonds and Sovereign Defaults. *Journal of International Economics 79*(1), 117–125.

Hofmann, B., M. Lombardi, B. Mojon, and A. Orphanides (2021). Fiscal and Monetary Policy Interactions in a Low Interest Rate World. CEPR Discussion Papers 16411, C.E.P.R. Discussion Papers.

International Monetary Fund, Bank for International Settlements and European Central Bank (2015). *Handbook on Securities Statistics*. International Monetary Fund, 2015.

Jiang, Z., H. Lustig, S. V. Nieuwerburgh, and M. Xiaolan (2019). Government Risk Premium Puzzle. 2019 Meeting Papers 437, Society for Economic Dynamics.

Jiang, Z., H. Lustig, S. V. Nieuwerburgh, and M. Xiaolan (2022). Measuring U.S. Fiscal Capacity Using Discounted Cash Flow Analysis. Brookings papers on economic activity draft, Brookings.

Jordà, Ò., M. Schularick, and A. M. Taylor (2016). Sovereigns versus Banks: Credit, Crises, and Consequences. *Journal of the European Economic Association 14*(1), 45–79.

Kim, D. H. and K. J. Singleton (2012). Term Structure Models and the Zero Bound: An Empirical Investigation of Japanese Yields. *Journal of Econometrics 170*(1), 32–49.

Krippner, L. (2013). Measuring the Stance of Monetary Policy in Zero Lower Bound Environments. *Economics Letters 118*(1), 135–138.

Laubach, T. (2010). Fiscal Policy and Interest Rates: The Role of Sovereign Default Risk. In *NBER International Seminar on Macroeconomics 2010*, NBER Chapters, pp. 7–29. National Bureau of Economic Research.

Leeper, E. M. (2013). Fiscal Limits and Monetary Policy. NBER Working Papers 18877, National Bureau of Economic Research.

Leland, H. (1998). Agency Costs, Risk Management, and Capital Structure. *Journal of Finance 53*, 1213–1243.

Litterman, R. and T. Iben (1991). Corporate Bond Valuation and the Term Structure of Credit Spreads. *Journal of Portfolio Management 17*(3), 52–64.

Longstaff, F. A., J. Pan, L. H. Pedersen, and K. J. Singleton (2011). How Sovereign Is Sovereign Credit Risk? *American Economic Journal: Macroeconomics 3*(2), 75–103.

Lorenzoni, G. and I. Werning (2019). Slow Moving Debt Crises. *American Economic Review 109*(9), 3229–63.

Mehrotra, N. R. and D. Sergeyev (2021). Debt Sustainability in a Low Interest Rate World. *Journal of Monetary Economics 124*, S1–S18.

Mendoza, E. G. and J. D. Ostry (2008). International Evidence on Fiscal Solvency: Is Fiscal Policy "Responsible"? *Journal of Monetary Economics 55*(6), 1081–1093.

Mendoza, E. G. and V. Z. Yue (2012). A General Equilibrium Model of Sovereign Default and Business Cycles. *Quarterly Journal of Economics 127*(2), 889–946.

Monfort, A. and J.-P. Renne (2014). Decomposing Euro-Area Sovereign Spreads: Credit and Liquidity Risks. *Review of Finance 18*(6), 2103–2151.

Moody's (2022). Sovereign Default and Recovery Rates, 1983-2021. Data report, Moody's Investors Service.

Pallara, K. and J.-P. Renne (2023). Fiscal Limits and the Pricing of Eurobonds. *Management Science forthcoming*.

Pan, J. and K. Singleton (2008). Default and Recovery Implicit in the Term Structure of Sovereign CDS Spreads. *Journal of Finance 63*(5), 2345–84.

Reinhart, C. M. and K. S. Rogoff (2011). The Forgotten History of Domestic Debt. *Economic Journal 121*(552), 319–350.

Trebesch, C. and M. Zabel (2017). The Output Costs of Hard and Soft Sovereign Default. *European Economic Review 92*, 416–432.

Verdelhan, A. and N. Borri (2010). Sovereign Risk Premia. 2010 Meeting Papers 1122, Society for Economic Dynamics.

Wachter, J. A. (2013). Can Time-Varying Risk of Rare Disasters Explain Aggregate Stock Market Volatility? *Journal of Finance 68*(3), 987–1035.

Wu, J. C. and F. D. Xia (2016). Measuring the Macroeconomic Impact of Monetary Policy at the Zero Lower Bound. *Journal of Money, Credit and Banking 48*(2-3), 253–291.