

I've got n problems but dynamics ain't one

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1 Introduction

The single pendulum is one of the most simple dynamic systems that can be easily analyzed and has a unique solution given any initial conditions. Its equation of motion can be found on paper with a few steps of math and physics. A 2-link pendulum is one of the simplest dynamic system which exhibits chaotic behavior. It is possible but quite difficult and time consuming to derive its equations of motion by hand. The 3-link pendulum is an even more chaotic system, even more impractical to derive on paper. These multi-link systems can be easily solved using various numerical methods, however. These methods are powerful and flexible enough that they can be generalized to an n -link system.

The first method utilizes Lagrange's equations with minimal coordinates. Conceptually it is the simplest since it only requires finding the position and velocity of each link to calculate kinetic and potential energy. The second method is the Newton-Euler approach, also using minimal coordinates. This technique relies on angular momentum balance (AMB) for each link. It is somewhat more complicated since it requires finding the acceleration of each link, and the AMB for any one link depends on the dynamics of all the links below it. Both of these methods use as minimal coordinates the angle from the fixed downward vertical to the center of mass (CoM) of each link.

The third method uses differential algebraic equations (DAE) with maximal coordinates, being both the angles and positions of the CoM in cartesian coordinates of each link. In this approach, both AMB and linear momentum balance (LMB) are necessary. Additionally, constraint equations are needed to obtain

a solvable system of equations. Though this method is conceptually difficult, it is the most efficient and flexible of the three. For example, by adding two additional constraint equations to fix the end of the last link, the n -link pendulum becomes an $n+1$ -bar linkage, where the new bar is simply a fixed link connecting the base and end of the pendulum.

All of these methods are explored below, showing for each how the system of ordinary differential equations is calculated. These methods were implemented using MATLAB's symbolic toolbox. The systems were then simulated and analyzed using the `ode45` function. In an n -link system, the initial conditions are given the angular position θ_i and velocity $\dot{\theta}_i$ of each link i . The system parameters are the mass m_i , length L_i , and distance to CoM d_i for each link. The moment of inertia about the CoM I_i^G is calculated as follows for a general case:

$$I^G = \int_0^L l^2 dm = \int_0^L l^2 \lambda(l) dl, \quad (1)$$

where

$$\lambda(l) = \frac{dm}{dl} = \begin{cases} \lambda_1, & \text{if } 0 < l \leq d \\ \lambda_2, & \text{if } d < l \leq L. \end{cases} \quad (2)$$

The linear mass density functions for each section λ_1, λ_2 can be found by integrating them along their sections and setting these equal to the masses of each section:

$$\begin{aligned} \int_0^d \lambda_1 dl &= \int_0^{m/2} dm \Rightarrow \lambda_1 = \frac{m}{2d}, \\ \int_d^L \lambda_2 dl &= \int_{m/2}^m dm \Rightarrow \lambda_2 = \frac{m}{2(L-d)}. \end{aligned} \quad (3)$$

Thus, the moment of inertia about the CoM can be found in terms of m, L , and d :

$$I^G = \int_0^d \lambda_1 l^2 dl + \int_d^L \lambda_2 l^2 dl = \frac{m}{2d} \int_0^d l^2 dl + \frac{m}{2(L-d)} \int_d^L l^2 dl, \quad (4)$$

$$I_i^G = \frac{m_i}{6} (L_i^2 - 2L_i d_i + 2d_i^2). \quad (5)$$

In the nominal case, all $d_i = 0.5$, and I_i^G reduces to $m_i L_i^2 / 12$, the moment of inertia about the center of mass for a simple rod.

2 Lagrange

The Lagrange approach to the n -link pendulum revolves around calculating the total kinetic energy E_k and potential energy E_p of the system, link by link. The difference of these scalar functions is the Lagrangian \mathcal{L} , which is used to solve the equations of motion of the system:

$$\mathcal{L}(\theta_1, \dots, \theta_n, \dot{\theta}_1, \dots, \dot{\theta}_n) = E_k - E_p, \quad (6)$$

where in general

$$E_k = \sum_{i=1}^n \left(\frac{1}{2} m_i v_{G_i}^2 + \frac{1}{2} I_i^G \dot{\theta}_i^2 \right) = \sum_{i=1}^n \left(\frac{1}{2} m_i \vec{v}_{G_i} \cdot \vec{v}_{G_i} + \frac{1}{2} I_i^G \dot{\theta}_i^2 \right) = \sum_{i=1}^n \left(\frac{1}{2} m_i (\dot{x}_{G_i}^2 + \dot{y}_{G_i}^2) + \frac{1}{2} I_i^G \dot{\theta}_i^2 \right), \quad (7)$$

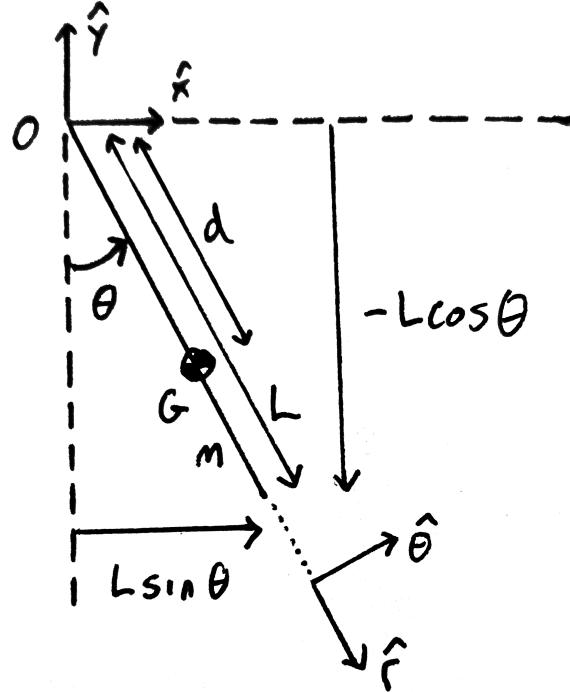


Figure 1: Simple Pendulum

and

$$E_p = \sum_{i=1}^n (m_i g y_{G_i}). \quad (8)$$

With this approach to finding the Lagrangian, no vector quantities or force balances are needed. The only challenging task is finding x_{G_i} and y_{G_i} , the \hat{x} and \hat{y} positions of the CoMs of each link, and \dot{x}_{G_i} and \dot{y}_{G_i} , their respective velocities. To begin, consider the first link (see Figure 1).

$$\begin{aligned} x_{G_1} &= d_1 \sin(\theta_1) & \dot{x}_{G_1} &= d_1 \dot{\theta}_1 \cos(\theta_1) \\ y_{G_1} &= -d_1 \cos(\theta_1) & \dot{y}_{G_1} &= d_1 \dot{\theta}_1 \sin(\theta_1) \end{aligned} \quad (9)$$

Here the chain rule has been employed to find \dot{x}_{G_1} and \dot{y}_{G_1} , because θ_i is a time-dependent quantity. A similar calculation is then used to find the position and velocity of the end of the first link.

$$\begin{aligned} x_{E_1} &= L_1 \sin(\theta_1) & \dot{x}_{E_1} &= L_1 \dot{\theta}_1 \cos(\theta_1) \\ y_{E_1} &= -L_1 \cos(\theta_1) & \dot{y}_{E_1} &= L_1 \dot{\theta}_1 \sin(\theta_1) \end{aligned} \quad (10)$$

To find the position and velocity of the next link, first calculate its position relative to the previous link.

$$\begin{aligned} x_{E_1 G_2} &= d_2 \sin(\theta_2) \\ y_{E_1 G_2} &= -d_2 \cos(\theta_2) \end{aligned} \quad (11)$$

These expressions can be added to the position of the end of the first link to find x_{G_2} and y_{G_2} .

$$\begin{aligned} x_{G_2} &= x_{E_1} + x_{E_1 G_2} = L_1 \sin(\theta_1) + d_2 \sin(\theta_2) \\ y_{G_2} &= y_{E_1} + y_{E_1 G_2} = -L_1 \cos(\theta_1) - d_2 \cos(\theta_2) \end{aligned} \quad (12)$$

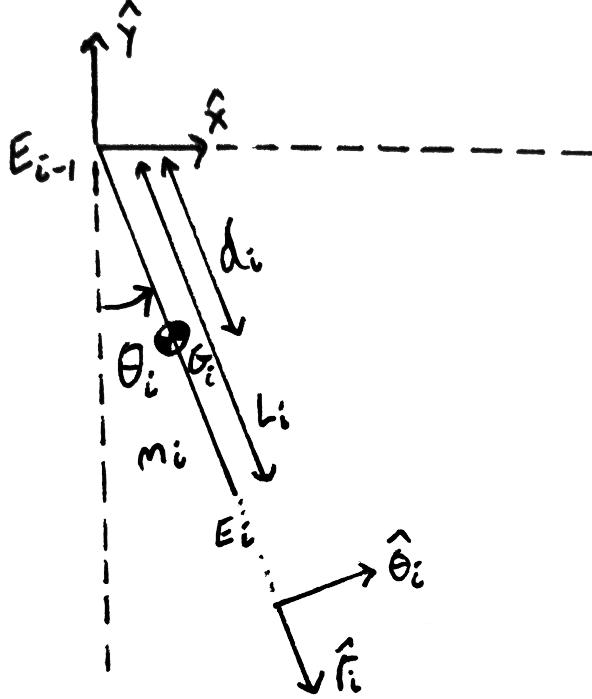


Figure 2: i th Link

This process is generalized as follows:

$$\begin{aligned} x_{G_i} &= x_{E_{i-1}} + x_{E_{i-1}G_i} = \sum_{j=1}^{i-1} \left(L_j \sin(\theta_j) \right) + d_i \sin(\theta_i) \\ y_{G_i} &= y_{E_{i-1}} + y_{E_{i-1}G_i} = \sum_{j=1}^{i-1} \left(-L_j \cos(\theta_j) \right) - d_i \cos(\theta_i). \end{aligned} \quad (13)$$

The velocities are then given by:

$$\begin{aligned} \dot{x}_{G_i} &= \sum_{j=1}^{i-1} \left(L_j \dot{\theta}_j \cos(\theta_j) \right) + d_i \dot{\theta}_i \cos(\theta_i) \\ \dot{y}_{G_i} &= \sum_{j=1}^{i-1} \left(-L_j \dot{\theta}_j \sin(\theta_j) \right) - d_i \dot{\theta}_i \sin(\theta_i). \end{aligned} \quad (14)$$

By plugging equations 13 and 14 into equations 7 and 8, the total kinetic and potential energies of the system are known. These results in turn are then plugged into Equation 6, giving the Lagrangian. Finally, Lagrange's equation (with no non-conservative forces) is applied.

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} - \frac{\partial \mathcal{L}}{\partial \theta_i} = \left(\frac{\partial}{\partial \theta_i} \frac{d\theta_i}{dt} + \frac{\partial}{\partial \dot{\theta}_i} \frac{d\dot{\theta}_i}{dt} \right) \frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} - \frac{\partial \mathcal{L}}{\partial \theta_i} = \frac{\partial}{\partial \theta_i} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} \dot{\theta}_i + \frac{\partial}{\partial \dot{\theta}_i} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} \ddot{\theta}_i - \frac{\partial \mathcal{L}}{\partial \theta_i} = 0 \quad (15)$$

This results in n differential equations giving motion of each link $\ddot{\theta}_i(\theta_1, \dots, \theta_n, \dot{\theta}_1, \dots, \dot{\theta}_n)$.

3 Newton-Euler

The Lagrange approach is relatively simple to understand and implement, but it becomes much more challenging when non-conservative forces are applied. The Newton-Euler approach is more robust in this respect, while still using minimal coordinates. In this derivation of the equations of motion for the n -link pendulum, friction (a non-conservative force) is included.

This method relies on vector algebra and force balance. To begin, the transformation from the cartesian basis $(\hat{x}, \hat{y}, \hat{z})$ to polar basis $(\hat{r}, \hat{\theta}, \hat{k})$ is defined.

$$\begin{bmatrix} \hat{r}_i \\ \hat{\theta}_i \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \sin(\theta_i) & -\cos(\theta_i) & 0 \\ \cos(\theta_i) & \sin(\theta_i) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \quad (16)$$

Unlike the cartesian unit vectors, the polar unit vectors exist in moving frames and are therefore time-dependent. Their time derivatives are given by

$$\begin{aligned} \dot{\hat{r}}_i &= \dot{\theta}_i \hat{\theta}_i, \\ \dot{\hat{\theta}}_i &= -\dot{\theta}_i \hat{r}_i, \\ \dot{\hat{k}}_i &= 0. \end{aligned} \quad (17)$$

To perform LMB and AMB, both the positions \vec{r}_{G_i} and accelerations $\ddot{\vec{r}}_{G_i}$ of the CoMs of each link are needed. First consider the kinematics of the CoM and end of the first link (points G_1 and E_1):

$$\begin{aligned} \vec{r}_{G_1} &= d_1 \hat{r}_1 & \dot{\vec{r}}_{G_1} &= d_1 \dot{\theta}_1 \hat{\theta}_1 & \ddot{\vec{r}}_{G_1} &= -d_1 \dot{\theta}_1^2 \hat{r}_1 + d_1 \ddot{\theta}_1 \hat{\theta}_1 \\ \vec{r}_{E_1} &= L_1 \hat{r}_1 & \dot{\vec{r}}_{E_1} &= L_1 \dot{\theta}_1 \hat{\theta}_1 & \ddot{\vec{r}}_{E_1} &= -L_1 \dot{\theta}_1^2 \hat{r}_1 + L_1 \ddot{\theta}_1 \hat{\theta}_1. \end{aligned} \quad (18)$$

Relative to E_1 , the kinematics of G_2 are given by

$$\begin{aligned} \vec{r}_{E_1 G_2} &= d_2 \hat{r}_2 & \dot{\vec{r}}_{E_1 G_2} &= d_2 \dot{\theta}_2 \hat{\theta}_2 & \ddot{\vec{r}}_{E_1 G_2} &= -d_2 \dot{\theta}_2^2 \hat{r}_2 + d_2 \ddot{\theta}_2 \hat{\theta}_2 \\ \vec{r}_{E_1 E_2} &= L_2 \hat{r}_2 & \dot{\vec{r}}_{E_1 E_2} &= L_2 \dot{\theta}_2 \hat{\theta}_2 & \ddot{\vec{r}}_{E_1 E_2} &= -L_2 \dot{\theta}_2^2 \hat{r}_2 + L_2 \ddot{\theta}_2 \hat{\theta}_2. \end{aligned} \quad (19)$$

Using the relation $\vec{r}_B = \vec{r}_A + \vec{r}_{AB}$, the absolute kinematics of G_2 and E_2 with respect to the origin are

$$\begin{aligned} \vec{r}_{G_2} &= \vec{r}_{E_1} + \vec{r}_{E_1 G_2} & \dot{\vec{r}}_{G_2} &= \dot{\vec{r}}_{E_1} + \dot{\vec{r}}_{E_1 G_2} & \ddot{\vec{r}}_{G_2} &= \ddot{\vec{r}}_{E_1} + \ddot{\vec{r}}_{E_1 G_2} \\ \vec{r}_{E_2} &= \vec{r}_{E_1} + \vec{r}_{E_1 E_2} & \dot{\vec{r}}_{E_2} &= \dot{\vec{r}}_{E_1} + \dot{\vec{r}}_{E_1 E_2} & \ddot{\vec{r}}_{E_2} &= \ddot{\vec{r}}_{E_1} + \ddot{\vec{r}}_{E_1 E_2}. \end{aligned} \quad (20)$$

The position, velocity, and acceleration of each CoM can be generalized as

$$\vec{r}_{G_i} = \sum_{j=1}^{i-1} \left(L_j \hat{r}_j \right) + d_i \hat{r}_i, \quad (21)$$

$$\dot{\vec{r}}_{G_i} = \sum_{j=1}^{i-1} \left(L_j \dot{\theta}_j \hat{\theta}_j \right) + d_i \dot{\theta}_i \hat{\theta}_i, \quad (22)$$

and

$$\ddot{\vec{r}}_{G_i} = \sum_{j=1}^{i-1} \left(-L_j \dot{\theta}_j^2 \hat{r}_j + L_j \ddot{\theta}_j \hat{\theta}_j \right) - d_i \dot{\theta}_i^2 \hat{r}_i + d_i \ddot{\theta}_i \hat{\theta}_i. \quad (23)$$

For LMB, the forces acting on each link are gravity

$$\vec{F}_{g_j} = -m_i g \hat{y} \quad (24)$$

and friction

$$\vec{F}_{f_j} = -\mu_f \dot{\theta}_j \hat{\theta}_j, \quad (25)$$

which must be equal to the force of the link defined by the product of its mass and acceleration,

$$\vec{F}_j = m_j \ddot{\vec{r}}_{G_j}. \quad (26)$$

The AMB of the system is done with respect to each joint or elbow of the pendulum, and is given by

$$\sum_{i=1}^n \vec{M}_E = \frac{d}{dt} \vec{H}_E. \quad (27)$$

This is done to avoid dealing with the reaction forces acting between links at each elbow. Since these forces go through the points in question, they do not contribute to AMB. The right-hand side of Equation 27 is the sum of torques. For each link, all forces acting below that link's upper elbow contribute. These forces are gravity and friction, where friction is assumed to act about a link's CoM.

$$\sum_{i=1}^n \vec{M}_E = \sum_{i=1}^n \sum_C \left(\vec{r}_{E_{i-1}C} \times \vec{F}_C \right) = \sum_{i=1}^n \sum_{j=1}^i \left((\vec{r}_{E_{i-1}G_j} \times \vec{F}_{g_j}) + (\vec{r}_{E_{i-1}G_j} \times \vec{F}_{f_j}) \right) \quad (28)$$

The left-hand side of Equation 27 is the sum of all change in angular momentum. For each link, all change in angular momentum below that link's upper elbow contribute.

$$\frac{d}{dt} \vec{H}_E = \sum_{i=1}^n \sum_C \left((\vec{r}_{E_{i-1}C} \times \vec{F}_C) + I_i^G \ddot{\theta}_i \hat{k} \right) = \sum_{i=1}^n \sum_{j=1}^i \left((\vec{r}_{E_{i-1}G_j} \times \vec{F}_j) + I_i^G \ddot{\theta}_i \hat{k} \right) \quad (29)$$

In these generalized formulae,

$$\vec{r}_{E_{i-1}G_j} = \vec{r}_{G_j} - \vec{r}_{E_{i-1}}, \quad (30)$$

and

$$\vec{r}_{E_0} = \vec{0} \quad (31)$$

when $i = 1$. The AMB calculations produce vector quantities purely in the \hat{k} direction, so dotting both sides by \hat{k} results in n differential equations giving motion of each link $\ddot{\theta}_i(\theta_1, \dots, \theta_n, \dot{\theta}_1, \dots, \dot{\theta}_n)$.

4 Differential Algebraic Equations

The DAE solutions to the n -link pendulum and n -bar linkage employ maximal coordinates: θ_i , x_{G_i} , and y_{G_i} . These give $3n$ differential equations, but involve the reaction forces at each elbow \vec{R}_i , which are unknown quantities. To produce solvable systems, constraint equations are used, which relate the motions of the elbows. The kinetics of the system are now analyzed about CoMs such that the reaction forces contribute to LMB and AMB.

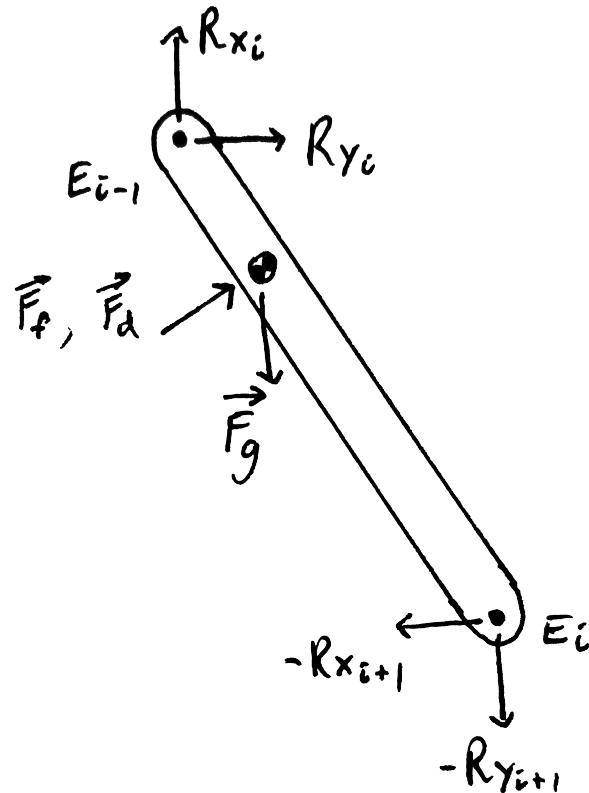


Figure 3: General Free Body Diagram

4.1 Pendulum

The n -link pendulum requires $2n$ constraint equations, resulting in a system of $5n$ differential equations with $5n$ unknowns. This can be represented as

$$[M] \cdot [x] = [b], \quad (32)$$

or

$$\begin{bmatrix}
 \begin{bmatrix}
 m_1 & 0 & \cdots & 0 & 0 \\
 0 & m_1 & \cdots & 0 & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & \cdots & m_n & 0 \\
 0 & 0 & \cdots & 0 & m_n
 \end{bmatrix} & [0] & \begin{bmatrix} \ddot{x}_{G_1} \\ \ddot{y}_{G_1} \\ \vdots \\ \ddot{x}_{G_n} \\ \ddot{y}_{G_n} \end{bmatrix} & \begin{bmatrix} F_{x_1} \\ F_{y_1} \\ \vdots \\ F_{x_n} \\ F_{y_n} \end{bmatrix} \\
 \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} I_1^G & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_n^G \end{bmatrix} & \begin{bmatrix} \text{LMB} \\ \text{AMB} \\ \vdots \\ \text{AMB} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \\
 \begin{bmatrix} \text{Constraint} \end{bmatrix} & \begin{bmatrix} \text{Constraint} \end{bmatrix} & [0] & \begin{bmatrix} R_{x_1} \\ R_{y_1} \\ \vdots \\ R_{x_n} \\ R_{y_n} \end{bmatrix} & \begin{bmatrix} \text{Constraint} \end{bmatrix}
 \end{bmatrix} = , \quad (33)$$

which has the solutions

$$[x] = [M]^{-1} \cdot [b]. \quad (34)$$

With reaction forces acting with at each elbow, the LMB about each CoM is

$$\text{LMB}_i = \sum \vec{F}_i = m_i \ddot{\vec{r}}_{G_i} = \begin{cases} \vec{R}_i - \vec{R}_{i+1} + \vec{F}_{g_i}, & \text{if } i < n. \\ \vec{R}_i + \vec{F}_{g_i}, & \text{if } i = n. \end{cases} \quad (35)$$

where

$$\vec{R}_i = R_{x_i} \hat{x} + R_{y_i} \hat{y}. \quad (36)$$

From this, $2n$ equations are produced:

$$\text{LMB}_{x_i} == \sum F_{x_i} = m_i \ddot{x}_{G_i} = \begin{cases} R_{x_i} - Rx_{i+1}, & \text{if } i < n. \\ R_{x_i}, & \text{if } i = n. \end{cases} \quad (37)$$

$$\text{LMB}_{y_i} == \sum F_{y_i} = m_i \ddot{y}_{G_i} = \begin{cases} R_{y_i} - Ry_{i+1} + F_{g_i}, & \text{if } i < n. \\ R_{y_i} - F_{g_i}, & \text{if } i = n. \end{cases} \quad (38)$$

Note that in the above equations, there is no lower reaction force for the last link.

Because the kinetics now involve the reaction forces, it is necessary to calculate the kinematics of each elbow instead of finding the acceleration of the CoMs:

$$\begin{aligned} \vec{r}_{E_{i-1}} &= \vec{r}_{G_i} - \vec{r}_{E_{i-1}G_i} \\ \vec{r}_{E_i} &= \vec{r}_{G_i} + \vec{r}_{G_iE_i}, \end{aligned} \quad (39)$$

and

$$\begin{aligned} \ddot{\vec{r}}_{E_{i-1}} &= \ddot{\vec{r}}_{G_i} - \ddot{\vec{r}}_{E_{i-1}G_i} \\ \ddot{\vec{r}}_{E_i} &= \ddot{\vec{r}}_{G_i} - \ddot{\vec{r}}_{G_iE_i}, \end{aligned} \quad (40)$$

where $\vec{r}_{E_{i-1}G_i}$ is as defined in Equation 30. Similarly

$$\vec{r}_{G_iE_i} = \vec{r}_{E_i} - \vec{r}_{G_i} \quad (41)$$

and

$$\ddot{\vec{r}}_{G_iE_i} = \ddot{\vec{r}}_{E_i} - \ddot{\vec{r}}_{G_i}, \quad (42)$$

where

$$\vec{r}_{E_i} = \sum_{j=1}^{i-1} \left(L_j \hat{r}_j \right) \quad (43)$$

and

$$\ddot{\vec{r}}_{E_i} = \sum_{j=1}^{i-1} \left(-L_j \dot{\theta}_j^2 \hat{r}_j + L_j \ddot{\theta}_j \hat{\theta}_j \right). \quad (44)$$

AMB is taken about the CoM of each link, producing n equations:

$$\text{AMB}_i == \sum \vec{M}_G = \frac{d}{dt} \dot{\vec{H}}_G = I_i^G \ddot{\theta}_i \hat{k} = \begin{cases} (-\vec{r}_{E_{i-1}} \times \vec{R}_i) + (\vec{r}_{E_i} \times -\vec{R}_{i+1}), & \text{if } i < n. \\ (-\vec{r}_{E_{i-1}} \times \vec{R}_i), & \text{if } i = n. \end{cases} \quad (45)$$

Again, in the above equations, there are no reaction forces on the bottom of the last link. As in the Newton-Euler method, though this is a vector equation, its only nonzero components are in the \hat{k} direction, so it is appropriate to use $\text{AMB}_i = \{\text{AMB}_i\} \cdot \hat{k}$.

The constraint equations are:

$$\text{Constraint}_i == \begin{cases} \ddot{\vec{r}}_{E_{i-1}} = \vec{0}, & \text{if } i = 1. \\ \ddot{\vec{r}}_{E_{i-1}} = \ddot{\vec{r}}_{E_i}, & \text{if } i > 1. \end{cases} \quad (46)$$

For the first link, the acceleration of the elbow above it must be $\vec{0}$. For all other links, the acceleration of the elbow above must be the same as the acceleration of the elbow below of the previous link. This is a convoluted way of saying that the lower elbow of one link is the upper elbow of the next link, and that they must forever coincide.

DAEs can also produce solutions for a pendulum with an unfixed base. The position of the base of the pendulum can be referred to as \vec{r}_A , and follows the path defined by

$$\vec{r}_A = \alpha \cos(\omega_f t) \hat{\phi} = \alpha \cos(\omega_f t) (\sin(\phi) \hat{x} + \cos(\phi) \hat{y}), \quad (47)$$

and assuming ϕ is constant the base accelerates at

$$\ddot{\vec{r}}_A = -\alpha \omega_f^2 \cos(\omega_f t) \hat{\phi} = -\alpha \omega_f^2 \cos(\omega_f t) (\sin(\phi) \hat{x} + \cos(\phi) \hat{y}). \quad (48)$$

With the base now accelerating, the $i = 1$ condition of Equation 46 becomes

$$\text{Constraint}_1 == \ddot{\vec{r}}_{E_0} = \ddot{\vec{r}}_A. \quad (49)$$

4.2 Linkage

The DAE approach to the n -bar linkage is extremely similar to that for the pendulum above. In this derivation, the n^{th} bar is neglected, as it contributes nothing to the dynamics of the system if both of its elbows are assumed to be fixed. Thus the only difference with the n -bar linkage is that the last ($n-1$) link has reaction forces at both elbows. This adds two constraint equations, so that Equation 33 becomes

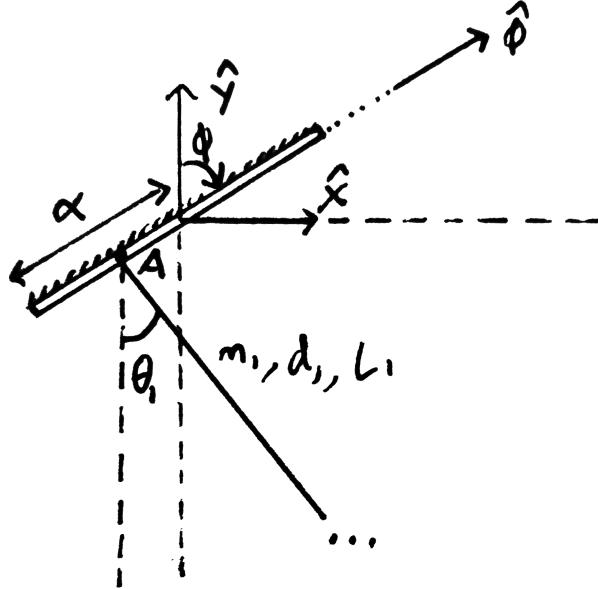


Figure 4: Driven Base

$$\begin{bmatrix}
 \begin{bmatrix} m_1 & 0 & \cdots & 0 & 0 \\ 0 & m_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & m_n & 0 \\ 0 & 0 & \cdots & 0 & m_n \end{bmatrix} & [0] & \begin{bmatrix} \text{LMB} \end{bmatrix} & \begin{bmatrix} \ddot{x}_{G_1} \\ \ddot{y}_{G_1} \\ \vdots \\ \ddot{x}_{G_n} \\ \ddot{y}_{G_n} \\ \theta_1 \\ \ddot{\theta}_n \end{bmatrix} & \begin{bmatrix} F_{x_1} \\ F_{y_1} \\ \vdots \\ F_{x_n} \\ F_{y_n} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
 [0] & \begin{bmatrix} I_1^G & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_n^G \end{bmatrix} & \begin{bmatrix} \text{AMB} \end{bmatrix} & \begin{bmatrix} R_{x_1} \\ R_{y_1} \\ \vdots \\ R_{x_n} \\ R_{y_n} \\ R_{x_{n+1}} \\ R_{y_{n+1}} \end{bmatrix} & \begin{bmatrix} \text{Constraint} \end{bmatrix} \\
 \begin{bmatrix} \text{Constraint} \end{bmatrix} & \begin{bmatrix} \text{Constraint} \end{bmatrix} & [0] & \begin{bmatrix} \text{Constraint} \end{bmatrix} & \begin{bmatrix} \text{Constraint} \end{bmatrix}
 \end{bmatrix} = . \quad (50)$$

The system now has $5n + 2$ equations with $5n + 2$ unknowns.

The LMB equations are now

$$\text{LMB}_{x_i} == \sum F_{x_i} = m_i \ddot{x}_{G_i} = R_{x_i} - R_{x_{i+1}}, \quad \text{for all } i < n. \quad (51)$$

and

$$\text{LMB}_{y_i} == \sum F_{y_i} = m_i \ddot{y}_{G_i} = R_{y_i} - R_{y_{i+1}} - F_{g_i}, \quad \text{for all } i < n. \quad (52)$$

The AMB equations are now

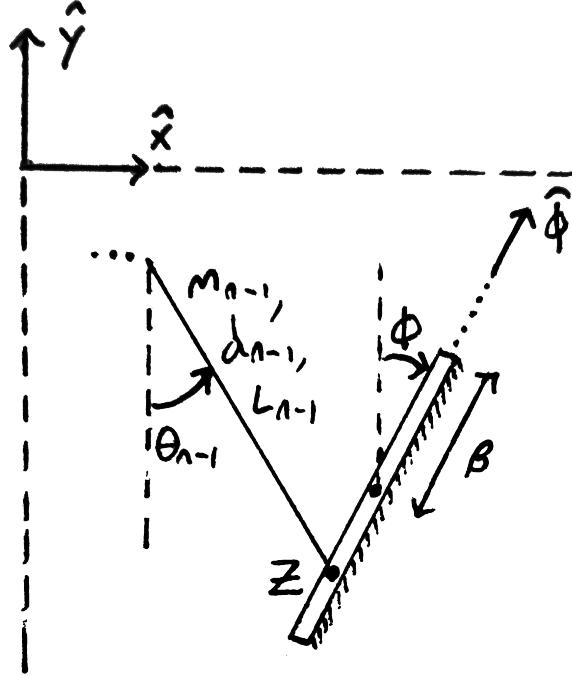


Figure 5: Driven End

$$\text{AMB}_i == \sum \vec{M}_G = \frac{d}{dt} \dot{\vec{H}}_G = I_i^G \ddot{\theta}_i \hat{k} = (-\vec{r}_{E_{i-1}} \times \vec{R}_i) + (\vec{r}_{E_i} \times -\vec{R}_{i+1}), \quad \text{for all } i < n. \quad (53)$$

The constraint equations are more complicated:

$$\text{Constraint}_i == \begin{cases} \ddot{\vec{r}}_{E_{i-1}} = \vec{0}, & \text{if } i = 1. \\ \ddot{\vec{r}}_{E_{i-1}} = \ddot{\vec{r}}_{E_i}, & \text{for } 1 < i < n-1. \\ \ddot{\vec{r}}_{E_{i-1}} = \ddot{\vec{r}}_{E_i} & \text{if } i = n-1. \\ \ddot{\vec{r}}_{E_i} = \vec{0}, & \end{cases} \quad (54)$$

These equations are nearly the same as in Equation 46, except that an additional constraint is added for $i = n - 1$, being that the acceleration of the end of the last link must be $\vec{0}$.

Where before an oscillating base was incorporated into the DAE pendulum, here the base is kept fixed and the end is allowed to oscillate. It is important to note that this breaks the condition that the n^{th} bar is fixed, so the system essentially becomes a pendulum with a driven end, being the $n - 1$ link. The motion of the end of this link is defined as

$$\vec{r}_Z = \beta \cos(\omega_f t) \hat{\phi} = \beta \cos(\omega_f t) (\sin(\phi) \hat{x} + \cos(\phi) \hat{y}), \quad (55)$$

and again assuming that ϕ is constant, it accelerates at

$$\ddot{\vec{r}}_Z = -\beta \omega_f^2 \cos(\omega_f t) \hat{\phi} = -\beta \omega_f^2 \cos(\omega_f t) (\sin(\phi) \hat{x} + \cos(\phi) \hat{y}). \quad (56)$$

The last condition of Equation 54 now becomes

$$\text{Constraint}_{n-1} == \begin{cases} \ddot{\vec{r}}_{E_{n-1}} = \ddot{\vec{r}}_{E_n} \\ \ddot{\vec{r}}_{E_n} = \ddot{\vec{r}}_Z \end{cases} \quad (57)$$

The n -link pendulum can have arbitrary initial conditions, but the n -bar linkage requires additional thought. The relative lengths, orientations, and angular velocities of the links must be consistent at all times. In the MATLAB implementation of the n -bar linkage, the n^{th} bar is not considered in the symbolic derivation because it remains stationary. The user may specify the initial lengths and angles of the first $n-2$ links, and then the last dynamic $n-1$ link will be automatically generated to satisfy the constraint.

Lastly, a driving force is considered which acts on only the first link. This is included for visualization purposes when simulating an n -bar linkage numerically. Many mechanical systems are based on 4-bar linkages, and these real-world systems are normally driven by a motor for some mechanical purpose. In a numerical simulation of one of these apparatus, a driving force is necessary to complete at least one cycle (an initial angular velocity could be given, but that is more challenging to model because it must obey the constraints). Therefore a driving force

$$\vec{F}_d = -\mu_d \dot{\theta}_i \hat{\theta}_i \quad (58)$$

is defined, and the AMB for the first link becomes

$$\text{AMB}_1 == I_1^G \ddot{\theta}_1 \hat{k} = (-\vec{r}_{E_0} \times \vec{R}_1) + (\vec{r}_{E_1} \times -\vec{R}_2) + (-\vec{r}_{T_1} \times \vec{F}_d). \quad (59)$$

As long as μ_d is very small, the driving force will allow an appropriate system to complete full cycles of its motion, though over long integration times, the system will be noticeably accelerated.

5 Results & Analysis

The three methods outlined above are each implemented in MATLAB's symbolic toolbox. Four symbolic derivation functions are created: three deriving the n -link pendulum solutions using each of the three methods, and one deriving the n -bar linkage solutions using DAEs. Each of these functions, when called with a number of links, produce two function files: one for the [M] matrix, and the other for the [b] vector. These structures are created using the `equationsToMatrix` function.

The initial conditions and simulation parameters are specified in `npend.m`, which calls and runs the symbolic function corresponding to the specified method. The [M] and [b] solution functions are then used to call the right-hand side functions corresponding to the specified method via `ode45`, producing the numerical solution for the simulation.

5.1 Energy Conservation

The numerical solutions for a simulation return the angles θ_i and angular velocities $\dot{\theta}_i$ at each discrete time step. From these and the simulation parameters, the positions and orientations of each link are calculated at each time step. Additionally, the kinetic and potential energy of the entire system are calculated as they are in the symbolic deriver for the Lagrange method. From these values, total energy is computed for the system at each time step.

$$\mathcal{H} = E_k + E_p \quad (60)$$

This is used to check the physical correctness of the solutions by plotting total energy as a function of simulation time. Setting `ode45`'s tolerances to 1×10^{-8} , total energy generally varied between 1×10^{-8} and 1×10^{-4} . Figure 6 shows the total energy of a 100-link pendulum using the DAE method, where all links are identical and begin at $\theta_0 = 90^\circ$. The small deviations in energy are due to numerical integration, and confirm that the derivations outlined above and implemented in MATLAB are physically sound.

The nominal pendulum setup used for analysis has 10 identical uniform density links, each initially at $\theta_0 = 90^\circ$ and $\dot{\theta}_0 = 0$, and `ode45`'s tolerances at 1×10^{-8} . Figure 7 shows the total energies of the nominal pendulum solved with each of the three methods. It is unclear which method has the greatest

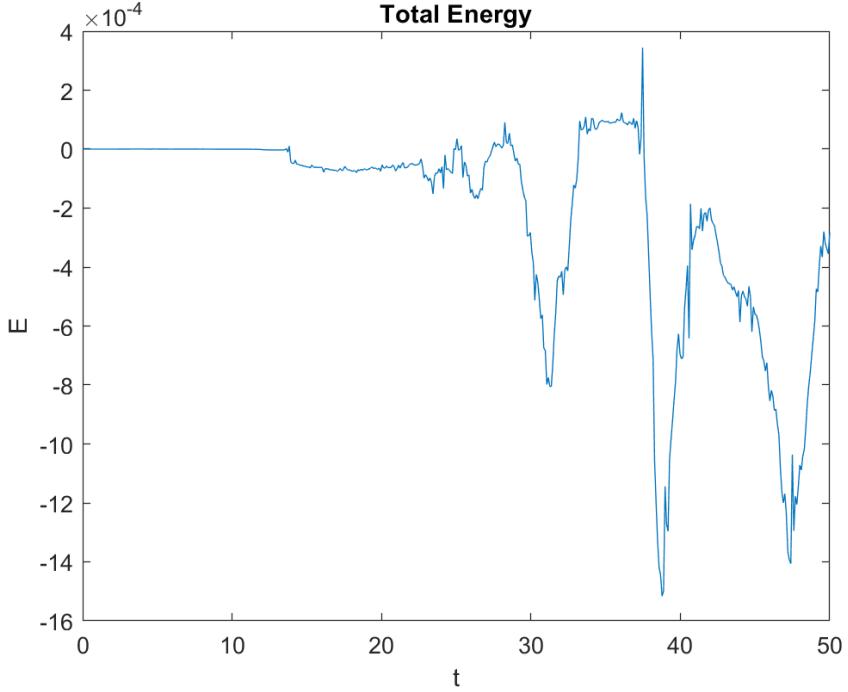


Figure 6: Total Energy of 100 Link Pendulum: DAE Method

numerical fidelity. The same nominal simulations are performed again, but with an acceleration due to gravity five times greater, and `ode45`'s tolerances set to 1×10^{-4} . These conditions reveal in Figure 8 that though all three numerical solution methods are fairly accurate, the DAE approach is slightly better than the Newton-Euler approach, which in turn slightly out-performs the Lagrange method.

5.2 Normal Modes of Oscillation

Lagrange's method produces n equations for the n -link pendulum (Equation 15). To simulate the system, the full n non-linear equations of motion are numerically calculated. This results in chaotic motion. Alternatively, the system can be linearized about a known equilibrium point. The only stable equilibrium is in the case that all $\theta_i = 0$. Linearizing about this point yields the normal modes of oscillation for the system.

In this regime, all θ_i are very small, so the following substitutions can be made within the lagrange equations:

$$\begin{aligned}
\sin(\theta_i) &\rightarrow \theta_i \\
\sin(\theta_i - \theta_j) &\rightarrow \theta_i - \theta_j \\
\cos(\theta_i) &\rightarrow 1 \\
\cos(\theta_i - \theta_j) &\rightarrow 1 \\
\theta_i^2 &\rightarrow 0 \\
\theta_i \theta_j &\rightarrow 0.
\end{aligned} \tag{61}$$

With these substitutions, all $\dot{\theta}_i$ terms drop out, and the equations are linear in $\ddot{\theta}_i$ and θ_i . The MATLAB function `equationsToMatrix` can be used again, this time solving once for $\ddot{\theta}_i$ to create the mass matrix $[M]$ and once for θ_i to create the stiffness matrix $[K]$. The linear system can now be expressed as

$$[M]\ddot{\theta}_i + [K]\theta_i = [0], \tag{62}$$

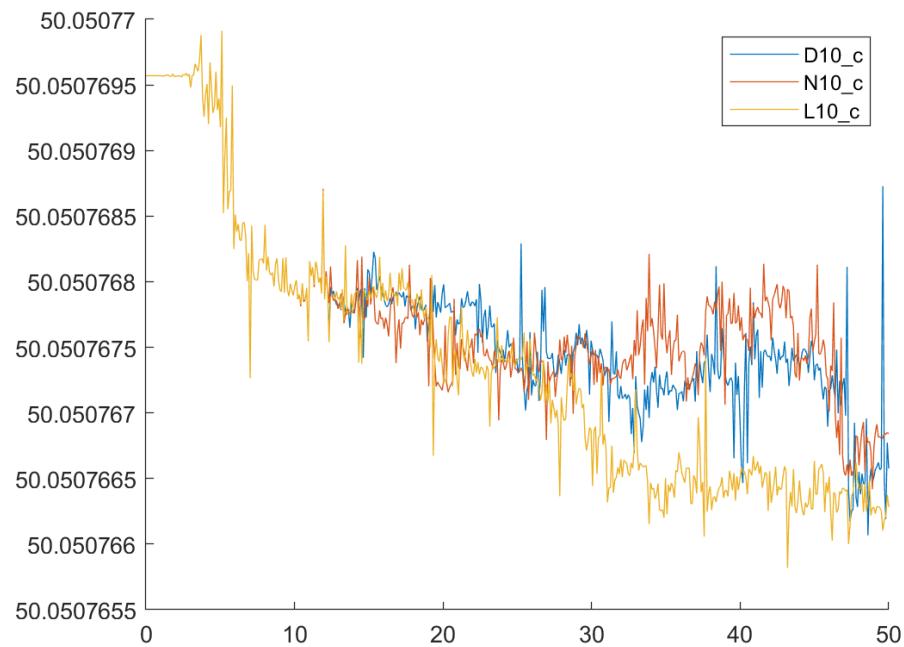


Figure 7: Total Energy Comparison

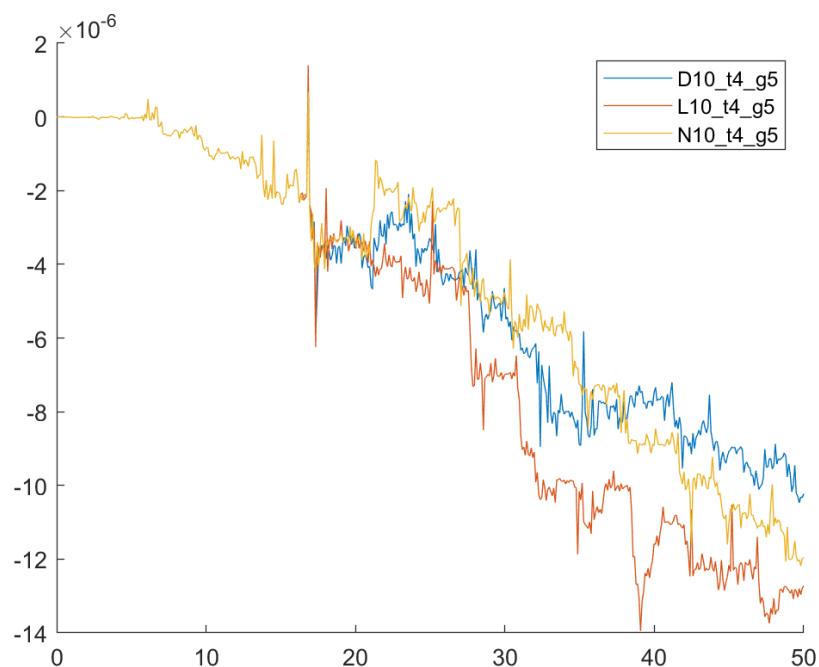


Figure 8: Total Energy Comparison

and the solutions define the normal modes of oscillation of the system. These solutions are found in MATLAB by `eig(M\K)`, which produces n eigenvectors \vec{v}_i of length n defining θ_{0i} for each mode and a diagonal $n \times n$ matrix whose elements are eigenvalues λ_i , defining the squared oscillation frequencies of the modes. To accurately simulate these modes, it was necessary to scale the eigenvector initial conditions down by a factor of 10. This ensures that the system remains close enough to the equilibrium position where this linearization is valid.

The deviation of the linearized models can be judged by comparing them to the actual normal modes, which are simply $\vec{v}_i \cos(\lambda_i)$. The motions of the links as well as their deviations from the actual normal modes are presented in Appendix A.

For visualizations of these normal modes, see:

$n = 2:$./animations/n2m1.mat or [here]
	./animations/n2m2.mat or [here]
$n = 3:$./animations/n3m1.mat or [here]
	./animations/n3m2.mat or [here]
	./animations/n3m3.mat or [here]
$n = 4:$./animations/n4m1.mat or [here]
	./animations/n4m2.mat or [here]
	./animations/n4m3.mat or [here]
	./animations/n4m4.mat or [here]
$n = 5:$./animations/n5m1.mat or [here]
	./animations/n5m2.mat or [here]
	./animations/n5m3.mat or [here]
	./animations/n5m4.mat or [here]
	./animations/n5m5.mat or [here]

5.3 Standing Wave Modes

A hanging n -link pendulum with identical, uniform density links as $n \rightarrow \infty$ is an approximation of a continuous rope or chain fixed at one end. This system can be modeled by a one-dimensional wave equation (Y. Verbin, 2014 [1]):

$$gy \frac{\partial^2 u(y)}{\partial y^2} + g \frac{\partial u(y)}{\partial y} + \omega^2 u(y) = 0, \quad (63)$$

where y is length from 0 to the end of the rope L_{tot} , $u(y)$ defines the normal modes of oscillation, and ω is the frequency of oscillation. The solution to this equation yields natural frequencies at which to drive the system ω_d :

$$\omega_d = \frac{\zeta_n}{2} \sqrt{\frac{g}{L_{tot}}}, \quad (64)$$

where ζ_n are the zeros of the Bessel function J_0 .

By driving an n -link pendulum (initially at the $\theta_i = 0$ equilibrium point) at these frequencies with small amplitudes relative to the total length, the standing waves begin to build. Interestingly, the waves' amplitudes grow and decay sinusoidally. Plotting the position of the end over time alongside the driving amplitude over time (Figure 9) shows that there is wave interference and a beat frequency. Similar plots for the first five standing wave modes are presented in Appendix B.

A beat effect is caused when there are two similar frequencies at play in a system. Since the simulated system is discrete and only approximates a rope, its actual natural frequency ω_n will be slightly different

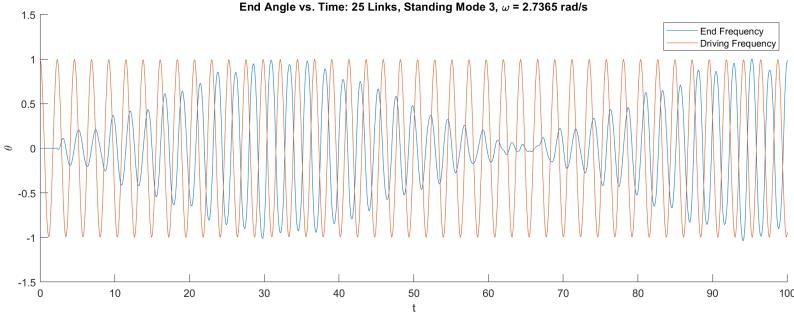


Figure 9: Standing Wave Interference Beat Frequency

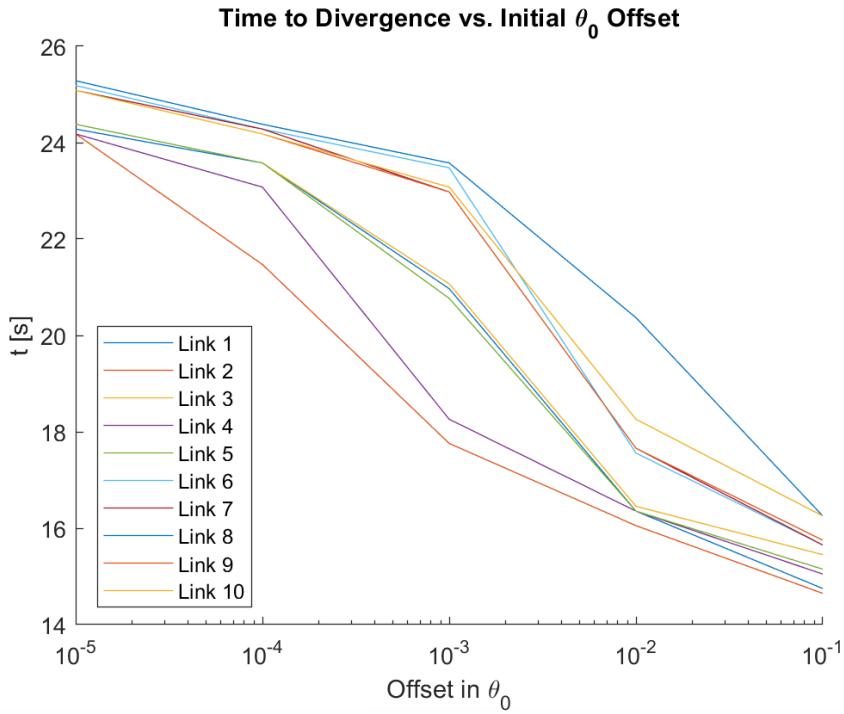


Figure 10: Performance Analysis 1-50 Links

than the calculated driving frequency ω_d . The beat frequency is therefore $|\omega_n - \omega_d|$. The better the simulation approximates an actual rope, the smaller this value will be, so beats will last longer. If ω_n could be found analytically, $\omega_d = \omega_n$ could be used such that the beat frequency would disappear, and the system would be in resonance. This would cause the standing waves to grow until the system becomes chaotic.

For visualizations of these standing wave modes, see:

Mode 1:	./animations/smode1_50.mat	or [here]
Mode 2:	./animations/smode2_50.mat	or [here]
Mode 3:	./animations/smode3_50.mat	or [here]
Mode 4:	./animations/smode4_50.mat	or [here]
Mode 5:	./animations/smode5_50.mat	or [here]

5.4 Additional Interesting Cases

- **100-Link Pendulum:** With many links, a pendulum becomes a model of a rope. At 100 links, it is difficult to identify the system as a pendulum rather than a rope. (See `./animations/DAEp100.mat` or [here]).
- **3-Link Inverted Pendulum:** A simple pendulum can be stabilized upright when a high frequency, low amplitude oscillation is applied at the base. It is possible to achieve a similar result using three links, though the last link is much less stable than a simple inverted pendulum. The masses of the links were set to 10, 1, and 0.1, their lengths to 0.01, 0.1, 1, their centers of mass were located at their ends, their initial angles were 0° , 0° , 180° , and their initial angular velocities were 0. Gravity was set to 0.5. (See `./animations/DAEp3_inv_stbl.mat` or [here]).
- **4-Bar Pumpjack Linkage:** With specific initial conditions, it is possible to produce circular motion with a 4-bar linkage. This is the crank-rocker system, where the combined length of two adjacent moving links is less than the combined length of the other moving link and the fixed link. Of the two shorter links, the one with one end fixed will move in a circle, if driven. This is the basis for an oil pumpjack. (See `./animations/DAE14_pumpjack.mat` or [here]).
- **Damped Pendulum:** A friction force is added to the pendulum using the Newton-Euler method, with $\mu_f > 0$. This clearly results in a damping effect of friction on a pendulum, and appears more realistic than a simulation which does not slow down. Additionally, the total energy versus time clearly shows that a non-conservative force is being applied. (See `./animations/NE10_fric10.mat` or [here]: 10-link pendulum; `./animations/NE3_fric10.mat` or [here]: 3-link pendulum).
- **Unstable Equilibrium:** A stationary n -link pendulum has equilibria with any combination of $\theta_0 = 0^\circ$ or 180° and $\dot{\theta}_0 = 0$ for each link. The only stable one of these will be when all $\theta_0 = 0^\circ$. All other equilibria are unstable. On paper, an n -link pendulum with such initial conditions will not move at all, though in the real world and with numerical solutions, slight perturbations will push the system away from these unstable equilibria. (See `./animations/N10_180.mat`, `./animations/L10_180.mat`, and `./animations/D10_180.mat` or [here], [here], and [here]).
- **Shaking End:** The n -bar linkage as outlined above and implemented in MATLAB can have an oscillating end (which as mentioned brakes the constraint that the n^{th} link be fixed). When the end of the 31-bar linkage is sinusoidally driven perpendicular to the direction between the base and end, it appears to shake the intermediate $n-1$ links as a string. If the frequency of oscillation is calibrated correctly, the intermediate links should exhibit standing wave oscillations. (See `./animations/DAE130_shake1.mat` or [here]: nearly 2nd order standing waves are found).
- **Stretched End:** When the end of a 21-bar linkage is sinusoidally driven in the direction between the base and end, it appears to stretch the intermediate $n-1$ links as a rubber band. When the amplitude of oscillation is calibrated to be close to the combined length of the intermediate links, the system is nearly stretched to its limit, and high frequency vibrations can be observed. (See `./animations/DAE120_stretch1.mat` or [here]; `./animations/DAE130_standing1.mat` or [here]: the end is driven for only half a period, resulting in oscillation of the intermediate links showing the 1st order standing wave).
- **No Gravity:** With gravity set to 0 but $\dot{\theta}_0 \neq 0$ for one, some, or all links, the n -link pendulum will still move chaotically. When only the first link is given an initial angular velocity, the system initially appears chaotic, but eventually approaches circular motion (See `./animations/DAEp10_1w.mat` or [here]). When only the last link is given an initial angular velocity, the system begins to behave as though the initial motion of the last link 'drags' the rest of the pendulum, but eventually, becomes increasingly chaotic (See `./animations/DAEp10_10w.mat` or [here]).

5.5 Perturbations and Divergence

Any multi-link pendulum is a chaotic system. The slightest deviation in initial condition will lead to exponential growth. To analyze this phenomenon, the 10-link pendulum is used. First, the nominal case is simulated, where each link is identical, initially at $\theta_0 = 90^\circ$, and with no angular acceleration. Then,

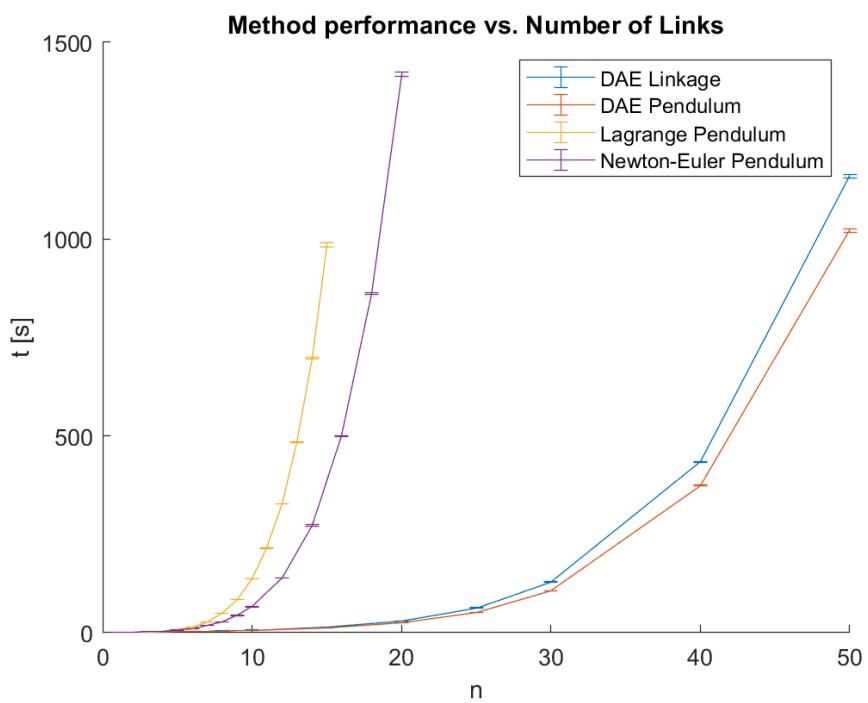


Figure 11: Performance Analysis 1-50 Links

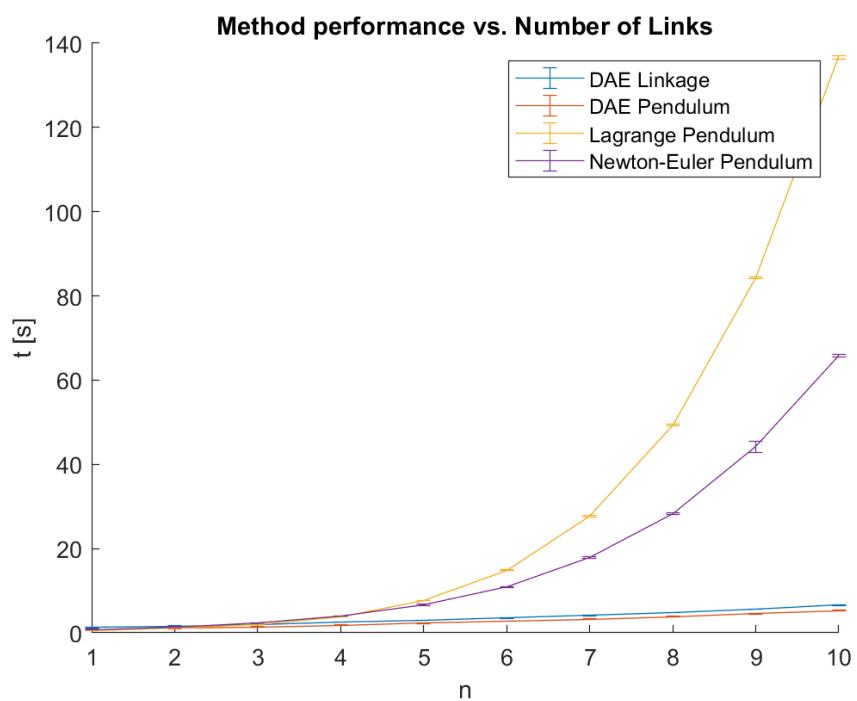


Figure 12: Performance Analysis 1-10 Links

fifty simulations are run in which $\theta_{0i} = \theta_0 + \epsilon_j$, where i is the link number and $\epsilon_j = 10^{-j}$ for $j = 1, 2, 3, 4$, and 5 .

To determine when the perturbed simulation diverges from the nominal simulation, an algorithm is written to compute the how separated each corresponding pair of link ends is between the two cases. The maximum end displacement of 10 links is calculated at each time step. Divergence is defined as when this value exceeds 0.1% of the uniform link length L (See `comp.m`). Though this threshold is based on visual inspection, it is used consistently within this experiment.

Each of the fifty perturbed simulations are compared in this way to the nominal case, and their times to divergence are recorded. The results are presented in Figure 9. It is clear that the larger the perturbation, the quicker the simulation is to diverge. Overall, it also appears that the further the perturbed link is from the base, the longer it takes to diverge, though this result is not as pronounced. Moreover, with even smaller perturbations applied (not shown in the figure), the time to divergence seems to converge.

5.6 Performance

Finally, the time to produce symbolic solutions using each numerical method is analyzed. For each value in a list of ns to solve for, a specified method is run m times under controlled conditions. The mean times for each method given n links to solve for are shown in Figures 10 and 11.

The DAE method clearly outperforms the Newton-Euler and Lagrange approaches. This may seem counter-intuitive, since DAEs use maximal coordinates and require $5n$ (or $5n + 2$) equations to solve. However, the equations created by the DAE method are less complicated because they only relate the motions of adjacent components, whereas the Lagrange approach creates one equation for the whole system, and in the Newton-Euler method, the torque on each link depends on all links below it. Therefore it is reasonable that the DAE approach is the most efficient. This fact and the earlier result concerning total energy conservation demonstrating the low error of the Differential Algebraic Equations show that this is the best numerical method to use to simulate complex, chaotic systems such as the n -link pendulum and n -bar linkage.

A Normal Modes

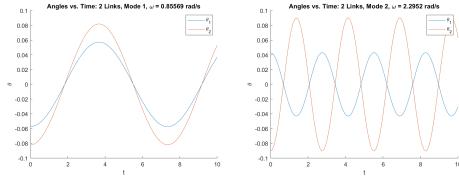


Figure 13: 2 Links

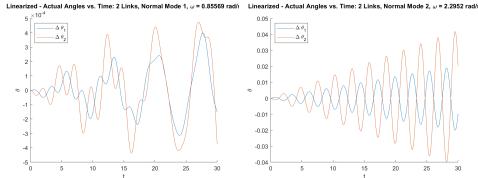


Figure 14: 2 Links, Linearization Divergence

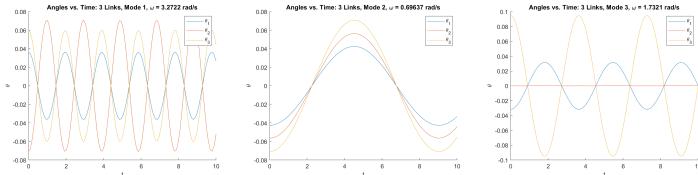


Figure 15: 3 Links

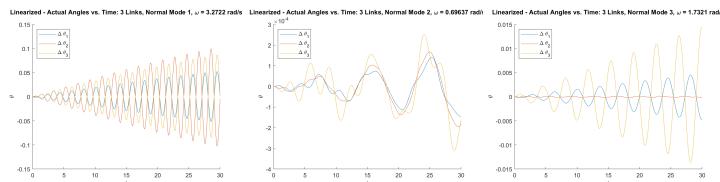


Figure 16: 3 Links, Linearization Divergence

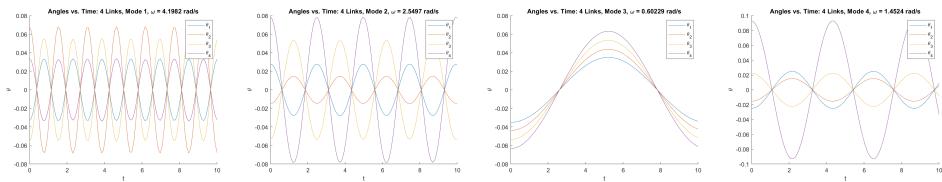


Figure 17: 4 Links

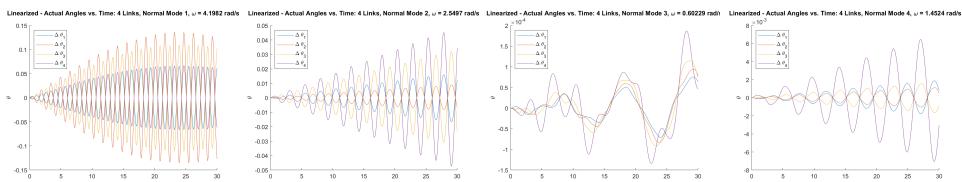


Figure 18: 4 Links, Linearization Divergence

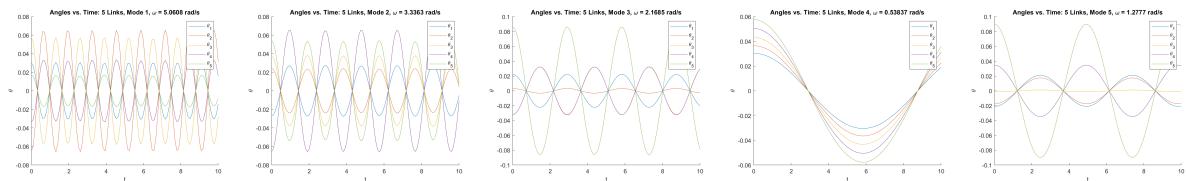


Figure 19: 5 Links

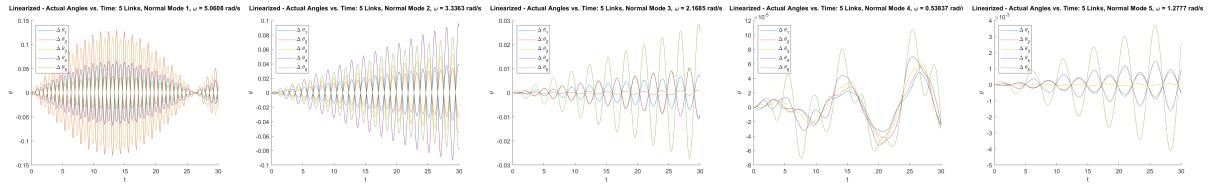


Figure 20: 5 Links, Linearization Divergence

B Standing Wave Modes

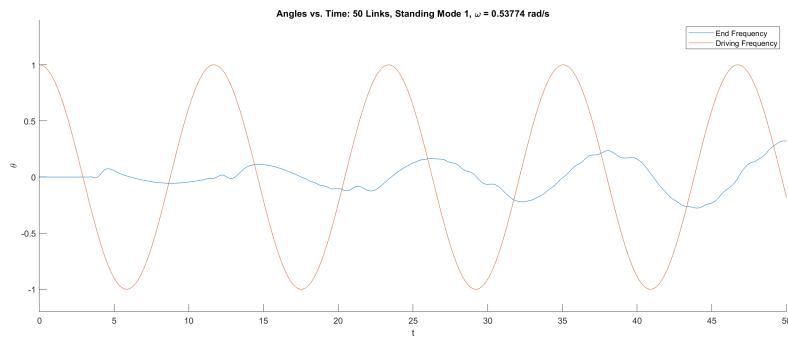


Figure 21: 1st Standing Wave Mode

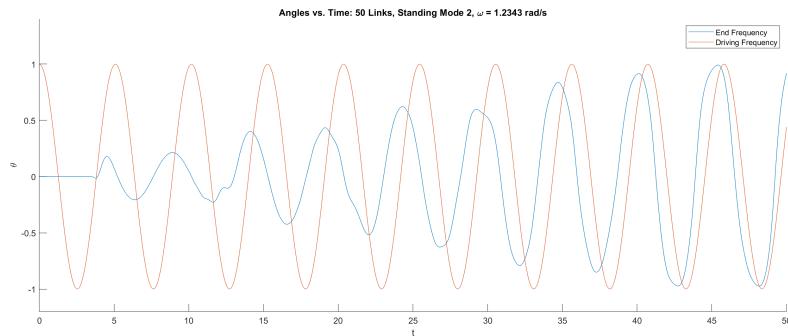


Figure 22: 2nd Standing Wave Mode

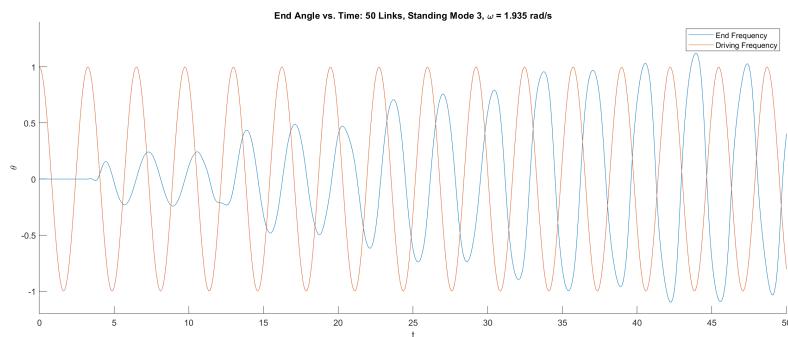


Figure 23: 3rd Standing Wave Mode

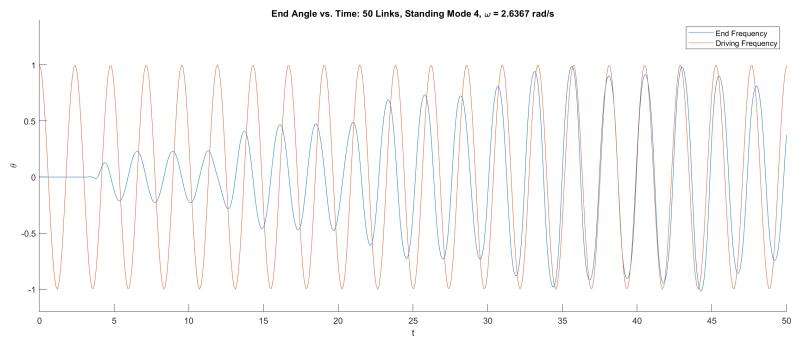


Figure 24: 4th Standing Wave Mode

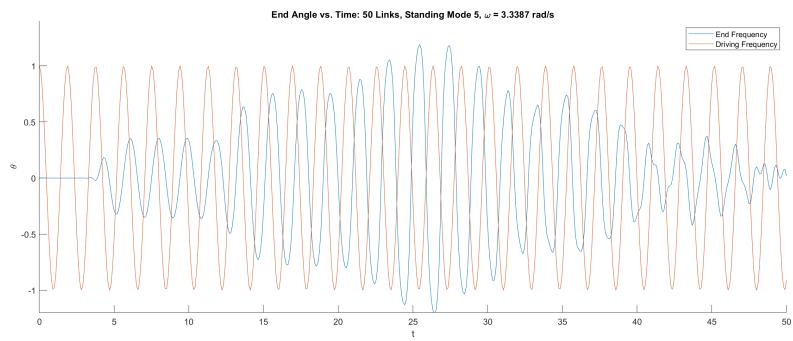


Figure 25: 5th Standing Wave Mode

References

- [1] Y. Verbin. “Boundary Conditions and Modes of the Vertically Hanging Chain”. In: (2014). DOI: 10.1088/0143-0807/36/1/015005. arXiv: 1412.1846.