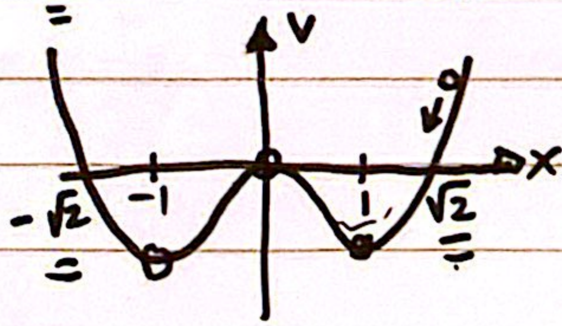
AME 522 Lecture 5

1/29/2025

Last time:

$$\to \text{Ex: } \dot{x} = x - x^3 = -\frac{dV}{dx} \Rightarrow V(x) = \underbrace{-\frac{1}{2}x^2 + \frac{1}{4}x^4}_{} + \overset{\uparrow 0}{C}$$



$V(x) = 0$

Normal forms for basic bifurcations:

$$\dot{x} = f(x, r)$$

parameter: r

state-space variable: x

(x^*, r_c)

Taylor expansion $f(x,r)$ around (x^*, r_c):

$$\dot{x} = \cancel{f(x^*, r_c)}^{\,0} + \overbrace{(x - x^*)}^{\text{small}} \underbrace{\frac{\partial f}{\partial x}\bigg|_{(x^*, r_c)}}_{0 \text{ for saddle-node}} + \overbrace{(r - r_c)}^{\text{small}} \underbrace{\frac{\partial f}{\partial r}\bigg|_{(x^*, r_c)}}_{a}$$

$$+ \frac{1}{2}(x - x^*)^2 \underbrace{\frac{\partial^2 f}{\partial x^2}\bigg|_{(x^*, r_c)}}_{b} + \cdots$$

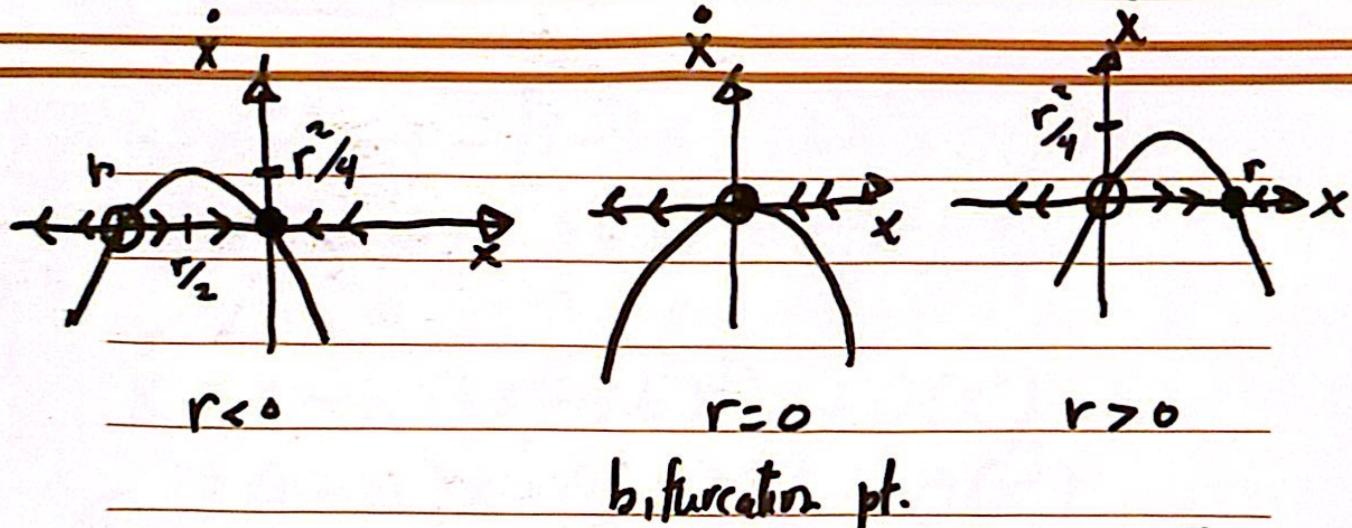
$$\boxed{\dot{x} = a(r - r_c) + b(x - x^*)^2}$$ Normal form for saddle-node bifurcation.

Transcritical bifurcation

Change of stability through a bifurcation pt.

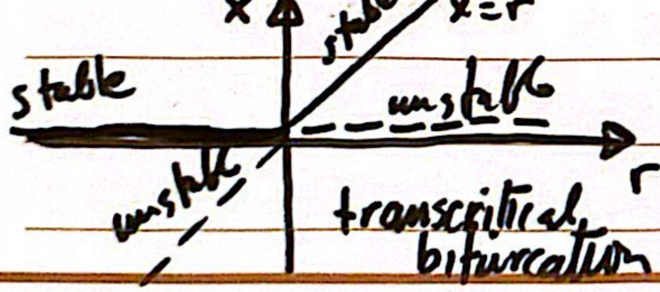
$\dot{x} = rx - x^2 = f(x)$ $\quad f(x) = 0 \Rightarrow x(r-x) = 0$

$x^* = 0, \; x^* = r$



$f' = r - 2x, \quad f'(x^*) = r - 2x^* = \begin{cases} r & x^* = 0 \\ -r & x^* = r \end{cases}$

↑ exchange of stability



Ex 4: $\dot{x} = x(1-x^2) - a(1-e^{-bx}) = f(x)$

- Show there is a transcritical bifurcation ✓ at $x=0$, when (a,b) satisfy a <u>certain</u> eqn.
- ✓ Find a formula for the fixed pt. that bifurcates from $x=0$.

fixed pts:

$f(x) = x(1-x^2) - a(1-e^{-bx}) = 0$

$\boxed{x^* = 0}$ fixed pt.

$$1-e^{-bx} = 1 - \left[1 - bx + \tfrac{1}{2}b^2x^2 + O(x^3)\right]$$
$$= bx - \tfrac{1}{2}b^2x^2 + O(x^3)$$

$$\dot{x} = x - a\left(bx - \tfrac{1}{2}b^2x^2\right) + O(x^3)$$
$$= \underbrace{(1-ab)}_{\equiv r}x + \left(\tfrac{1}{2}ab^2\right)x^2 + O(x^3)$$

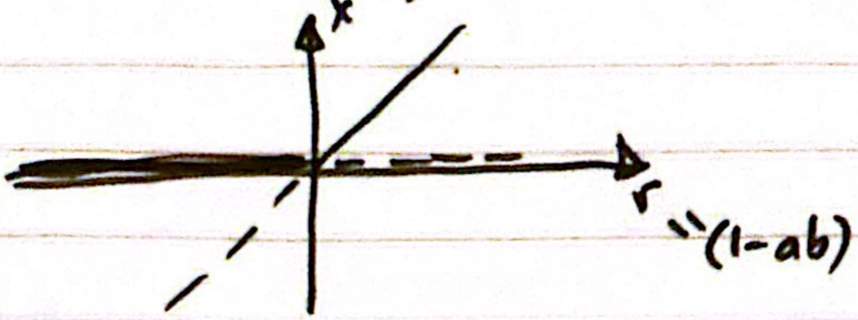
transcritical bifurcation occurs when $r=0 \Rightarrow ab=1$

$$\Big[\underbrace{(1-ab) + \tfrac{1}{2}ab^2x}_{=0}\Big]x = 0$$

$$\boxed{x^* = \frac{2(ab-1)}{ab^2}}$$

$(ab-1)$ must be small

bifurcation diagram:



Ex5: $\dot{x} = r \ln(x) + x - 1 = f(x)$

$\uparrow$ bifurcation parameter

$f(x) = 0 \Rightarrow \underline{x^* = 1}$ is a fixed pt $f(1) = 0$.

Analyze the dynamics near $x=1$ and show the system undergoes a transcritical bifurcation at r_c. Then, find new variables X, R so that $\underbrace{\dot{X} \cong RX - X^2}_{\text{normal form.}}$

Let $u = x - 1$ (shift fixed pt. to origin).

$\dot{u} = \dot{x}$

$= r \ln(1+u) + u$

$= r\left[u - \frac{1}{2}u^2 + O(u^3)\right] + u$ small u expansion

$\approx (r+1)u - \frac{1}{2}ru^2 + O(u^3)$

Transcritical bifurcation occurs at $r_c = -1$

Normal form:

Let $u = av$

$$a\dot{v} = (r+1)av - \tfrac{1}{2}ra^2v^2 + O(v^3).$$

divide by a:

$$\dot{v} = (r+1)v - \underbrace{\tfrac{1}{2}ra}_{=1}v^2 + O(v^3).$$

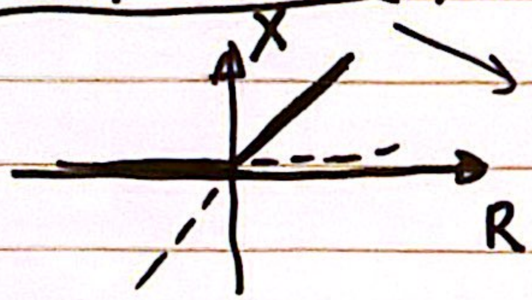
let $a = 2/r$

$$\dot{v} = \underbrace{(r+1)}_{R \equiv r+1}v - v^2 + O(v^3)$$

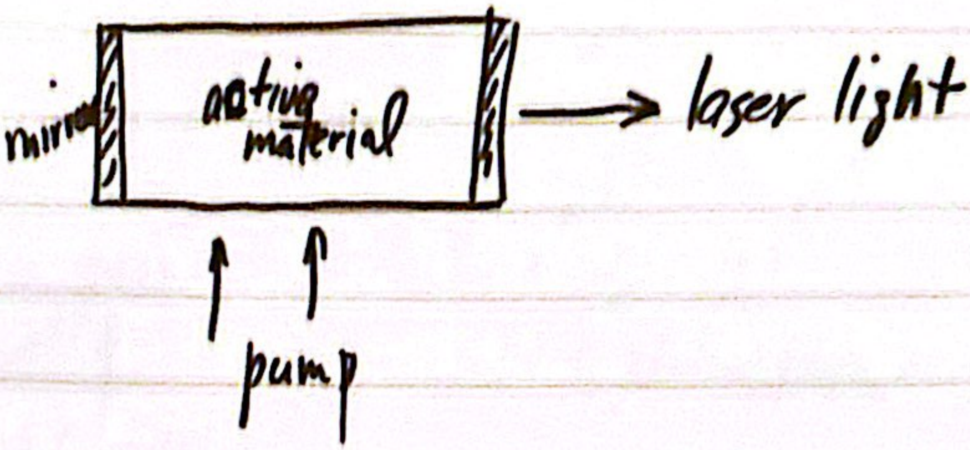
$R \equiv r+1$, $X = v$

$$\boxed{\dot{X} \cong RX - X^2} \quad X = v = u/a = \underline{\underline{\tfrac{1}{2}r(x-1)}}$$

normal form



Ex6: Solid-state laser model:



- Collection of "laser-active" atoms embedded in a solid state matrix, bounded partially by mirrors at each end.

- External energy source "pump" atoms out out of their ground states - each atom radiates energy:
- When pumping is too weak, the laser is an ordinary lamp with excited atoms oscillating independently emiting randomly phased light waves.
- Above some pumping threshold, the atoms begin to oscillate in phase, light becomes a laser, more coherent and intense.

Simplified model:

$n(t)$ # of photons in the laser field ≥ 0

$$\dot{n} = \text{gain} - \text{loss} = \overset{\text{gain}}{GnN} - \overset{\text{loss}}{kn}$$

escape of photons thru ends

stimulated emission
(photons stimulate excited atoms to emit additional photons)

$N(t)$: number of excited atoms ≥ 0

Key: After an excited atom emits a photon, it drops to a lower energy. N decreases by emission of photons:

$$N(t) = N_0 - \alpha n \quad (\alpha > 0)$$

$$\dot{n} = Gn(N_0 - \alpha n) - kn$$

$$\boxed{\dot{n} = (GN_0 - k)n - (\alpha G)n^2}$$

quadratic
nonlinear.

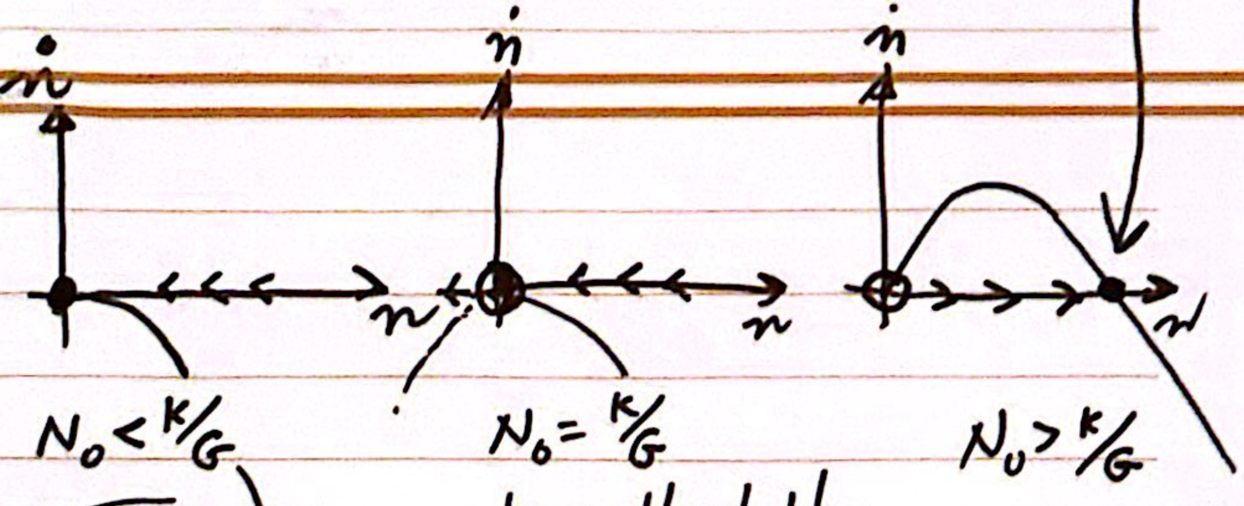
photons

$$f(n) = \left[(GN_0 - K) - (\alpha G)n\right]n = 0$$

$$n^* = 0$$
$$n^* = (GN_0 - K)/(\alpha G)$$

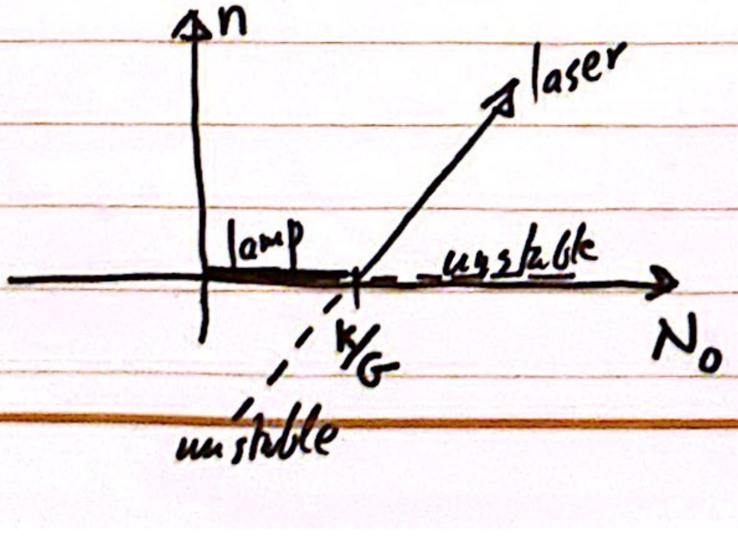
$$GN_0 - K = 0 \Rightarrow N_0 = K/G \quad (K>0,\ G>0)$$

↑ bifurcation parameter



$N_0 < K/G$ $N_0 = K/G$ $N_0 > K/G$

laser threshold.



transcritical bif. at $N_0 = K/G$

Pitchfork bifurcation

Arises in problem that have symmetry

Supercritical pitchfork:

$\dot{x} = rx - x^3$

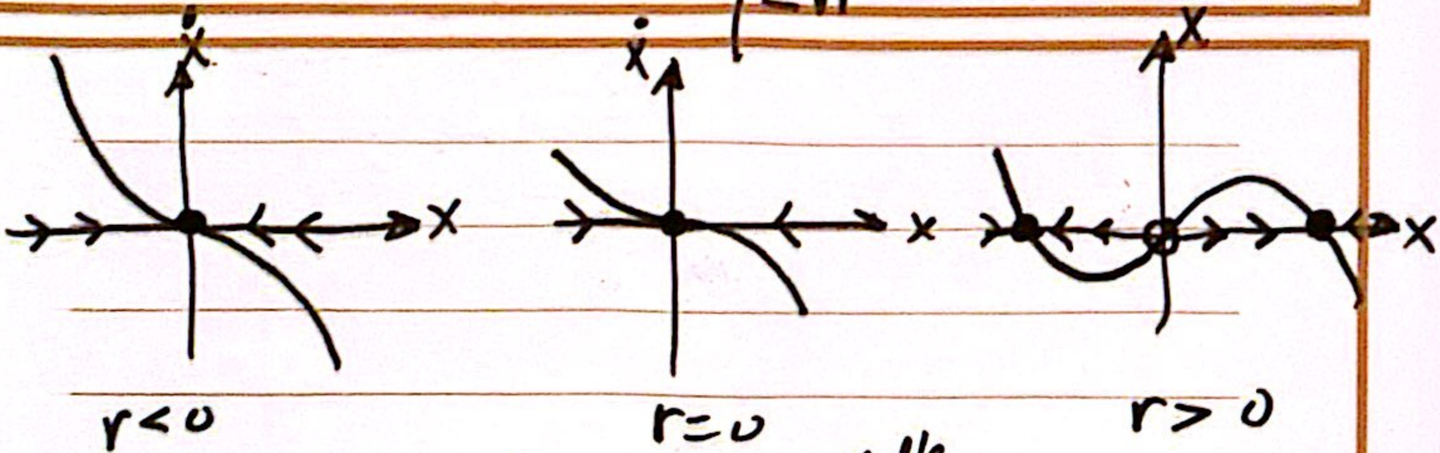
bifurcation parameter

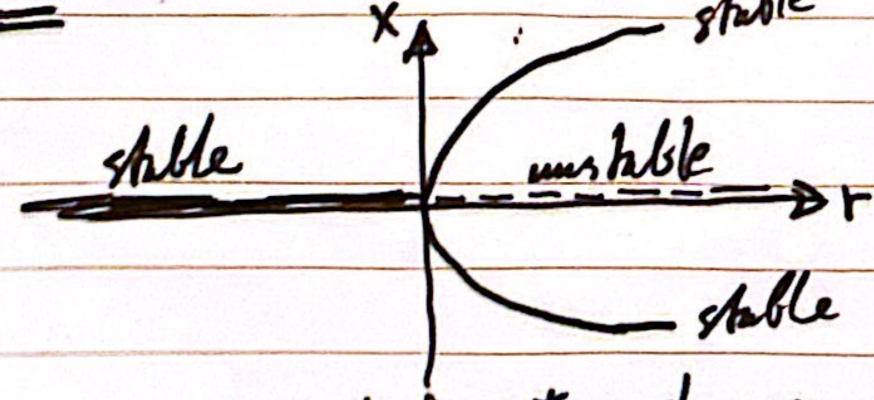
Symmetry: change $x \to -x$

$-\dot{x} = -rx + x^3$

$\boxed{\dot{x} = rx - x^3}$

$f(x) = rx - x^3$
$= x(r - x^2) \Rightarrow x^* = \begin{cases} 0 \\ \pm\sqrt{r} \end{cases}$ fixed pts.





bifurcation diagram

supercritical pitchfork bifurcation

forward facing