AME 522 Lecture 4 1/27/2025

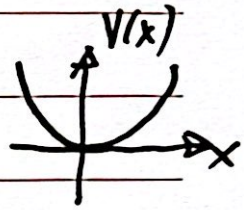
- HW1 due next Sunday (Feb 2 11:59pm)

Today: finishing up Chapter 2 ✓
Starting Chapter 3.

$$\dot{x} = f(x) \xrightarrow{= -\frac{dV}{dx}} \frac{dx}{dt} = f(x) \Rightarrow \int \frac{dx}{f(x)} = \int dt$$

$x(t)$: solution curve.

Potential functions: Introduce $V(x)$

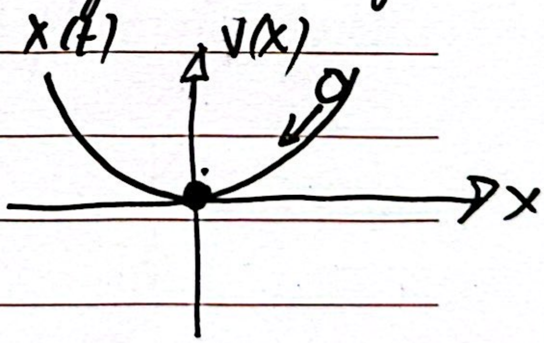


$V(x(t))$: $$\frac{dV}{dt} = \frac{dV}{dx}\underbrace{\frac{dx}{dt}}_{-\frac{dV}{dx}} = -\left(\frac{dV}{dx}\right)^2 \le 0$$

$V(t)$ decreases in time along solution trajectories

Ex 10: $\boxed{\dot{x} = -x} = -\frac{dV}{dx}$

$\frac{dV}{dx} = x \Rightarrow V(x) = \frac{1}{2}x^2 + \underset{=0}{C}$

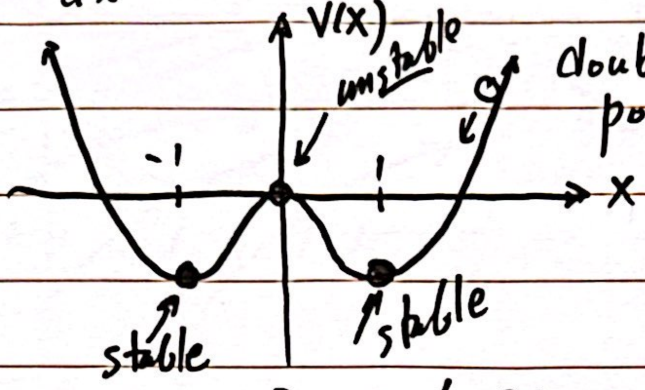


ball rolls down to the minimum point

Ex 11: $\dot{x} = x - x^3 = -\frac{dV}{dx}$

$\frac{dV}{dx} = -x + x^3 \Rightarrow V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + C$

0



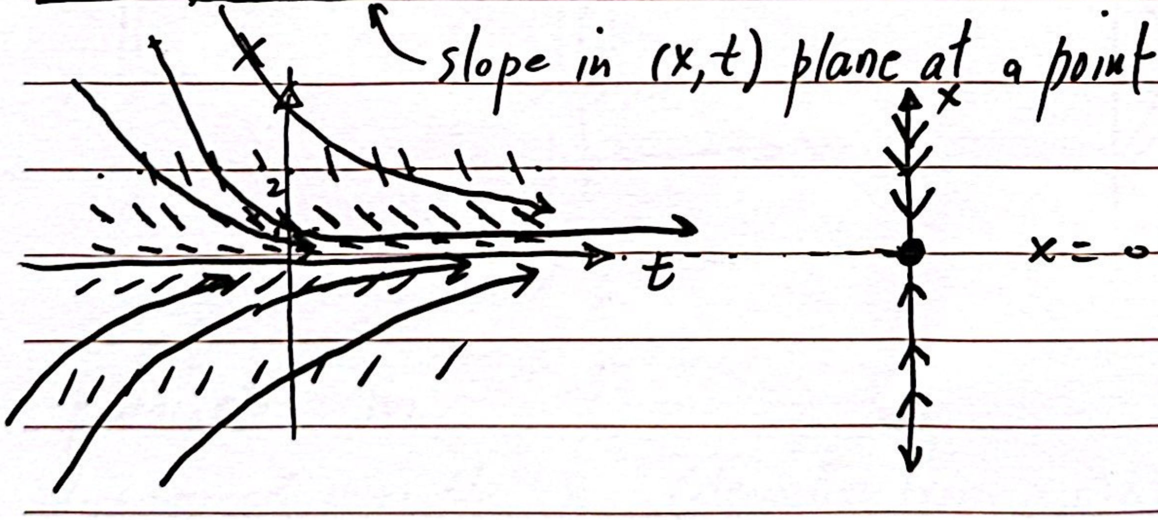
$V(x) = 0 = x^2\left(-\frac{1}{2} + \frac{1}{4}x^2\right) \Rightarrow x = 0$

$\frac{1}{2} = \frac{1}{4}x^2 \Rightarrow x = \pm\sqrt{2}$

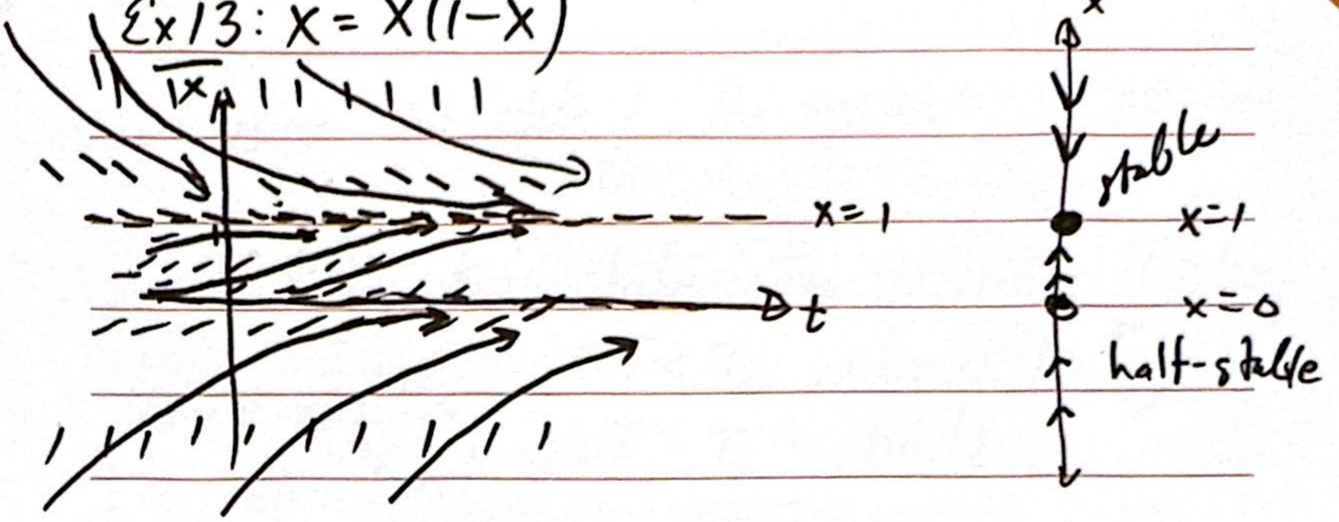
Direction fields: slope field: tangent to solution curves

Ex 12: $\boxed{\dot{x} = -x}$

slope in (x,t) plane at a point



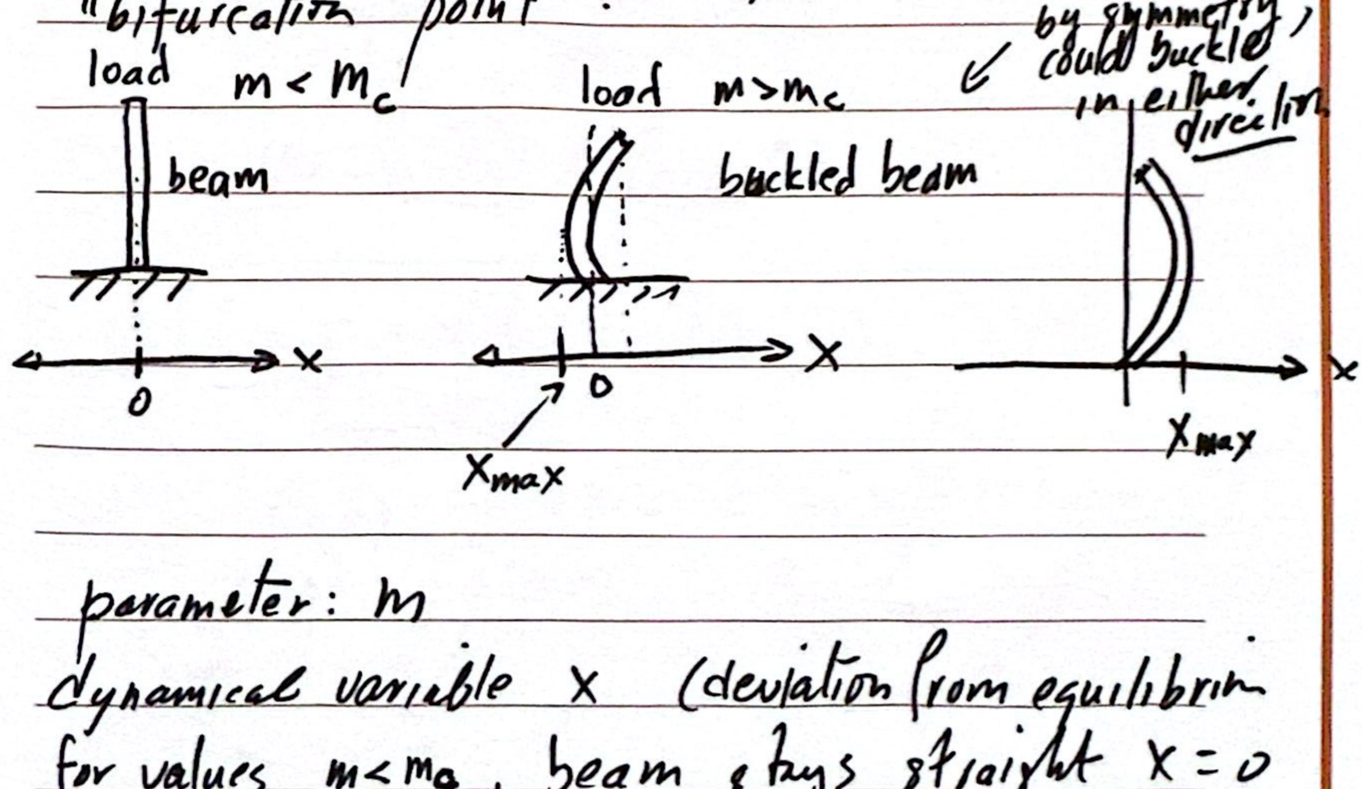
Ex 13: $\dot{x} = x(1-x)$



~~Lecture~~

Chapter 3: Bifurcations (part 1)

What is a bifurcation? Qualitative change in the systems behavior as a parameter crosses a "bifurcation point".



parameter: m

dynamical variable x (deviation from equilibrium)

for values $m < m_c$, beam stays straight $x = 0$

$m > m_c$, beam buckles $x = |x_{max}|$

Many problems have parameters that determine the state of the system:

- heating water: bifurcation parameter T: temp.

phase transition $T < T_c$ water is heating

bifurcation point $T > T_c$ boils

- fluid turbulence: Reynolds # parameter

As Reynolds # increases, the flow goes from laminar flow, to turbulent flow.

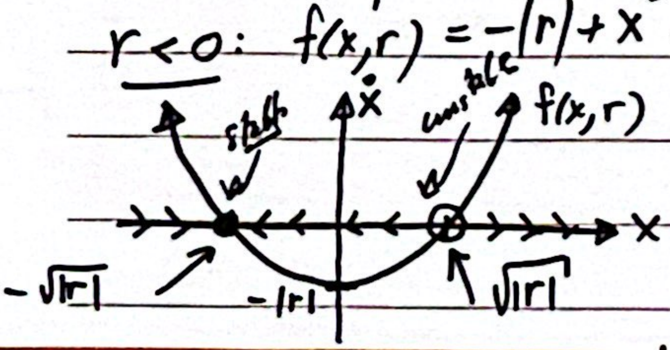
- Key technique: Taylor series expansions

The saddle-node bifurcation

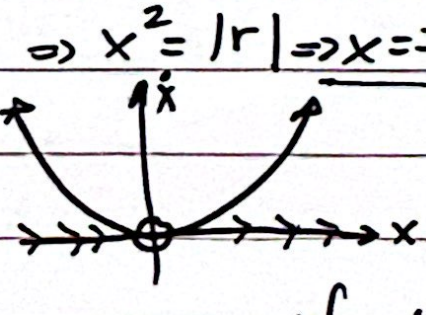
Ex 1: $\boxed{\dot{x} = r + x^2} = f(x, r)$

↑ parameter (r)

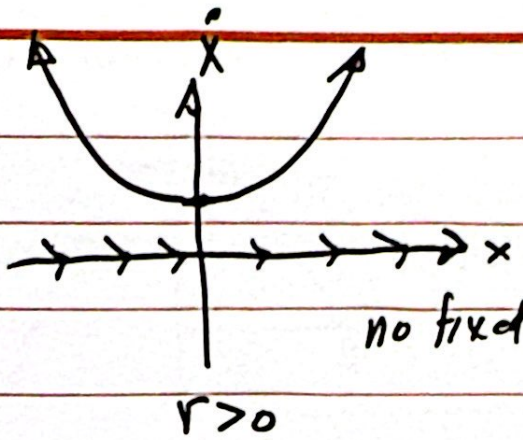
$r < 0$: $f(x,r) = -|r| + x^2 = 0 \Rightarrow x^2 = |r| \Rightarrow x = \pm\sqrt{|r|}$



$r < 0$ 2 fixed pts

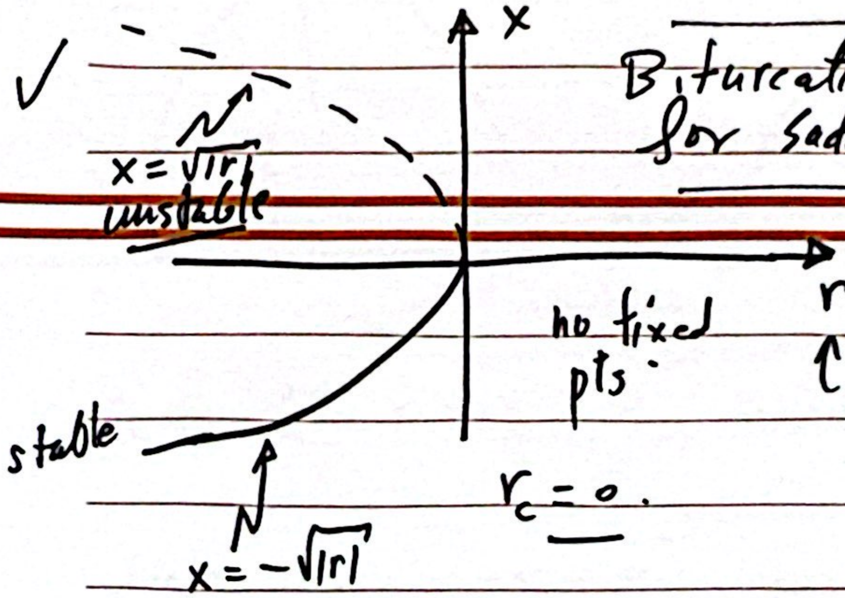


$r = 0$ 1 fixed pt.



Bifurcation occurs at $r=0$. since the vector fields for $r<0$ and $r>0$ are qualitatively different!

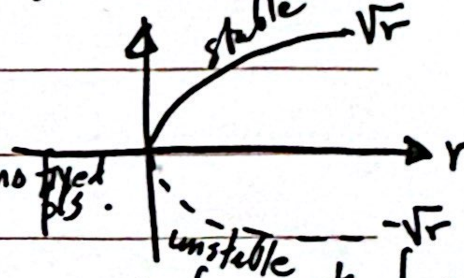
What is the bifurcation diagram for a saddle-node bifurcation:



✓

Bifurcation diagram for saddle node

✓



Ex2: $\dot{x}=r-x^2=f(x,r)$

$f(x)=r-x^2=0 \Rightarrow x^*=\pm\sqrt{r}$

$f'=-2x$

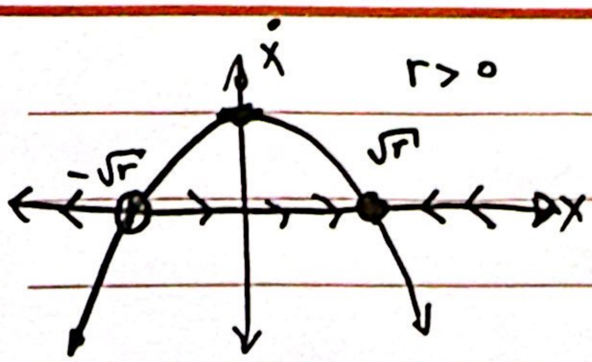
$f'(x^*)=-2x^*=-2(\pm\sqrt{r})=\mp 2\sqrt{r}$

2 fixed pts for $r>0$

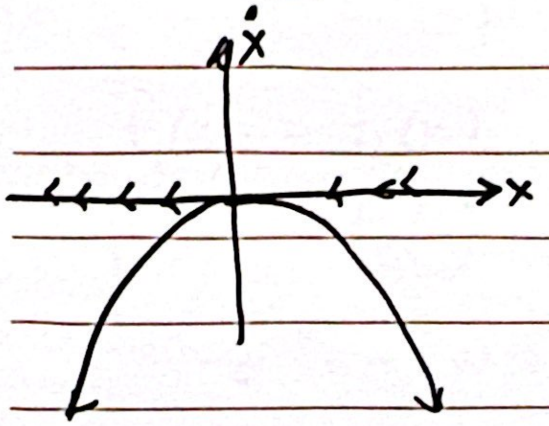
No fixed pts for $r<0$

$r=0$: bifurcation pt.

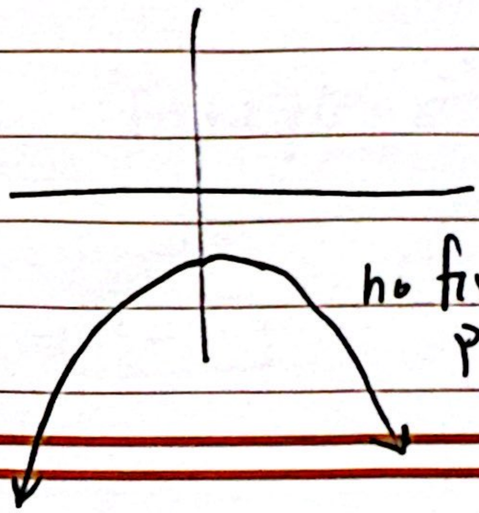
$r > 0$



As r decreased:



$r \leq 0$

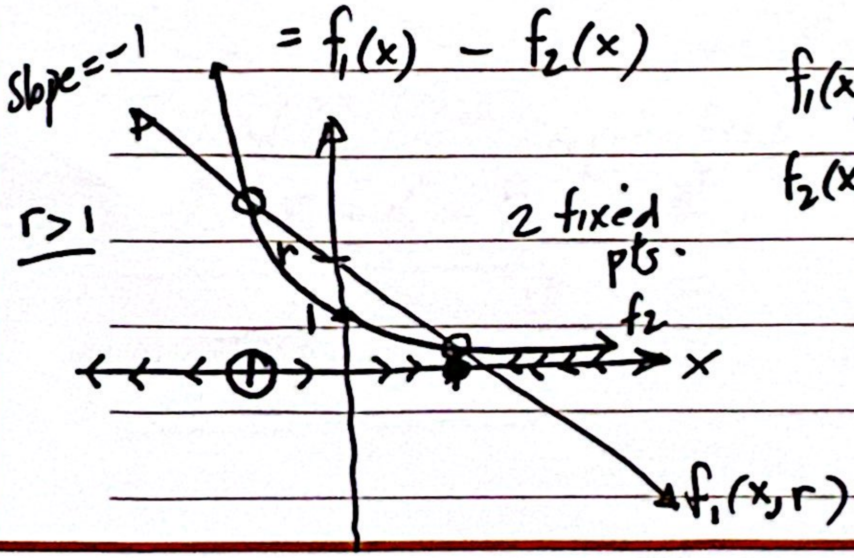


no fixed pts.

Ex3: $\dot{x} = r - x - e^{-x} = f(x, r)$

$f(x,r) = 0 = r - x - e^{-x}$ cannot solve explicitly

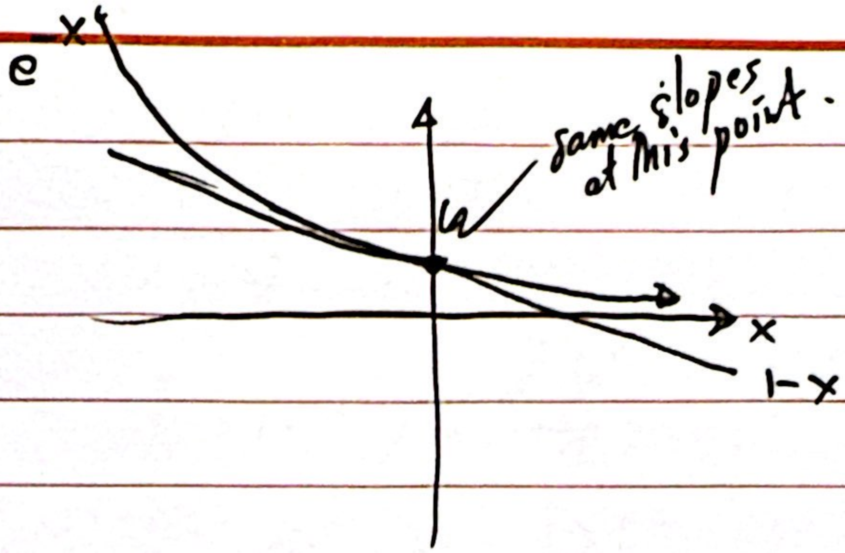
$= f_1(x) - f_2(x)$



$f_1(x) = r - x$, $f_1 = 0 \Rightarrow x = r$

$f_2(x) = e^{-x}$, $f_2 = 1 \Rightarrow x = 0$

bifurcation at $r = 1$ when f_1 and f_2 have same tangents.



$r=1$

$$f_1(x) = f_2(x)$$
$$f_1'(x) = f_2'(x)$$
$$\Rightarrow -e^{-x} = -1 \Rightarrow e^{-x} = 1 \Rightarrow \boxed{x=0}$$

fixed pts. $e^{-x} = r - x$

$\boxed{r=1}$ bifurcation

Bifurcation pt. $r_c = 1$, at $x^* = 0$.

What is the "normal form" for this problem?

$$\dot{x} = r - x - e^{-x} \qquad \left.\begin{array}{l} r = r_c = 1 \\ x = x^* = 0 \end{array}\right\} \text{bifurcation values.}$$

Expand in that neighborhood:

$$\dot{x} = r - \cancel{x} - \left[1 - \cancel{x} + \frac{x^2}{2!} + \dots\right]$$

$$\boxed{\dot{x} = (r-1) - \frac{x^2}{2}} + \dots$$

Re-scale the variables

$$\dot{x} = (r-1) - \frac{x^2}{2}$$ original variables around critical pts.

$$x = \alpha \hat{x}$$

$$\alpha \dot{\hat{x}} = (r-1) - \frac{\alpha^2 \hat{x}^2}{2}$$

$$\dot{\hat{x}} = \left(\frac{r-1}{\alpha}\right) - \frac{\alpha \hat{x}^2}{2}$$

Let $\alpha = 2$

$$\frac{r-1}{2} \equiv \hat{r}$$

$$\boxed{\dot{\hat{x}} = \hat{r} - \hat{x}^2}$$ Normal form for saddle-node bifurcation.