Homework 1

AME 522, Spring 2025

Due: Wednesday January 29 at 11:59 p.m.

Analyze the following equations (2.2.4 and 2.2.7) graphically. In each case, sketch the vector field on the real line, find all the fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions. Then try for a few minutes to obtain the analytical solution for $x(t)$; if you get stuck, don't try for too long since in several cases it's impossible to solve the equation in closed form!

2.2.4 $\dot{x} = e^{-x} \sin x$

2.2.7 $\dot{x} = e^{x} - \cos x$ (Hint: Sketch the graphs of e^x and $\cos x$ on the same axes, and look for intersections. You won't be able to find the fixed points explicitly, but you can still find the qualitative behavior.)

2.3.5 (Dominance of the fittest) Suppose X and Y are two species that reproduce exponentially fast: $\dot{X} = aX$ and $\dot{Y} = bY$, respectively, with initial conditions X_0, $Y_0 > 0$ and growth rates $a > b > 0$. Here X is "fitter" than Y in the sense that it reproduces faster, as reflected by the inequality $a > b$. So we'd expect X to keep increasing its share of the total population $X + Y$ as $t \to \infty$.The goal of this exercise is to demonstrate this intuitive result, first analytically and then geometrically.

a) Let $x(t) = \frac{X(t)}{X(t)+Y(t)}$ denote X's share of the total population. By solving for $X(t)$ and $Y(t)$, show that $x(t)$ increases monotonically and approaches 1 as $t \to \infty$.

b) Alternatively, we can arrive at the same conclusions by deriving a differential equation for $x(t)$. To do so, take the time derivative of $x(t) = \frac{X(t)}{X(t)+Y(t)}$ using the quotient and chain rules. Then substitute for $\dot{X}$ and $\dot{Y}$ and thereby show that $x(t)$ obeys the logistic equation $\dot{x} = (a - b)x(1 - x)$. Explain why this implies that $x(t)$ increases monotonically and approaches 1 as $t \to \infty$.

2.4.4 Use linear stability analysis to classify the fixed points of the following system. If linear stability analysis fails because $f'(x^*) = 0$, use a graphical argument to decide the stability.

$$\dot{x} = x^2(6 - x)$$

2.7.3 For the following vector field, plot the potential function $V(x)$ and identify all the equilibrium points and their stability.

$$\dot{x} = \sin x$$

3.1.1 For the following exercise, sketch all the qualitatively different vector fields that occur as r is varied. Show that a saddle-node bifurcation occurs at a critical value of r, to be determined. Finally, sketch the bifurcation diagram of fixed points x^* versus r.

$$\dot{x} = 1 + rx + x^2$$

3.2.4 For the following exercise, sketch all the qualitatively different vector fields that occur as r is varied. Show that a transcritical bifurcation occurs at a critical value of r, to be determined. Finally, sketch the bifurcation diagram of fixed points x^* vs. r.

$$\dot{x} = x(r - e^x)$$

3.3.1 (An improved model of a laser) In the simple laser model considered in Section 3.3, we wrote an *algebraic* equation relating N, the number of excited atoms, to n, the number of laser photons. In more realistic models, this would be replaced by a *differential* equation. For instance, Milonni and Eberly (1988) show that after certain reasonable approximations, quantum mechanics leads to the system

$$\begin{aligned}\dot{n} &= GnN - kn \\ \dot{N} &= -GnN - fN + p\end{aligned}$$

Here G is the gain coefficient for stimulated emission, k is the decay rate due to loss of photons by mirror transmission, scattering, etc., f is the decay rate for spontaneous emission, and p is the pump strength. All parameters are positive, except p, which can have either sign. This two-dimensional system will be analyzed in Exercise 8.1.13. For now, let's convert it to a one-dimensional system, as follows.

a) Suppose that N relaxes much more rapidly than n. Then we may make the quasi-static approximation $\dot{N} \approx 0$. Given this approximation, express $N(t)$ in terms of $n(t)$ and derive a first-order system for n. (This procedure is often called **adiabatic elimination**, and one says that the evolution of $N(t)$ is slaved to that of $n(t)$. See Haken (1983).)

b) Show that $n^* = 0$ becomes unstable for $p > p_c$, where p_c is to be determined.

c) What type of bifurcation occurs at the laser threshold p_c?

d) (Hard question) For what range of parameters is it valid to make the approximation used in (a)?

3.4.3 In the following exercise, sketch all the qualitatively different vector fields that occur as r is varied. Show that a pitchfork bifurcation occurs at a critical value of r (to be determined) and classify the bifurcation as supercritical or subcritical. Finally, sketch the bifurcation diagram of x^* vs r.

$$\dot{x} = rx - 4x^3$$

3.7.3 (A model of a fishery) The equation $\dot{N} = rN\left(1 - \frac{N}{K}\right) - H$ provides an extremely simple model of a fishery. In the absence of fishing, the population is assumed to grow logistically. The effects of fishing are modeled by the term $-H$, which says that fish are caught or "harvested" at a constant rate $H > 0$, independent of their population N. (This assumes that the fishermen aren't worried about fishing the population dry—they simply catch the same number of fish every day.)

a) Show that the system can be rewritten in dimensionless form as

$$\frac{dx}{d\tau} = x(1 - x) - h$$

for suitably defined dimensionless quantities x, τ and h.

b) Plot the vector field for different values of h.

c) Show that a bifurcation occurs at a certain value h_c, and classify this bifurcation.

d) Discuss the long-term behavior of the fish population for $h < h_c$ and $h > h_c$, and give the biological interpretation in each case.

There's something silly about this model — the population can become negative! A better model would have a fixed point at zero population for all values of H. See the next exercise for such an improvement.