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AME 522 – Nonlinear Dynamics Spring 2025

Lecture 1:

- HW 60% (6 assignments)
- Midterm 20% (Before Spring break)
- Final 20% (Friday May 9, 11–1)

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Office hrs:
MW 11–12 and following class. Zoom + office

Book: S.H. Strogatz: Nonlinear Dynamics and Chaos
2nd Ed.
(PDF is posted ~~on~~ Brightspace)

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12 lectures on nonlinear dynamics:

Lecture 1: Introduction to nonlinear dynamics
Lecture 2: Flows on a line
Lecture 3: Bifurcations (Part I)
Lecture 4: Flows on a circle
Lecture 5: Linear systems, phase planes
Lecture 6: Limit cycles
Lecture 7: Bifurcations (Part II)
Lecture 8: The Lorenz equations
Lecture 9: One-dimensional maps

Lecture 10: Fractals

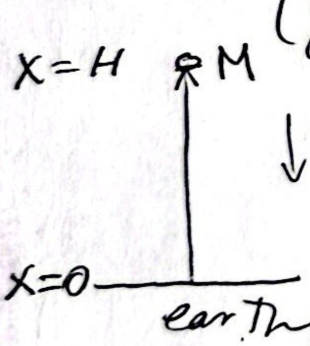
Lecture 11: Strange attractors

Lecture 12: Evolutionary game theory

Lecture 1: Introduction to nonlinear dynamics

Most people in the class are familiar with some of the basic equations and techniques in the field of dynamics:

Ex 1: Free-fall of an object of mass M from height H under gravitational attraction (g (9.8 m/sec^2)). Isaac Newton 1643–1727



$$Ma = F \leftarrow \text{forces (could be linear or nonlinear)}$$

$$M\frac{d^2x}{dt^2} = -gM \Rightarrow \ddot{x} = -g$$

$$\dot{x} = -gt + V_0$$

$$\boxed{x(t) = -\tfrac{1}{2}gt^2 + V_0 t + x_0}$$

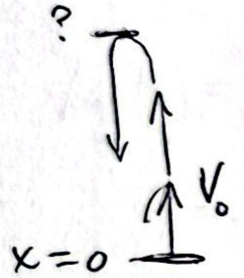
Drop from height H ($x_0 = H$)
zero velocity ($V_0 = 0$)

$$\boxed{x(t) = -\tfrac{1}{2}gt^2 + H}$$

- position decreases quadratically
- velocity increases linearly
- acceleration is constant (g)

This formula was known before Newton by taking data and measuring.

<u>More complicated</u>: How high will a ball of mass M rise if we throw it from ground straight up with velocity V_0?



At the peak, $\dot{x} = 0$, $0 = -g t_{max} + V_0$

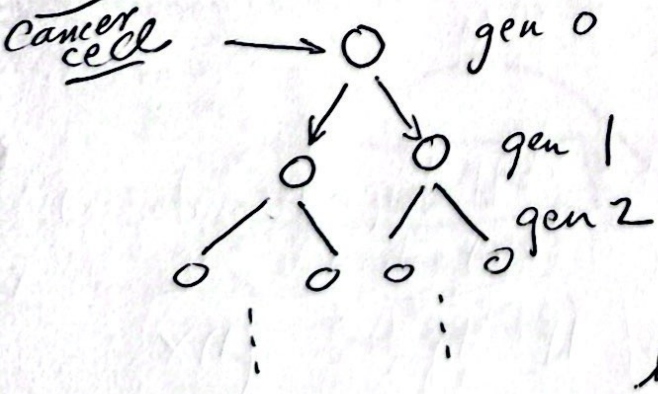
$$t_{max} = V_0/g$$

$$x(t_{max}) = -\tfrac{1}{2} g \left(V_0/g\right)^2 + V_0 \left(V_0/g\right) + \cancel{X_0}^{\,0}$$

$$\boxed{X_{max} = V_0^2/2g}$$

<u>But</u>: Forces are not always constant, uni-directional, might not be linear in x.

<u>Ex 2</u>: <u>Cell division</u> (i.e. tumor growth)



$N(0) = 1$

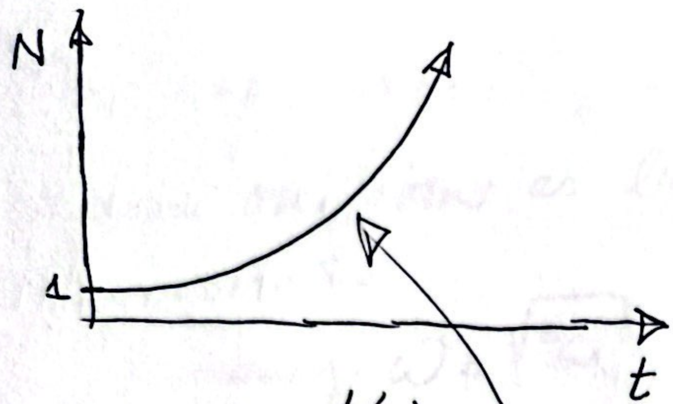
$N(n) = 2^n$

number of cells ↑ ↑ generation (discrete time)

$$\ln(N) = \ln(2^n) = n \ln(2)$$

$$\boxed{N = \exp(n \ln(2))}$$

exponential growth rate.



$N(n) = 2^n$

or $N(t) = 2^{t/\tau}$

t: time (cell cycle splitting time)

τ: doubling time

$t : \tau \to 2\tau$

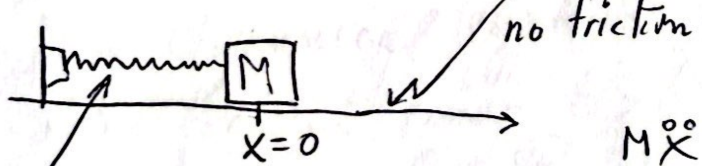
$N : 2^1 \to 2^2$

$$\ln(N) = \ln\left(2^{t/\tau}\right) = \frac{t}{\tau}\ln(2) \Rightarrow \boxed{N(t) = \exp\left(\frac{t}{\tau}\ln(2)\right)}$$

$$\frac{dN}{dt} = \left(\frac{\ln(2)}{\tau}\right)N \quad = N(t) = \quad , N(0) = 1$$

But eventually, the tumor growth must slow down, due to limited resources, competition, crowding, ...

Ex 3: Harmonic oscillations



spring

spring const. K.

$Ma = F$ ← restoring force

$$M\ddot{x} = \underbrace{-Kx}$$

Hooke's law: restoring force is linear in x, usually accurate when x is not too large

$$\ddot{x} = -\frac{K}{M}x$$

Let $\boxed{x(t) = \exp(\alpha t)}$

$\dot{x} = \alpha \exp(\alpha t)$

$\ddot{x} = \alpha^2 \exp(\alpha t)$

$$\alpha^2 \cancel{\exp(\alpha t)} = -\frac{K}{M}\cancel{\exp(\alpha t)}$$

$$\boxed{\alpha^2 = -K/M} \to \boxed{\alpha = \pm i\sqrt{K/M}}$$

$$x(t) = A\underbrace{\exp\left(it\sqrt{\tfrac{K}{M}}\right)}_{\cos(t\sqrt{K/M}) + i\sin(t\sqrt{K/M})} + B\underbrace{\exp\left(-it\sqrt{\tfrac{K}{M}}\right)}_{\cos(t\sqrt{K/M}) - i\sin(t\sqrt{K/M})}$$

$$= (A+B)\cos\left(t\sqrt{K/M}\right) + i(A-B)\sin\left(t\sqrt{K/M}\right) \leftarrow$$

harmonic oscillations as linear combinations of sin's and cosines.

frequency: $\omega \equiv \sqrt{K/M}$, T: period $= 2\pi/\omega = 2\pi\sqrt{M/K}$

But: No spring follows Hooke's law exactly.
The further we pull, the greater the restoring force (rubber band)

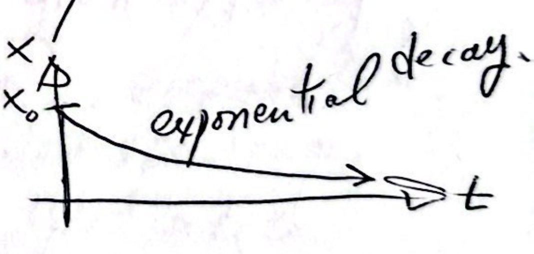
$$F = -k_0 X - k_1 X^3 + \dots$$

nonlinear oscillators.

Some general observations:

1) Some dynamical systems have explicit solutions valid for all times $t > 0$.

$(\cdot \equiv \frac{d}{dt})$



Ex 4: $\dot{x} = -x$
$x(0) = x_0$ $\Rightarrow x(t) = x_0 \exp(-t)$

2) Others have solutions that are singular in finite-time:

Ex 5: $\boxed{\dot{x} = x^2}$, $x(0) = x_0$ ✓

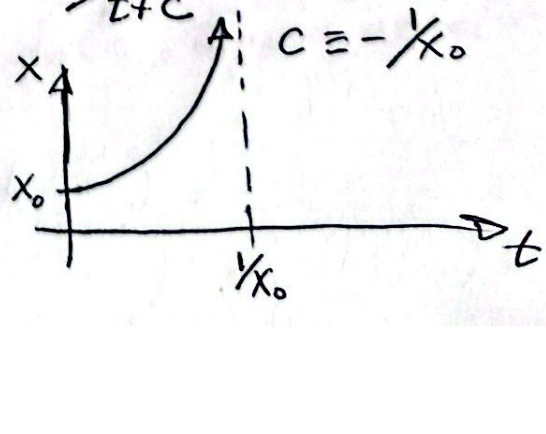
$\int \frac{dx}{x^2} = \int dt \Rightarrow -x^{-1} = t + C$

$x = -\frac{1}{t+C}$ $\Rightarrow x(0) = -1/C = x_0$

$C \equiv -1/x_0$

$$\boxed{x(t) = \frac{-1}{\left(t - \frac{1}{x_0}\right)} = \frac{x_0}{(1 - x_0 t)}}$$

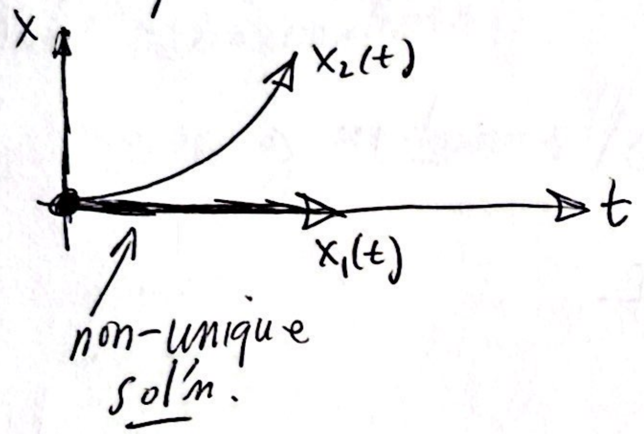
x(t) is defined only for times $0 \le t < 1/x_0$



3) Still others have multiple solutions for the same initial condition (non-unique)

Ex 6: $\begin{cases} \dot{x} = \sqrt{x} \\ x(0) = 0 \end{cases}$ Solution 1: $x_1(t) = 0$

Solution 2: $x_2(t) = t^2/4$



This cannot happen with a linear equation.

Once we leave the safe confines of linear dynamical systems, where key concepts like: harmonic oscillations, exponential growth/decay, linear superposition (Fourier series), eigenvalues, eigenfunctions, Doppler shifts, color spectrum, acoustical freqs, ..., things get more complicated and interesting.

Keep in mind, there is nothing linear about

$ma = F$

forces do not have to be linear, usually they are not.

(e.g. inverse square gravitational force law)

Newton opened up a Pandora's box that we are still trying to sort out.
This course is about techniques, models, intuition, on what ~~to~~ happens when linearity is no longer a valid approximation.

Some history of mechanics / dynamics:

1666 Newton (Leibniz) — Invention of calculus, explanation of planetary motion (nonlinear)

1700's — Flowering of calculus and classical mechanics

1800's — Analytical studies of planetary motion

1890's Poincaré — Geometric approach to chaos theory (3-body problem)

1920-1950 — Linear and nonlinear oscillators in physics and engineering, Radio, radar, lasers, transistors, ..

1920-1960's · Birkhoff, Kolmogorov, Arnold, Moser — Complex behavior in Hamiltonian systems

1963 Lorenz — Strange attractors in a simple model of fluid convection

1970's Ruelle & Takens
- Turbulence, chaos

May
- Chaos in logistic map

Feigenbaum
- Universality and renormalization, connection
- between chaos and phase transitions

1980's Mandelbrot
- Fractals
- Widespread interest in chaotic systems....

Some more basic review:

Ex 7: $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$ damped harmonic oscillator

damping term

acceleration, velocity, Hooke's law

$x(t)$: dependent variable
2nd order (2 time derivs)
linear in x.

$\boxed{\text{Linear superposition:}}$

2 sols: $x_1(t)$, $x_2(t)$

$$\left.\begin{aligned} m\ddot{x}_1 + b\dot{x}_1 + kx_1 = 0 \\ m\ddot{x}_2 + b\dot{x}_2 + kx_2 = 0 \end{aligned}\right\} \text{add} \Rightarrow m\frac{d^2}{dt^2}\overbrace{(x_1+x_2)}^{x} + b\frac{d}{dt}\overbrace{(x_1+x_2)}^{x} + k\underbrace{(x_1+x_2)}_{x} = 0$$

The sum $x_1 + x_2$ satisfies the same eqn.

Ex 8: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $u(x,t)$ unknown.

heat equation: linear pde. linear in u.

A general framework for nonlinear ODEs is the following:

$$\begin{cases} \dot{x}_1 = f_1(x_1, \dots x_n) \\ \dot{x}_2 = f_2(x_1, \dots, x_n) \\ \vdots \\ \dot{x}_n = f_n(x_1, \dots x_n) \end{cases}$$

$x_1(t), x_2(t), \dots, x_n(t)$ are the dependent variables

t: is the independent variable

Stop for today. Finish intro Wed, on to Lecture 2.