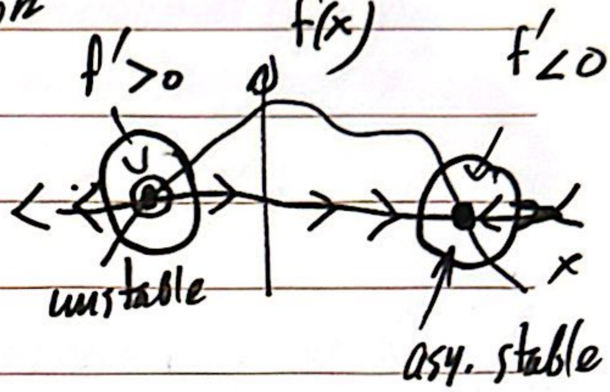
AME 522 — 1/22/2025

Lecture 3.

Chapter 2 Strogatz book
Next Chapt. 3

$\dot{x} = f(x)$:



Linear stability analysis.

Fixed pt: $x = x^*$ $\quad f(x^*) = 0$

$\boxed{\dot{x}^* = \cancel{f(x^*)}^{\,0}}$ $\quad x^* = \text{const}.$

Taylor expand $f(x)$ around x^* :

$$f(x^* + \eta) = f(x^*) + \eta f'(x^*) + \frac{\eta^2}{2} f''(x^*) + O(\eta^3)$$

x^*: fixed pt

η: perturbation

$$(x^* + \eta)^{\cdot} = f(x^* + \eta) \approx \cancel{f(x^*)} + \eta f'(x^*) + O(\eta^2)$$

$$\cancel{\dot{x}^*} + \dot{\eta} = $$

$$\underline{\underline{\dot{\eta}}} = \underbrace{f'(x^*)}_{\text{const.}} \underline{\underline{\eta}}$$

· linearized eqn. around x^*.
· variational eqn associated with $f(x)$, and x^*.

$\underline{\eta(t)} = \eta_0 \exp(at) \quad a \equiv f'(x^*)$

η_0 ← small.

$f'(x^*) > 0$: η grows exponentially $\Rightarrow x^*$ is linearly unstable

$f'(x^*) < 0$: η decays exponentially $\Rightarrow x^*$ is linearly <u>stable</u>

$$\dot{\eta} = a\eta \rightarrow \frac{\dot{\eta}}{\eta} = a$$

$$\boxed{\frac{d}{dt}(\ln(\eta)) = a}$$

$f'(x^*)$ is the log growth rate.

$\frac{1}{|a|}$ characteristic time scale associated w/ growth/decay of perturbations.

$\frac{1}{|f'(x^*)|}$ characteristic timescale.

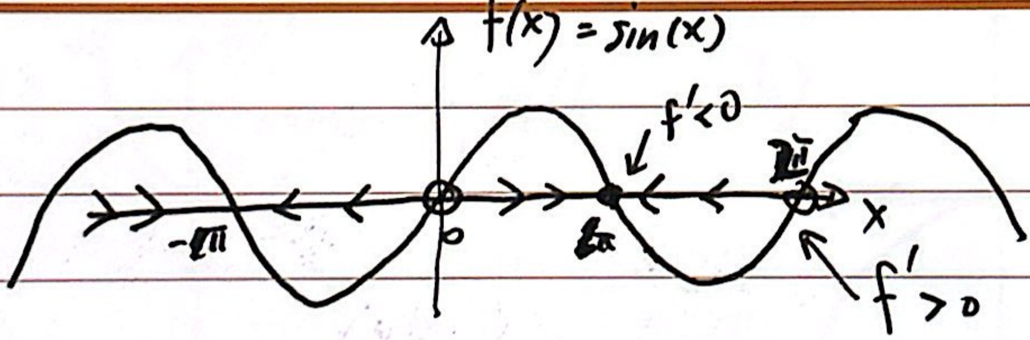
Linear stability analysis gives <u>no</u> information when $f'(x^*) = 0$.

Ex 5: Determine the linear stability of the fixed pts. for
$\dot{x} = \sin(x)$.

$f(x) = \sin(x) = 0 \Rightarrow x^* = k\pi$ k integers

$$f'(x) = \cos(x) \Rightarrow f'(x^*) = \cos(k\pi) = \begin{cases} 1 & k \text{ even} \\ -1 & k \text{ odd.} \end{cases}$$

$x^* \equiv k\pi$ is linearly unstable if k is even
& stable if k is odd.



Ex6: Use linear stability analysis for logistic eqn:

$\dot{N} = f(N) = rN(1 - N/K)$ r, K consts.

N(t)

$f(N) = 0 = rN(1 - N/K) = 0$, $\boxed{N^* = 0,\ N^* = K}$

$$f'(N) = r - \frac{2rN}{K} \qquad f'(0) = r$$

$$f'(K) = -r$$

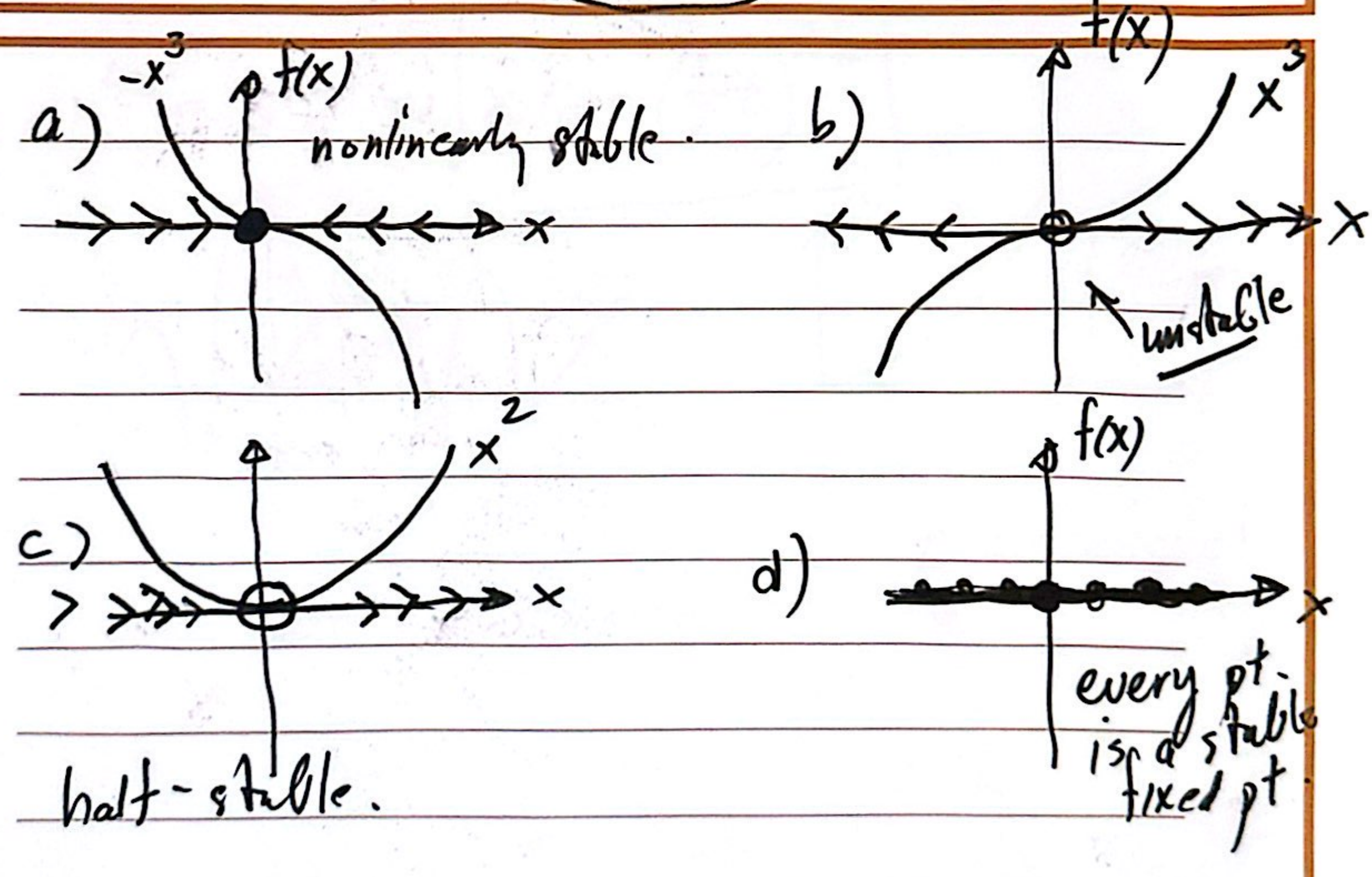
$N^* = 0$: unstable $r > 0$
$N^* = K$: stable $r > 0$

$\frac{1}{r}$ charact. timescale of growth/decay

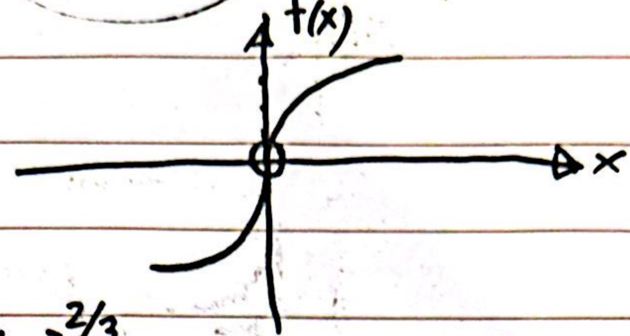
Ex 7: What if $f'(x^*) = 0$ linear stability analysis doesn't work.

a) $\dot{x} = -x^3$ b) $\dot{x} = x^3$ c) $\dot{x} = x^2$ d) $\dot{x} = 0$

All cases: $x^* = 0$, $f'(0) = 0$



Ex 8: $\dot{x} = x^{1/3} = f(x)$ $\quad f(0) = 0$

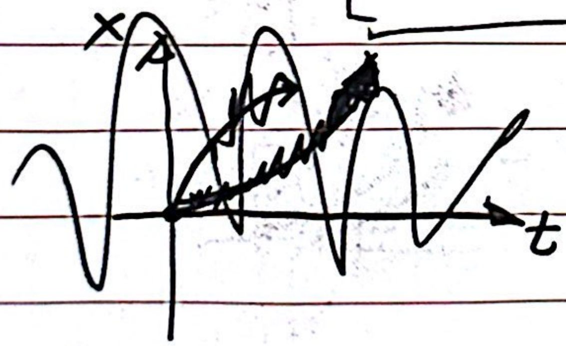


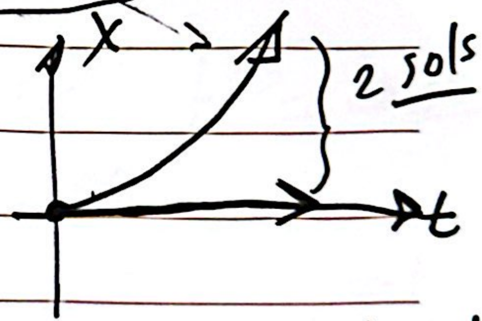
$f' = \frac{1}{3}x^{-2/3}$ $\quad f'(0) = \infty$ $\quad x(0) = 0$

$$\frac{dx}{dt} = x^{1/3} \Rightarrow \int \frac{dx}{x^{1/3}} = \int dt$$

$$\Rightarrow \frac{3}{2}x^{2/3} = t + C \Rightarrow C = 0$$

$$x^{2/3} = \frac{2}{3}t \Rightarrow \boxed{x(t) = \left(\frac{2}{3}t\right)^{3/2}}$$





Non-unique!

$x \equiv 0$ also solves the eqn!

Existence and uniqueness thm:

Consider the initial value problem $\boxed{\dot{x} = f(x),\ x(0) = x_0}$

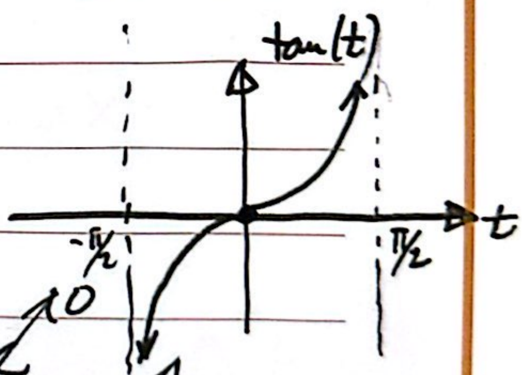
Suppose $f(x)$ and $f'(x)$ are continuous functions on an open interval R of the x-axis, and let x_0 be a point in R. Then, the IVP has a sol'n $x(t)$ on some interval $(-\underbrace{\tau, \tau}_{\text{might not exist forever.}})$ around $t=0$ and the sol'n is unique.

Ex 9: $\dot{x} = 1 + x^2 \qquad x(0) = x_0$

$\underbrace{f(x) = 1 + x^2}_{\text{continuous function of } x} \qquad f' = 2x \qquad f'' = 2$

$f''' = 0 \ldots$

Let $x(0) = 0$: $\underbrace{\int \frac{dx}{1+x^2}}_{} = \int dt$



$\tan^{-1}(x) = t + \cancel{c}\ 0$

$\tan^{-1}(0) = c = 0$

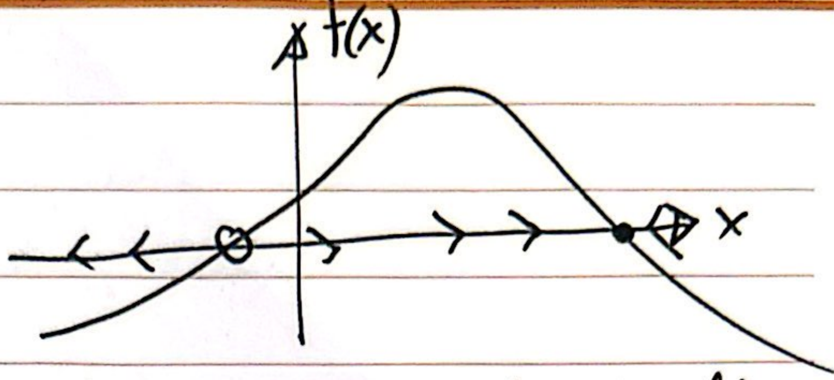
$\tan^{-1}(x) = t$

$\boxed{x(t) = \tan(t)}$

$\tau = \frac{\pi}{2}$

Sol'n only exists for finite time.

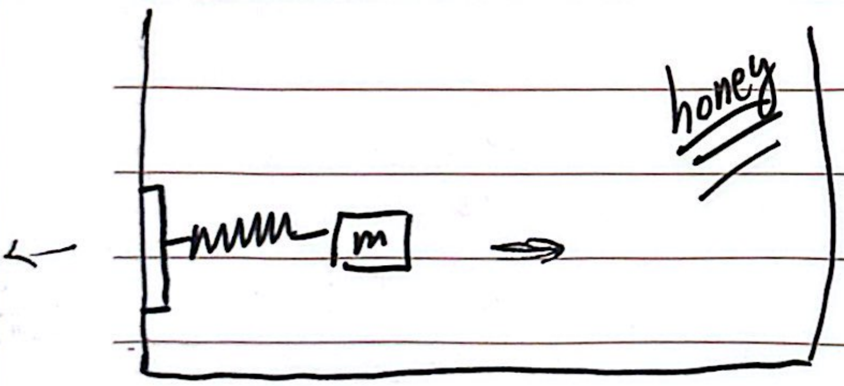
$\underline{\dot{x} = f(x)}$



All that can happen is evolution to fixed pts. No oscillations are possible! No periodic sols to $\dot{x} = f(x)$.

Think of Newton's law: $m\ddot{x} + b\dot{x} = F(x)$

inertia term, viscous damping, restoring force



spring-mass system in a vat of honey

Viscous damping term $b\dot{x} >> m\ddot{x}$

$$\cancel{m\ddot{x}} + b\dot{x} = F(x)$$

$$\boxed{\dot{x} = \frac{1}{b}F(x)}$$

overdamped limit. Mass is slowly dragged by spring to equilibrium.

No oscillations

Potentials: View $\boxed{\dot{x} = f(x)}$ by introducing a potential function $V(x)$.

Think of $V(x(t))$:

$$\boxed{\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}}$$

chain rule

$$\underbrace{\dot{x} = f(x) = -\frac{dV}{dx}}$$

$$\frac{dV}{dt} = \frac{dV}{dx}\left(-\frac{dV}{dx}\right)$$

$$\boxed{\frac{dV}{dt} = -\left(\frac{dV}{dx}\right)^2}$$