AME 522 1/15/2025.

Lecture 2

Introduction (continued...)

General framework for ODE's:

$$\begin{cases} \dot{x}_1 = f_1(x_1, \ldots, x_n) \\ \dot{x}_2 = f_2(x_1, \ldots x_n) \\ \vdots \\ \dot{x}_n = f_n(x_1, \ldots, x_n) \end{cases}$$

$x_1(t), x_2(t), \ldots$ are dependent variables

Ex 9: Damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$x_1 = x, \ \dot{x}_1 = x_2, \ \dot{x}_2 \equiv \ddot{x}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{b}{m}\dot{x} - \frac{k}{m}x = -\frac{b}{m}x_2 - \frac{k}{m}x_1$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1 \end{cases}$$

$x_1(0)$, $x_2(0)$ given

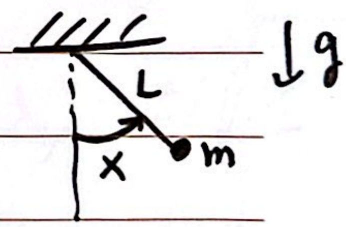
$f_1(x_1, x_2) = x_2$

$f_2(x_1, x_2) = -\frac{k}{m}x_1 - \frac{b}{m}x_2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{\bullet} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \boxed{\dot{\vec{x}} = A\vec{x}} \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

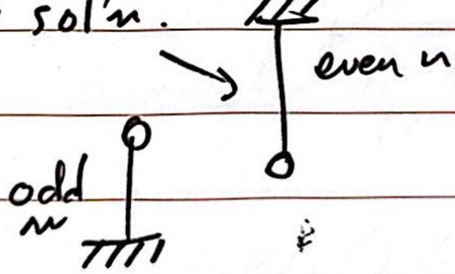
eigenvalues & eigenvectors of A are key.

<u>Ex 10</u>: $\ddot{x} + \frac{g}{L}\sin(x) = 0$



<u>Note</u>: $x = 0$ is a sol'n.

$x = \pm n\pi$



If x_1 is a sol'n, x_2 is a sol'n, is $x_1 + x_2$ a solution? <u>No</u>

$$\ddot{x}_1 + \frac{g}{L}\sin(x_1) = 0$$
$$\ddot{x}_2 + \frac{g}{L}\sin(x_2) = 0$$

$$(x_1+x_2)^{\cdot\cdot} + \frac{g}{L}\Big(\underbrace{\sin(x_1) + \sin(x_2)}_{\neq \sin(x_1+x_2)}\Big) = 0$$

$$\sin(x) = x - \frac{1}{3}x^3 + \cdots$$ sufficiently small angles X

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{L}x_1 \end{cases}$$ "Linearized" around $X=0$.

$\longrightarrow \dot{\vec{X}} = A\vec{X}$, $A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix}$

$$\det(A - \lambda I) = 0$$

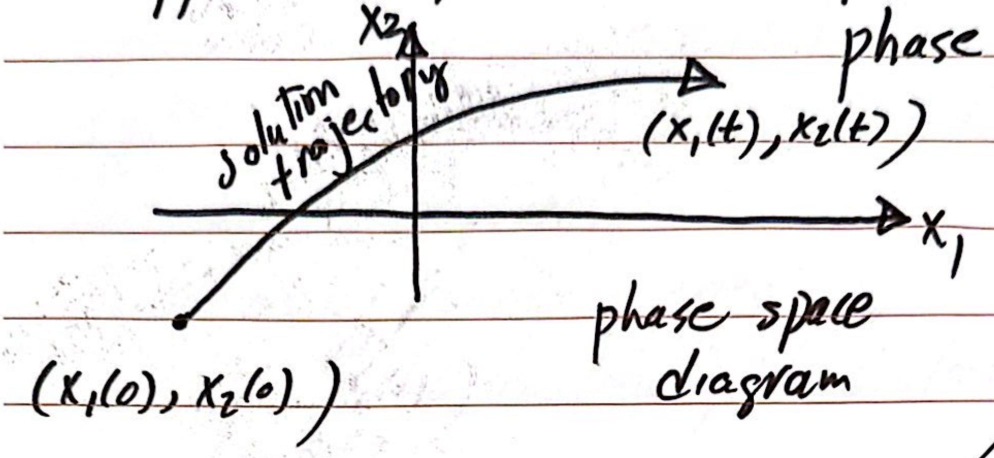
$$\begin{vmatrix} -\lambda & 1 \\ -\frac{g}{L} & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + \frac{g}{L} = 0 \Rightarrow \lambda_{1,2} = \pm i\sqrt{g/L}$$

eigenvectors $\vec{v}_1, \vec{v}_2$ oscillation frequencies.

general sol'n

$$\boxed{\vec{X} = \alpha \vec{v}_1 \exp(\lambda_1 t) + \beta \vec{v}_2 \exp(\lambda_2 t)}$$

Suppose we plot x_2 vs. x_1:



non-autonomous forcing

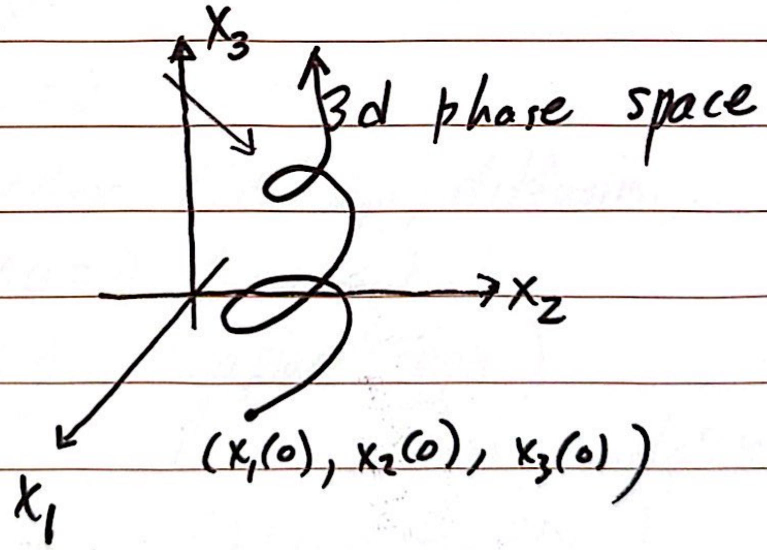
Ex 11: $m\ddot{x} + b\dot{x} + kx = F\cos(t)$

forced linear oscillator

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{1}{m}\left(-kx_1 - bx_2 + F\cos(x_3)\right) \\ \dot{x}_3 = 1 \end{cases}$$

$x_3 = t$

$\dot{x}_3 = 1$



Semi-final point:

The most basic tool exploited from linear systems are harmonic oscillations ($\cos(\omega t)$, $\sin(\omega t)$) and linear superposition:

Sol'n as a Fourier-series:

$$x(t) = \sum_{n=0}^{\infty} \left(a_n \cos(n\omega t) + b_n \sin(n\omega t) \right)$$

Does not work for nonlinear systems!

 ← what to do??

 ← opportunities

Linear hall of fame:

1) $\dot{x} = \alpha x$
2) $\ddot{x} + ax = 0$
3) $u_t = \alpha u_{xx}$ (heat eqn, diffusion, ...)
4) $u_{tt} - \Delta u = 0$ (waves)
5) $\Delta u = 0$ (Laplace's eqn)
6) $\dot{\vec{x}} = A\vec{x}$ linear systems analysis
7.) $i\psi_t \mp \Delta\psi$ (Schröding eqn.)
8.) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, $\nabla \times \vec{B} = \mu_0 j + \frac{1}{c^2}\frac{\partial E}{\partial t}$ Maxwell's eqns.

Some important science problems that are nonlinear:

1) Orbital mechanics and planetary motion
2) Fluid mechanics and turbulence
3) Tumor growth
4) High powered lasers
5) Shock waves and gas dynamics
6) Combustion chemical kinetics
7.) Atmospheric + oceanographic dynamics, hurricanes, tornadoes, tsunamis

8.) Climate models
9.) Collective dynamics - traffic flows, microbial populations
10.) Economic forecasting models
11.) Synchronization
12.) Evolution by natural selection (fitness)

Lecture 2: Flows on a line

Scalar nonlinear 1^{st} order ode:

(1) $\dot{x} = f(x) \qquad x(0) = x_0$

If $f(x) \equiv ax \Rightarrow x(t) = x_0 \exp(at)$ (linear).
$x(0) = x_0$

More generally:

Ex 1: $\dot{x} = \sin(x) \qquad x(0) = x_0$

Approach 1: $\frac{dx}{dt} = \sin(x)$

$$\int \frac{dx}{\sin(x)} = \underbrace{\int dt}_{t}$$

$$\int \csc(x)\,dx = -\ln\left|\csc(x) + \cot(x)\right| + C$$

$t=0, x=x_0 \Rightarrow \ln\left|\csc(x_0) + \cot(x_0)\right| = C$

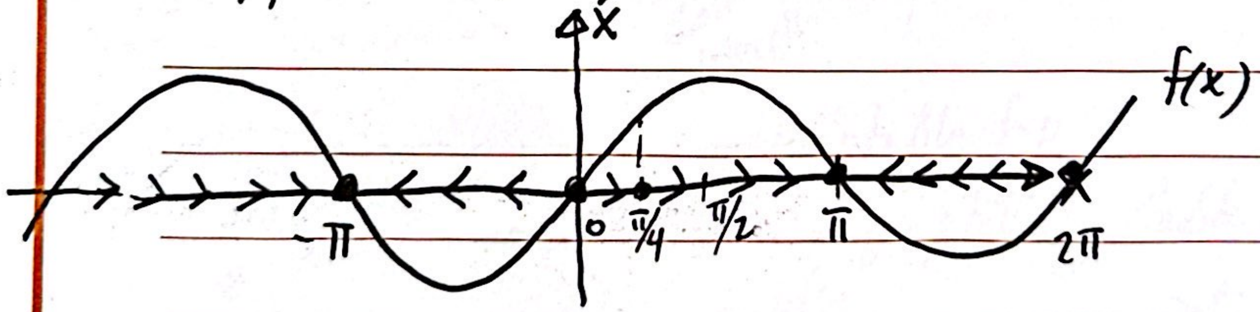
$$t = \ln\left|\frac{\csc(x_0)+\cot(x_0)}{\csc(x)+\cot(x)}\right|$$ exact sol'n. very hard to interpret

Try to answer the following questions:

1) Suppose $x_0 = \pi/4$. Describe the qualitative features of the sol'n $x(t)$ for all $t>0$. What happens as $t\to\infty$?

2.) For any arbitrary x_0, what is the behavior of $x(t)$ as $t\to\infty$?

Approach 2: $\dot{x} = \sin(x) = f(x)$



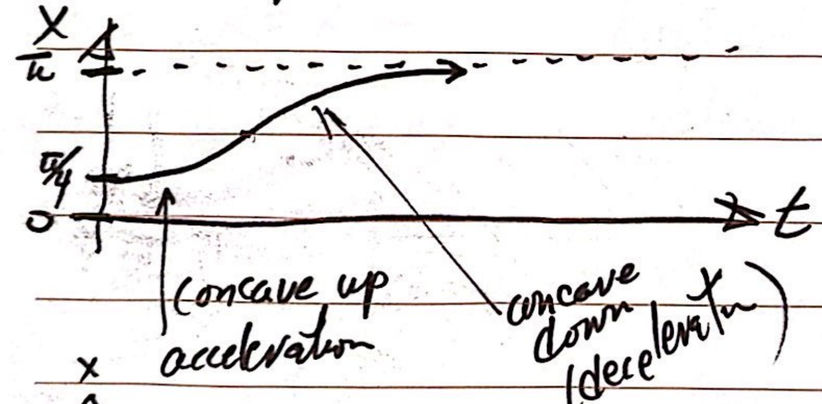
Think of sol'n as a flow on real axis (x-axis)
When $x = 0, \pm\pi, \pm 2\pi, \dots, \pm n\pi$, $\dot{x} = 0$, no flow.
Fixed points of the system.

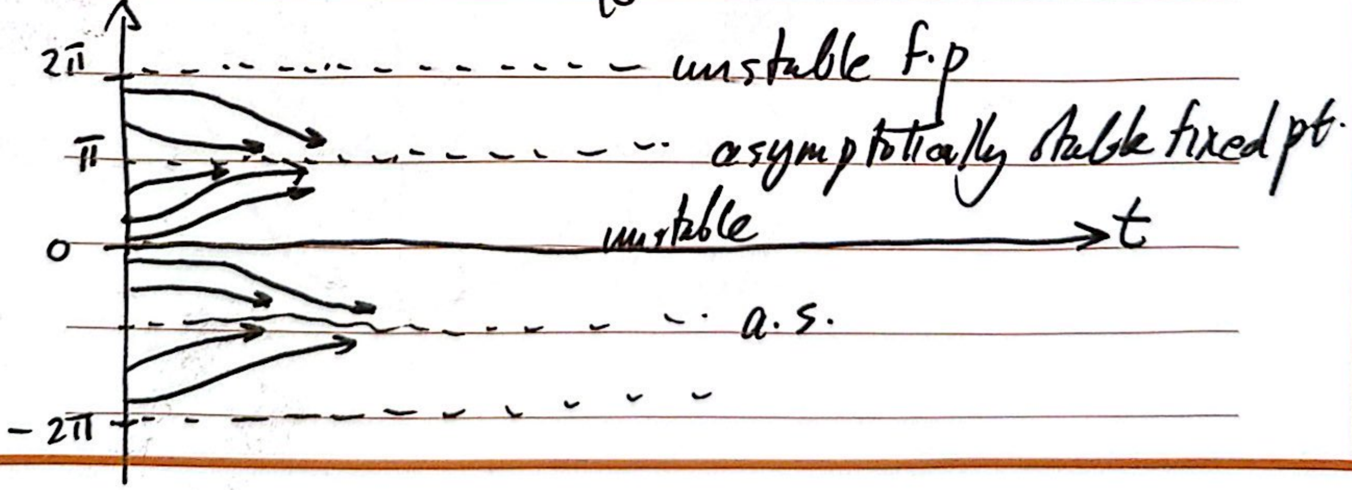
flow moves away from $x = 0, \pm 2\pi, \pm 4\pi, \ldots$ on both sides: unstable fixed points

flow moves towards $x = \pm\pi, \pm 3\pi, \ldots$ These are stable fixed points.

Also from this description, we can answer:

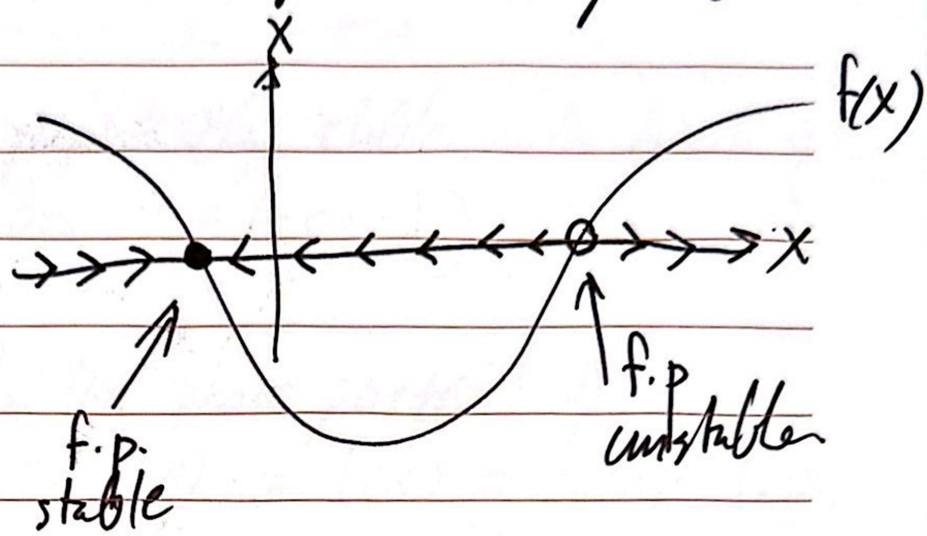
1.) A "particle" starting at $x_0 = \frac{\pi}{4}$ moves right faster and faster until it crosses $x = \frac{\pi}{2}$ (where $\sin(x)$ is maximum). Then it slows down and eventually approaches the stable fixed pt $x = \pi$ from left:





Notice, the same ideas hold for any $f(x)$.

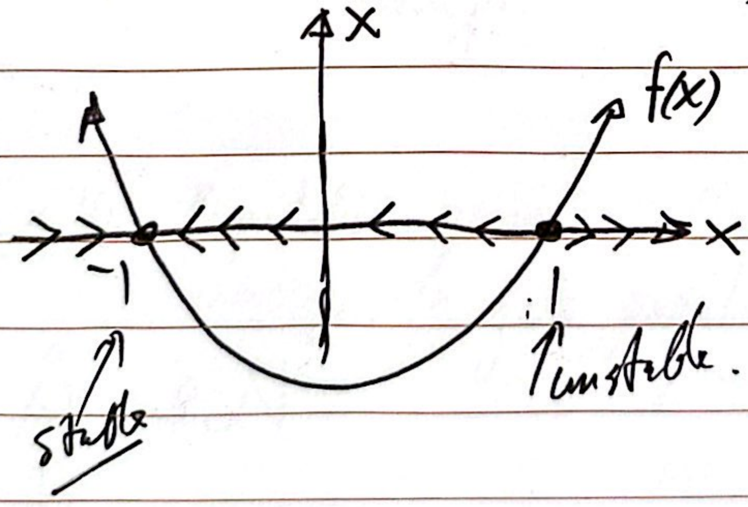
$\dot{x} = f(x)$



Fixed pts. are sols x^*, where $\boxed{f(x^*) = 0}$

Ex 2: $\dot{x} = \underbrace{x^2 - 1}_{f(x)}$ Find all fixed pts. and classify the stability

$f(x) = x^2 - 1 = 0 \Rightarrow x^2 = 1$, $\underbrace{\boxed{x^* = \pm 1}}_{f.p.'s}$



All $x_0 < 1 \to -1$ as $t \to \infty$

All $x_0 > 1 \to \infty$ as $t \to \infty$

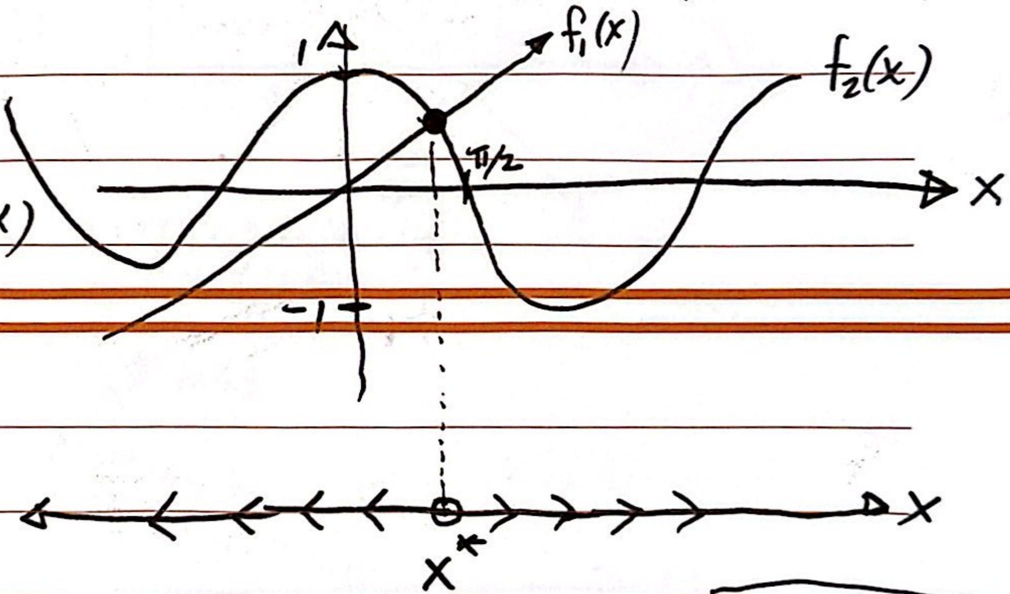
$x = -1$ is asymptotically stable with basin of attraction $x_0 \in (-\infty, 1)$.

Ex 3: Sketch the phase portrait of:

$$\dot{x} = x - \cos(x) = f(x) = f_1(x) - f_2(x)$$

$f_1(x) = x$

$f_2(x) = \cos(x)$



unstable fixed pt.

Sol'n of $\boxed{x^* = \cos(x^*)}$

Ex 4: Population growth: Consider a model for the growth of a population of organisms:

$\dot{N} = rN$ — $r > 0$ growth rate

$N(t)$ is the population

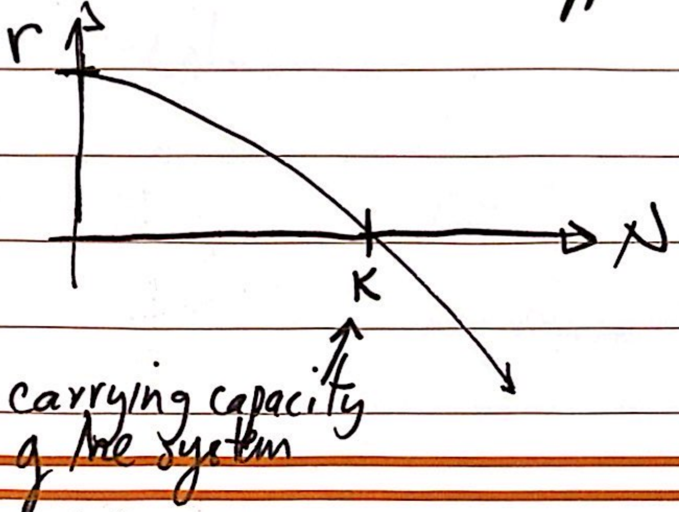
population at time t

$N(t) = N_0 e^{rt}$ grows forever

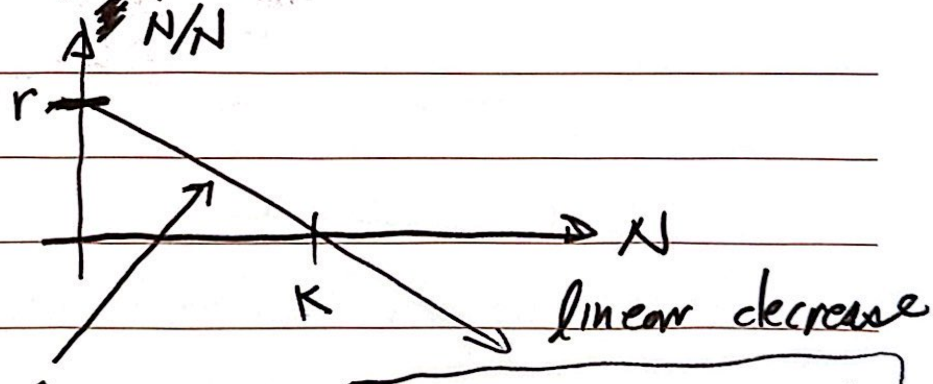
But in many systems, the growth rate

$$r = \dot{N}/N = \frac{d}{dt}\ln(N) = r$$

decreases when N becomes sufficiently large because the environment cannot support the populatn



Simplest

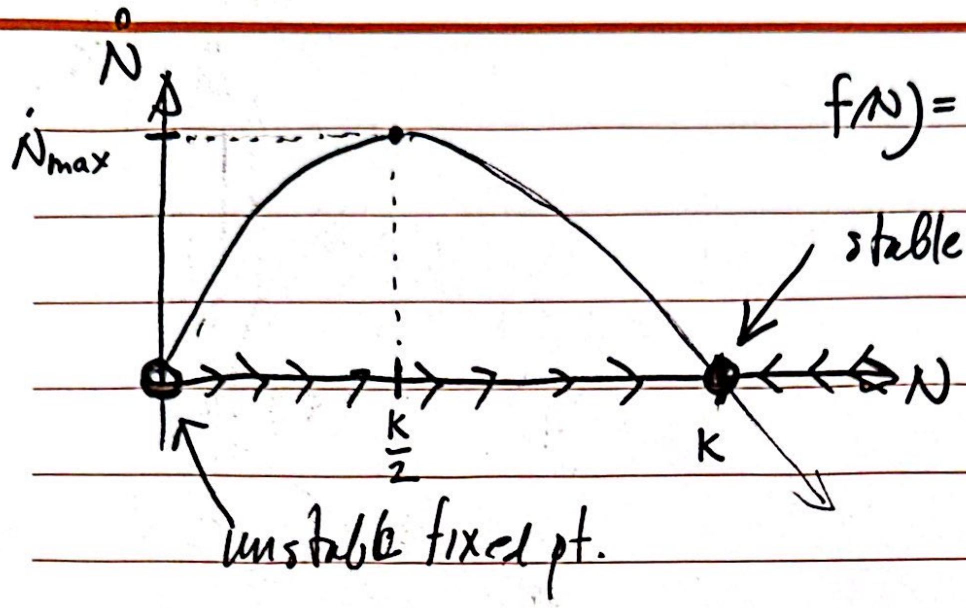


$$\frac{\dot{N}}{N} = \underbrace{r\left(1 - N/K\right)} \Rightarrow \boxed{\dot{N} = rN(1 - N/K)}$$

famous logistic eqn. for population growth and saturation.

tumors: N: Volume of the tumor, # of cancer cells.

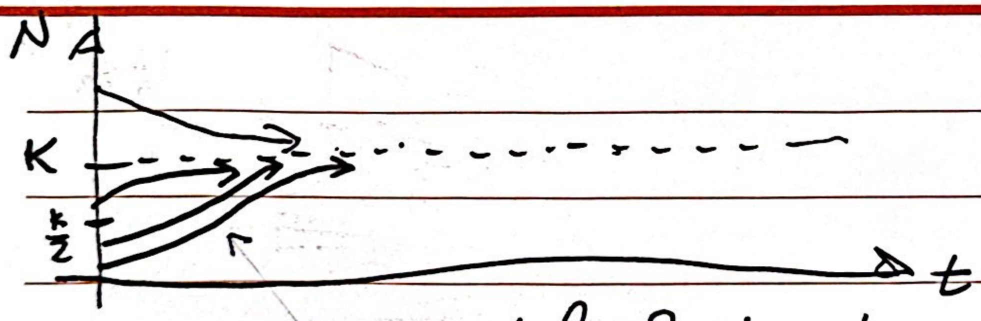
$f(N) = rN(1 - N/K)$



$$f(N) = rN - \frac{r}{K}N^2, \quad f' = r - \frac{2r}{K}N = 0$$

$$\Rightarrow \dot{N}_{max} = f\left(\frac{K}{2}\right)$$

$f(0) = 0, \ f(K) = 0.$



sigmoidal S shaped growth curves for $N_0 < \frac{K}{2}$.

More generally, can have $K(t)$, i.e. time-dependent carrying capacities, time-dependent $r(t)$, ..