Homework 1

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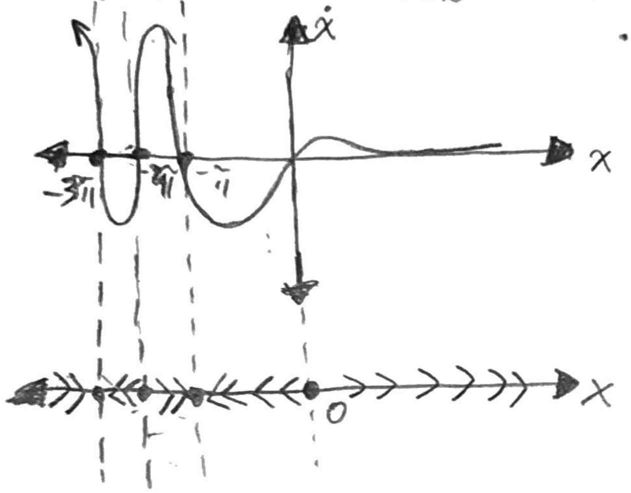
(2.2.4) $\dot{x} = e^{-x}\sin x$

- Sketch the vector field



$0 = e^{-x}\sin x$

$0 = \sin(x^*)$

$\sin^{-1}(0) = x^*$

$x^* = \pm n\pi$ where $n = 0, 1, 2, \ldots$

but because of e^{-x}

$x^* = -n\pi$

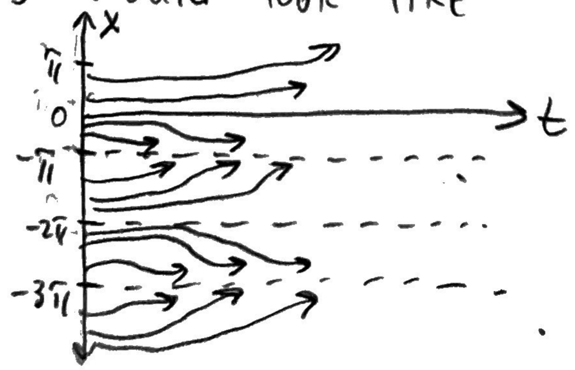
→ to find $x(t)$, we try

$\frac{dx}{dt} = e^{-x}\sin x \Rightarrow \int \frac{dx}{e^{-x}\sin x} = \int dt$

would probably need to solve by parts!

- All fixed points are $x^* = -n\pi$ where $n = 0, 1, 2, \ldots$
- At $x^* = 0$, it is unstable because the vector field goes <u>away</u> from this point
- At $x^* = -n\pi$ where $n = 1, 3, 5, \ldots$ [odd], the fixed points are asymptotically stable because the vector field goes <u>toward</u> this point
- At $x^* = -n\pi$ where $n = 2, 4, \ldots$ [even], the fixed points are unstable because the vector field goes <u>away</u> from this point

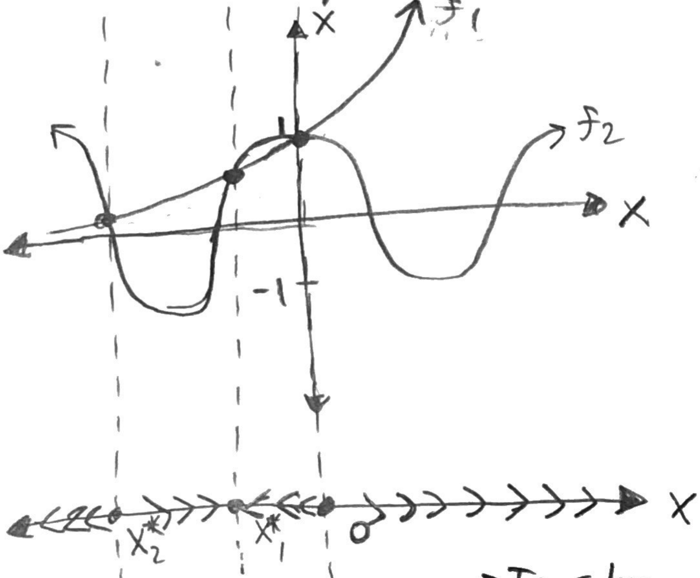
→ the graph of $x(t)$ would look like



2.2.7 $\dot{x} = e^x - \cos(x)$

let $f_1 = e^x$ and $f_2 = \cos(x)$ so $f(x) = f_1 - f_2$

- sketch the vector field



$e^{x^*} = \cos(x^*)$

solve for x^* to get fixed/equilibrium points, particularly $\cos(x^*) = 0$

$x^* = -n\frac{\pi}{2}$

$n = 3, 4, \ldots$

At $x^* = 0$ from right

$f_1 > f_2$, so $f(x) > 0$, meaning vector field goes to the right

At $x^* = 0$ from left

$f_2 > f_1$, so $f(x) < 0$, meaning vector field goes to the left

→ To solve analytically for $x(t)$

$\frac{dx}{dt} = e^x - \cos(x)$

$\int \frac{dx}{e^x - \cos(x)} = \int dt$

↓

Would probably need to solve by parts!

At x_1^* from left

$f_1 > f_2$, so $f(x) > 0$

meaning vector field goes to the right

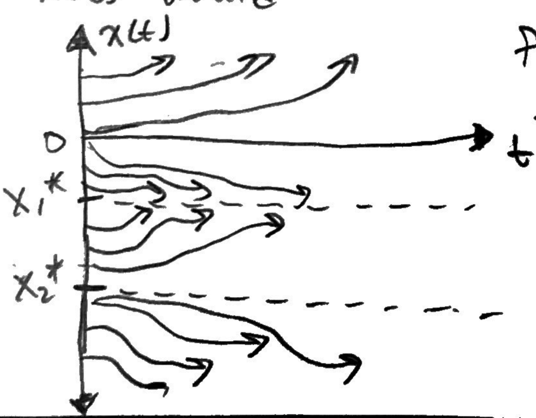
At x_2^* From left

$f_1 < f_2$, so $f(x) < 0$

meaning vector field goes to the left

- So at $x^* = 0$ and every other point of $\cos(x^*) = 0$, $x^* = -n\frac{\pi}{2}$ where $n = 3, 7, 11$ etc., the fixed points are unstable.
- At x_1^* and every other point of $\cos(x^*) = 0$, $x^* = -n\frac{\pi}{2}$ where $n = 5, 9, 13$, etc, the fixed points are asymptotically stable

- the graph of $x(t)$ would look like



2.3.5 $\dot{X} = aX$, $\dot{Y} = bY$ $\quad a > b > 0$

$X_0 > 0 \quad Y_0 > 0$

a) let $x(t) = \dfrac{X(t)}{X(t) + Y(t)}$

- By solving for $X(t)$ and $Y(t)$, show that $x(t)$ increases monotonically and approaches 1 as $t \to \infty$

let, $X(t) = X_0 e^{at} \qquad Y(t) = Y_0 e^{bt}$

so, $x(t) = \dfrac{X_0 e^{at}}{X_0 e^{at} + Y_0 e^{bt}}$

→ divide by e^{at} to numerator and denominator

$$x(t) = \frac{X_0}{X_0 + Y_0 \frac{e^{bt}}{e^{at}}} = \frac{X_0}{X_0 + Y_0 e^{(b-a)t}}$$

→ since $a > b > 0 \Rightarrow e^{(b-a)\cdot\infty} = e^{-(\infty)} \approx 0$

so $\boxed{x(t) = \dfrac{X_0}{X_0} = 1}$

b) $x(t) = \dfrac{X(t)}{X(t)+Y(t)}$ $\qquad x(t) = \dfrac{f}{g} \Rightarrow \dot{x}(t) = \dfrac{f' \cdot g - f \cdot g'}{g^2}$

→ take $\dot{x}(t)$

$$\dot{x}(t) = \frac{\dot{X}(t) \cdot [X(t)+Y(t)] - X(t)\,[\dot{X}(t)+\dot{Y}(t)]}{[X(t)+Y(t)]^2}$$

$$\dot{x}(t) = \frac{aX(t)\,[X(t)+Y(t)] - X(t)[aX(t)+bY(t)]}{[X(t)+Y(t)]^2}$$

$$\dot{x}(t) = \frac{\cancel{aX(t)^2} + aX(t)Y(t) - \cancel{aX(t)^2} - bX(t)Y(t)}{[X(t)+Y(t)]^2}$$

$$\dot{x}(t) = \frac{aX(t)Y(t) - bX(t)Y(t)}{[X(t)+Y(t)]^2}$$

$$\dot{x}(t) = \frac{X(t)Y(t)\,(a-b)}{[X(t)+Y(t)]^2}$$

$$x(t) = \frac{X(t)}{X(t)+Y(t)}$$

$$x(t)X(t) + x(t)Y(t) = X(t)$$

$$\underline{x(t)Y(t)} = X(t) - x(t)X(t)$$

$$\dot{x}(t) = \frac{X(t)}{X(t)+Y(t)} \cdot \frac{Y(t)}{X(t)+Y(t)} \cdot (a-b)$$

$$\dot{x}(t) = x(t) \cdot \frac{Y(t)}{X(t)+Y(t)} \cdot (a-b)$$

$$\dot{x}(t) = (a-b) \cdot (X(t) - x(t)X(t))$$

$$\dot{x}(t) = (a-b) \cdot (1 - x(t))\,X(t)$$

→ since $a > b$

$$\dot{x}(t) = X(t)\,(1 - x(t))$$

→ $x(t)$ increases monotonically and approaches 1 as $t \to \infty$ because $\dot{x}(t)$ is always positive since $a > b > 0$

2.4.4 Use linear stability analysis to classify the fixed points of the following system. If linear stability analysis fails because $f'(x^*) = 0$, use a graphical argument to decide the stability

$$\dot{x} = x^2(6-x)$$

- First obtain fixed/equilibrium points

$0 = x^2(6-x)$ $\quad$ $f(x) = x^2(6-x) = 6x^2 - x^3$

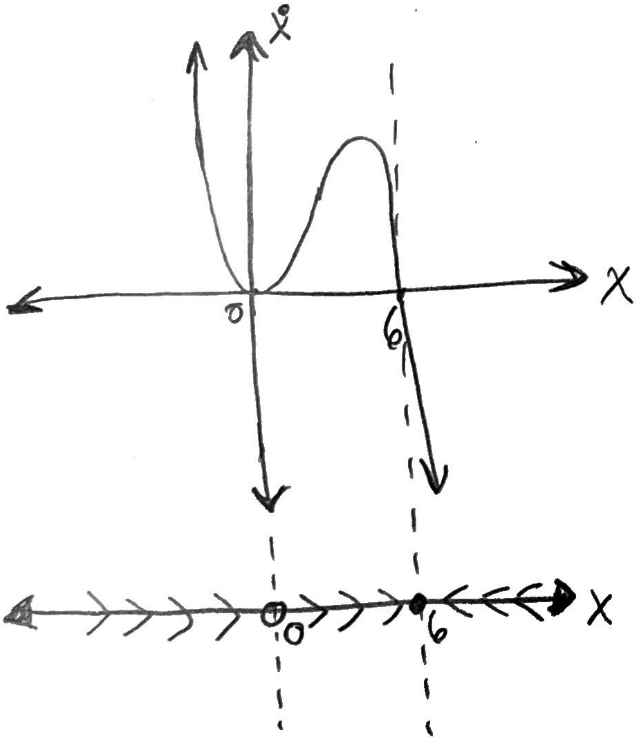
$\boxed{0 = x^*}$ and $\boxed{x^* = 6}$ $\quad$ $f(x^*) = 0$

- Then take derivative of $x^2(6-x)$: $f'(x)$

$f'(x^*) = 12x - 3x^2$ $\quad$ $f'(0) = 0$ & $f'(6) = 72 - 108 = -36$

- Because one of our stability points is zero, lets plot $f(x)$ to determine stability

$f'(6) < 0$, decays so it is stable



- At $x^* = 6$, it is locally asymptotically stable
- At $x^* = 0$, it is half-stable
  - → Stable from left side
  - → Unstable from right side

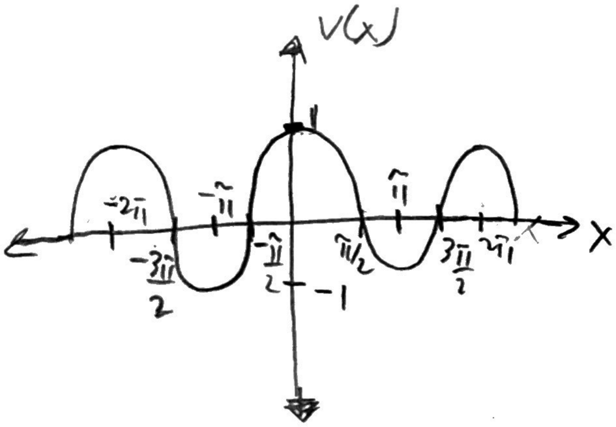
2.7.3 For the following vector field, plot the potential function $V(x)$ and identify all the equilibrium points and their stability.

$$\dot{x} = \sin(x)$$

$$-\frac{dV}{dx} = \sin(x)$$

$$V(x) = \cos(x) + C \Rightarrow V(x) = \cos(x)$$

$(2k-1) \rightarrow$ odd



Local minima: $x = \pm k\pi$, k is odd

Local maxima: $x = \pm k\pi$, k is even

- Equilibrium points at $x^* = \pm k\pi$ where k is odd, they are stable
- Equilibrium points at $x^* = \pm k\pi$ where k is even, they are unstable

(3.1.1) For the following exercise, sketch all the qualitatively different vector fields that occur as r is varied. Show that a saddle-node bifurcation occurs at a critical value of r, to be determined. Finally, sketch the bifurcation diagram of fixed points x^* versus r.

$$\dot{x} = 1 + rx + x^2$$

$$f(x) = 1 + rx + x^2$$

$$0 = 1 + rx^* + (x^*)^2$$

$$x^* = \frac{-r \pm \sqrt{r^2 - 4(1)(1)}}{2(1)}$$

$$x^* = \frac{-r \pm \sqrt{r^2-4}}{2}$$ → valid for $-2 \le r \le 2$

* saddle-node bifurcation occurs at $f'(x^*) = 0$

$$f'(x) = r + 2x$$

$$f'(x^*) = r_c + 2\left(\frac{-r_c \pm \sqrt{r_c^2-4}}{2}\right) = 0$$

$$0 = r_c - r_c \pm \sqrt{r_c^2 - 4}$$

$$0 = \pm\sqrt{r_c^2-4}$$

$$0 = \sqrt{r_c^2-4}$$

$$0 = r_c^2 - 4$$

$$4 = r_c^2$$

$$\boxed{r_c = \pm 2}$$

Find x^*, $r_c = -2$

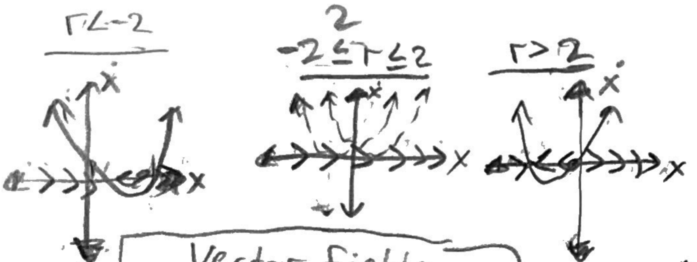
$$x^* = \frac{2 \pm \sqrt{4-4}}{2}$$

$$x^* = 1$$

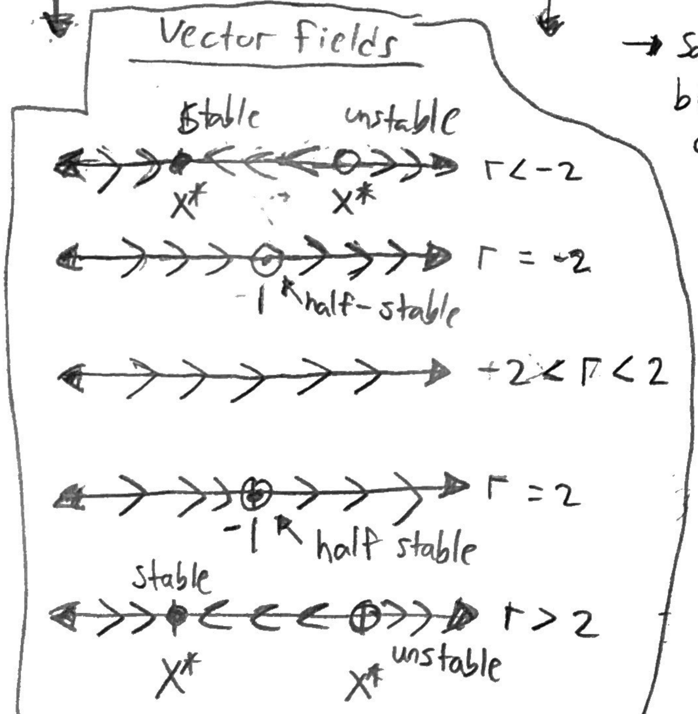
Find x^*, $r_c = 2$

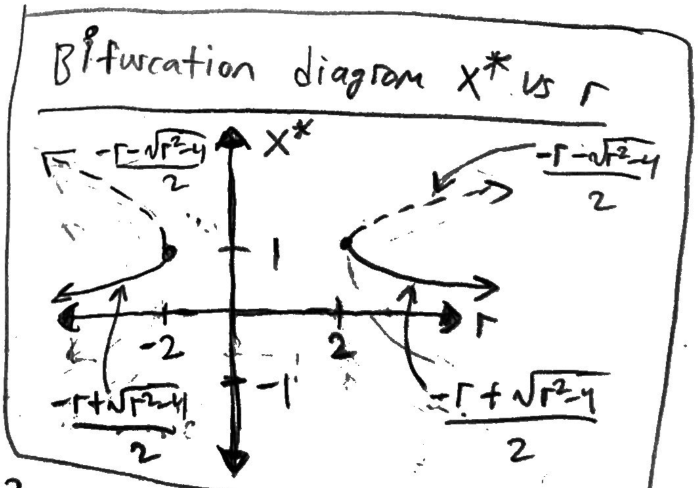
$$x^* = \frac{-2 \pm \sqrt{4-4}}{2}$$

$$x^* = -1$$



→ Saddle-node bifurcation occurs at $r_c = \pm 2$ with $x^* = \pm 1$ [Look at matlab plot]





For $r < -2$

$f'(x^*) = \sqrt{r^2-4}$ & $-\sqrt{r^2-4}$

$\Downarrow$ > 0 [unstable] $\quad$ $\Downarrow$ < 0 [stable]

For $r > 2$

$f'(x^*) = \sqrt{r^2-4}$ & $-\sqrt{r^2-4}$

$\Downarrow$ > 0 [unstable] $\quad$ $\Downarrow$ < 0 [stable]

3.2.4 For the following exercise, sketch all the qualitatively different vector fields that occur as r is varied. Show that a transcritical bifurcation occurs at a critical value of r, to be determined. Finally, sketch the bifurcation diagram of fixed points x^* versus r.

Normal form $\Rightarrow rx - x^2$

$x(r-x^2)$

$$\dot{x} = x(r-e^x) = rx - xe^x$$

$f(x) = x(r-e^x)$

$x^* = 0$ but for $0 = r - e^x$

lets Taylor expand e^x to get into normal form

so $e^x = 1 + x + \frac{1}{2}x^2 + \underbrace{H.O.T.(x^3)}_{\text{Higher order terms can ignore}}$

So,

$$\dot{x} = x\left(r - 1 - x - \frac{x^2}{2} + H.O.T.(x^3)\right)$$

$$\dot{x} = rx - x - x^2 + H.O.T.(x^3)$$

so, $\dot{x} = rx - x - x^2$

$\dot{x} = x(\underbrace{r-1}_{\text{let be } R} - x^2)$

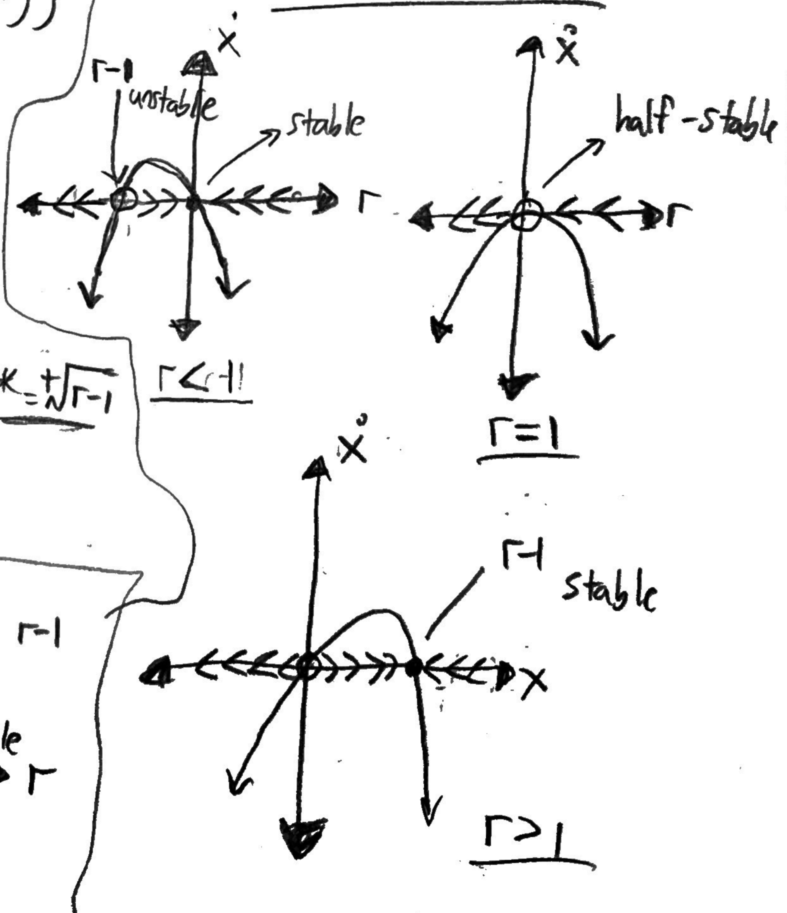
$x^* = r - 1 - x^2$

$0 = r - 1 - x^2$

$x^2 = r - 1$

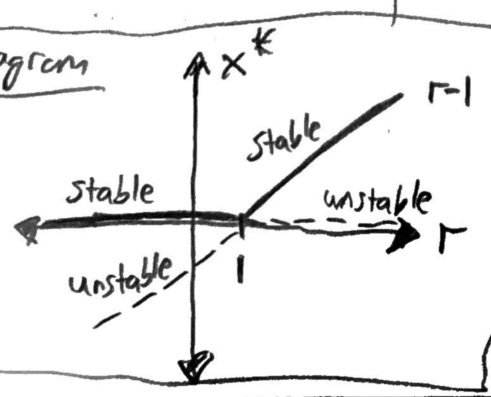
$x^* = \pm\sqrt{r-1}$

So vector fields



$\boxed{\text{a transcritical bifurcation occurs when } R=0\text{, so } r_c = 1}$

Bifurcation diagram



(3.3.1) $\dot{n} = GnN - kn$

$\dot{N} = -GnN - fN + p$

$G>0, k>0, f>0,$

p either $\pm$

a) Can approximate $\dot{N} \approx 0$, express $N(t)$ in terms of $n(t)$ and derive a first-order system for n.

$0 = -GnN - fN + p$

$GnN + fN = p$

$p = N(Gn+f)$

$N(t) = \dfrac{p}{Gn(t)+f}$

plug into $\dot{n}$

$\dot{n} = Gn(t) \cdot \left[\dfrac{p}{Gn(t)+f}\right] - kn(t)$

$$\boxed{\dot{n} = \frac{pGn(t)}{(Gn(t)+f)} - kn(t)}$$

b) Show that $n^* = 0$ becomes unstable for $p > p_c$, where p_c is to be determined.

-use linear stability analysis then set to zero

- Find n^* by setting $\dot{n} = 0$

$0 = \dfrac{pGn}{Gn+f} - kn$

$0 = n\left[\dfrac{pG}{Gn+f} - k\right]$

$n^* = 0$

$0 = \dfrac{pG}{Gn+f} - k$

$k = \dfrac{pG}{Gn+f}$

$kGn + kf = pG$

$kGn = pG - kf$

$n = \dfrac{pG}{kG} - \dfrac{kf}{kG}$

$n^* = \dfrac{p}{k} - \dfrac{f}{G} = \dfrac{pG - fk}{kG}$

→ Next do $f'(n^*) = 0$, for $n^* = 0$

$f'(n) = \dot{n}$

$f'(n) = \dfrac{(pG)(Gn(t)+f) - (pGn(t))(G)}{(Gn(t)+f)^2} - k$

$f'(n) = \dfrac{\cancel{pG^2n} + pGf - \cancel{pG^2n}}{(Gn+f)^2} - k = \dfrac{pGf}{(Gn+f)^2} - k$

Since $f'(0) = \frac{pGf}{(G(0)+f)^2} - k = \frac{pGf}{f^2} - k = \frac{pG}{f} - k$

$f'(0) = \frac{pG - kf}{f}$, for n^* to be unstable, $f'(0) > 0$
which happens when $p_cG - kf > 0$

$\boxed{p_c > kf/G}$

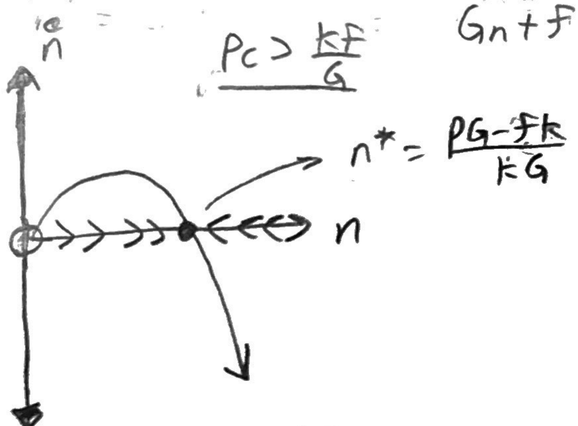
c) What type of bifurcation occurs at the laser threshold p_c?

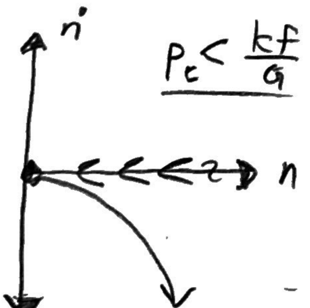
→ Based on the other fixed points found and stability known of $n^* = 0$ when $p_c > \frac{kf}{G}$, we can draw vector fields for $\dot{n} = \frac{pGn}{Gn+f} - kn$

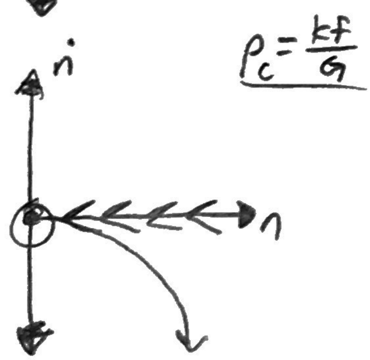
$f'(n) = \frac{pGf}{(Gn+f)^2} - k$

$f'\left(\frac{pG - fk}{kG}\right) = \frac{pGf}{\left(\frac{pG}{k}\right)^2} - k$

$f'\left(\frac{pG - fk}{kG}\right) = \frac{fk^2}{pG} - k$







⇒ This is a transcritical bifurcation with critical point $\frac{kf}{G}$

d) For what range of parameters is it valid to make the approximation used in (a)?

- If we let $a = \frac{\dot{n}}{n} = \frac{1}{n}\left(\frac{pGn}{Gn+f} - kn\right)$ to relate $\dot{n}$ and $\dot{N}$ based on the approximation that was made

$a = \frac{pG}{Gn+f} - k$, and a is valid if $p \cdot G > 0$ and $k > 0$ for the approximation to be valid.

3.4.3 In the following exercise, sketch all the qualitatively different vector fields that occur as r is varied. Show that a pitchfork bifurcation occurs at a critical value of r (to be determined) and classify the bifurcation as supercritical or subcritical. Finally, sketch the bifurcation diagram of x^* vs r.

$$\dot{x} = rx - 4x^3$$

$f(x) = rx - 4x^3$

$f(x) = x(r - 4x^2) = 0$

$x^* = 0$ $\qquad r - 4x^2 = 0$

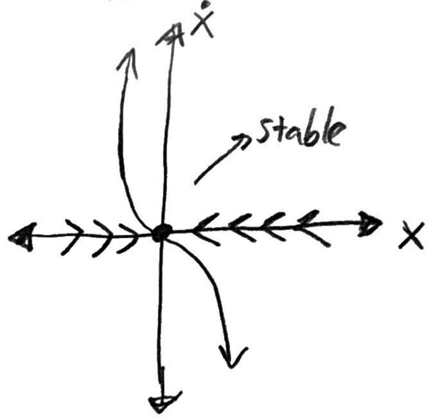
$4x^2 = r$

$x^2 = \frac{r}{4}$

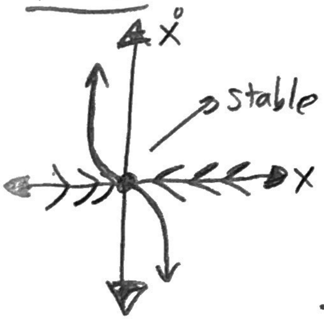
$x^* = \pm\sqrt{\frac{r}{4}}$ → valid for $r \geq 0$

Plotting vector fields

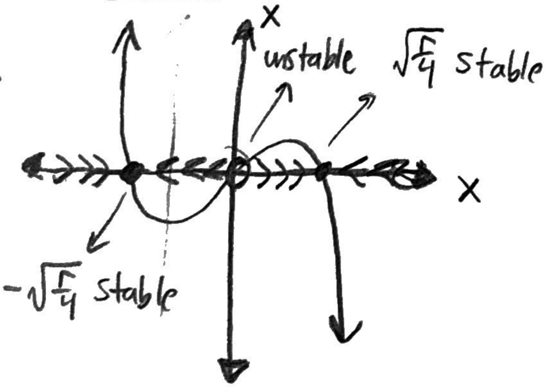
$r < 0$



$r = 0$



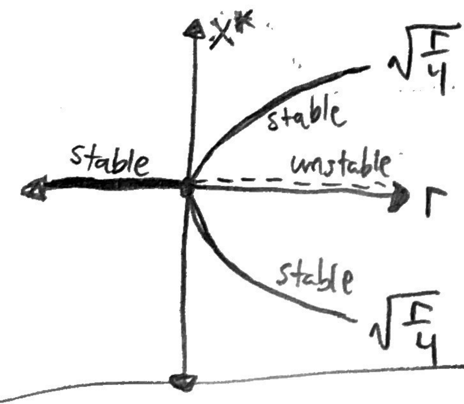
$r > 0$



Critical point $r_c = 0$

→ this is a supercritical pitchfor bifurcation because of the two stabilizing fixed points

Bifurcation Diagram



3.7.3 $\dot{N} = rN\left(1-\frac{N}{K}\right) - H$ $\quad -H \to$ effects of fishing

$H>0$
$\hookrightarrow$ rate

a) Show that the system can be rewritten in dimensionless form as

$$\frac{dx}{d\tau} = x(1-x)-h$$

for suitably defined dimensionless quantities x, τ and, h.

$\to$ Need to rescale N and t (for τ)

Let $x = \alpha N$, so $N = \frac{x}{\alpha}$ and $\dot{N} = \frac{x'}{\alpha} \cdot \frac{d}{dt} \Rightarrow \dot{N} = \frac{x'}{\alpha} \cdot B\frac{d}{d\tau}$

$\tau = Bt$, so $\frac{d\tau}{dt} = B$

Also, $\frac{d}{dt} = \frac{d\tau}{dt} \cdot \frac{d}{d\tau} \Rightarrow B \cdot \frac{d}{d\tau}$

$$\frac{B}{\alpha}x' = r\left(\frac{x}{\alpha}\right)\left(1-\frac{\frac{x}{\alpha}}{K}\right) - H$$

$$\frac{B}{\alpha}x' = \frac{r}{\alpha}x\left(1-\frac{x}{\alpha K}\right) - H$$

$$x' = \frac{r}{B}x\left(1-\frac{x}{\alpha K}\right) - \frac{\alpha H}{B}$$

$\to$ Now choose $\alpha = \frac{1}{K}$ and $B = r$

so that $\frac{\alpha}{B} = \frac{1}{Kr}$

$$x' = \frac{r}{B}x\left(1-\frac{x}{\frac{1}{K}\cdot K}\right) - \frac{H}{Kr}$$

$$x' = x(1-x) - \frac{H}{Kr}$$

Now let $h = \frac{H}{Kr}$

$$\boxed{\begin{gathered}\text{So, } x' = x(1-x)\,h \\ \tau = rt, \; x = \frac{N}{K} \\ \text{and } h = \frac{H}{Kr}\end{gathered}}$$

b) Plot the vector field for different values of h.

So now we have

$$\dot{x} = x(1-x) - h = f(x)$$

Find x^* so, $0 = x - x^2 - h \Rightarrow 0 = -x^2 + x - h$

$$x^* = \frac{-1 \pm \sqrt{1 - 4(-1)(-h)}}{-2}$$

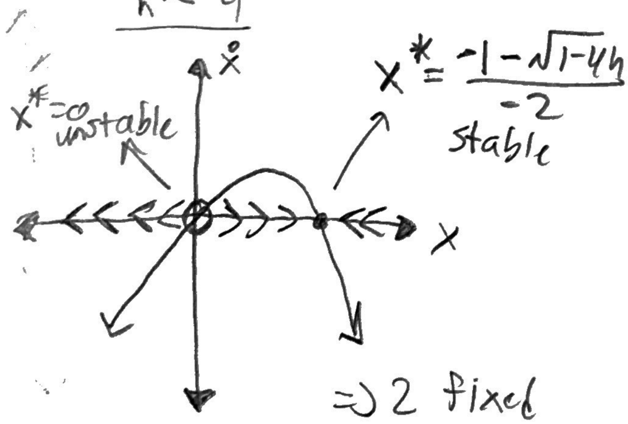
$$x^* = \frac{-1 \pm \sqrt{1-4h}}{-2}$$

$$1 - 4h \geq 0$$
$$1 \geq 4h$$
$$\frac{1}{4} \geq h$$

So, x^* is only valid for $h \leq \frac{1}{4}$

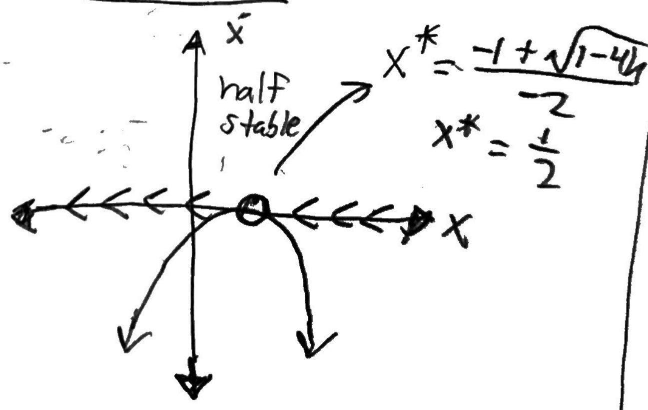
So the vector fields

$h < \frac{1}{4}$



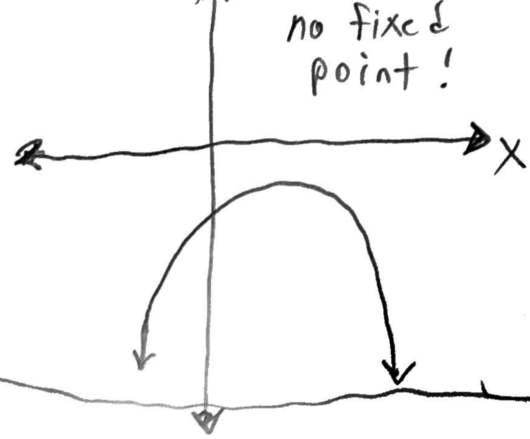
$\Rightarrow$ 2 fixed points

$h = \frac{1}{4}$



$\Rightarrow$ 1 fixed point

$h > \frac{1}{4}$



no fixed point!

c) Show that a bifurcation occurs at a certain value h_c, and Classify this bifurcation

→ From part b, the critical point was found to be at

$$h_c = \frac{1}{4}$$

→ Any values less than this, there are 2 fixed points $x^* = 0$ and $x^* = \frac{-1+\sqrt{1-4h}}{-2}$, where at $x^* = 0$ it is unstable and at $x^* = \frac{-1+\sqrt{1-4h}}{-2}$ it is stable

→ At the critical point, there is 1 fixed point at $x^* = \frac{-1+\sqrt{1-4h}}{-2}$ where it is half stable

→ Any values above the critical point, there are no fixed points.

→ This is a saddle node bifurcation because the fixed points appear and disappear at a certain critical point.

d) Discuss the long-term behavior of the fish population for $h < h_c$ and $h > h_c$, and give the biological interpretation in each case.

$H = h \cdot kr$ $\quad h_c = \frac{1}{4}$

as h goes from $(\frac{1}{4}, -\infty)$, H gets smaller which means the rate goes down and there is less fishing

as h goes from $\frac{1}{4}$ to ∞, H gets bigger and there is more fishing.

So, for $h < \frac{1}{4}$, the population of fishing will grow exponentially as the rate of fishing decreases to control the population

for $h > \frac{1}{4}$, the population of fishing will decrease exponentially as the rate of fishing increases to almost deplete the population of fish.