For §1.1, narrow focus to immediate needs.
Think of this as a future resource

def: A - set, $\boxed{[A]^k}$:= the set of all k-element subsets of A.

Ex| $A = \{a,b,c,d,e\}$, then $\{a,b,c\} \in [A]^3$

$\left|[A]^3\right| = \binom{5}{3} = \frac{5!}{3!2!} = 10$ $\qquad \{a,b\} \notin [A]^3$

def: A graph $G=(V,E)$ is a pair of sets such that $\boxed{E \subseteq [V]^2}$

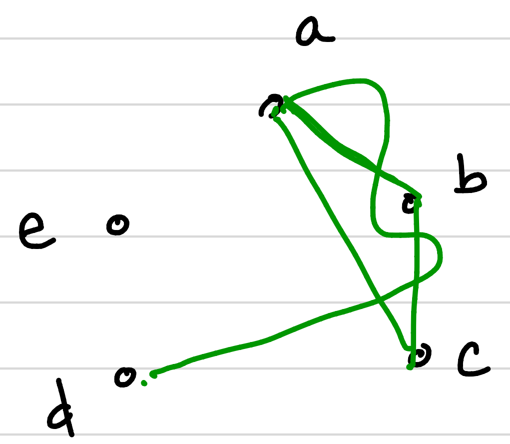
all of elements in E are unordered pairs of elents in V

Ex| Let $G=(V,E)$ be defined as

$V = \{a,b,c,d,e\}$, $E = \{\{a,b\},\{b,c\},\{a,c\},\{a,d\}\}$

$= \{ab, bc, ac, ad\}$

$ab \in E$



def: $G = (V,E)$

$|G| = |V|$ the order of G

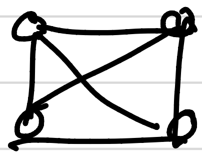
$\|G\| = |E|$ the index of G

def: Given $G=(V,E)$, if $a,b \in V$ and $ab \in E$, then we say

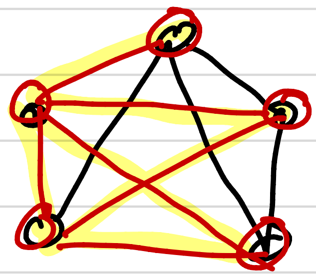
a and b are adjacent
or are neighbors.

($ab \notin E$, then a and b are nonadjacent, nonneighbors
nonedges

def: A complete graph on n vertices, denoted K^n, has edge set equal to $[V]^2$.

Ex| K^4 is 

$\bar{G}$ G

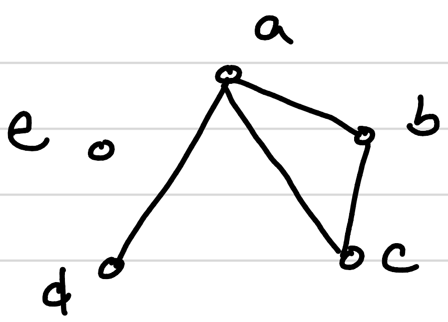


def: Given $G=(V,E)$, the complement of G, denoted $\bar{G}$ is $\bar{G} = (V, [V]^2 - E)$

def: Let $G=(V,E)$ and $G'=(V',E')$. We say G' is a subgraph of G if $\boxed{V' \subseteq V \text{ and } E' \subseteq E}$

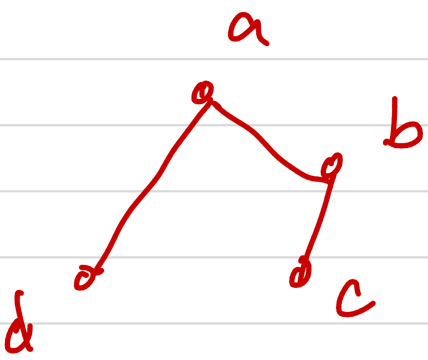
$G' \subseteq G$

Ex| $V = \{a,b,c,d,e\}$
$E = \{ab, bc, ab, ad\}$



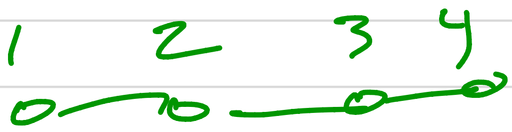
$G = (V, E)$

$V' = \{a,b,c,d\}$
$E' = \{ab, ad, bc\}$



$G' = (V', E')$

$V'' = \{1,2,3,4\}$
$E'' = \{12, 23, 34\}$



$G''(V'', E'')$

$\phi : V' \to V''$ is $d \to 1$, $a \to 2$, $b \to 3$, $c \to 4$

G'' is isomorphic to a subgraph of G
$G'' \subseteq G$

Q| What does it mean for two graphs to be the same?

def: $G=(V,E)$, $G'=(V',E')$. We say G and G' are $\boxed{\text{isomorphic}}$ if $\exists$ bijection $\phi : V \to V'$ and $\forall a, b \in V$

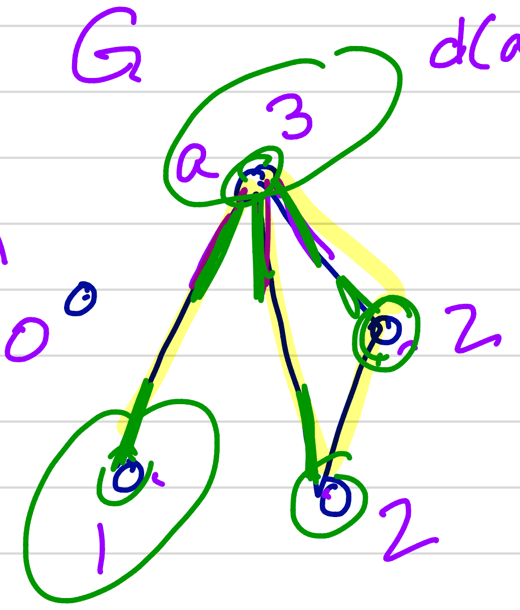
$$ab \in E \iff \phi(a)\phi(b) \in E'$$

$\Rightarrow$
$\Leftarrow$

§1.2 The Degree of a Vertex

defs : $G=(V,E)$ s.t. $|V|>0$; $v \in V$

- The degree of v, $d(v)$, is
  \# edges incident to v // $d(v) = |\{w \in V : vw \in E\}|$
  \# neighbors of v, \# of vert. adjacent to v
- The minimum degree of G, $\delta(G)$, is
  $\min\{d(v) : v \in V\}$
- The maximum degree of G, $\Delta(G)$, is
  $\max\{d(v) : v \in V\}$
- The average degree of G, $d(G)$,
  $\left(\sum_{v \in V} d(v)\right) / |V|$
  average # of incidences per vertex

G 

$d(a)=3$

$\delta(G)=0$

$\Delta(G)=3$

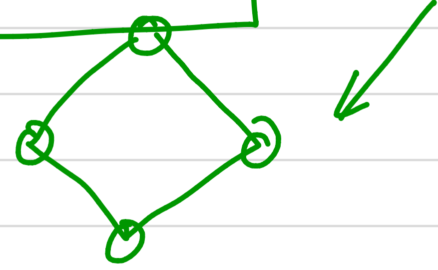
$d(G) = \frac{3+2+2+1+0}{5} = \frac{8}{5}$

$= 1.6$

$\varepsilon(G) = \frac{4}{5} = 0.8$

- A graph is k-regular if
  $d(v)=k \quad \forall v \in G$.

$\boxed{d(H)=2}$



- \#edges per vertex is $\varepsilon(G) = \frac{|E|}{|V|}$

$$\varepsilon(G) = \frac{|E|}{|V|} = \frac{\frac{1}{2}\sum_v d(v)}{|V|} = \frac{1}{2} d(G)$$

$\delta(G) \leq d(v)$

- Show $\delta(G) \leq d(G) \leq \Delta(G)$.

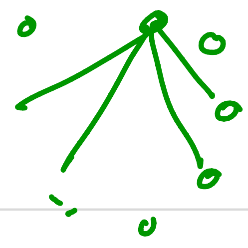
$$\delta(G) = \min\{d(v) : v \in V\} \leq \max\{d(v) : v \in V\} = \Delta(G)$$

$$\delta = \frac{|V| \cdot \delta}{|V|} = \frac{\sum_v \delta(G)}{|V|} \leq d(G) = \frac{\sum_v d(v)}{|V|} \leq \text{ Same}$$

- How to relate $|E|$ and vertex degrees?

$$\boxed{|E| = \frac{1}{2} \sum_{v \in V} d(v)}$$

Not possible $4, 4, 4, 3, 2, 2, 2, 2, 2, 2, 2$



* **Prop 1.2.1** In any graph G, the number of vertices of odd degree is **even**.

Pf: $|E| = \frac{1}{2}\sum_{v} d(v)$. Since $|E|$ and $\sum d(v)$ are integers, then $\sum_{v} d(v)$ is even. But $\sum d(v)$ is a sum of integers that is even, so it must have an even # of odd entries.

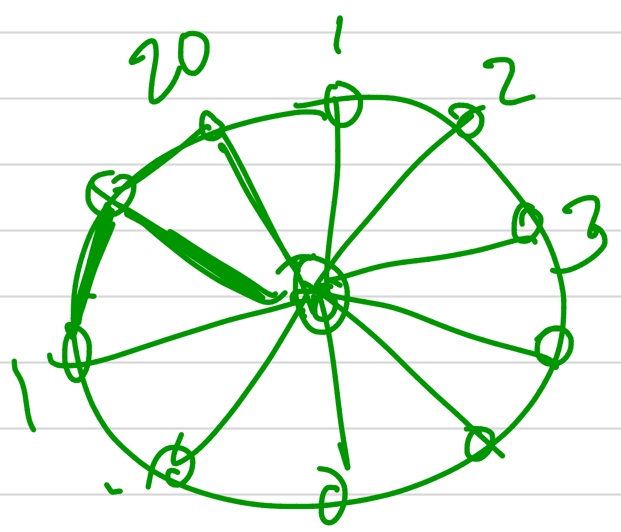
Prop 1.2.2 For $G=(V,E)$ s.t. $|E|>0$, $\exists\, H \subseteq G$ s.t.

$$\delta(H) > \varepsilon(H) \geq \varepsilon(G) \quad \text{OR}$$

$$\delta(H) > \tfrac{1}{2} d(H) \geq \tfrac{1}{2} d(G).$$

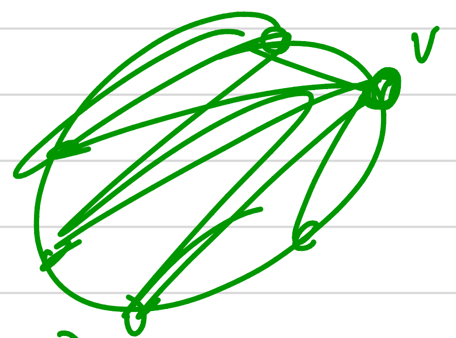
Build a graph such that $\varepsilon(G) \geq 6$ and $\delta(G) = 2$.
In addition, G has 10 vertices w/ $d(v) = 2$.

$$\tfrac{1}{2} d(G) = \varepsilon(G) \geq 6 \iff d(G) \geq 12$$



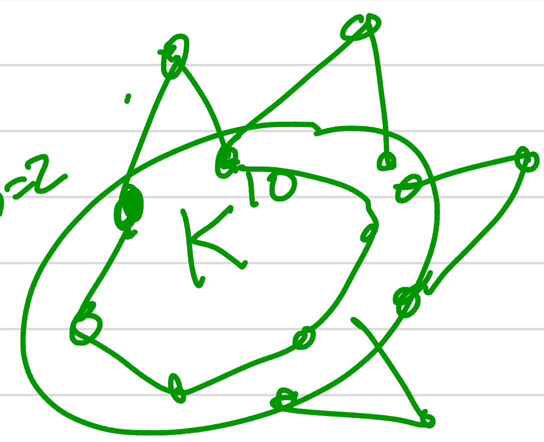
$$\frac{20 + 20\cdot 3}{21} = \frac{80}{21} \approx 4$$

K^{10} $d(v)=9$



G

K^{10} w/ 10 $\delta(v)=2$ added



$d(K^{10}) = 9$

$$\varepsilon(G) = \frac{20 + \binom{10}{2}}{10+10} = \frac{65}{20}$$

$$\frac{10\cdot 9}{2}$$