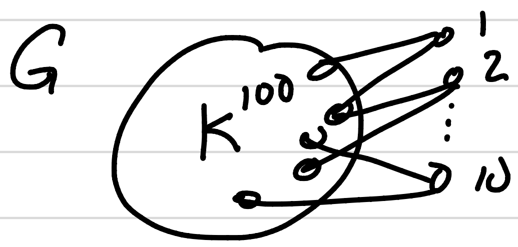
From Wed

Ex Find G so that $\varepsilon(G) \geq 6$ and G has 10 vertices of degree 2

or equivalently

find G so that $d(G) = \dfrac{\sum d(v)}{|V|} \geq 12$ and 10 vert of degree 2.

① Construct G by starting with K^{100} and append 10 vertices



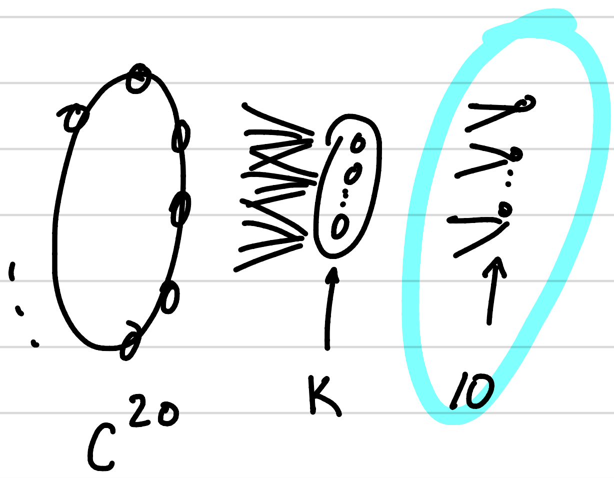
$$|E| = \binom{100}{2} + 10 \cdot 2 = 4970$$

$$|V| = 110$$

$$\varepsilon(G) = \frac{4970}{110} = 45.\overline{18}$$

$$d(G) = 90.36$$

②



$$|E| = 20 + K \cdot 20 + 20$$

$$|V| = 20 + K + 10$$

$$\varepsilon(G) = \frac{20K + 40}{K + 30} \to 20 \text{ as } K \to \infty$$

for $K = 100$, $\varepsilon = 15.530$

$$d(G) = 31.07$$

- We know $\delta(G) \leq d(G)$.

  For these examples $\delta(G) \ll d(G)$

- Prop 1.2.2 says you can ALWAYS find a subgraph $H \subseteq G$ s.t. $d(H) \geq d(G)$ but

$$\delta(H) > \frac{1}{2} d(H) \geq \frac{1}{2} d(G)$$

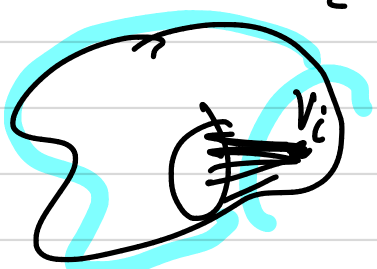
$\varepsilon(G) > 0$

Prop 1.2.2 For $G=(V,E)$ s.t. $|E|>0$, $\exists\, H \subseteq G$ s.t.

$$\delta(H) > \underset{\frac{1}{2}d(G)}{\underset{\|}{\varepsilon(H)}} \geq \varepsilon(G).$$

Pf (by construction)

Find a seq. of graphs $G=G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots \supseteq G_k$ $\quad G_i$



by iteratively deleting vertex $v_i \in G_i$ s.t.

$$d_{G_i}(v_i) \leq \varepsilon(G_i) = \frac{1}{2} d(G_i)$$

terminate? yes. $|V| < \infty$.

What if @ G_i no v_i exists? $\forall v_i$, $d_{G_i}(v_i) > \varepsilon(G_i) = \frac{1}{2}d(G_i)$

N.t.s. $\varepsilon(G_{i+1}) \geq \varepsilon(G_i)$.

$$\varepsilon(G_{i+1}) = \frac{|E(G_{i+1})|}{|V(G_{i+1})|} = \frac{|E(G_i)| - d(v_i)}{|V(G_i)|-1}$$

$$\geq \frac{|E(G_i)| - \varepsilon(G_i)}{|V(G_i)|-1}$$

b/c

$$\varepsilon(G_i) = \frac{|E(G_i)|}{|V(G_i)|}$$

$$\varepsilon(G_i)\cdot|V(G_i)| = |E(G_i)|$$

$$= \frac{\varepsilon(G_i)\cdot|V(G_i)| - \varepsilon(G_i)}{|V(G_i)| - 1}$$

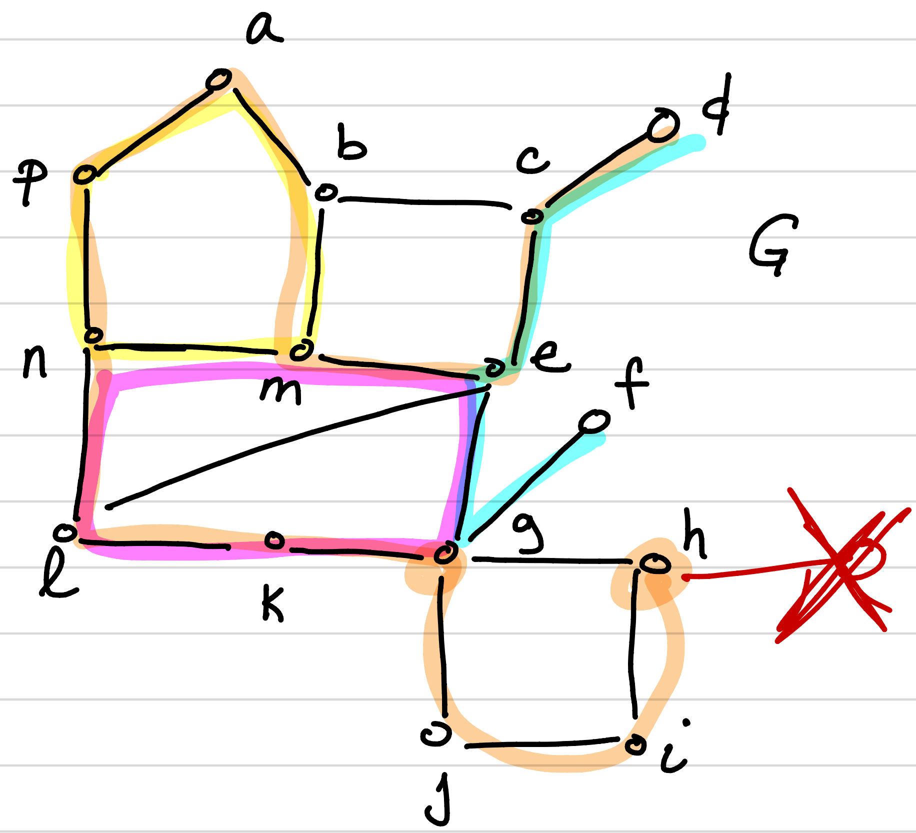
$$= \varepsilon(G_i)$$

Section 1.3

Path: $P^k = x_0 x_1 x_2 \dots x_k$, x_i's distinct $x_i x_{i+1} \in E$

cycle: $C^k = P^{k-1} + x_{k-1}x_0$

length



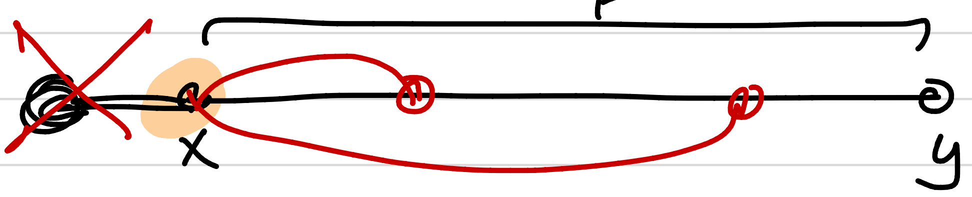
Prop 1.3.1 For a graph G with $\delta = \delta(G)$, G has

(i) a path of length at least δ

and

(ii) a cycle of length at least $\delta + 1$ (provided $\delta \geq 2$)

Pf: (i) Find a longest path, P

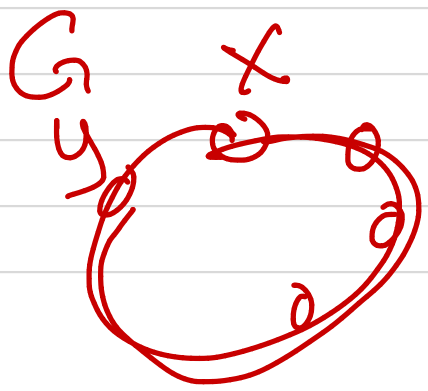


We know $d(x) \geq \delta(G)$

We know nbhs of x lie on P otherwise P is not longest.

So P has to contain at least $1 + d(x)$ vertices. Thus it has at least $1 + \delta(x)$ vertices.

So, P has LENGTH δ.



(ii) $\delta(G) \geq 2 \Rightarrow G$ has cycle of length $\delta+1$

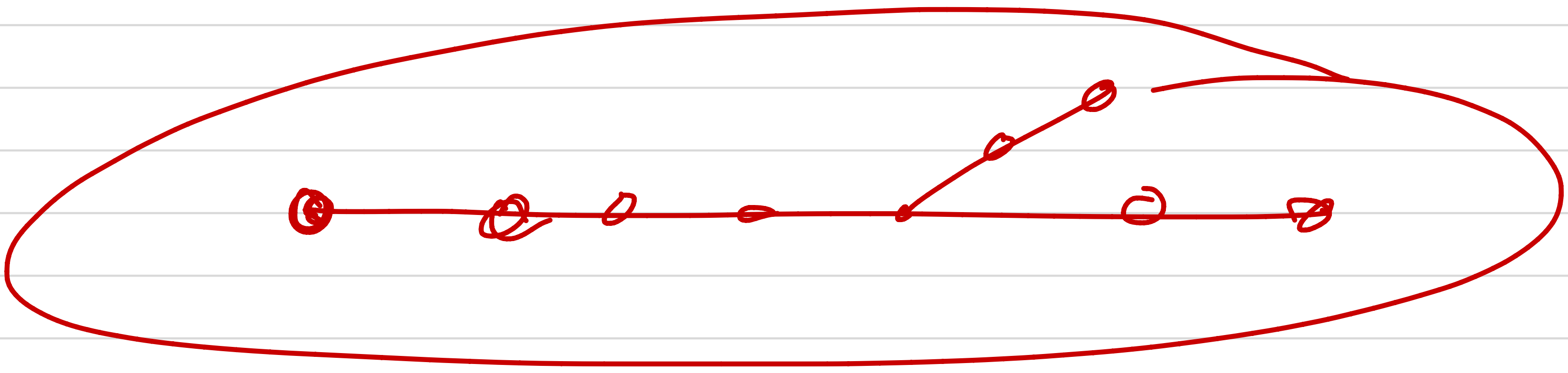
P a longest path

$N(x)$ on Path P.

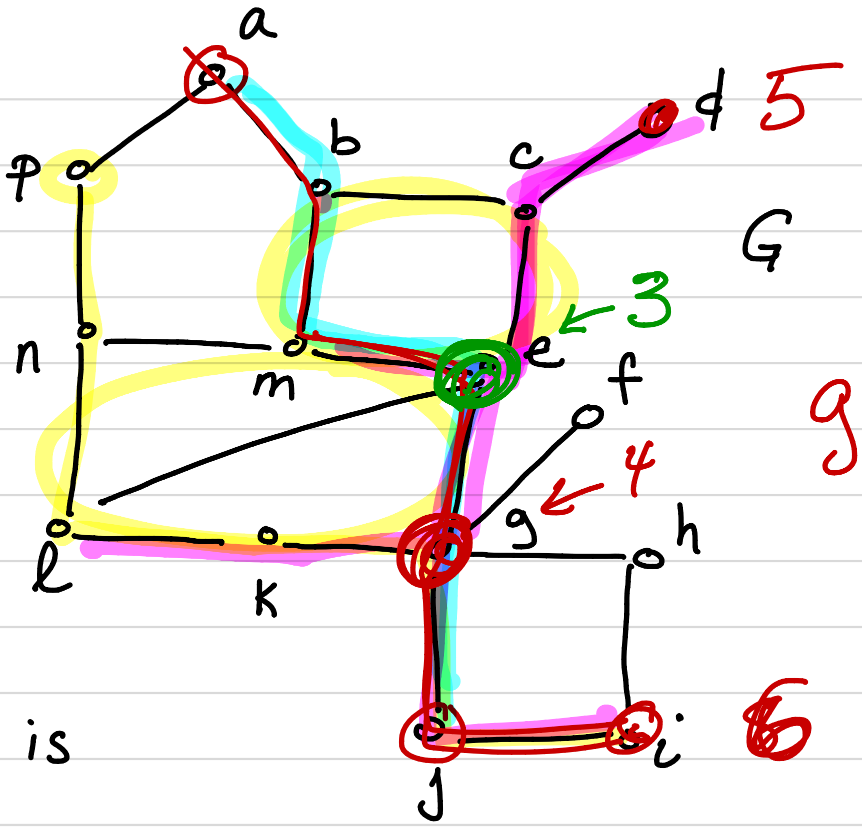
Let z be nbh of x closest to y (possibly $z=y$)

Then $C = xPzx$ has length at least

$d(x)+1 \geq \delta(G)+1$.



- girth of G, $g(G)$, is
  Smallest ↑induced cycle in G
- circumference of G is
  largest cycle

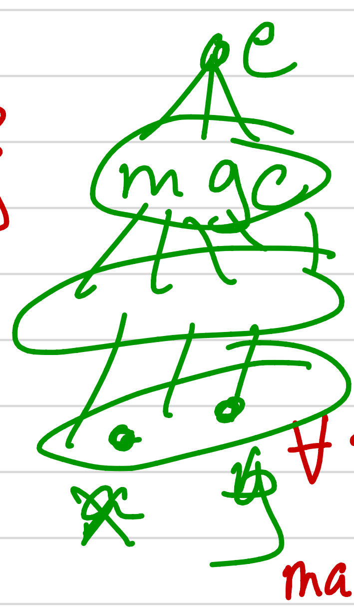


$g(G)=4$

- distance from x to y, $d(x,y)$ is
  # edges on a shortest xy path

$\text{circum}(G) = 9$

- diameter of G, $\text{diam}(G)$, is
  $\max\{d(x,y) : x,y \in V\}$



$d(a,j) = 5$

$\text{diam}(G) = \quad \geq 6$

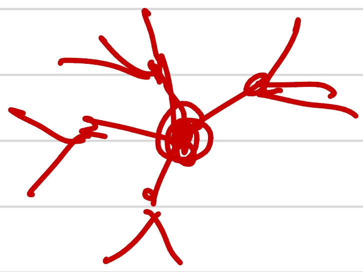
$\text{rad}(G) = 3$

- radius of G, $\text{rad}(G)$ is
  $\min\limits_{v \in G} \left(\max\limits_{w \in G} d(v,w) \right)$

$\forall v$ s.t. $\max\{d(v,w) : w \in V\} = \text{rad}(G)$

we say v is central

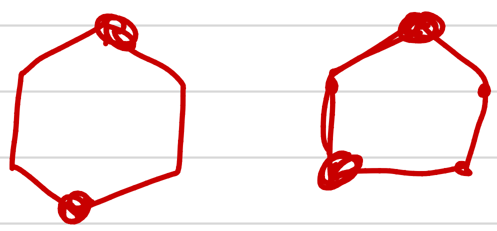
Prop 1.3.2 Every graph that contains a cycle satisfies $g(G) \leq 2\,\text{diam}(G)+1$.



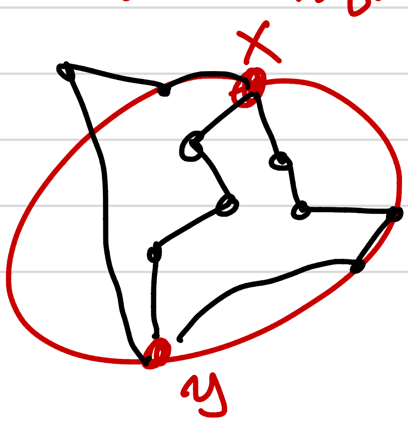
Pf: (by contradiction) Sppse $g(G) > 2\,\text{diam}(G)+1$

$\geq 2\,\text{diam}(G)+2$

$= 2(\text{diam}(G)+1)$



Let C have length $g(G)$



$d_C(x,y) \geq \text{diam}(G)+1 \Rightarrow \exists$ a shorter xy-path in G say P.

Prop 1.3.3 G graph s.t. $\mathrm{rad}(G) \le k$ and $\Delta(G) \le d$ (where $d \ge 3$)

then $|V| \le \frac{d}{d-2}(d-1)^k$.

Pf: (direct counting proof).

P leaves + returns to C. That portion along w/ the shortest path on cycle is a cycle that is smaller than C. $\Rightarrow\Leftarrow$ Since C had girth $g(G)$.