Jill's Solutions

1. Prove that every automorphism of a tree fixes a vertex or an edge.

   Proof: (by induction on the number of vertices). Let T be a tree on n vertices.
   Base Step: If $n = 1$, every automorphisms fixes the one vertex and the statement holds. If $n = 2$, then each of the two automorphisms fixes the one edge and the result holds.
   Inductive Step: Suppose the result hold for all trees on less than n vertices where $n \geq 3$. Let $f : V \to V$ be an automorphism of T and let L be the set of leaves of T. Observe that since f is an automorphism, $f(L) = L$ and $f(V - L) = V - L$. Thus, $f|_{V-L} : (V - L) \to (V - L)$ is an automorphism of the tree $T - L$ which by the inductive hypothesis must fix a vertex or an edge. Thus, T must fix the same vertex or edge.

2. Let G be a simple planar graph on n vertices with girth k. Prove that G has at most $(n-2)\frac{k}{k-2}$ edges.

   Proof: Suppose G be a simple planar graph on n vertices with girth k. Observe that is is sufficient to demonstrate that upper bound holds for each component of G, say C. Since C is planar and connected, Euler's formula applies. That is, $2 = n - m + f$ where n is the number of vertices, m is the number of edges, and f is the number of faces.
   Since G has girth k, we know $kf \leq 2m$ or $f \leq 2m/k$. Plugging into Euler's formula gives: $2 \leq n - m + 2m/k$. Solving for m gives: $m \leq k(n-1)/(k-2)$.

3. Let $G = (A \cup B, E)$ be a bipartite graph with partite sets A and B such that $|A| = |B|$ and $E \neq \emptyset$. Prove that if $|N(X)| > |X|$ for every nonempty $X \subseteq A$, then every edge of G lies on a 1-factor.

   Proof: Let $ab \in E$. It is sufficient to show that $G - \{a, b\}$ has a 1-factor. Let $A' = A - a$, $B' = B - b$, and $G' = G - \{a, b\}$. Our strategy is to apply Konig's Theorem. Specifically, we need to show that for every $X' \subseteq A'$, $|N_{G'}(X')| \geq |X'|$.

   Let X' be an arbitrary subset of A'. Using the fact that $X' \subset A$ and the hypothesis, we know that $|N_G(X')| > |X'|$. Moreover, $N_G(X') \subseteq B = B' \cup b$. Thus, $|N_{G'}(X')| \geq |N_G(X')| - 1 > |X'| - 1$. Since all the numbers here are integers, $|N_{G'}(X')| \geq |X'|$. and the result follows.

4. Let G be a graph on n vertices.

   (a) Prove that if $\delta(G) \geq 3$, then G contains a cycle with a chord. Recall that a **chord** in a cycle is an edge between two vertices on the cycle that is not a cycle edge.

   Proof: Let $P = v_0 v_1 \cdots v_k$ be a longest path in G. Since $\delta(G) \geq 3$, it follows that v_0 has two neighbors not including v_1. Since P is a longest path, those neighbors must lie on P, say v_i and v_j where $i < j$. Then, edge $v_0 v_i$ is a chord in cycle $v_0 v_1 v_2 \cdots v_j v_0$.

   (b) Prove that if $n \geq 4$ and $|E(G)| \geq 2n - 3$, then G contains a chord.

   Proof: (by induction on n) Suppose $n \geq 4$ and $|E(G)| \geq 2n - 3$.
   Base Step: If $n = 4$, then G has at least 5 edges. Thus, $G = K_4 - e$ and the result follows.

Inductive Step: Suppose the result holds for all graphs on fewer than n vertices. If $\delta(G) \geq 3$, then G has a cycle with a chord by part (a). If $\delta(G) \leq 2$, then there exists some vertex x such that $d(x) \leq 2$. Thus, $|E(G-x)| \geq (2n-3)-2 = 2n-5 = 2(n-1)-3$.

By the inductive hypothesis, the graph $G-x$ has a cycle with a chord and the result follows.

5. Prove that in e very 2-coloring of the edges of K_n (for $n \geq 3$), ther is either a monochromatic hamiltonian cycle or a hamiltonian cycle with exactly two monochromatic arcs. (By "exactly two monochromatic arcs" we mean that the hamiltonian cycle can be labelled $C = v_1v_2\cdots v_nv_1$ such that all the edges on the path $v_1v_2\cdots v_i$ are one color and the remaining edges $v_iv_{i+1}\cdots v_nv_1$ are the same color.)

   Proof: (induction on n)
   Base Step: The result holds by inspection for K_3.
   Inductive Step: Suppose the result holds for all complete graphs on fewer than n vertices. Let $c : E(K_n) \to \{0,1\}$ be a 2-coloring of K_n. Let x be an arbitrary vertex of K_n. By the inductive hypothesis, the induced coloring $K_n - x$ must contain a monochromatic cycle or one with exactly two monochromatic arcs.

   Suppose $K_n - x$ contains a monochromatic cycle, C. Pick an arbitrary pair of consecutive vertices on C, say y and z. Then, no matter how edges xy and xz are colored, the result will follow.

   Suppose $K_n - x$ contains a cycle with exactly two monochromatic arcs, say $C = v_1v_2\cdots v_nv_1$ such that all the edges on the path $v_1v_2\cdots v_i$ are colored red and the remaining edges $v_iv_{i+1}\cdots v_nv_1$ are colored blue.

   Consider the edge $e = xv_i$. If e is colored red, then no matter how edge $e^+ = xv_{i+1}$ is colored, x can be added to C and maintain two monochromatic arcs. On the other hand, if e is colored blue, then no matter how $e^- = xv_{i-1}$ is colored, x can be added to C and maintain two monochromatic arcs.

6. Let $S_1, S_2, \cdots, S_m$ be a collection of finite sets such that $2 \leq |S_1| \leq |S_2| \leq \cdots \leq |S_m|$. Define a graph G with vertex set $V = S_1 \times S_2 \times \cdots \times S_m$ such that m-tuples u and v are adjacent if and only of they differ in every coordinate. Determine $\chi(G)$.

   Claim: $\chi(G) = |S_1|$

   Let $k = |S_1|$ and $a_1, a_2, \cdots, a_k$ be the set of elements in S_1. Define $f : V(G) \to \{a_1, a_2, \cdots, a_k\}$ by

$$f(\mathbf{x}) = f(x_1, x_2, \cdots, x_m) = x_1.$$

   First, we show that f is a **proper** k-coloring of G.

   Let $\mathbf{x}, \mathbf{y} \in V$ such that $\mathbf{xy} \in E(G)$. Then, by the definition of G, $x_1 \neq y_1$. Thus, $f(\mathbf{x}) = x_1 \neq y_2 = f(\mathbf{y})$.
   Last, we show that G is not $k-1$-colorable.

It is sufficient to find a K_k. Observe that the set of vertices below form a clique on k vertices, where s_{ij} is the jth element in set S_i :

$$\{(s_{11}, s_{21}, s_{31}, \cdots, s_{m1}), (s_{12}, s_{22}, s_{32}, \cdots, s_{m2}), (s_{13}, s_{23}, s_{33}, \cdots, s_{m3}), \cdots, (s_{1k}, s_{2k}, s_{3k}, \cdots, s_{mk})\}$$