The $n \times n$ case

def: Let A be an $n \times n$ matrix and A_{ij} is the number in i^{th} row and j^{th} column of A.

our book's notation

Let S_{ij} denote the $(n-1)\times(n-1)$ submatrix of A obtained by deleting the i^{th} row and the j^{th} column from A.
Typically, S_{ij} is called a <u>minor</u> of A, or, the ij'th minor of A.

Let $C_{ij} = (-1)^{i+j} \det(S_{ij})$ be the <u>cofactor</u> of A associated with element A_{ij}.

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix}, \quad A_{11} = 1, \quad S_{11} = \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix}$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} -2 & 3 \\ 0 & 4 \end{vmatrix} = -8$$

$\boxed{\text{You}}$ find C_{12} and C_{13}.

$$C_{12}: \ A_{12} = 2, \ S_{12} = \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}, \ C_{12} = (-1)^{1+2} \begin{vmatrix} -1 & 3 \\ 2 & 4 \end{vmatrix} = (-1)(-4-6)$$

$$= 10$$

$$C_{13}: \ A_{13} = 3, \ S_{13} = \begin{bmatrix} -1 & -2 \\ 2 & 0 \end{bmatrix}, \ C_{13} = (-1)^{1+3} \begin{vmatrix} -1 & -2 \\ 2 & 0 \end{vmatrix} = 4$$

Thm: A is $n \times n$ matrix with entries A_{ij}.

For any row i (so $i = 1, 2, \dots, n$)

$$\det(A) = \sum_{j=1}^{n} A_{ij} C_{ij} = A_{i1} C_{i1} + A_{i2} C_{i2} + \dots + A_{in} C_{in}$$

Return to

Example $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix}$

$$\det(A) = A_{11} C_{11} + A_{12} C_{12} + A_{13} C_{13} = 1 \cdot (-8) + 2(10) + 3(4)$$
$$= 24$$

- How we usually write/think about this computation.
- Where this comes from.