SOME SAMPLE GRAPH THEORY PICS USING TIKZ

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Abstract. Some sample graph theory pictures using tikz. I will include more explanation later, but for now the reader will need to infer/guess/test.

1. Sample Pics

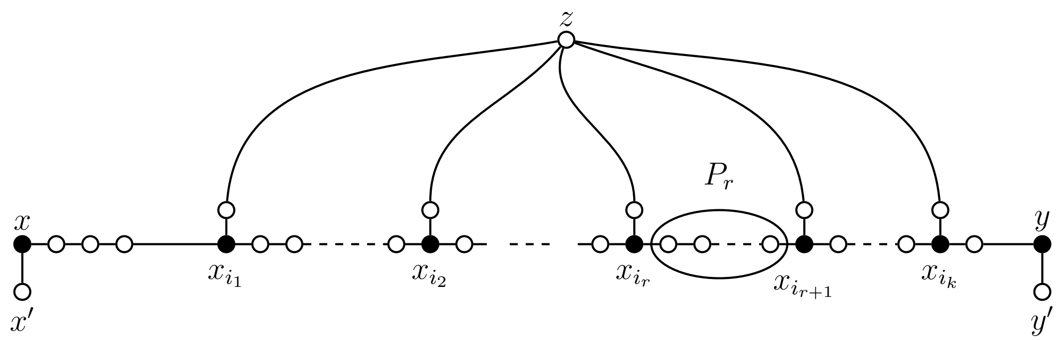

Figure 1. This figure illustrates the paths from vertex z to P, a shortest path dominating a maximum number of vertices. Note the circled section denotes P_r, the section of the path strictly between two consecutive paths, x_r and x_{r+1}.

Let $P_r = P(x_{i_r}, x_{i_{r+1}})$ denote the subpath of P strictly between two consecutive endpoints of paths from z. (See Figure 1.)

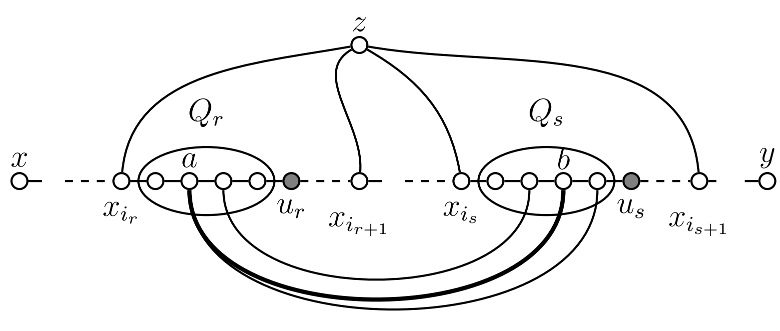

Figure 2. For each subpath of P, u_i is the first *nonmovable* vertex and, so, those in Q_i are all movable. Observe that any edges between distinct Q_i's results in a path dominating more vertices by using the edge between vertices of smallest indices (or, alternatively, left-most vertices). New path: x to x_{i_r} to z to x_{i_s} to a to b to y. Note that all vertices of Q_r and Q_s not on the new path are dominated elsewhere.

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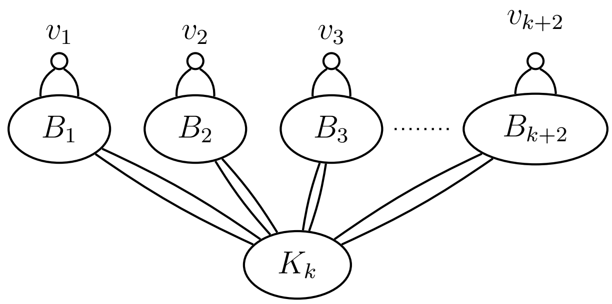


FIGURE 3. Example ??

The following corollary follows immediately from the statement of the theorem:

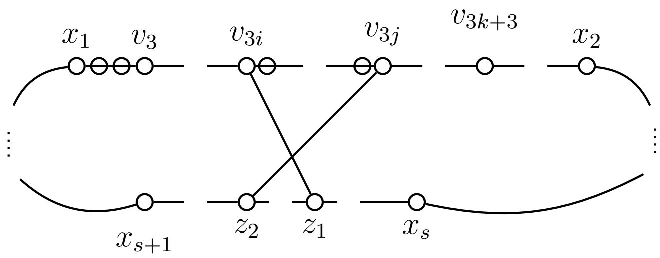


FIGURE 4. A single pair of crossing edges results in a smaller cycle. Follow $x_1 = v_0$ to v_{3i}, down to z_1, around to v_{3j} via x_s, down to z_2 and back to x_1.

Now, using the previous upper and lower bounds on the number of excess chords, we produce the following contradiction:

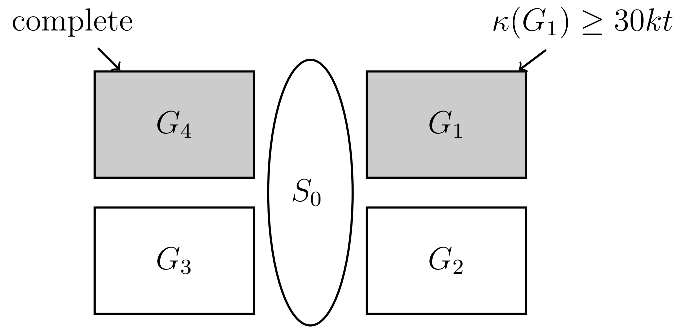
$$\begin{aligned} \frac{1}{k+2}n &= \frac{(k+3)n}{k+2} - n \\ &\leq \frac{(k+3)n}{k+2} - (b+c+3k+7) \\ &< \frac{(k+3)n}{k+2} - b - 2k - 6 - c \\ &\leq (2k+2)|X| \\ &= o(n). \end{aligned} \tag{1}$$

Now, for n sufficiently large,

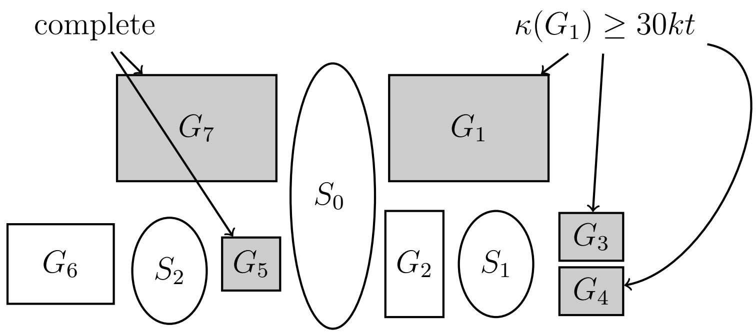
$$\begin{aligned} \sigma_2(G_{j_1}) &\geq \frac{2n}{k+2} + f(k) - 2s \geq \frac{2|V(G_{j_1})| + 2(n - |V(G_{j_1})|)}{k+2} + f(k) - 2s \\ &> \frac{2|V(G_{j_1})|}{k+2} + \frac{2(\sigma_2(G)/2)}{k+2} - 2s = \frac{2|V(G_{j_1})|}{k+2} + \frac{2n}{(k+2)^2} - 2s \\ &> \frac{2|V(G_{j_1})|}{k+2} \end{aligned} \tag{2}$$

since $s = o(n)$.

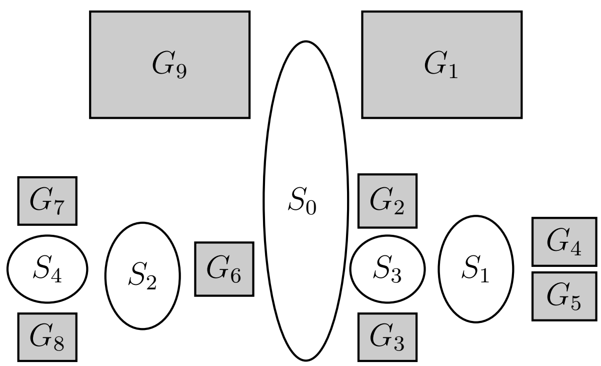
TABLE 1. Cut-Set Selection Algorithm



Iteration 1: A minimum cut set S_0 results in four connected components, G_1, G_2, G_3, and G_4. So $\mathcal{S} = S_0$ and $k \leq s < t$.



Iteration 2: Minimum cut sets S_1 and S_2 are found in noncomplete components of $G - \mathcal{S}$ with connectivity less than $30kt$. Now, $\mathcal{S} = S_0 \cup S_1 \cup S_2$, $s < t + 2(30kt)$ and $G - \mathcal{S}$ results in seven connected components.

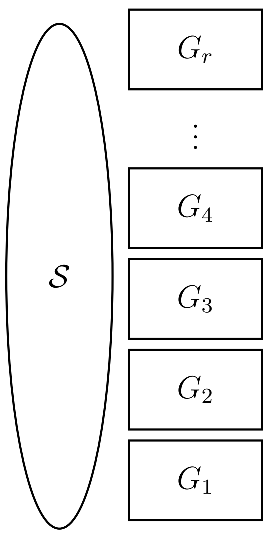


Iteration 3: Minimum cut sets S_3 and S_4 are found in noncomplete components of $G - \mathcal{S}$ with connectivity less than $30kt$. Now, $\mathcal{S} = \cup_{i=0}^{4} S_i$, $s < t + 4(30kt)$ and $G - \mathcal{S}$ results in 9 connected components. The algorithm would terminate with all components either complete or with connectivity at least $30kt$.

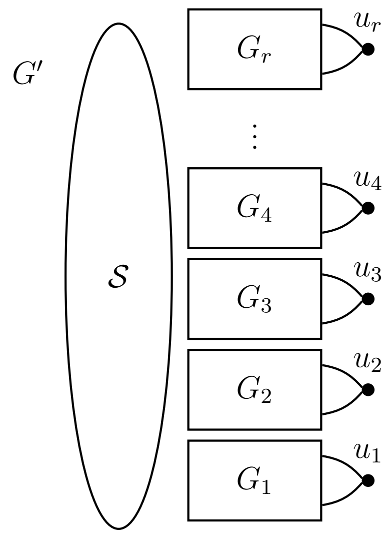
Because the steps used in Case 1 will be essentially the same in Cases 2 and 3, the steps are given names for ease of reference later. To aid the reader, pictures of the various steps are shown in Table 2.

Theorem 1.1 (Turán, '40)**.** *If $3 \leq p \leq n$, then $\boldsymbol{ex}(n, K_p) = \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}$ and the unique K_p -saturated graph of size $\boldsymbol{ex}(n, K_p)$ is the balanced (or nearly balanced) complete $(p-1)$-partite graph.*

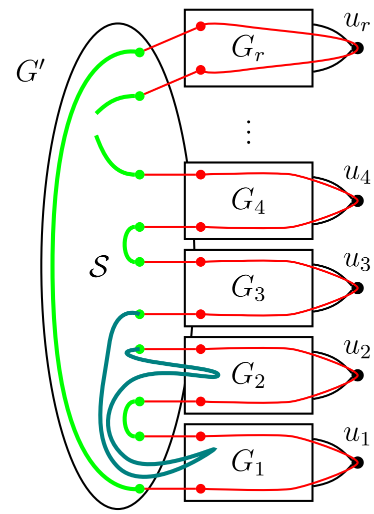
Table 2. Cartoon Illustrating the Proof of Theorem **??**



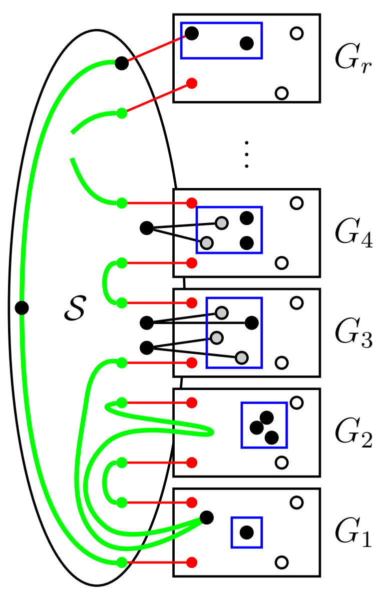
This shows $\mathcal{S}$ and the components of $G - \mathcal{S}$



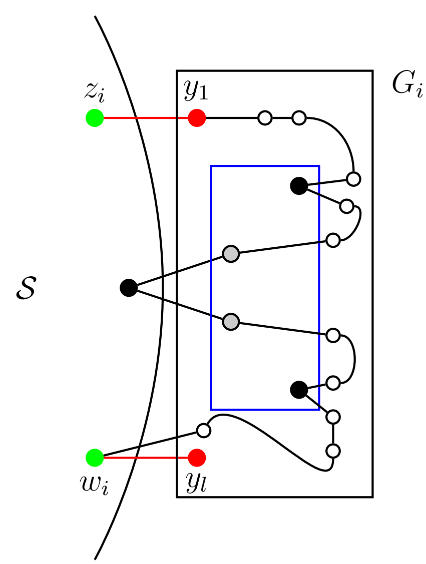
This shows G' with added vertices u_i.



This shows C' (in red, green, and teal) in G'. Note that the the Q_i's are depicted in green (and teal) which is the portion of C' left intact. The teal section (Q_3') illustrates how complicated these connector sections may be. They may include vertices from G outside $\mathcal{S}$ and possibly vertices from T.

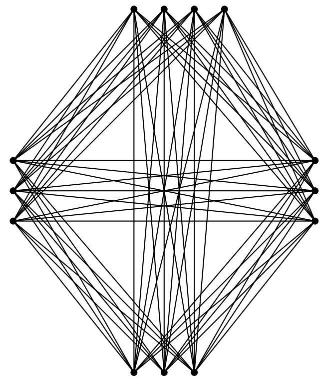


In the diagram above, black vertices are in T, gray vertices are designated neighbors of vertices of $T_{\mathcal{S}}$ not on any green path. For each G_i, the set T_i is in a blue box. Observe that vertices of T on any Q_j' (in green) are not included in any T_i.

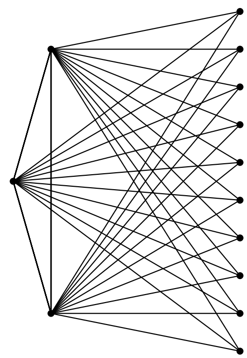


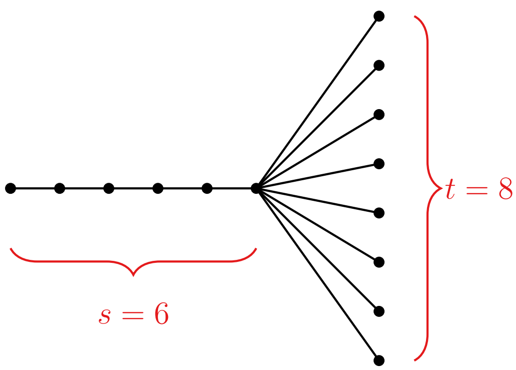
This diagram illustrates how the red portion of C' is replaced with a path containing the vertices of T_i. As before, vertices of T are black, associated neighbors of vertices that belong to G_i are in gray. Observe that y_1 and y_l may or may not be on this new path. It is enough to know that z_i and w_i have distinct neighbors in G_i.

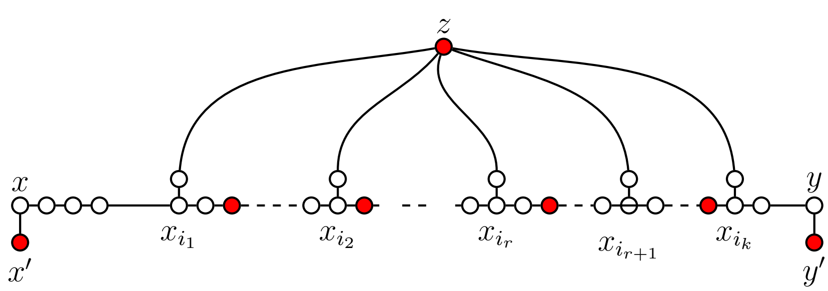
The unique K_5-free graph with $\mathbf{ex}(13, K_5)$ edges



The unique K_5-free graph with $\mathbf{sat}(13, K_5)$ edges







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