

SECTION 3-1: DEFINING THE DERIVATIVE

1. Definitions of the Derivative

Version 1

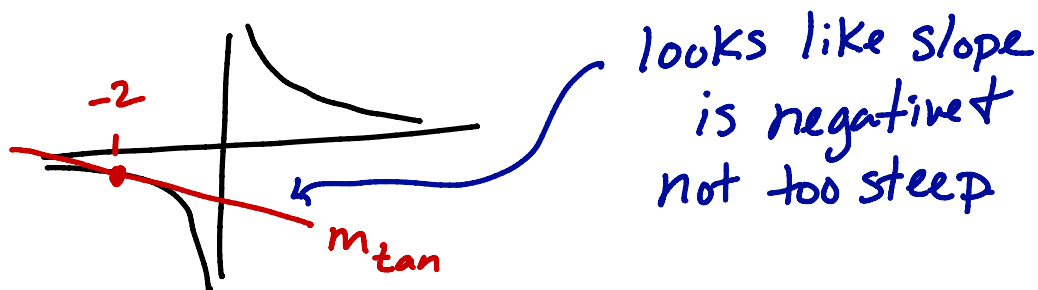
$$f'(a) = m_{\text{tan}} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

Version 2

$$f'(a) = m_{\text{tan}} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

2. In the problems below, let $f(x) = \frac{1}{x}$.

- (a) Using a rough sketch of $f(x)$ make a rough estimate of the slope of the tangent to $f(x)$ when $x = -2$.



- (b) Use version 1 of the definition to find m_{tan}

$$\begin{aligned} f'(-2) &= m_{\text{tan}} = \lim_{x \rightarrow -2} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x \rightarrow -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2} = \lim_{x \rightarrow -2} \left(\frac{2+x}{2x} \right) \\ &= \lim_{x \rightarrow -2} \left(\frac{1}{x+2} \right) \left(\frac{x+2}{2x} \right) = \lim_{x \rightarrow -2} \frac{1}{2x} = \frac{1}{-4} = \left(-\frac{1}{4} \right) \end{aligned}$$

when $a = -2$

- (c) Using version 1 of the definition to find m_{tan}

$$\begin{aligned} f'(-2) &= \lim_{h \rightarrow 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{-2+h} + \frac{1}{2}}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{1}{h} \right) \left(\frac{2 + (-2+h)}{(-2+h)(2)} \right) = \lim_{h \rightarrow 0} \frac{h}{2(h-2)h} = \lim_{h \rightarrow 0} \frac{1}{2(h-2)} = \frac{-1}{4} \end{aligned}$$

- (d) Write the equation of the line tangent to $f(x)$ when $x = -2$. Plausible?

point $(-2, -\frac{1}{2})$
 $m = -\frac{1}{4}$

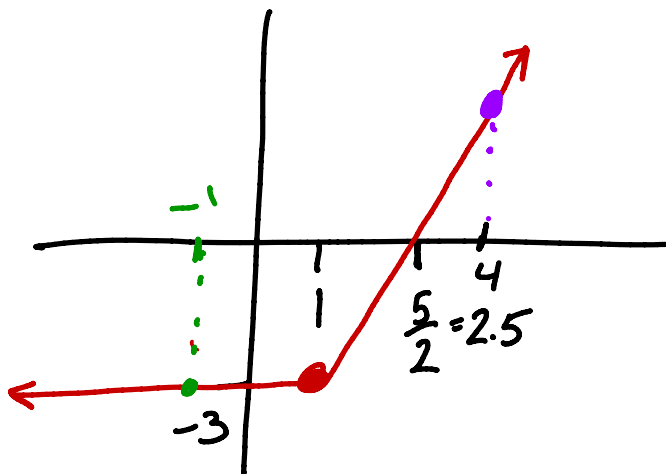
line: $y + \frac{1}{2} = -\frac{1}{4}(x + 2)$ or $y = -\frac{1}{4}x - 1$ (Looks roughly correct)

3. Graph the function $G(t) = \begin{cases} -3 & x \leq 1 \\ 2x - 5 & 1 < x \end{cases}$.

(a) Use the graph to determine $G'(-1)$ and $G'(4)$

• $G'(-1) = 0$

• $G'(4) = 2$



(b) Explain – using the definition – why $G'(1)$ fails to exist.

The definition of the derivative involves a TWO-sided limit. On the left side of $x=1$, all secant lines have a slope of zero, but on the right side, all secant lines have a slope of 2.

OR: $\lim_{x \rightarrow 1^-} \frac{G(x) - G(1)}{x - 1} = 0$, but $\lim_{x \rightarrow 1^+} \frac{G(x) - G(1)}{x - 1} = 2$. So $\lim_{x \rightarrow 1} \frac{G(x) - G(1)}{x - 1} = \text{DNE}$.

4. A rock is dropped from a height of 100 feet. Its height above ground at time t seconds later is given by $s(t) = -16t^2 + 100$.

(a) Find and interpret $s(0)$ and $s(1)$.

$s(0) = -16 \cdot 0^2 + 100 = 100$ feet.

When time starts, the rock is 100 feet above the ground (like the problem says...)

$s(1) = -16(1)^2 + 100 = 84$ feet.

One second later, the rock is only 84 feet above the ground (i.e. it has fallen, which is expected.)

(b) Given $s'(1) = -32$, determine the units of $s'(1)$ and interpret it in the context of the problem.

Recall that $s'(1) = \lim_{t \rightarrow 1} \frac{s(t) - s(1)}{t - 1} = \frac{\Delta s}{\Delta t}$; ← velocity!

so (units of s') = $\frac{\text{units of } s}{\text{units of } t} = \frac{\text{ft}}{\text{sec}}$ ← velocity!! $\frac{\text{change in position}}{\text{change in time}}$

When 1 second has passed, the velocity of the rock is -32 ft/s

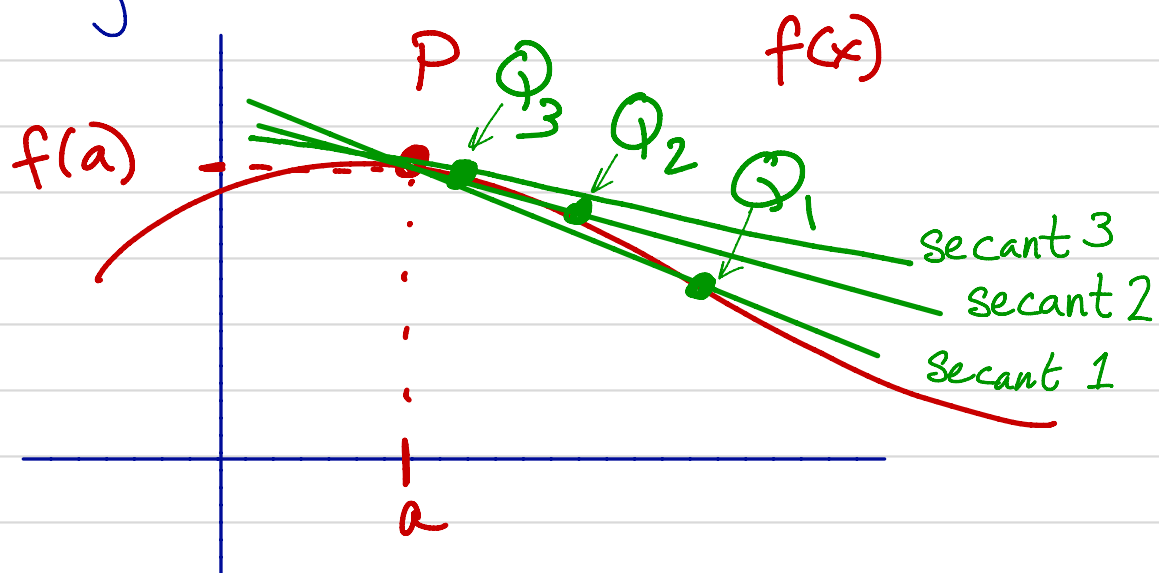
Plan

This is a one-day sheet.

- ① Motivate the two definitions.
- ② Get definitions on the board.
- ③ Students need to think in advance what $f'(-2)$ should be for $f(x) = \frac{1}{x}$.
- ④ All remaining problems are
Jill starts, you keep going,
check w/ neighbor. Jill finishes
on board
- ⑤ If time permits, go back to
 $f(x) = \frac{1}{x}$ → ask them to
anticipate what $f'(x)$ would
be (v.u. roughly) for $x = -1, -3$
 $x = +1, +2, +3$.
(This foreshadows § 3.2)

Notes

Return to §2.1 where we approximated the slope of the tangent to $f(x)$ at point P using secant lines



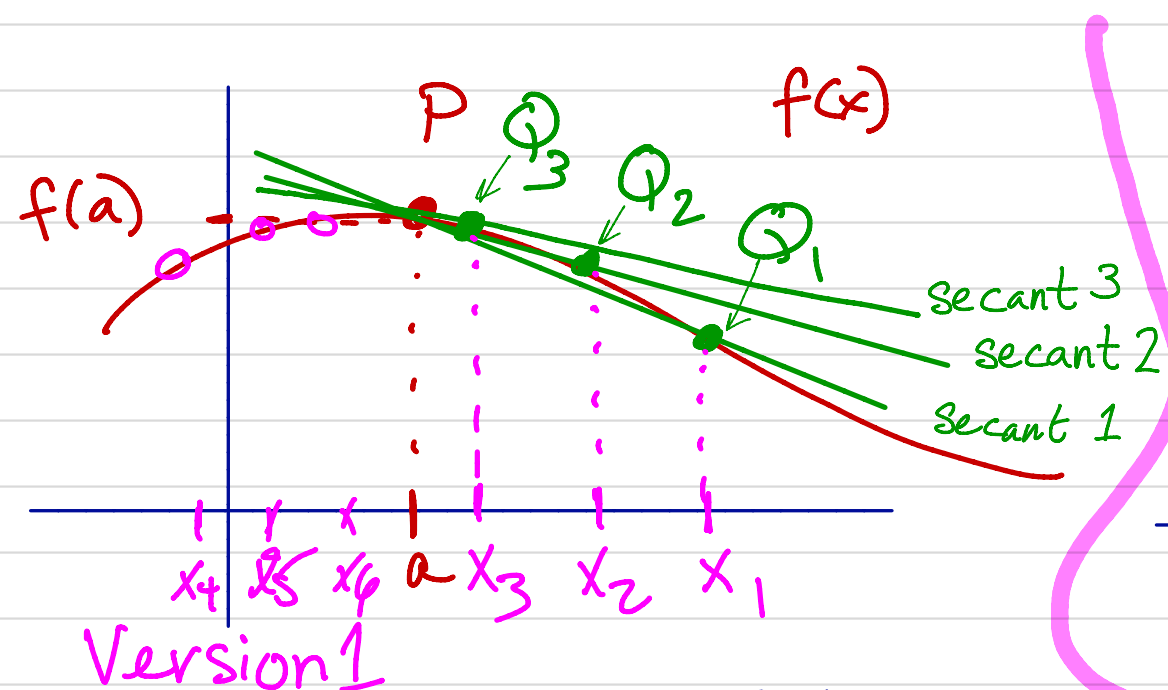
We now can see this as a limit.

As the Q 's get close to the P 's,
 m_{sec} gets close to m_{tan} .

OR

$$\lim_{Q \rightarrow P} m_{\text{sec}} = m_{\text{tan}}$$

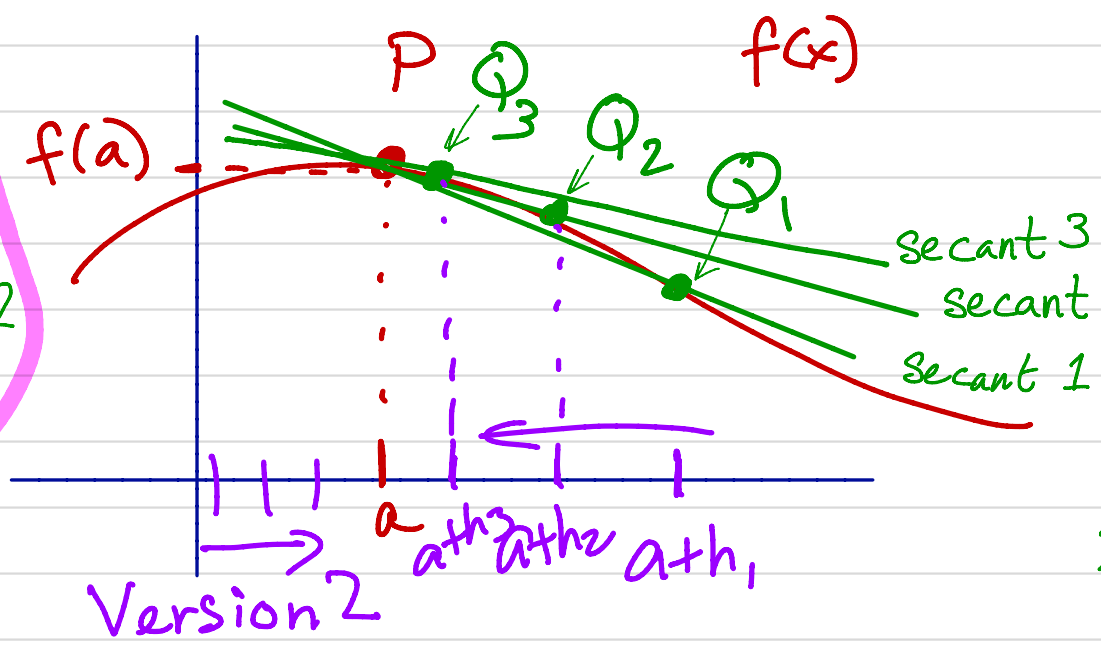
How do we make this precise?



Version 1

$$\lim_{x \rightarrow a} \left(\frac{f(x) - f(a)}{x - a} \right) = m_{\text{tan}}$$

\uparrow slope
 secant $\frac{\Delta y}{\Delta x}$



Version 2

$$\lim_{h \rightarrow 0} \left(\frac{f(a+h) - f(a)}{(a+h) - ()} \right) = m_{\text{tan}}$$

\uparrow or $a+h-a=h$