Your Name (print clearly)

Wednesday, December 15, 2015

Problem	Total Points	Score
1	20	
2	10	
3	10	
4	15	
5	10	
6	15	
7	10	
8	10	
Total	100	

Instructions and information:

- Please turn off cell phones or any other thing that will go BEEP.
- You are allowed to use your textbook
- Read the directions for each problem.

1. (5 points each) Find the number of ways to distribute all 30 books to 10 libraries in each situation.

   (a) Assume the books are distinct and the libraries are distinct and there are no restrictions on where the books go.

   10 choices for each of 30 books
   different

   So 10^{30}

   (b) Assume 5 of the 30 books are identical and that these 5 books should go to different libraries, but there are no other restrictions.

   $\boxed{A}$ Pick libraries to get special 5: $\binom{10}{5}$
   $\boxed{B}$ Distribute remaining books: 10^{25}

   ANS: $\binom{10}{5} 10^{25}$

   (c) Assume all 30 books are identical and every library should get at least one book.

   Distribute 10 books to 10 libraries: 1 way
   Distribute remaining ~~30~~ 20 books to 10 librarys: $\binom{10+20-1}{20} = \binom{29}{20}$

   Answer: $\binom{29}{20}$ or $\left(\!\binom{10}{20}\!\right)$

   (d) Assume all 30 books are distinct and each library will receive exactly three books.

   Answer 1: $\cancel{\prod_{i=1}^{10}\binom{30-3(i+1)}{3}}$ $\prod_{i=0}^{9}\binom{30-3i}{3}$
   $= \binom{30}{3}\binom{27}{3}\binom{24}{3}\cdots\binom{6}{3}\binom{3}{3}$

   Answer 2 permute books: $30!$, divide by overcount $(3!)^{10}$
   answer $\dfrac{30!}{(3!)^{10}}$

2. (10 points) How many numbers must be chosen from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ to guarantee that at least one pair of these numbers adds up to 9? You must prove your answer is correct. (Hint: Use the Pigeon Hole Principle)

ANS: 5 numbers.

Just FYI clear 4 is too small since the set $\{1,2,3,4\}$ doesn't have this property

Proof: Partition the set into following blocks: $\{1,8\}$, $\{2,7\}$, $\{3,6\}$ and $\{4,5\}$.

Observe that the pair in each block sums to 9. If 5 numbers are chosen, by PHP, two must come from same block.

3. (10 points) Give a combinatorial proof that $\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1}$ for all positive integers, n.

Let S be the of all ordered pairs (A, x) where $A \subseteq [n]$ and $x \in A$. (In language of committees, let S be the set of all possible nonempty committees w/ a designated chair from a set of n people)

Count 1: Pick committee chair: n ways. Then fill committee in 2^{n-1} ways.

Count 2: Partition committees according to the number of members. For $i \in [n]$, $\binom{n}{i}$ counts the number of committees w/ i members and then there are i ways to choose the chair

Thus, $n2^{n-1} = \sum_{i=1}^{\infty} i\binom{n}{i}$

4. (10 points) Let $\mathcal{C}$ be a q-ary code with codewords of length n. (So $\mathcal{C}$ is a subset of all n-length words using the alphabet $X = \{0, 1, 2, \cdots, q-1\}$.) As with the Hamming distance on binary words, assume that the distance between two q-ary words is the number of positions in which the two words differ. Prove that if code $\mathcal{C}$ can correct up to e errors, than

$$|\mathcal{C}| \leq \frac{q^n}{\sum_{i=0}^{e} \binom{n}{i}(q-1)^i}$$

5. (15 points) Theorem 7.2.4 (page 288) says that if $\mathcal{D}$ is a symmetric BIBD with parameters (v, k, λ), then $\mathcal{D}'$,the derived design obtained from $\mathcal{D}$, has parameters $(v-1, k, k-1, \lambda, \lambda-1)$.

(a) Explain directly (not using the other parameters of $\mathcal{D}'$) why the derived design has $v-1$ blocks.

To make $\mathcal{D}'$, one block is deleted; so one is lost. Moreover, the other blocks must have λ elements in common with the deleted block. Thus, the other $v-1$ blocks remain.

(K-1)

typo

(b) Explain directly (not using the other parameters of $\mathcal{D}'$) why the derived design is $\cancel{\lambda}$ regular.

~~Because $\mathcal{D}$ is symmetric, it is λ-linked.~~

~~Thus the intersect~~

Every variety in $\mathcal{D}$ appears k times. Since exactly one block is deleted, the remaining varieties appear in one fewer block, namely $k-1$.

6. (15 points) Define a partially ordered set $P(X, R)$ where $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and for all $a, b \in X$, $(a, b) \in R$ if and only if $a \mid b$.

   (a) Draw the Hasse Diagram for P.

   (b) Find all maximal elements of P.

   (c) Find the meet of 4 and 6 (i.e. $4 \wedge 6$) and show that 4 and 6 have a lower bound distinct from $4 \wedge 6$.

   (d) Find a maximal chain in P that is not a maximum chain.

7. (15 points)

(a) Give ONE example of a partition of 40 into 3 parts.

1, 1, 38

(b) Give ONE example of a partition of 40 into 3 parts each of which is an even number (i.e. a partition with even parts)

2, 2, 36

(c) Give TWO examples of a partition of 40 into even parts each of which is at most 8.

4 4 4 4 4 4 4 4 4 4

8 8 8 8 8

(d) Express the number of partitions of 40 into even parts each of which is at most 8 as a suitable coefficient of a certain generating function. (That is, you must specify a generating function and a coefficient of the generating function.)

$$\cancel{\left[(x^2+x^4+x^6+x^8)^{20}\right]_{x^{40}}}$$ parts

$$(1+x^2+x^4+\dots)(1+x^4+x^8+\dots)(1+x^6+x^{12}+\dots)(1+x^8+x^{16}+\dots)$$

$$= \frac{1}{(1-x^2)(1-x^4)(1-x^6)(1-x^8)}$$

ANS: $\left[\frac{1}{(1-x^2)(1-x^4)(1-x^6)(1-x^8)}\right]_{x^4}$

8. (10 points) Show that if $\delta(G) \geq k$ then G must contain a cycle on at least $k+1$ vertices.

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