1. Make a list of statements equivalent to: **The $n \times n$ matrix $A$ is nonsingular.**

   - $\mathrm{rref}(A) = I_n$
   - $A\vec{x} = \vec{0}$ has a unique solution
   - $A^{-1}$ exists. (or $A$ is invertible.)
   - $A\vec{x} = \vec{b}$ has a unique solution
   - $A$ is the matrix representation of an isomorphism

2. Make the analogous list for: **The $n \times n$ matrix $A$ is singular.**

   - $\mathrm{rref}(A)$ will have a row of zeros
   - $A^{-1}$ does not exist. (or $A$ is not invertible)
   - $A\vec{x} = \vec{0}$ will always have an infinite number of solutions
   - $A\vec{x} = \vec{b}$ will either have no solution or an infinite number of solutions.
   - $A$ is the matrix representation of a linear map that is not 1-1 and not onto.

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

3. Cor 4.11 (pg 259) The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is nonsingular if and only if $ad - bc \neq 0$

why? Need $(a\ b) \neq k(c\ d)$. But $a = kc$ and $b = kd$ means $k = \frac{a}{c} = \frac{b}{d}$ or $ad = bc$ or $ad - bc = 0$.

4. **definition:** Let $A = (a_{ij})$ be an $n \times n$ matrix. Then $A_{ij}$ is the $(n-1) \times (n-1)$ matrix obtained by deleting the $i$th row and $j$th column from $A$.

5. **example:** Find $B_{23}$ for $B = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 1 & -1 \\ 2 & -3 & 5 \end{pmatrix}$.  $B_{23} = \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix}$

6. **definition:** The *determinant* of the $n \times n$ matrix $A = (a_{ij})$, denoted by $det(A)$, is defined as follows.

- For $A = (a_{11})$, $det(A) = a_{11}$
- For $n \geq 2$, $det(A)$ is the sum of $n$ terms of the form $\pm a_{1j}\, det(A_{1j})$, with the plus and minus signs alternating starting with a plus. Specifically,

$$det(A) = a_{11}\, det(A_{11}) - a_{12}\, det(A_{12}) + a_{13}\, det(A_{13}) - \cdots + (-1)^{n-1} a_{1n}\, det(A_{1n})$$

7. Find $det(B)$.

$$det(B) = 1 \cdot det(B_{11}) - (2)\,det(B_{12}) + (4)\,det(B_{13})$$

$$= 1 \cdot \begin{vmatrix} 1 & -1 \\ -3 & 5 \end{vmatrix} - 2 \begin{vmatrix} 4 & -1 \\ 2 & 5 \end{vmatrix} + 4 \begin{vmatrix} 4 & 1 \\ 2 & -3 \end{vmatrix}$$

$$= 1(5-3) - 2(20+2) + 4(-12-2)$$

$$= 2 \quad -44 \quad -56 \quad = -98$$

Comments

- Calc III definition
- Octave/Matlab for checking
- Does coincide w/ 2×2 def from #3 on other page.
- Can be used for any $n$ but is clearly recursive.
- Now $|A| \equiv det(A)$, for matrix $A$.
- Want $det(A) \neq 0 \iff A$ nonsingular