Chapter 11

Basic Multivariate Regression

This chapter extends the bivariate model to a multivariate model, i.e., the case with more than one independent variable. There is also a YouTube Video associated with this section:

- Multivariate Regression with R - Video
- Multivariate Regression with R - Slides

The following topics associated with multivariate regression models are covered in this chapter:

- Dummy variables
- Natural logarithm
- Functional forms
- Interaction Terms
- Multicollinearity

11.1 Introduction

Recall the bivariate regression model with one independent and one dependent variable:

$$y = \beta_0 + \beta_1 \cdot x_1 + \epsilon$$

The multivariate linear regression model includes more than one independent variable and is simply an extension of the bivariate regression model:

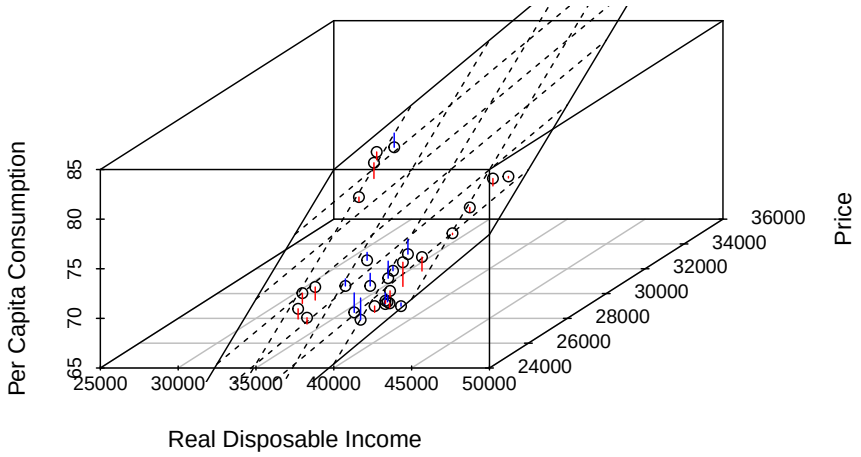
$$y = \beta_0 + \beta_1 \cdot x_1 + \beta_2 \cdot x_2 + \cdots + \beta_k \cdot x_k + \epsilon$$

Whether we consider the bivariate or multivariate model, the objective is always to minimize the sum of squared errors which has led to the name ordinary least

square (OLS) model. The equation of a line can be determined using slope (β_0) and the intercept ($\beta 1$), i.e.:

$$E(y|x_1) = \beta_0 + \beta_1 \cdot x_1$$

The case of a regression model with two independent variable can still be represented in a 3-dimensional graph as depicted below



The purpose of the multivariate regression model is to measure the effect of independent variables on the dependent variable. It is crucial to control for everything else that could influence the dependent variable. For example, measuring the weekly grocery bill as a function of years of education might give you a statistically significant effect for education but if income is included, the effect for education might (most likely) disappear.

The first example involves estimating home values based on square footage and number of garage spots of a house in the 46268 ZIP code in Indianapolis. The data is contained in `indyhomes`.

```
indyhomes46268 = subset(indyhomes,zip==46268)
bhat = lm(price~sqft+garage,data=indyhomes46268)
summary(bhat)
```

```
##
## Call:
## lm(formula = price ~ sqft + garage, data = indyhomes46268)
##
```

```
## Residuals:
##    Min     1Q Median     3Q    Max
## -58780  -7817   1582   7886  51803
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 81733.141  15896.004   5.142 5.20e-06 ***
## sqft           40.897      4.383   9.331 2.85e-12 ***
## garage      16580.964   7136.866   2.323   0.0245 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 20710 on 47 degrees of freedom
## Multiple R-squared:  0.675,  Adjusted R-squared:  0.6611
## F-statistic:  48.8 on 2 and 47 DF,  p-value: 3.388e-12
```

Depending on the nature of the variables, it might be necessary to scale your variables for ease of interpretation. This might be necessary if coefficients are very large or very small. A rescaling, e.g., dividing income by 1000, does affect the coefficients and the standard errors but has no effect on the t-statistics.

11.2 Dummy Variables

So far, independent variables were quantitative such as price, income, square footage, miles, and so on. But very often, a qualitative characteristic such as religion or gender must be modeled. For this purpose, dummy variables that can be either 0 or 1 are used. Dummy variables represent a single qualitative characteristic. For example, consider the price (y_i) of a car depending on miles (x_i) and whether the car has all-wheel drive (AWD) or rear-wheel drive (RWD). This characteristic can be modeled using a dummy variable (d_i). If $d_i = 1$, the car has AWD and if $d_1 = 0$, the car has RWD. The regression equation can be written as follows:

$$y_i = \beta_0 + \beta_1 \cdot x_i + \beta_2 \cdot d_i + \epsilon_i$$

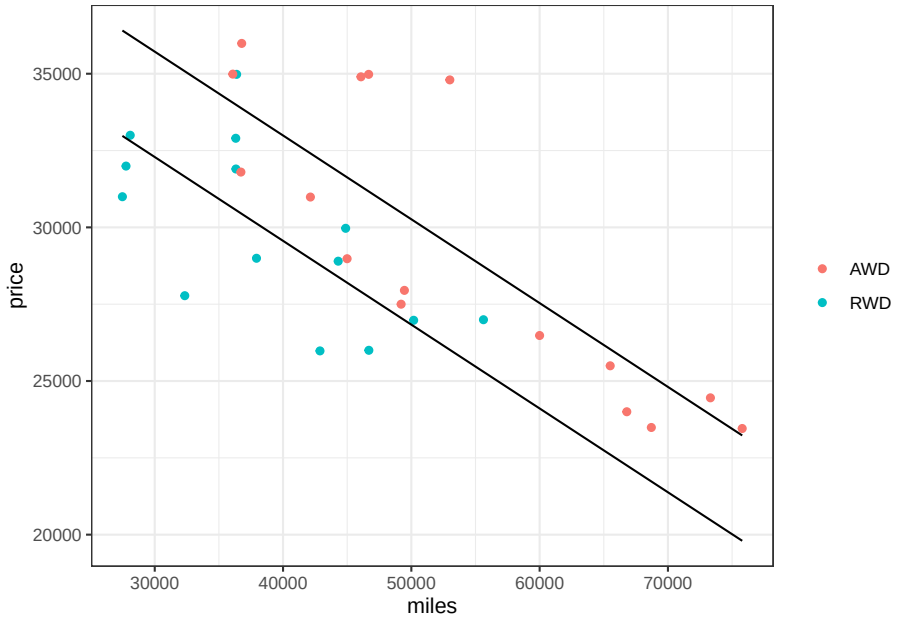
his regression can theoretically be separated into two single equations:

- RWD: $y_i = \beta_0 + \beta_1 \cdot x_i + \epsilon_i$
- AWD: $y_i = (\beta_0 + \beta_2) + \beta_1 \cdot x_i + \epsilon_i$

To interpret the dummy variables, it is necessary to know how it is coded. In the above case, if the coefficient β_2 is positive, then the dummay variable adds to the price. That is, the coefficient β_2 represents the value of all-wheel drive.

```
##
## Call:
## lm(formula = price ~ miles + allwheeldrive, data = bmw)
##
## Residuals:
```

```
##     Min      1Q  Median      3Q     Max
## -3874.1 -1724.0  -176.5  1604.5  5355.0
##
## Coefficients:
##                 Estimate Std. Error t value Pr(>|t|)
## (Intercept)    4.047e+04  1.711e+03  23.660  < 2e-16 ***
## miles         -2.728e-01  4.044e-02  -6.745 3.05e-07 ***
## allwheeldrive  3.429e+03  1.063e+03   3.227  0.00327 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2449 on 27 degrees of freedom
## Multiple R-squared:  0.6287, Adjusted R-squared:  0.6012
## F-statistic: 22.86 on 2 and 27 DF,  p-value: 1.553e-06
```



11.3 Natural Logarithm

Transforming the dependent and/or independent variables using the natural logarithm has some important and useful interpretations. Consider the following simple consumption equation in which both variables are in logarithmic form:

$$ln(consumption) = \beta_0 + \beta_1 \cdot \ln(income) + \epsilon$$

In this case, β_1 is the elasticity of consumption with respect to income, i.e., a 1% increase in income leads to a $\beta_1 \cdot 1\%$ increase in consumption. For example,

if $\beta_1 = 0.4$, then a 1% increase in income will rise consumption by 0.4%. Note that the percentage increase is only an approximation for small changes.

Dep. Var.	Indep. Var	Interpretation
y	x	1 dollar change in x changes y by $\hat{\beta}$ dollars
$\ln(y)$	x	1 dollar change in x changes y by 100 $\times\hat{\beta}$ percent
$\ln(y)$	$\ln(x)$	1 percent change in x changes y by $\hat{\beta}$ percent
y	$\ln(x)$	1 percent change in x changes y by $\hat{\beta}/100$ dollars

```
bhat = lm(log(total)~yards+att+exp+draft1+veteran+changeteam+pbowlever,data=nfl)
summary(bhat)
```

```
##
## Call:
## lm(formula = log(total) ~ yards + att + exp + draft1 + veteran +
##     changeteam + pbowlever, data = nfl)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -2.4899 -0.4998 -0.0801  0.4554  3.1959
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.9289322  0.0767846 -12.098  < 2e-16 ***
## yards        0.0003566  0.0001411   2.527 0.011783 *
## att          0.0003927  0.0009657   0.407 0.684408
## exp          0.0108812  0.0160213   0.679 0.497312
## draft1       0.8876564  0.1132374   7.839 2.30e-14 ***
## veteran      0.6735244  0.1144567   5.885 6.88e-09 ***
## changeteam  -0.3095919  0.0893125  -3.466 0.000568 ***
## pbowlever    0.4093324  0.0936078   4.373 1.46e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.7909 on 560 degrees of freedom
##   (441 observations deleted due to missingness)
## Multiple R-squared:   0.55,  Adjusted R-squared:  0.5444
## F-statistic:  97.8 on 7 and 560 DF,  p-value: < 2.2e-16
```

11.4 Functional Form

To model non-linear relationships, an independent variable can be transformed by squaring it. For example, consider the relationship between income and food

expenditure. The regular OLS assumes a linear relationship in the sense that an increase in income always leads to a proportional increase in food expenditure. In realty, there is likely a flattening out of food expenditure for high incomes because only so much money can be spent on food.

```
bhat = lm(log(total)~yards+att+exp+exp2+draft1+veteran+changeteam+pbowlever,data=nfl)
summary(bhat)
```

```
##
## Call:
## lm(formula = log(total) ~ yards + att + exp + exp2 + draft1 +
##     veteran + changeteam + pbowlever, data = nfl)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -2.4477 -0.5010 -0.0807  0.4452  3.1638
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) -1.0270821  0.1228294  -8.362 4.93e-16 ***
## yards        0.0003353  0.0001426   2.351 0.019087 *
## att          0.0005137  0.0009728   0.528 0.597691
## exp          0.0702541  0.0601679   1.168 0.243452
## exp2        -0.0037576  0.0036704  -1.024 0.306398
## draft1       0.8910570  0.1132812   7.866 1.90e-14 ***
## veteran      0.5752784  0.1493617   3.852 0.000131 ***
## changeteam  -0.3221519  0.0901474  -3.574 0.000383 ***
## pbowlever    0.4158535  0.0938203   4.432 1.12e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.7909 on 559 degrees of freedom
##   (441 observations deleted due to missingness)
## Multiple R-squared:  0.5509, Adjusted R-squared:  0.5445
## F-statistic: 85.71 on 8 and 559 DF,  p-value: < 2.2e-16
```

11.5 Interaction Effects

Interaction effects are used when the influence of one independent variable depends on the level of another independent variable. Suppose that you want to measure time spent volunteering (y) and you think that it depends on the marital status (x_1), the number of children (x_2), and some other independent variables (X). So you could have the following regression equation

$$y = \beta_1 \cdot x_1 \cdot x_2 + \beta \cdot X + \epsilon$$

$$\frac{dy}{dx} = \beta_1 x_2 + \beta_2$$

11.6 Exercises

1. ***Ohio Schools III*** (***): Consider the data sets `ohioincome` and `ohioscore`. In a first step, merge the two data sets by IRN. For all questions below, interpret the coefficients in terms of direction, magnitude, and statistical significance.

   a. Estimate the following equation and report the output.

   $$score = \beta_0 + \beta_1 \cdot medianincome + \beta_2 \cdot enrollment$$

   b. Estimate the following equation. Compare your answer to the previous part. Do the coefficients change in magnitude? How do you interpret the squared term?

   $$score = \beta_0 + \beta_1 \cdot medianincome + \beta_2 \cdot enrollment + \beta_3 \cdot medianincome^2$$

2. ***Honda vs. BMW*** (***): The data sets `honda` and `bmw` contain prices and mileage of used Honda and BMW cars in the Indianapolis area. For BMW, you have a dummy variable which indicates all-wheel drive ($allwheeldrive = 1$) or rear-wheel drive ($allwheeldrive = 0$).

   a. Run a regression with price as the dependent variable and miles as the independent variable for both cars (separately). Report the intercept and slope coefficients. Interpret your results, e.g., how does an increase in miles affect the price of the cars.
   b. Generate two scatter plots of the data and the fitted lines. For each car, I want the scatter plot and the fitted line in the same graph. What can you say about the difference in depreciation of the two cars.

3. ***WDI*** (**): Using the data in `wdi`, estimate the equation below for the year 2018. Report and interpret the results:

   $$fertrate = \beta_0 + \beta_1 \cdot gdp + \beta_2 \cdot litrate$$

4. ***Retail*** (***): This exercise will demonstrate the use of dummy variables to model so-called seasonality in the data `retail`. Note that time series analysis is a fairly complex topic and this question only serves as an introduction. Using the data in `retail`, estimate the following regression model:

   $$retail = \beta_0 + \beta_1 \cdot t + \sum_{m=1}^{11} \beta_m \cdot D_m$$

where t represents a simple time trend and D_m are monthly dummy variables. Make sure to only include 11(!) monthly dummy variables. Is there seasonality in the data? Interpret.

5. ***Indy Homes I*** (***): The data `indyhomes` contains home values of two ZIP codes in Indianapolis. In this exercise, you will estimate the value of homes (dependent variable) based on a set of independent variables. The variables are mostly self-explanatory. The variables *levels* and garage refers to the number of *stories* and the garage parking spots, respectively.

   a. Create a dummy variable called *northwest* for the 46268 ZIP code.
   b. Report the results of the following regression equation

   $$\ln(price) = \beta_0+\beta_1\cdot\ln(sqft)+\beta_2\cdot northwest+\beta_3\cdot\ln(lot)+\beta_4\cdot bed+\beta_5\cdot garage+\beta_6\cdot levels+\beta_7\cdot north$$

   c. Interpret each coefficient from the previous part and how it affects $ln(price)$. How do you interpret the interaction term?
   d. What is the expected home value of a house in the 46228 ZIP code area with the following characteristics: 1800 sqft, 0.54 acres lot, 4 bedrooms, 3 bathrooms, 2 garage spots, and 1 story.

6. ***Pork Demand*** (***): In this exercise, you will estimate the per-capita pork demand as a function of pork prices and the prices of substitutes (beef and chicken) as well as real disposable income. Use the date `meatdemand` for this exercise. Estimate the following equation and interpret the coefficients. Are the signs of the coefficients what you would expect?

   $$\ln(q_{pork}) = \beta_0 + \beta_1 \cdot \ln(p_{pork}) + \beta_2 \cdot \ln(p_{chicken}) + \beta_3 \cdot \ln(p_{beef}) + \beta_4 \cdot \ln(rdi)$$

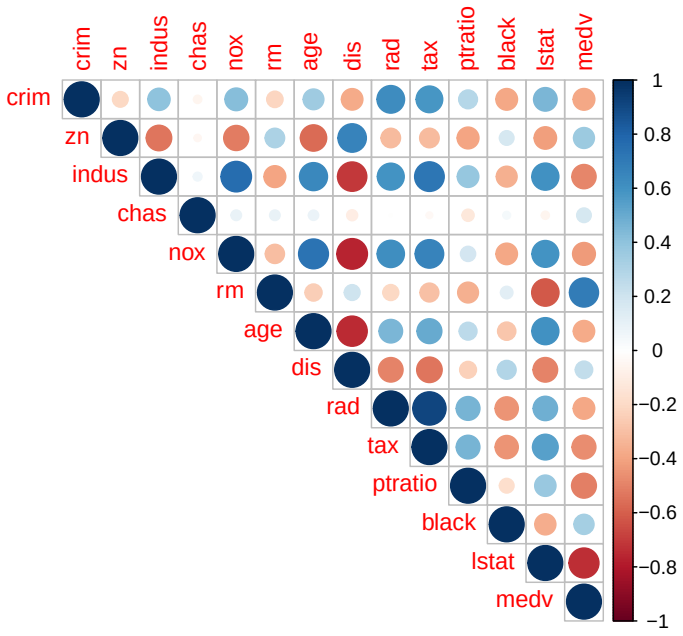
7. ***NFL I*** (***): This question will have you create a similar analysis to the one found in Berri et al. (2011). The corresponding data is in `nfl`:

   $$\ln(total) = \beta_0+\beta_1\cdot yards+\beta_2\cdot att+\beta_3\cdot exp+\beta_4\cdot exp^2+\beta_5\cdot draft1+\beta_6\cdot draft2+\beta_7\cdot veteran+\beta_8\cdot changet$$

   Report the output and interpret the coefficients in terms of statistical significance and direction (i.e., sign).

8. ***Boston*** (***): For this exercise, use the data set `boston`. In a first step, execute the following code:

```
library(corrplot)
corr_matrix = cor(boston)
corrplot(corr_matrix,type="upper")
```



What does the resulting plot represent? In a second step, estimate the following model:

$$medv = \beta_0 + \beta_1 \cdot lstat + \beta_2 \cdot crim + \beta_3 \cdot age$$

Explain what exactly you estimated and what hypotheses are underlying the model. Lastly, estimate the model including all remaining independent variables. Are any of the results surprising?

9. ***BLM I*** (***): The following question is based on the article Black Lives Matter: Evidence that Police-Caused Deaths Predict Protest Activity. Note that we use a simplified version of the data set for this question. The dependent variable for this exercise is protest frequency (*totprotests*) and the independent variables are city population (*pop*), population density (*popdensity*), percent Black (*percentblack*), black poverty rate (*blackpovertyrate*), percent of population with at least a bachelor (*percentbachelor*), college enrollment (*collegeenrollpc*), share of democrats (*demshare*), and black police-caused deaths per 10,000 people (*deathsblackpc*). Interpret the output.

10. ***Furnished Apartments*** (***): Long-term furnished apartments are usually managed by companies. Some of those companies are more expensive than others. The dependent variable is *rent*. Estimate a regression model using all other columns as independent variables. Which companies are more expensive than others and by how much? Is there a difference between living in Berlin (city) or one of its close suburbs (Potsdam). On a side note, people in Potsdam prefer the city not to be called a suburb

since it is fairly sizable state capital.

11. ***U.S. Labor Market*** (***): Understanding wage determinants is essential for policymakers working on labor market policies. In this exercise, you will analyze the factors affecting individual wages, including education, experience, gender, and union membership using the data set `wages` from the Ecdat package. Estimate a multiple linear regression model where the dependent variable is *wage* (hourly wage), and the independent variables include *education* (years of schooling), *experience* (years of labor market experience), *gender* (female dummy variable: 1 for female, 0 for male), and *union* (union dummy variable: 1 for union member, 0 otherwise). Interpret the coefficient estimates regarding the effects of education and union membership on wages. Is there a significant gender wage gap after controlling for education and experience? Test for interaction effects between gender and education. Does the return to education differ between men and women?

12. ***Dry January*** (***): The data set `dj` contains the real per-capita retail sales of alcoholic beverages. Estimate a simple model including a time trend, dummy variables for the months, and a dummy variable indicating the beginning of the pandemic in the United States. For that date, I suggest to use April 2020. Interpret your results. Based on the data, how would you estimate the effects of Dry January? That is, the movement which started in 2014 to abstain from alcohol in January.