Probability

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Basic Concepts

Sample space

- A sample space is a list of all possible outcomes of an experiment.
- Notation: Ω

Examples of a sample space:

- Rolling a single die: $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Tossing a coin: $\Omega = \{H, T\}$
- Grades: $\Omega = \{A+, A, A-, \ldots, F\}$
- Number of calls to a fire station in a 24-hour period: $\Omega = \{0, 1, 2 \ldots\}$

What about tomorrow's temperature?

Event

- Subset of the sample space

Examples

- Event E: Rolling a die and getting an even number

$$E = \{2, 4, 6\}$$

- Event S: Rolling a number less or equal to four

$$S = \{1, 2, 3, 4\}$$

- Event F: More than five calls to the fire station

$$F = \{5, 6, \dots\}$$

Probability

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Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

Set Notation and Set Operations

Intersection

- The intersection W of two sets X and Y is the set of elements that are in both X and Y. We write $W = X \cap Y$.

Empty or Null Sets

- The empty set or the null set $(\emptyset)$ is the set with no elements. For example, if the sets A and B contain no common elements then these two sets are said to be disjoint, e.g., odd and even numbers: $A \cap B = \emptyset$.

Unions

- The union of two sets A and B is the set of all elements in one or the other of the sets. We write $C = A \cup B$.

Complements

- The complement of a set X is the set of elements of the universal set U that are not elements of X, and is written X^c.

Probability Concepts

Probability defined for a discrete sample space:

- The probability of an event is a non-negative number, i.e., $P(A) \geq 0$, for any subset A of Ω.
- $P(\Omega) = 1$: All the probabilities of the outcomes in the sample space sum up to 1.

If $A, B, C, \ldots$ is a finite or infinite sequence of mutually exclusive events of Ω, i.e., events that cannot happen at the same time, then we have

$$P(A \cup B \cup C \cup \cdots) = P(A) + P(B) + P(C) + \cdots$$

Example: Coin flip

- Sample space: $\Omega = \{H, T\}$
- $P(H) = 0.5$ and $P(T) = 0.5$

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Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

Probability Example: Flipping a Coin Three Times

The eight events in Ω are:

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Probability of each event is equally likely, i.e., $P(E_i) = 1/8$ for $i = 1, 2, 3, \ldots, 8$. If the event of interest (A) is exactly two heads, then it is written as:

$$A = \{E_2, E_3, E_5\}$$

We find $P(A)$ by summing as follows

$$P(A) = P(E_2) + P(E_3) + P(E_5) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

Probability of a Union

Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

For any two events A and B, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Examples:

- Alcohol and cocaine consumption
- Safety checks by the police
- On-time arrival of airplanes

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Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

Alcohol and Cocaine Consumption

Consider the blood content of randomly selected people. Define the following events: $A = \{Alcohol\}$, $B = \{Cocaine\}$, and $A \cap B = \{Both\}$. The probabilities are as follows: $P(A) = 0.86$, $P(B) = 0.35$, and $P(A \cap B) = 0.29$. To calculate the probability of finding either alcohol or cocaine or both in the blood stream, you need to perform the following calculation:

$$P(A \cup B) = 0.86 + 0.35 - 0.29 = 0.92$$

Intrigued by this example? Check out this EU Project.

Probability

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Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

Safety checks by the police

Highway patrols are randomly checking the safety of trucks. Assume the following events:

$$A = \{\text{faulty breaks}\}$$

$$B = \{\text{bad tires}\}$$

$$A \cup B = \{\text{faulty breaks and/or bad tires}\}$$

Let $P(A) = 0.23$, $P(B) = 0.24$, and $P(A \cap B) = 0.09$. Thus, we can determine that $P(A \cup B) = 0.23 + 0.24 - 0.09 = 0.38$.

Probability

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Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

On-time arrival of airplanes

If the events are mutually exclusive, the term $P(A \cap B)$ is equal to 0. Gate arrival of airplanes during a week at a mid-sized airport. Everyting not within $+/-$ 10 minutes is considered "Not on Time."

Arrival	Event	Flights	Probability
Less than 10 minutes early	A	55	0.20
Within $+/-$ 10 minutes	B	121	0.44
More than 10 minutes late	C	99	0.36

What is the probability that an airplane is not arriving on time at the gate.

Probability of an Intersection

Independent Events

To find the probability that events A and B occur, we have to use the multiplication rule (i.e., probability of the intersection) which is written as

$$P(A \cap B) = P(A) \cdot P(B)$$

For the multiplication rule to hold, the two events must be independent!

Probability

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Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

Rolling a Die and Drawing Cards

Rolling a die

- Suppose you are interested in the probability of getting a 6 on roll 1 (event A) and a 6 on roll 2 (event B). This is written as $P(A) \cdot P(B) = 1/6 \cdot 1/6 = 1/36$.

Drawing cards

- Let $A = \{Hearts\}$ and $B = \{Queen\}$. The joint probability is the likelihood of drawing the Queen of Hearts and is written as:

$$P(A) = \frac{1}{4}$$

$$P(B) = \frac{4}{52}$$

$$P(A \cap B) = P(A) \cdot P(B) = \frac{1}{52}$$

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Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

Dependent Events

For the multiplication rule to hold, the two events must be independent! The multiplication rule for dependent events will be introduced in more detail later but can be written as $P(A \cap B) = P(A) \cdot P(B|A)$ where $P(B|A)$ is the probability of A given that even B occurred.

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Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

Dependent Events: Example

Suppose you have 16 polo shirts in your closet with your company's logo. Nine of them are green and seven are blue. In the morning, you get dressed when it is dark and you randomly grab a shirt two days in a row (without doing laundry). What is the probability that both shirts are blue.

$$P(B_1) = 7/16$$

$$P(B_2|B_1) = 6/15$$

Thus, $P(B_1 \cap B_2) = P(B_1) \cdot P(B_2|B_1) = 7/16 \cdot 6/15 = 0.175$.

Conditional Probability

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Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

Examples

Conditional probability

- Probability of event A given that event B happened
- Notation: $P(A|B)$

Examples:

- Probability of a person earning more than \$150,000 given graduation from Harvard Law School
- Probability of a person getting arrested given a prior arrest
- Probability of getting an "A" in graduate statistics given an undergraduate degree in mathematics
- Probability of receiving a grant from a funding agency given prior funding from the same agency

Concept

Given event B such that $P(B) > 0$ and any other event A, we define the conditional probability of A given B as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

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Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

Example

	Good service	Bad Service
Over 10 years	16	4
Below 10 years	10	20

Questions:

- What is the probability of receiving good service, i.e., $P(G)$?
- What is the probability of receiving good service given more than 10 years of service, i.e., $P(G|\text{Over 10 years})$?

Multiplication Rule

Rearrangement of the terms from the conditional probability definition leads to:

$$P(A \cap B) = P(A|B) \cdot P(B)$$

Multiplication rule example:

- $P(\text{Over 10 years})$: Probability of being in business for over 10 years
- $P(G|\text{Over 10 years})$: Probability of receiving good service given more than 10 years of service

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Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

Example 1: Rolling a Die

What is the probability of a 1, given that an odd number was obtained?

- Event A: Observe a 1.
- Event B: Observe an odd number.

We seek the probability of A given that the event B has occurred. The event $A \cap B$ requires the observance of both a 1 and an odd number. In this instance, $A \subset B$ so $A \cap B = A$ and $P(A \cap B) = P(A) = 1/6$. Also, $P(B) = 1/2$ and, using the definition,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{1/2} = \frac{1}{3}$$

Probability

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Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

Example 2: Box and Balls

Suppose a box contains r red balls labeled $1, 2, 3, \cdots, r$ and b black balls labeled $1, 2, 3, \cdots, b$. If a ball from the box is known to be red, what is the probability it is the red ball labeled 1, i.e., $P(B|A)$?

- Event A: Observe a red ball.
- Event B: Observe a 1.

Probability of A:

$$P(A) = \frac{r}{r + b}$$

Probability of a red ball with the number 1 on it:

$$P(A \cap B) = \frac{1}{r + b}$$

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Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

Example 2: Box and Balls (continued)

Then the probability that the ball is red and labeled 1 given that it is red is given by

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/(r+b)}{r/(r+b)} = \frac{1}{r}$$

This differs from the probability of B (a 1 on the ball) which is given by

$$P(B) = \frac{2}{r+b}$$

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Probability

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Concept

Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

Two events are said to be independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

If $P(B) > 0$ (or $P(A) > 0$), this can be written in terms of conditional probability as

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

The events A and B are independent if knowledge of B does not affect the probability of A.

Probability

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Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

Example I: Setup and Calculation

Rolling a red die and a green die

- Event A: 4 on the red die.
- Event B: Sum of the dice is odd.

Are A and B independent? Consider the table on the next slide illustrating the probabilities and calculations below:

$$P(A) = 6/36 = 1/6$$
$$P(B) = 18/36 = 1/2$$
$$P(A \cap B) = 3/36 = 1/2$$

This leads to

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3/36}{1/2} = \frac{1}{6}$$

The events A and B are thus independent.

Example I: Visual Representation

			Green			
Red	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

Example II: Setup and Calculation

Rolling a red die and a green die

- Event C: At least three dots
- Event D: Sum equal to seven

Are C and D independent? Consider the table on the next slide illustrating the probabilities and calculations below:

$$P(C) = \frac{32}{36}$$

$$P(D) = \frac{1}{6}$$

This leads to

$$P(C|D) = \frac{P(C \cap D)}{P(D)} = \frac{6/36}{6/36} = 1$$

Thus, the two events are dependent.

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Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

Example II: Visual Representation

Red	Green					
	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

Table 3: All pairs except the lightgray ones are in event C. The pairs in blue are in event D.

Bayes Rule

Probability

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Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

Law of Total Probability and Bayes Rule I

Cows and bovine spongiform encephalopathy (BSE). Let the events be as follows:

- B: Cow has BSE
- T: Cow tests positive

Assume the following probabilities:

- $P(T|B) = 0.7$
- $P(T|B^C) = 0.1$
- $P(B) = 0.02$
- $P(B^C) = 0.98$

What is $P(T) = P(T|B) \cdot P(B) + P(T|B^C) \cdot P(B^C)$?

Law of Total Probability and Bayes Rule II

Remember from conditional probability

- $P(T|B) = \frac{P(T \cap B)}{P(B)}$
- $P(T|B^C) = \frac{P(T \cap B^C)}{P(B^C)}$

Question

- What is the probability that a cow has BSE if it tests positive, i.e., $P(B|T)$?

Solution

$$P(B|T) = \frac{P(T \cap B)}{P(T)} = \frac{P(T|B) \cdot P(B)}{P(T|B) \cdot P(B) + P(T|B^C) \cdot P(B^C)}$$

Permutations and Combinations

Probability

Jerome
Dumortier

Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

Permutations: Ordered Arrangement

A ordered arrangement of k distinct objects is called a permutation. The number of ways to order n distinct objects taken k at a time is distinguished by the symbol P_k^n

$$P_k^n = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots (n-k+1) = \frac{n!}{(n-k)!}$$

where

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots 2 \cdot 1$$

and $0! = 1$.

Probability

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Dumortier

Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

Permutations: Example

Consider a bowl containing six balls with the letters A, B, C, D, E, F on the respective balls. Now consider an experiment where you draw one ball from the bowl and write down its letter and then draw a second ball and write down its letter. The outcome is than an ordered pair, i.e., $BA \neq AB$. The number of distinct ways of doing this is given by

$$P_2^6 = \frac{6!}{4!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 6 \cdot 5 = 30$$

Number of ways to arrange 6 items if $k = 6$

Combinations: Ordering Does Not Matter

The number of unordered subsets of size k chosen (without replacement) from n available objects is:

$$\binom{n}{k} = C_k^n = \frac{P_k^n}{k!} = \frac{n!}{k! \cdot (n-k)!}$$

Probability

Jerome
Dumortier

Basic
Concepts

Probability of
a Union

Probability of
an Intersection

Conditional
Probability

Independence

Bayes Rule

Permutations
and
Combinations

Combinations: Example

Consider a bowl containing six balls with the letters A, B, C, D, E, F on the respective balls. Now consider an experiment where you draw two balls from the bowl and write down the letter on each of them, not paying any attention to the order in which you draw the balls so that AB is the same as BA. The number of distinct ways of doing this is given by

$$C_2^6 = \frac{6!}{2! \cdot 4!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{6 \cdot 5}{2} = 15$$