Violating Assumptions
Jerome Dumortier
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Violating Assumptions

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Required Packages

The following packages are needed for the material presented in the slides

- car
- lmtest
- MASS
- nlme
- orcutt
- prais
- sandwich

Assumptions

Key assumptions underlying the ordinary least square (OLS) model

1. **Linear in coefficients**: Linear relationship between y and $x_1, \ldots, x_k$
2. **Zero mean of the error terms**: $E(\epsilon | x_1, \ldots, x_k) = 0$ and normally distributed error terms
3. **Homoscedasticity**: $Var(\epsilon_i) = \sigma^2$
4. **No autocorrelation between error terms**: $Cov(\epsilon_i, \epsilon_j) = 0$ for all $i \neq j$
5. **Exogeneity of independent variables**: $E(\epsilon_i | x_1, \ldots, x_k)$, i.e., independent variables contain no information to predict error terms
6. **Full rank** (linear independence of all columns) of X (matrix of independent variables): Perfect multicollinearity (i.e., one independent variable being perfectly predicted from a linear combination of one or more other independent variables) leads to a rank deficiency of X

Topics Covered

Non-constant error (ϵ_i) variance, i.e., heteroscedasticity

- Testing for heteroscedasticity using the Goldfeld-Quandt Test (1965) and the Breusch-Pagan-Godfrey Test (1979)
- Correcting for heteroscedasticity by using heteroskedasticity-consistent (robust) standard errors

Multicollinearity

- Detecting multicollinearity with Variance Inflation Factors (VIF)

Autocorrelation

Overview

Homoscedasticity

$$Var(\epsilon_i) = \sigma^2$$

Heteroscedasticity

$$Var(\epsilon_i) = \sigma_i^2$$

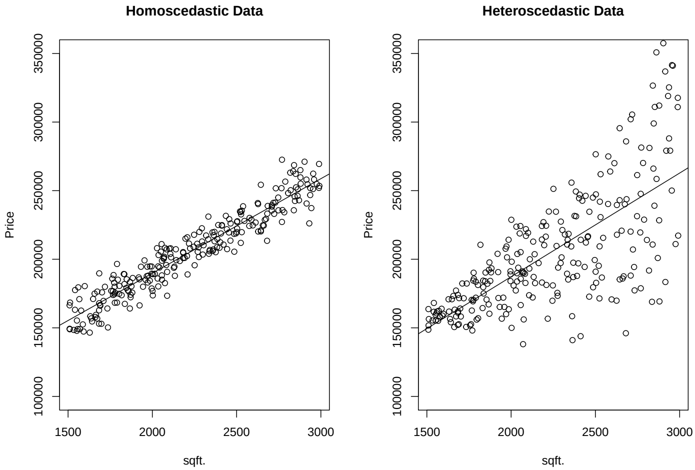
It can be shown that

$$Var(\hat{\beta}_1) = \underbrace{\frac{\sigma_i^2}{\sum x_i^2}}_{Hetero.} \neq \underbrace{\frac{\sigma^2}{\sum x_i^2}}_{Homo.}$$

Notes

- Coefficient estimates and R^2 are unaffected by heteroscedasticity
- Variance of β_1 is larger

Homoscedastic vs. Heteroscedastic Data



Examples and Effects of Heteroscedasticity I

Examples

- *Income, savings, and consumption*: People with higher incomes tend to have more variability in their savings and expenditures whereas low-income individuals spend close to their income
- *Firms and dividends*: Companies with larger profits show more variability in dividend payments
- *Education and income*: Wages may be more predictable for lower education levels while higher education degrees introduce greater variability due to differences in occupation, industry, and experience
- *House price and square footage*: Small price variations for smaller homes compared to larger homes

Examples and Effects of Heteroscedasticity II

Examples

- *Municipal budget variability and city size*: Larger cities experience greater fluctuations in budget expenditures due to the complexity and unpredictability of managing diverse public services
- *Public program effectiveness and demographics*: Policy interventions show more variable outcomes in diverse populations (e.g., in terms of socio-economics) compared to more homogeneous communities, leading to inconsistent program effectiveness

Effects of heteroscedasticity

- Requirement of homoscedasticity for t-test, F-test, and confidence intervals
- F-statistics no longer have the F-distribution
- Bottom line: Hypothesis tests on the β coefficients are no longer valid

Generalized Least Squares (GLS) I

If σ_i^2 was known:

$$y_i = \beta_0 + \beta_1 \cdot x_i + \epsilon_i$$

Dividing both sides by the known variance:

$$\frac{y_i}{\sigma_i} = \beta_0 \cdot \frac{1}{\sigma_i} + \beta_1 \cdot \frac{x_i}{\sigma_i} + \frac{\epsilon_i}{\sigma_i}$$

If $\epsilon_i^* = \epsilon_i / \sigma_i$, then it can be shown that $Var(\epsilon_i^*) = 1$, i.e., constant.

Generalized Least Squares (GLS) II

Regular OLS

$$\sum_{i=1}^{N} \epsilon_i^2 = \sum_{i=1}^{N} (y_i - \beta_0 + \beta_1 \cdot x_i)^2$$

GLS with $w_i = 1/\sigma_i$

$$\sum_{i=1}^{N} w_i \cdot \epsilon_i^2 = \sum_{i=1}^{N} w_i \cdot (y_i - \beta_0 + \beta_1 \cdot x_i)^2$$

GLS: Minimization of the weighted sum of squared residuals

Generalized Least Squares (GLS) III

Implementation of GLS

1. Estimate the heteroscedasticity structure, e.g., using a White test or Breusch-Pagan test
2. Model the variance function σ_i^2, e.g., as a function of explanatory variables
3. Compute weights $w_i = 1/\hat{\sigma}_i$.
4. Transform the dependent and independent variables using those weights
5. Perform weighted least squares (WLS) regression on the transformed data

Goldfeld-Quandt Test: Steps

Steps for the Goldfeld-Quandt Test

1. Sorting observations in ascending order of an independent variable likely introducing heteroscedasticity
2. Choosing c as the number of central observations to drop resulting in sample sizes $n_1 = n_2 = (n-c)/2$
3. Running two separate regression equations
4. Compute λ with k as the number of coefficients to be estimated including the intercept
$$\lambda = \frac{RSS_2/(n_2-k)}{RSS_1/(n_1-k)}$$
5. λ follows F-distribution and a hypothesis test can be conducted

Goldfeld-Quandt Test: Manual Implementation

Setup

- Example using gqdata with *sqft* being sorted in ascending order,
- $C = 4$

```
gqdata1          = gqdata[1:20,]
gqdata2          = gqdata[31:50,]
bhat             = lm(price~sqft,data=gqdata)
bhat1            = lm(price~sqft,data=gqdata1)
bhat2            = lm(price~sqft,data=gqdata2)
sum(bhat2$residuals^2)/sum(bhat1$residuals^2)
```

```
## [1] 2.826607
```

Goldfeld-Quandt Test: R Function

```
gqtest(bhat,fraction=10)
```

```
##
##  Goldfeld-Quandt test
##
## data:  bhat
## GQ = 2.8266, df1 = 18, df2 = 18, p-value = 0.01665
## alternative hypothesis: variance increases from segment 1 to 2
```

Breusch-Pagan-Godfrey Test: Setps

Steps for the Breusch-Pagan-Godfrey Test

1. Run a regular OLS model and obtain the residuals
2. Calculate
$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{N} \epsilon_i^2}{N}$$
3. Construct the variable $p_i = \epsilon_i^2 / \hat{\sigma}^2$
4. Run a regression as follows with x_i as the independent variables from the original regression
$$p = \alpha_0 + \alpha_1 \cdot x_1 + \alpha_2 \cdot x_2 + \dots$$
5. Obtain the explained sum of squares (ESS) and define $\Theta = 0.5 \cdot ESS$. Then $\Theta \sim \chi^2_{m-1}$.

Or simply use `bptest(bhat)` in R

Breusch-Pagan-Godfrey Test: R Function

```r
bptest(bhat)
```

```
##
##  studentized Breusch-Pagan test
##
## data:  bhat
## BP = 3.8751, df = 1, p-value = 0.04901
```

Robust Standard Errors: Steps

Robust standard (heteroscedasticity-consistent) errors

- Estimation of a covariance matrix (usually denoted Ω in books)

Steps in R

1. Estimation of a regular OLS model
2. Estimation of a covariance matrix using `vcovHC()` from the sandwich package
3. Applying the function `coeftest()` from the nlme package

Simultaneous execution of steps 2 and 3

Robust Standard Errors: Methods

HC0: Default heteroscedasticity-consistent (HC) standard error estimator

- Uses squared residuals without any adjustment
- Suitable for large samples

HC1: Adjusts HC0 for small sample bias by scaling the residuals

- Equivalent to HC0 multiplied by $n/(n-k)$ where k is the number of independent variables

HC2: Corrects for leverage effects in small samples

- Division of squared residuals by $1-h_i$ where $0 \leq h_i \leq 1$ is the leverage of observation i (i.e., influence of i on regression coefficients)

HC3: Additional adjustment compared to HC2 for small sample size

- Division of squared residuals by $(1-h_i)^2$

Robust Standard Errors: Implementation

```
bhat = lm(price~sqft,data=gqdata)
b1   = coeftest(bhat,vcov=vcovHC(bhat,type="HC0"))
b2   = coeftest(bhat,vcov=vcovHC(bhat,type="HC1"))
b3   = coeftest(bhat,vcov=vcovHC(bhat,type="HC2"))
b4   = coeftest(bhat,vcov=vcovHC(bhat,type="HC3"))
```

Robust Standard Errors: Implementation

```
##
## ===================================================================================
##                                          Dependent variable:
##                    ---------------------------------------------------------------
##                            price
##                             OLS                         coefficient
##                                                             test
##                             (1)           (2)       (3)       (4)       (5)
## -----------------------------------------------------------------------------------
## sqft                     73.911***     73.911*** 73.911*** 73.911*** 73.911***
##                           (8.062)       (6.894)   (7.036)   (7.088)   (7.289)
## -----------------------------------------------------------------------------------
## Observations                50
## R2                         0.636
## Adjusted R2                0.629
## Residual Std. Error  24,873.760 (df = 48)
## F Statistic         84.045*** (df = 1; 48)
## ===================================================================================
## Note:                                              *p<0.1; **p<0.05; ***p<0.01
```

Problem with Multicollinearity

From the book Basic Econometrics by Gujarati:

"*If multicollinearity is perfect [. . .], the regression coefficients of the X variables are indeterminate and their standard errors are infinite. If multicollinearity is less than perfect [. . .], the regression coefficients, although determinate, possess large standard errors (in relation to the coefficients themselves), which means the coefficients cannot be estimated with great precision or accuracy.*"

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Perfect multicollinearity with λ_i representing constants that are not all zero simultaneously

$$\lambda_1 \cdot x_1 + \lambda_2 \cdot x_2 + \cdots + \lambda_k \cdot x_k = 0$$

Example

$$x_1 = \{8, 12, 15, 45\}$$

$$x_2 = \{24, 36, 15, 51\}$$

$\lambda_1 = 1$ and $\lambda_2 = -1/3$ are such that $x_1 - 1/3 \cdot x_2 = 0$. Multicollinearity refers to linear relationships and including a squared or cubed term does not represent multicollinearity

Examples

Estimation of energy consumption based on income and home size

- Likely high correlation between income and house size

Estimation of education quality (e.g., test scores, graduation rates) based on public spending (e.g., per-capita education budget, teacher salary, and number of schools)

- Correlation between education budget and teacher salary as well as education budget and number of schools

Estimation of crime based on crime prevention policies and public safety expenditures

- Likely correlation of public safety expenditures and the ability to fund crime prevention policies

Over-determined model

- Number of variables k larger than number of observations n

Indications of Multicollinearity

Signs of multicollinearity

- High R^2 but few significant variables
- Failure to reject H_0 (i.e.,$\beta_i = 0$) based on t-values but rejection of F-test (i.e., all slopes being simultaneously zero)
- High correlation among explanatory variables
- Variation of statistically significant variables between models that include different sets of independent variables

Consequences of multicollinearity

- Increase in variances of coefficients

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VIF: Overview

Purpose

- Identification of possible correlation among multiple independent variables and not just two as in the case of a correlation coefficient
- Detect inflated variance based on multicollinearity

Theoretical aspects

- Existence of a VIF for each independent variable in the model

Regressing each independent variable on all other independent variables

VIF: Calculation and Interpretation

Calculation

- VIF for variable k

$$VIF_k = \frac{1}{1 - R_k^2}$$

Interpretation

- $VIF = 1$: No relationship between the variable x_k and the remaining independent variables
- $VIF > 1$: Some degree of multicollinearity
- $VIF > 4$: Warrants attention
- $VIF > 10$: Indication of serious problem

The latter two are rules of thumb

Setup

Data used: `bloodpressure`

- Patient ID (*pt*), blood pressure (*bp*), body surface area (*bsa*), and duration of hypertension (*dur*)

Correlation matrix

```
##          bp  age weight  bsa  dur pulse stress
## bp     1.00 0.66   0.95 0.87 0.29  0.72   0.16
## age    0.66 1.00   0.41 0.38 0.34  0.62   0.37
## weight 0.95 0.41   1.00 0.88 0.20  0.66   0.03
## bsa    0.87 0.38   0.88 1.00 0.13  0.46   0.02
## dur    0.29 0.34   0.20 0.13 1.00  0.40   0.31
## pulse  0.72 0.62   0.66 0.46 0.40  1.00   0.51
## stress 0.16 0.37   0.03 0.02 0.31  0.51   1.00
```

Regular OLS Regression Results

```
##
## ===========================================
##                    Dependent variable:
##                ----------------------------
##                            bp
## -------------------------------------------
## age                  0.703*** (0.050)
## weight               0.970*** (0.063)
## bsa                   3.776** (1.580)
## dur                     0.068 (0.048)
## pulse                  -0.084 (0.052)
## stress                  0.006 (0.003)
## -------------------------------------------
## Observations              20
## R2                       0.996
## F Statistic    560.641*** (df = 6; 13)
## ===========================================
## Note:          *p<0.1; **p<0.05; ***p<0.01
```

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VIF Calculation

Using the function `vif` from the package car:

```
vif(bhat1)
```

```
##      age   weight      bsa      dur    pulse   stress
## 1.762807 8.417035 5.328751 1.237309 4.413575 1.834845
```

VIF: Calculation of VIF for `weight`

```
##
## ========================================
##                    Dependent variable:
##              ---------------------------
##                          weight
## ----------------------------------------
## age                   -0.145 (0.206)
## bsa                  21.422*** (3.465)
## dur                    0.009 (0.205)
## pulse                 0.558*** (0.160)
## stress                -0.023 (0.013)
## ----------------------------------------
## Observations                20
## R2                         0.881
## F Statistic     20.768*** (df = 5; 14)
## ========================================
## Note:        *p<0.1; **p<0.05; ***p<0.01
```

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VIF: Manual Calculation

The results indicate that $R^2 = 0.881$ then

$$VIF = \frac{1}{1 - 0.881} = 8.403361$$

Solution:

- Eliminate `BSA` because `weight` is easier to obtain.
- `Pulse` may be an issue as well.

VIF: Final Regression

```
##
## ============================================================
##                           Dependent variable:
##              -----------------------------------------------
##                                   bp
##                         (1)                     (2)
## ------------------------------------------------------------
## age               0.703*** (0.050)       0.732*** (0.056)
## weight            0.970*** (0.063)       1.099*** (0.038)
## bsa                3.776** (1.580)
## dur                 0.068 (0.048)          0.064 (0.056)
## pulse              -0.084 (0.052)       -0.137** (0.054)
## stress              0.006 (0.003)         0.007* (0.004)
## ------------------------------------------------------------
## Observations             20                      20
## R2                      0.996                   0.994
## F Statistic  560.641*** (df = 6; 13) 502.503*** (df = 5; 14)
## ============================================================
## Note:                            *p<0.1; **p<0.05; ***p<0.01
```