Second Borel-Cantelli lemma

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1 Introduction

This project is concerned with the proof of the second Borel-Cantelli lemma which states that:

Lemma 1.1 (Second Borel-Cantelli) *For sets E_n which are independent and measurable with respect to the probability measure Pr, if $\sum_{i=1}^{\infty} Pr(E_n) = \infty$, then $Pr(limsup E_n) = 1$*

This lemma has yet to be proven in the Mathlib library and remains in the list of undergraduate results that have yet to be formalised. Despite this, the majority of the preliminary lemmas that form the steps of the proof listed on wikipedia are in the Mathlib library, meaning that I was able to construct most of the proof. Due to time constraints, I left a number of helpful lemmas with no proof and struggled in particular with the later steps involving the exponential function. This is because the results I needed were very specific and so were not in the library. Indeed, had I had more time, I do believe I could have fixed these problems and made my proof complete. However, the proof's length is still extremely long and parts remain inefficient and sub-optimal. For example, proving that the family of sets E_{n+i} is independent was implemented in a long, laborious and incomplete way with lots of unfolding to base definitions which is generally bad practice. A major difficulty I encountered was that Mathlib lacked much API implementation for independent sets. In the above case, if the library had a result stating that a subset of of independent sets is independent, then the previous lemma could be solved in a few lines rather than the many it currently takes. In the future, as this API is built up, my project could likely be shortened down considerably in this manner.

The remainder of this document will describe the steps and problems I encountered along the way.

2 First stage

This section will be dedicated to proving the first few steps of the proof, beginning by showing that its sufficient to prove that $1 - Pr(\bigcap_{i=0}^{\infty} \bigcup_{j=0}^{\infty}) = 0$. This simple result is complicated in lean by the fact that measures are functions from sets of an arbitrary type to the type called the extended non-natural reals, ennreal for short. This type is not a ring so we can't rely on the ring tactic to solve steps like this. Instead I had to use a theorem that works towards the goal for monoids which have ordered subtraction. On reflection, I now know that the ennreals have tonnes of lemmas which are there to do this for you, so if I were to comeback to this with more time, I would exchange this proof for using a result from Mathlib.

The next step, proving $1-Pr(\bigcap_{i=0}^{\infty}\bigcup_{j=0}^{\infty}) = Pr((\bigcap_{i=0}^{\infty}\bigcup_{j=0}^{\infty})^c)$, was straightforward by using the lemma measure_compl, which proves exactly what we need given that the sets in question are measurable. This measurability comes from the definition of a sigma algebra and Jason KY helped me in proving this.

Following this, we now move the complement inside the set operators. This readily followed from using simp on a few basic lemmas. Additionally at this step, we use that if the measure of a union equals zero then the measure of each set is zero, simplifying our goal slightly.

3 Independence

We now want to show that the sequence of sets defined by $n \to E_{n+i}^c$ a fixed i in the naturals is independent. As stated above, if we had a lemma that a subset of independent sets is independent, this would follow directly. However because of the lack of API produced for independence, we go against good practice and break the goal down into the basic statements which it is definitionally equal to. We then use that the sigma algebra generated by a set is equal to the sigma algebra generated by the set's compliment to change our goal to sets $n \to E_{n+i}$ being independent.

We now introduce an arbitrary finite set s_1, function f, and a hypothesis hyp. We then translate our finite set by +i to a new set s_2, and specialise the independence of $n \to E_n$ to s_2 and the function $n \to f(n-i)$.

To progress, we'll show that the statement

$$Pr(\bigcap_{n\in s_1} fn) = \prod_{n\in s_1} Pr(fn)$$

is true if and only if

$$Pr(\bigcap_{n\in s_2} f(n-i)) = \prod_{n\in s_2} Pr(n-i)$$

Using the congr' tactic, which attempts to show things are equal by recursion up to a specified number of steps, our goal simplifies to

$$\prod_{n\in s_1} Pr(fn) = \prod_{n\in s_2} Pr(f(n-i))$$

and,

$$\bigcap_{n\in s_1} fn = \bigcap_{n\in s_2} f(n-i)$$

These are both true by how s_2 is defined, but I don't have the time to find the combination of lemmas that would prove them.

Exchanging our previous goal for the new one its implied by, we apply our specialised hsI (independence of E_ns) and extract a goal showing that for t in s_2, $f(t-n)$ belongs to the measure space generated by E_t. This again follows from the definition of s_2.

4 Changing form to one suitable for a useful lemma

The next section will concern changing the form of our goal so that we can apply the lemma ennreal.tendsto_at_top_zero, which states that the proposition that f tendsto 0 with the filter at top if and only if f converges to 0 with the epsilon-delta notion of convergence. Since the epsilon-delta convergence here though uses $\leq$ rather than $<$, we will need to start by showing we can change the $<$ to a $\leq$ in our current goal. We do this by introducing b in the reals and hb which states b is greater than 0. Naturally we will specialise the less-equal convergence with epsilon being $\frac{b}{2}$ and provide a proof that $\frac{b}{2}$ is greater than 0 using a lemma from ennreal. Now, since b can take any value in the ennreals, we do cases on b being equal to or not equal to $+\infty$. When b isn't $+\infty$, we have that $\frac{b}{2} < b$ so we can quickly produce our goal. When b is $+\infty$, the previous argument breaks down as $\frac{+\infty}{2}$ is not less than ∞. Instead we use that the probability measure is bounded above by 1 so our goal is necessarily true as $1 < \infty$.

We now make our goal more general to meet the form of ennreal.tendsto_at_top_zero. Our current goal is that for all $b > 0$, there exists and n such that the intersection of the naturals up to n has measure less than b. We change this to the following, that for all $b > 0$, there exists an n such that the intersection over the naturals up to any natural above n has measure less than b. The new goal clearly implies the previous goal, allowing us to make the substitute.

We have now finished adjusting our goal to one that ennreal.tendsto_at_top_zero will accept and so reduce our goal to a tendsto proposition.

5 Final part

The remaining steps depend on results from filter theory which i have minimal knowledge on. As such, some parts haven't been formalised but the proofs are still true.

To begin with, we use the independence of the sets $n \to e_{n+i}$ to show that for all n, the measure of intersections of the finite sets equals the product of the measures. This yields from the earlier work we did but during this process, talking to some people from the Xena server revealed that probability_theory.Indep was defined wrong in Mathlib! It was producing a goal along the lines of `s ∈ set α → Prop`. This doesn't make much sense but is definitionally equal to `set α → Prop s`, so after changing our goal to one of this form, the result follows.

We next turn the measure of the compliment into one minus the measure of the set, then convert our statement of convergence in the ennreals into a statement about convergence in the reals. This is done by noting that if neither the function nor the limit is $+\infty$, then turning them into the corresponding real and then back again constitutes the identity. Then, by using the lemma ennreal.tendsto_of_real, we obtain a goal of convergence in the reals.

Next, since our function is non-negative, its sufficient to show that the function is bounded above by a function that tends to 0 as n goes to infinity. The function we use is the one provided by the wikipedia article, namely the one which sends,

$$n \to \prod_{x \in (1, \ldots n)} exp(-Pr(E_{x+i}))$$

This bounds our function as each term in the new product is greater than each term in the old product. We then exchange the product of the exponentials with the exponential of the sum, the property which made us choose this function.

Now we want to show its sufficient for the exponential of the sum to go to 0 if the sum goes to negative infinity. This is almost accomplished by the lemma filter.tendsto_exp, which states that if a function, f, tends to a limit z, then the function e^f will tend to the limit e^z. Had our limit not been zero, we could have applied this function and not worry. However, real.exp is only defined on the reals, not on the extended reals so it cant be applied to $-\infty$. Moreover, there is even a lemma that states that $\forall (x :), f(x) \neq 0$, so we really can't make progress this way.

All that remains is to firstly change our goal by taking the negative of the function and showing it tends to positive infinity, and secondly by monotonicity show that this is implied by $\sum_{n=0}^{\infty} Pr(E_n)$ tending to ∞. Finally, this leaves us with our assumption. As such we have completed our proof.

6 Remarks

6.1 Starting assumtion

Initially, I wanted the assumption hs to be the statement

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∑' (n : ℕ), Pr(E n) = +∞
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rather than its current form which is

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tendsto (λ (a : ℕ), ∑ (x : ℕ) in finset.range a, μ (s x)) at_top at_top.
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In lean, the statement that `∑' = x` means that as we sum over bigger and bigger finite sets, approaching the whole type, we limit towards x. Specifically, for each neighbourhood of x, there is a finite set such that for all finite sets containing this set, the sum over these finite sets belongs to the neighbourhood. In the ennreals, the neighbourhoods of the top (i.e. infinity) are the open intervals (a, T) for all ennreals a which aren't T. I struggled to show that this is equal to the at_top topological filter which has a basis equal to the intervals $[a, T)$ for all ennreals a due to a lack of time and knowledge of filters. Additionally, I haven't proven that the finite sets $\{0, 1, 2, ..., n\}$ tend to the whole type N yet. For this reason, I have left the lemma in its current state but I'm sure with more time and knowledge it could be adapted to the equivalent state.

7 Conclusion

Overall, I'm happy with the amount of progress I made on proving the second Borel-Cantelli lemma. I think it was slightly too ambitious to try and prove this lemma, however I have learned a lot about how to construct arguments in lean and how to navigate the mathlib library. Compared to my previous project, I was able to work backwards throughout my proof which made things significantly easier and simplified the argument compared to working forwards. The main difficulty I had was the limited API of independence in lean, which forced me to prove lots of results from elementary lemmas, requiring lots of time, thought and effort. Hopefully in the future, more results will be added to this API as currently I believe its size might be a limiting factor in what results can be proven about independent sets.