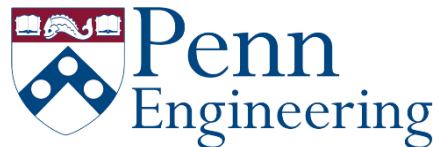


Robotics

Estimation and Learning
with Dan Lee

Supplementary Notes on MLE for Univariate Gaussian



Maximum Likelihood Estimate of Gaussian Model Parameters

- Objective

Estimate the mean and the variance given observed data

$$\hat{\mu}, \hat{\sigma} = \arg \max_{\mu, \sigma} p(\{x_i\} | \mu, \sigma)$$

- Assuming independence of observations,

$$\hat{\mu}, \hat{\sigma} = \arg \max_{\mu, \sigma} \prod_{i=1}^N p(x_i | \mu, \sigma)$$

Maximum Likelihood Estimate of Gaussian Model Parameters

- To obtain $\hat{\mu}, \hat{\sigma} = \arg \max_{\mu, \sigma} \prod_{i=1}^N p(x_i | \mu, \sigma)$

$$(1) \quad \arg \max_{\mu, \sigma} \prod_{i=1}^N p(x_i | \mu, \sigma) = \arg \max_{\mu, \sigma} \sum_{i=1}^N \ln p(x_i | \mu, \sigma)$$

$$(2) \quad p(x_i | \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{(x_i - \mu)^2}{2\sigma^2} \right\} \quad (\text{Gaussian})$$

Maximum Likelihood Estimate of Gaussian Model Parameters

$$\hat{\mu}, \hat{\sigma} = \arg \max_{\mu, \sigma} \prod_{i=1}^N p(x_i | \mu, \sigma)$$

$$= \arg \max_{\mu, \sigma} \sum_{i=1}^N \ln p(x_i | \mu, \sigma) \quad \text{from (1)}$$

$$= \arg \max_{\mu, \sigma} \sum_{i=1}^N \ln \left\{ \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{(x_i - \mu)^2}{2\sigma^2} \right\} \right\} \quad \text{from (2)}$$

Maximum Likelihood Estimate of Gaussian Model Parameters

(continued)

$$\hat{\mu}, \hat{\sigma} = \arg \max_{\mu, \sigma} \sum_{i=1}^N \ln \left\{ \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{(x_i - \mu)^2}{2\sigma^2} \right\} \right\}$$

$$= \arg \max_{\mu, \sigma} \sum_{i=1}^N \left\{ -\frac{(x_i - \mu)^2}{2\sigma^2} - \ln \sqrt{2\pi}\sigma \right\}$$

$$= \arg \min_{\mu, \sigma} \sum_{i=1}^N \left\{ \frac{(x_i - \mu)^2}{2\sigma^2} + \ln \sigma + \text{const} \right\}$$

Maximum Likelihood Estimate of Gaussian Model Parameters

$$\text{Let } J(\mu, \sigma) = \sum_{i=1}^N \left\{ \frac{(x_i - \mu)^2}{2\sigma^2} + \ln \sigma \right\}$$

$$\text{Then } \hat{\mu}, \hat{\sigma} = \arg \min_{\mu, \sigma} J(\mu, \sigma)$$

$$\textcircled{1} \quad \frac{\partial J}{\partial \mu} = 0 \longrightarrow \hat{\mu} \qquad \textcircled{2} \quad \frac{\partial J(\hat{\mu}, \sigma)}{\partial \sigma} = 0 \longrightarrow \hat{\sigma}$$

Maximum Likelihood Estimate of Gaussian Model Parameters

$$\textcircled{1} \quad \boxed{\frac{\partial J}{\partial \mu} = 0 \longrightarrow \hat{\mu}} \quad \frac{\partial J}{\partial \mu} = \frac{\partial}{\partial \mu} \sum_{i=1}^N \left\{ \frac{(x_i - \mu)^2}{2\sigma^2} + \ln \sigma \right\}$$

$$= \sum_{i=1}^N \left\{ \frac{\partial}{\partial \mu} \frac{(x_i - \mu)^2}{2\sigma^2} \right\}$$

$$\hat{\mu} = \frac{1}{N} \sum_{i=1}^N x_i \quad \longleftarrow \quad = \frac{1}{\sigma^2} \sum_{i=1}^N (x_i - \mu) = 0$$

Maximum Likelihood Estimate of Gaussian Model Parameters

$$\begin{aligned} \textcircled{2} \quad \boxed{\frac{\partial J(\hat{\mu}, \sigma)}{\partial \sigma} = 0} &\rightarrow \hat{\sigma} \quad \frac{\partial J}{\partial \sigma} = \frac{\partial}{\partial \sigma} \sum_{i=1}^N \left\{ \frac{(x_i - \hat{\mu})^2}{2\sigma^2} + \ln \sigma \right\} \\ &= \left(\frac{\partial}{\partial \sigma} \frac{1}{2\sigma^2} \right) \left(\sum_{i=1}^N (x_i - \hat{\mu})^2 \right) - \frac{N}{\sigma} \\ \hat{\sigma}^2 &= \frac{1}{N} \sum_{i=1}^N (x_i - \hat{\mu})^2 \leftarrow = \frac{1}{\sigma} \left(N - \frac{1}{\sigma^2} \sum_{i=1}^N (x_i - \hat{\mu})^2 \right) = 0 \end{aligned}$$

Maximum Likelihood Estimate of Gaussian Model Parameters

- In summary, we have

$$\hat{\mu} = \frac{1}{N} \sum_{i=1}^N x_i$$

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^N (x_i - \hat{\mu})^2$$