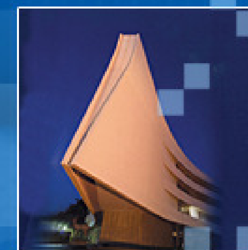
ERP 420

Research Project

Markov Processes and Kendall's Notation



Dr A. De Freitas

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA
Denkleiers • Leading Minds • Dikgopolo tša Dihlalefi

Stochastic Process

Let ξ denote the random outcome of an experiment. To every such outcome suppose a waveform $X(t,\xi)$ is assigned. The collection of such waveforms form a stochastic process. The set of $\{\xi_k\}$ and the time index t can be continuous or discrete (countably infinite or finite) as well.

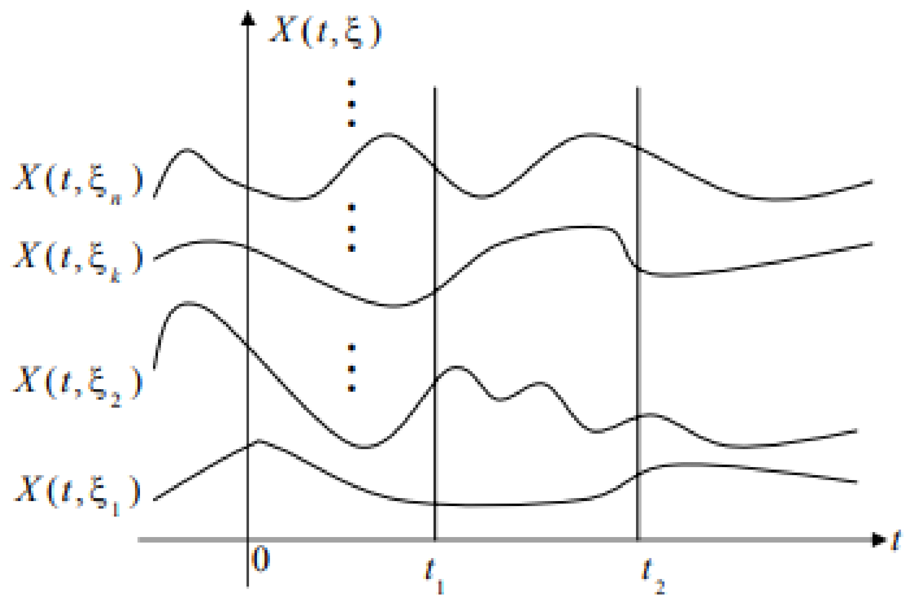


Fig. 14.1

For fixed $\xi_i \in S$ (the set of all experimental outcomes), $X(t,\xi)$ is a specific time function. For fixed t,

$$X_1 = X(t_1,\xi_i)$$

is a random variable. The ensemble of all such realizations $X(t,\xi)$ over time represents the stochastic process $X(t)$.

Stochastic Process

State space: the set of possible values of X_t

Parameter space: the set of values of t

Stochastic processes can be classified according to whether these spaces are discrete or continuous:

	State space	
Parameter space	Discrete	Continuous
Discrete	$*$	$**$
Continuous	$***$	$****$

According to the type of the parameter space one speaks about discrete time or continuous time stochastic processes.

Markov Process

Markov process

A stochastic process is called a Markov process when it has the Markov property:

$$\boxed{\mathrm{P}\{X_{t_n} \leq x_n \,|\, X_{t_{n-1}} = x_{n-1}, \ldots X_{t_1} = x_1\} = \mathrm{P}\{X_{t_n} \leq x_n \,|\, X_{t_{n-1}} = x_{n-1}\}} \quad \forall n, \ \forall t_1 < \cdots < t_n$$

- The future path of a Markov process, given its current state ($X_{t_{n-1}}$) and the past history before t_{n-1}, depends only on the current state (not on how this state has been reached).
- The current state contains all the information (summary of the past) that is needed to characterize the future (stochastic) behaviour of the process.
- Given the state of the process at an instant its future and past are independent.

In the following we additionally assume that the process is time homogeneous.

A Markov process of this kind is characterized by the (one-step) transition probabilities (transition from state i to state j):

$$\boxed{p_{i,j} = \mathrm{P}\{X_n = j \,|\, X_{n-1} = i\}}$$

time homogeneity: the transition probability does not depend on n

Markov Process

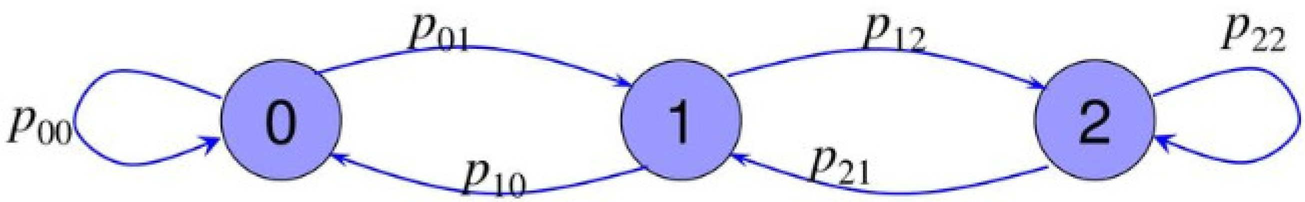
	State Space	
Parameter Space	**Discrete**	**Continuous**
Discrete	(Discrete-time) Markov chain	Discrete-time Markov Process
Continuous	Continuous-time Markov Chain	Continuous-time Markov Process (Stochastic Process with the Markov property)

Markov Chain properties

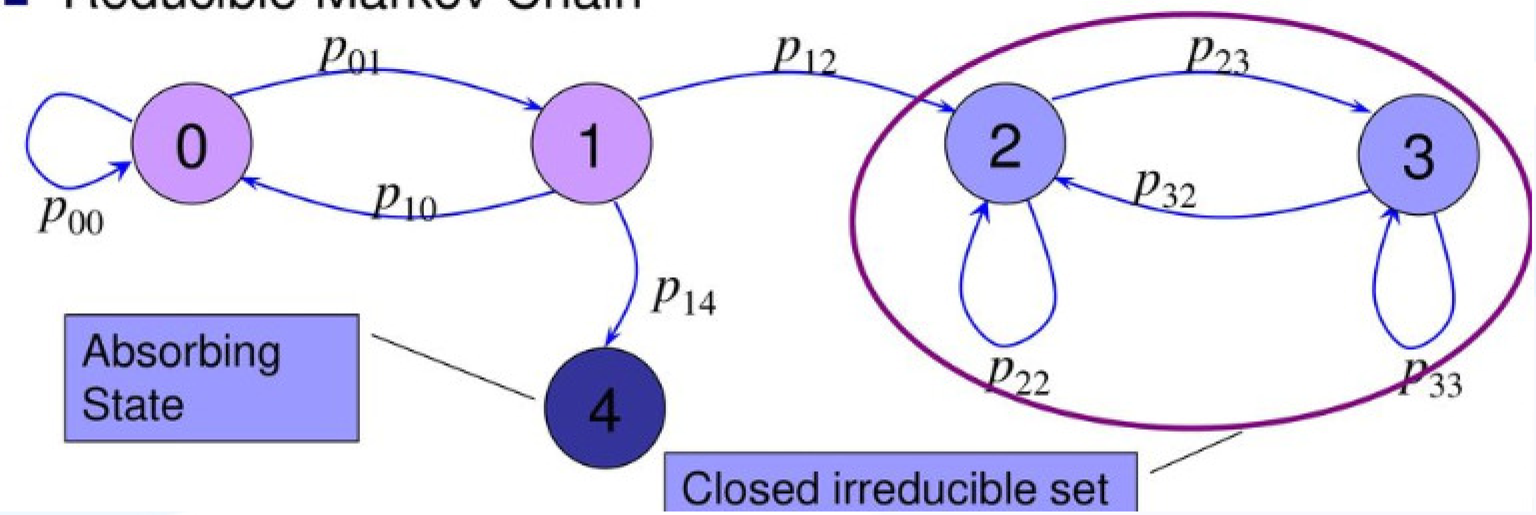
- Reducibility - A Markov chain is said to be irreducible if it is possible to get to any state from any other state.
- Transience & Recurrence - A state *i* is said to be transient if, given that we start in state *i*, there is a non-zero probability that we will never return to *i*.

Markov Chain

- Irreducible Markov Chain



- Reducible Markov Chain

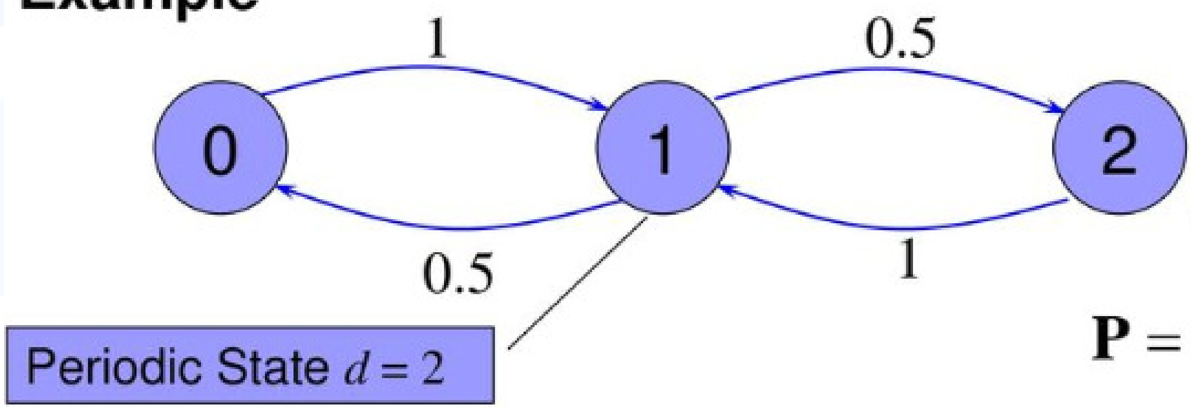


Markov Chain properties

- Reducibility - A Markov chain is said to be irreducible if it is possible to get to any state from any other state.
- Transience & Recurrence - A state *i* is said to be transient if, given that we start in state *i*, there is a non-zero probability that we will never return to *i*.
- Periodicity - A state *i* has period *k* if any return to state *i* must occur in multiples of *k* time steps.

Markov Chain

Example



$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$$

Markov Chain properties

- Reducibility - A Markov chain is said to be irreducible if it is possible to get to any state from any other state**.**
- Transience & Recurrence - A state *i* is said to be transient if, given that we start in state *i*, there is a non-zero probability that we will never return to *i*.
- Periodicity - A state *i* has period *k* if any return to state *i* must occur in multiples of *k* time steps.
- Ergodicity –
  - DTMC: A state *i* is said to be ergodic if it is **aperiodic** and positive **recurrent**. In other words, a state *i* is ergodic if it is recurrent, has a period of *1*, and has finite mean recurrence time. If all states in an **irreducible** Markov chain are ergodic, then the chain is said to be ergodic. /
  - CTMC: An **irreducible**, **homogeneous** CTMC is called ***ergodic*** if and only if the unique steady-state (stationary) probability distribution exists.
- Time-homogeneity – The probability of any state transition is independent of time

Continuous time Markov Chain

Markov processes (Continuous time Markov chains)

Consider (stationary) Markov processes with a continuous parameter space (the parameter usually being time). Transitions from one state to another can occur at any instant of time.

- Due to the Markov property, the time the system spends in any given state is memoryless: the distribution of the remaining time depends solely on the state but not on the time already spent in the state ⇒ the time is exponentially distributed.

A Markov process X_t is completely determined by the so called generator matrix or transition rate matrix

$$q_{i,j} = \lim_{\Delta t \to 0} \frac{\mathrm{P}\{X_{t+\Delta t} = j \,|\, X_t = i\}}{\Delta t} \qquad i \neq j$$

- probability per time unit that the system makes a transition from state i to state j

- transition rate or transition intensity

$$q_{ij}(t) = \lim_{\Delta t \to 0} \frac{p_{ij}(t, t+\Delta t)}{\Delta t}, \quad i \neq j,$$

$$q_{ii}(t) = \lim_{\Delta t \to 0} \frac{p_{ii}(t, t+\Delta t) - 1}{\Delta t}.$$

Continuous time Markov Chain

Transition rate matrix and time dependent state probability vector

The transition rate matrix in full is

$$\mathbf{Q} = \begin{pmatrix} q_{0,0} & q_{0,1} & \cdots \\ q_{1,0} & q_{1,1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} -q_0 & q_{0,1} & \cdots \\ q_{1,0} & -q_1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

row sums equal zero:
the probability mass flowing out of state i
will go to some other states (is conserved)

State probability vector $\boldsymbol{\pi}(t)$ is a function of time evolving as follows

$$\boxed{\frac{d}{dt}\boldsymbol{\pi}(t) = \boldsymbol{\pi}(t) \cdot \mathbf{Q}}$$

Continuous time Markov Chain

often the *steady-state probability vector* of a CTMC is of primary interest. The required properties of the steady-state probability vector, which is also called the *equilibrium probability vector*, are:
For all states $i \in S$, the steady probabilities π_i

1. Independent of time t
2. Independent of the initial state probability vector $\boldsymbol{\pi}(0)$
3. Strictly positive, $\pi_i > 0$
4. Given as the time limits, $\pi_i = \lim_{t \to \infty} \pi_i(t) = \lim_{t \to \infty} p_{ji}(t)$, of the state probabilities $\pi_i(t)$ and of the transition probabilities $p_{ji}(t)$, respectively

Continuous time Markov Chain

If existing for a given CTMC, the steady-state probabilities are independent of time, we immediately get

$$\lim_{t \to \infty} \frac{\mathrm{d}\,\boldsymbol{\pi}(t)}{\mathrm{d}\,t} = 0\,. \tag{2.56}$$

Under condition (2.56), the differential equation (2.51) for determining the unconditional state probabilities resolves to a much simpler *system of linear equations*:

$$0 = \sum_{i \in S} q_{ij}\pi_i\,, \quad \forall j \in S\,. \tag{2.57}$$

In vector–matrix form, we get accordingly

$$\mathbf{0} = \boldsymbol{\pi}\mathbf{Q}\,. \tag{2.58}$$

Continuous time Markov Chain

$$\mathbf{0} = \boldsymbol{\pi}\mathbf{Q}$$

- To obtain a **unique** solution, normalization condition is imposed:

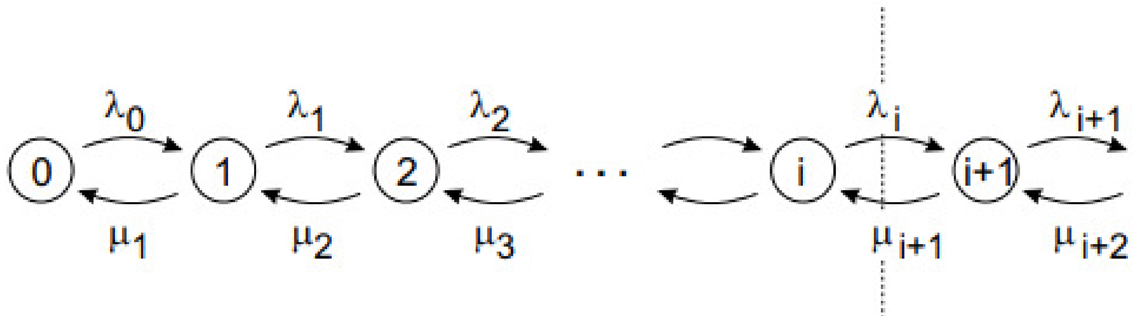
$$\boldsymbol{\pi}\mathbf{1} = \sum_{i \in S} \pi_i = 1$$

- Direct or iterative *numerical* methods
- Closed-form results

Birth-death Process

A birth-death (BD process) process refers to a Markov process with

- a discrete state space
- the states of which can be enumerated with index i=0,1,2,...such that
- state transitions can occur only between neighbouring states, $i \rightarrow i+1$ or $i \rightarrow i-1$



Transition rates

$$q_{i,j} = \begin{cases} \lambda_i & \text{when} \quad j = i+1 \\ \mu_i & \text{if} \quad j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

probability of birth in interval Δt is $\lambda_i \Delta t$
probability of death in interval Δt is $\mu_i \Delta t$
when the system is in state i

Birth-death Process

The equilibrium probabilities of a BD process

We use the method of a cut = global balance condition applied on the set of states $0, 1, \ldots, k$.

In equilibrium the probability flows across the cut are balanced (net flow =0)

$$\boxed{\lambda_k \pi_k = \mu_{k+1}\pi_{k+1}} \qquad k = 0, 1, 2, \ldots$$

We obtain the recursion

$$\pi_{k+1} = \frac{\lambda_k}{\mu_{k+1}}\pi_k$$

By means of the recursion, all the state probabilities can be expressed in terms of that of the state 0, π_0,

$$\boxed{\pi_k = \frac{\lambda_{k-1}\lambda_{k-2}\cdots\lambda_0}{\mu_k\mu_{k-1}\cdots\mu_1}\pi_0 = \prod_{i=0}^{k-1}\frac{\lambda_i}{\mu_{i+1}}\pi_0}$$

The probability π_0 is determined by the normalization condition π_0

$$\boxed{\pi_0 = \frac{1}{1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \cdots} = \frac{1}{1 + \sum_{k=1}^{\infty}\prod_{i=0}^{k-1}\frac{\lambda_i}{\mu_{i+1}}}}$$

Birth-death Process

The time-dependent solution of a BD process

Above we considered the equilibrium distribution $\boldsymbol{\pi}$ of a BD process.

Sometimes the state probabilities at time 0, $\boldsymbol{\pi}(0)$, are known

- usually one knows that the system at time 0 is precisely in a given state k; then $\pi_k(0) = 1$ and $\pi_j(0) = 0$ when $j \neq k$

and one wishes to determine how the state probabilities evolve as a function of time $\boldsymbol{\pi}(t)$

- in the limit we have $\lim_{t\to\infty} \boldsymbol{\pi}(t) = \boldsymbol{\pi}$.

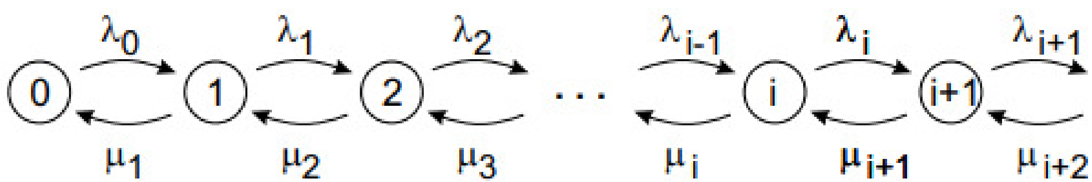
This is determined by the equation

$$\boxed{\frac{d}{dt}\boldsymbol{\pi}(t) = \boldsymbol{\pi}(t) \cdot \mathbf{Q}} \qquad \text{where}$$

$$\mathbf{Q} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \dots & \dots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 \\ \vdots & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \lambda_3 \\ \vdots & \vdots & 0 & \mu_4 & -(\lambda_4 + \mu_4) \end{pmatrix}$$

Birth-death Process

The time-dependent solution of a BD process (continued)



The equations component wise

$$
\begin{cases}
\dfrac{d\pi_i(t)}{dt} = \underbrace{-(\lambda_i+\mu_i)\pi_i(t)}_{\text{flows out}} + \underbrace{\lambda_{i-1}\pi_{i-1}(t)+\mu_{i+1}\pi_{i+1}(t)}_{\text{flows in}} & i=1,2,\ldots \\
\dfrac{d\pi_0(t)}{dt} = \underbrace{-\lambda_0\pi_0(t)}_{\text{flow out}} + \underbrace{\mu_1\pi_1(t)}_{\text{flow in}}
\end{cases}
$$

Pure Birth Process Example

$$\begin{cases} \lambda_i = \lambda \\ \mu_i = 0 \end{cases} \quad i = 0, 1, 2, \ldots \qquad\qquad \pi_i(0) = \begin{cases} 1 & i = 0 \\ 0 & i > 0 \end{cases}$$

birth probability per time unit is constant λ

initially the population size is 0

$$0 \xrightarrow{\lambda} 1 \xrightarrow{\lambda} 2 \xrightarrow{\lambda} \cdots \xrightarrow{\lambda} i\text{-}1 \xrightarrow{\lambda} i \xrightarrow{\lambda}$$

All states are transient

$$\begin{cases} \frac{d}{dt}\pi_i(t) = -\lambda\pi_i(t) + \lambda\pi_{i-1}(t) & i > 0 \\ \frac{d}{dt}\pi_0(t) = -\lambda\pi_0(t) & \Rightarrow \quad \pi_0(t) = e^{-\lambda t} \end{cases}$$

$$\frac{d}{dt}(e^{\lambda t}\pi_i(t)) = \lambda\pi_{i-1}(t)e^{\lambda t} \qquad \Rightarrow \qquad \pi_i(t) = e^{-\lambda t}\lambda\int_0^t \pi_{i-1}(t')e^{\lambda t'}dt'$$

$$\pi_1(t) = e^{-\lambda t}\lambda\int_0^t \underbrace{e^{-\lambda t'}e^{\lambda t'}}_{1}dt' = e^{-\lambda t}(\lambda t)$$

Recursively $\boxed{\pi_i(t) = \frac{(\lambda t)^i}{i!}e^{-\lambda t}}$ Number of births in interval $(0, t) \sim \text{Poisson}(\lambda t)$

Kendall's Notation

- The following notation, known as Kendall's notation, is widely used to describe elementary queuing systems:
- A/B/m - queuing discipline
- where A indicates the distribution of the arrival process, *B* denotes the distribution of the service process, and *m* is the number of servers (m≥1).
- The following denotes:
- *M* Exponential distribution (memoryless property)
- E_K Erlang distribution with *k* phases
- H_K Hyperexponential distribution with *k* phases
- C_K Cox distribution with *k* phases
- *D* Deterministic distribution, i.e., the interarrival time or service tinie is constant
- G General distribution
- G_K General distribution with independent interarrival time

Kendall's Notation

- The queuing discipline or service strategy determines which job is selected from the queue for processing when a server becomes available. Some commonly used queueing disciplines are:
- ***FCFS* (First-Come-First-Served):** If no queueing discipline is given in the

  Kendall notation, then the default is assumed to be the FCFS discipline.

  The jobs are served in the order of their arrival.
- ***LCFS* (Last-Come-First-Served):** The job that arrived last is served next.
- ***SIR0* (Service-In-Random-Order):** The job to be served next is selected at
- random.

Kendall's Notation

- ***RR*** **(Round Robin):** If the servicing of a job is not completed at the end of a time slice of specified length, the job is preempted and returns to the queue, which is served according to FCFS. This action is repeated until the job service is completed.
- ***PS*** **(Processor Sharing):** This strategy corresponds to round robin with infinitesimally small time slices. It is as if all jobs are served simultaneously and the service time is increased correspondingly.
- ***IS*** **(Infinite Server):** There is an ample number of servers so that no queue ever forms.
- ***Static Priorities:*** The selection depends on priorities that are permanently assigned to the job. Within a class of jobs with the same priority, FCFS is used to select the next job to be processed.

Kendall's Notation

- ***Dynamic Priorities****:* The selection depends on dynamic priorities that alter
- with the passing of time.
- ***Preemption****:* If priority or LCFS discipline is used, then the job currently
- being processed is interrupted arid preempted if there is a job in the
- queue with a higher priority.

Kendall's Notation-Example

- As an example of Kendall's notation, the expression
- M/G/1-LCFS preemptive resume (PR)
- describes an elementary queueing system with exponentially distributed interarrival times, arbitrarily distributed service times, and a single server. The queueing discipline is LCFS where a newly arriving job interrupts the job currently being processed and replaces it in the server. The servicing of the job that was interrupted is resumed only after all jobs that arrived after it have completed service.

Kendall's Notation-Example

- Kendall's notation can be extended in various ways. An additional parameter is often introduced to represent the number of places in the queue (if the queue is finite) and we get the extended notation **A/B/m/K-queueing discipline,**
- where K is the capacity of the station (queue + server). This means that if the number of jobs at server and queue is K , a newly arriving job is lost.