1 Cartesian notes

1.1 Two dimensional cartesian stream function

We wish to simplify

$$\mathbf{u}' = A\left(-\nabla' P' + T'\hat{\mathbf{z}}\right) \tag{1}$$

The equation $\nabla \cdot \mathbf{u}' = 0$ is immediately satisfied by introducing $\mathbf{u}' = (\frac{\partial \psi}{\partial z}, -\frac{\partial \psi}{\partial x})$, so we substitute this in and take the curl.

$$\nabla \times \mathbf{u}' \quad = \quad A\left(0 + -T'\nabla \times \hat{z}\right)$$

Leaving

$$\nabla^2 \psi \quad = \quad -A\frac{\partial T'}{\partial x'}$$

1.2 Two dimensional axisymmetric stream function

In axisymmetric flow $\mathbf{u}(r, \theta, z)$ there is no u_θ component and none of the other components depend on θ. Therefore

$$\nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{u_z}{z} \tag{2}$$

which is satisfied by

$$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} \qquad\qquad u_z = \frac{1}{r}\frac{\partial \psi}{\partial r} \tag{3}$$

Where ψ is known as the Stokes streamfunction

2 Solving poisson's equation

Poisson's equation

$$\nabla^2 \psi = -f(x, y) \tag{4}$$

Can be discretized and solved by successive over-relaxation (SOR) using the iterative formula

$$\psi_{i,j}^{n+1} = (1-w)\psi_{i,j}^{n} + \frac{w}{4}\left(\psi_{i-1,j}^{n} + \psi_{i+1,j}^{n} + \psi_{i,j-1}^{n} + \psi_{i,j+1}^{n} + f_{i,j}\right) \tag{5}$$

We wish to solve for $0 < x < 1, 0 < y < 1$ with the boundary conditions

$$\psi(0, y) = 0 \qquad \psi(1, y) = 0 \qquad \psi(x, 0) = 0 \qquad \psi(x, 1) = \sin(\pi x) \tag{6}$$

and

$$f(x, y) = \beta(y(1-y) + x(1-x)) \tag{7}$$

The analytic solution in this case is

$$\psi(x, y) = \sin(\pi x)\frac{\sinh(\pi y)}{\sinh(\pi)} + \frac{1}{2}\beta y(1-y)x(1-x) \tag{8}$$

where we set $\beta = 10$ so that the forcing function has a noticable effect.

2.1 Results

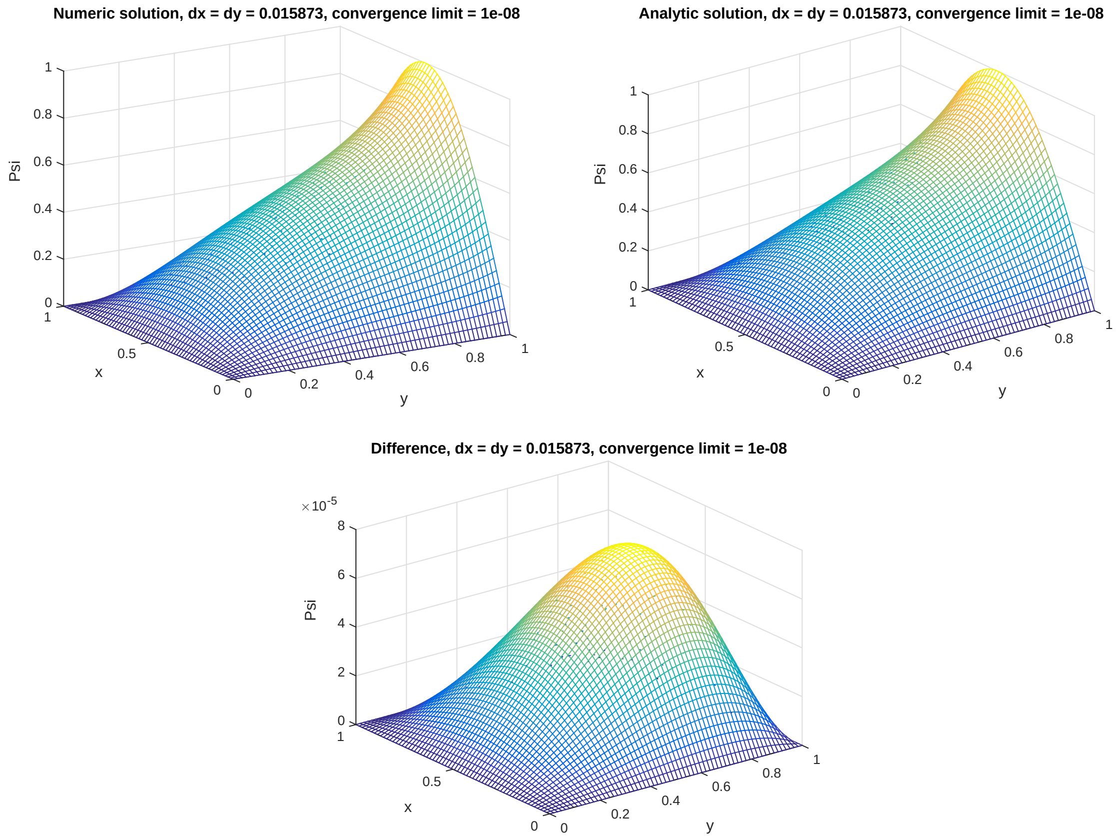

Figure 1:

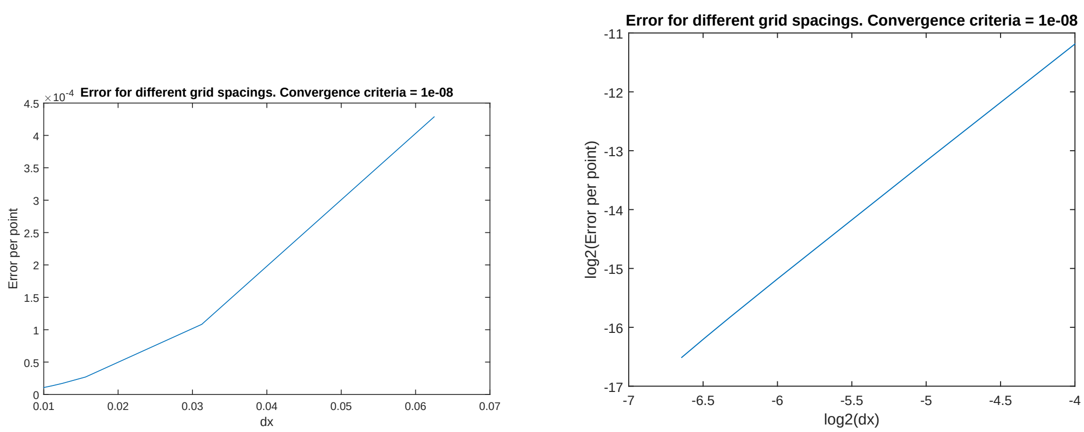

Figure 2:

3 Coupled heat and momentum equations

Following the discussion in Nield and Bejan (2006) chapter 5, we start with

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9}$$

$$\mathbf{u} = -\frac{K}{\mu}\left[\nabla P' - \rho g \hat{\mathbf{x}} \beta (T - T_\infty)\right] \tag{10}$$

$$\frac{\partial P'}{\partial y} = 0 \tag{11}$$

$$\sigma \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_m \left(\frac{\partial^2 T}{\partial^2 y} + \frac{\partial^2 T}{\partial^2 x}\right) \tag{12}$$

introducing the streamfunction ψ with

$$u = \frac{\partial \psi}{\partial y} \qquad\qquad v = -\frac{\partial \psi}{\partial x} \tag{13}$$

we can reduce these four equations to

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{g\beta K}{\nu}\frac{\partial T}{\partial y} \tag{14}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha_m}\left(\sigma\frac{\partial T}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y}\right) \tag{15}$$

3.1 Discretization and iteration

We start at $t = 0$, with known initial and boundary temperatures. At each timestep, n, the first equation is solved iteratively using SOR:

$$\psi^n_{i,j} = (1-w)\psi^{n-\frac{1}{2}}_{i,j} + \frac{w}{2((\Delta x)^2 + (\Delta y)^2)}\left[(\Delta y)^2(\psi^{n-\frac{1}{2}}_{i-1,j} + \psi^{n-\frac{1}{2}}_{i+1,j}) + (\Delta x^2)(\psi^{n-\frac{1}{2}}_{i,j-1} + \psi^{n-\frac{1}{2}}_{i,j+1}) - (\Delta x)^2(\Delta y)^2 f_{i,j}\right] \tag{16}$$

where

$$f_{i,j} = \frac{g\beta K}{\nu}\frac{T^{n-\frac{1}{2}}_{i,j+1} - T^{n-\frac{1}{2}}_{i,j-1}}{2\delta y} \tag{17}$$

We will then calculate $T^{n+1/2}$ from the second equation using ψ^n and a timestep of $dt/2$ in the Alternating Direction Implicit (ADI) scheme.
The equation is discretized by

$$\frac{T^{n+1/2} - T^n}{\Delta t/2} = \frac{1}{2\sigma}\left(-\frac{\partial \psi^n}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \psi^n}{\partial x}\frac{\partial}{\partial y} + \alpha_m\frac{\partial^2}{\partial x^2} + \alpha_m\frac{\partial^2}{\partial y^2}\right)(T^{n+1/2} + T^n) \tag{18}$$

This is then split into two separate equations, each implicit in one direction and explicit in the other

$$\frac{T^{n+1/4} - T^n}{\Delta t/4} = \frac{1}{2\sigma}\left(-\frac{\partial \psi^{n+1/4}}{\partial y}\frac{\partial}{\partial x} + \alpha_m\frac{\partial^2}{\partial x^2}\right)T^{n+1/4} + \frac{1}{2\sigma}\left(\frac{\partial \psi^n}{\partial x}\frac{\partial}{\partial y} + \alpha_m\frac{\partial^2}{\partial y^2}\right)T^n \tag{19}$$

$$\frac{T^{n+1/2} - T^{n+1/4}}{\Delta t/4} = \frac{1}{2\sigma}\left(-\frac{\partial \psi^{n+1/4}}{\partial y}\frac{\partial}{\partial x} + \alpha_m\frac{\partial^2}{\partial x^2}\right)T^{n+1/4} + \frac{1}{2\sigma}\left(\frac{\partial \psi^{n+1/2}}{\partial x}\frac{\partial}{\partial y} + \alpha_m\frac{\partial^2}{\partial y^2}\right)T^{n+1/2} \tag{20}$$

Discretizing the spatial derivatives using 2nd order central differences, we have

$$(1+2H_x)T^{n+1/4}_{i,j} + (-U^n_{i,j} - H_x)T^{n+1/4}_{i-1,j} + (U^n_{i,j} - H_x)T^{n+1/4}_{i+1,j} = (1-2H_y)T^n_{i,j} + (V^n_{i,j} + H_y)T^n_{i,j-1} + (-V^n_{i,j} + H_y)T^n_{i,j+1} \tag{21}$$

$$(1+2H_y)T^{n+1/2}_{i,j} + (-V^n_{i,j} - H_y)T^{n+1/2}_{i,j-1} + (V^n_{i,j} - H_y)T^{n+1/2}_{i,j+1} = (1-2H_x)T^{n+1/4}_{i,j} + (U^n_{i,j} + H_x)T^{n+1/4}_{i-1,j} + (-U^n_{i,j} + H_x)T^{n+1/4}_{i+1,j} \tag{22}$$

Where

$$H_x = \frac{\alpha_m \Delta t/4}{2\sigma \Delta x^2} \qquad H_y = \frac{\alpha_m \Delta t/4}{2\sigma \Delta y^2} \tag{23}$$

$$U^n_{i,j} = \frac{u^n_{i,j}\Delta t/4}{4\sigma\Delta x} \qquad V^n_{i,j} = \frac{-v^n_{i,j}\Delta t/4}{4\sigma\Delta y} \tag{24}$$

So to solve for $T^{n+1/2}$ we solve the first set of equations for all j, then the second set of equations for all i.

Written as matrices for an arbitrary timestep δ, this means solving firstly

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ -U^n_{2,j}-H_x & 1+2H_x & U^n_{2,j}-H_x & 0 & 0 & \dots & 0 \\ 0 & -U^n_{3,j}-H_x & 1+2H_x & U^n_{3,j}-H_x & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & -U^n_{N_x-1,j}-H_x & 1+2H_x & U^n_{N_x-1,j}-H_x \\ 0 & \dots & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T^{n+\delta/2}_{1,j} \\ T^{n+\delta/2}_{2,j} \\ T^{n+\delta/2}_{3,j} \\ \vdots \\ T^{n+\delta/2}_{N_x,j} \end{pmatrix} \tag{25}$$

$$= \begin{pmatrix} T^n_{1,j} \\ (1-2H_y)T^n_{2,j} + (V^n_{2,j}+H_y)T^n_{2,j-1} + (-V^n_{2,j}+H_y)T^n_{2,j+1} \\ (1-2H_y)T^n_{3,j} + (V^n_{3,j}+H_y)T^n_{3,j-1} + (-V^n_{3,j}+H_y)T^n_{3,j+1} \\ \vdots \\ T^n_{N_x,j} \end{pmatrix} \tag{26}$$

for $j = 2...(N_y-1)$, and then solving

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ -V^n_{i,2}-H_y & 1+2H_y & V^n_{i,2}-H_y & 0 & 0 & \dots & 0 \\ 0 & -V^n_{i,3}-H_y & 1+2H_y & V^n_{i,3}-H_y & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & -V^n_{i,N_y-1}-H_y & 1+2H_y & V^n_{i,N_y-1}-H_y \\ 0 & \dots & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T^{n+\delta}_{i,1} \\ T^{n+\delta}_{i,2} \\ T^{n+\delta}_{i,3} \\ \vdots \\ T^{n+\delta}_{i,N_y} \end{pmatrix} \tag{27}$$

$$= \begin{pmatrix} T^n_{i,1} \\ (1-2H_x)T^{n+\delta/2}_{i,2} + (U^n_{i,2}+H_x)T^{n+\delta/2}_{i-1,2} + (-U^n_{i,2}+H_x)T^{n+\delta/2}_{i+1,2} \\ (1-2H_x)T^{n+\delta/2}_{i,3} + (U^n_{i,3}+H_x)T^{n+\delta/2}_{i-1,3} + (-U^n_{i,3}+H_x)T^{n+\delta/2}_{i+1,3} \\ \vdots \\ T^n_{i,N_y} \end{pmatrix} \tag{28}$$

for $i = 2...(N_x-1)$. This then gives us $T^{n+1/2}$.
Thirdly, we calculate $\psi^{n+\frac{1}{2}}$ from the poisson equation using SOR again.
Finally we calculate T^{n+1} with a timestep dt, using $\psi^{n+\frac{1}{2}}$ for the velocity and using exactly the same method as the second step.

3.2 Boundary Conditions

We require boundary conditions on ψ and T. Some of these are given by Nield & Bejan, the rest are chosen ourselves.

On the wall

$$y = 0 \qquad T = T_\infty + Ax \tag{29}$$

$$\frac{\partial \psi}{\partial x} = 0 \implies \psi = \text{const} = 0 \tag{30}$$

On the left

$$y = \infty \qquad T = T_\infty \qquad \frac{\partial \psi}{\partial y} = 0 \tag{31}$$

At the bottom

$$x = 0 \qquad T = T_\infty \qquad \psi = 0 \tag{32}$$

At the top

$$x = \infty \qquad T = T_{analytic}(x,y) \qquad \psi = \psi_{analytic}(x,y) \tag{33}$$

Where the analytic solution is given by

$$\psi(\eta) = \alpha_m \sqrt{Ra_x} f(\eta) \tag{34}$$

$$T(\eta) = Ax\theta(\eta) + T_\infty \tag{35}$$

$$\tag{36}$$

where we have defined

$$\eta = y\sqrt{\frac{g\beta K}{\nu\alpha_m}} \tag{37}$$

$$Ra_x = \frac{g\beta K}{\nu\alpha_m} Ax^2 \tag{38}$$

Such that, in terms of x and y,

$$\psi(x,y) = \alpha_m \sqrt{\frac{g\beta K}{\nu \alpha_m}} x f(\eta(y)) \quad (39)$$

$$T(x,y) = Ax\theta(\eta(y)) + T_\infty \quad (40)$$

The functions $f(\eta)$, and $\theta(\eta)$ are the solutions of the differential equations

$$f'' - \theta' = 0 \quad (41)$$

$$\theta'' + f\theta' - f'\theta = 0 \quad (42)$$

$$f(0) = 0, \qquad \theta(0) = 1 \quad (43)$$

$$f'(\infty) = 0, \qquad \theta(\infty) = 0 \quad (44)$$

equation (41) can be integrated, with the boundary conditions, to give

$$f' = \theta \quad (45)$$

leaving just

$$f''' + ff'' - (f')^2 = 0 \quad (46)$$

to be solved.

This is done numerically in matlab. An initial attempt was made using the bvp4c routine, but it couldn't handle the boundaries at infinity so we convert the boundary value problem to an initial value problem. We already know that

$$f(0) = 0, f'(0) = 0 \quad (47)$$

and we then calculate $f''(0)$ by solving (46) using the Mathematica function NDSolve. We find that $f''(0) = -1$. With this, we can re-formulate (46) as

$$f = f_3, f' = f_2, f'' = f_1 \quad (48)$$

$$f_1' = f_3^2 - f_3 f_1 \quad (49)$$

$$f_2' = f_1 \quad (50)$$

$$f_3' = f_2 \quad (51)$$

$$f_1(0) = -1, f_2(0) = 1, f_3(0) = 0 \quad (52)$$

Integrating this using ode45 gives us a numerical solution for f, which is plotted in figure 3.2. Satisfied that this agrees with the $\lambda = 1$ case presented in figure 5.1 of Nield & Bejan, we apply the calculated solutions at the top boundary and solve numerically for ψ and T everywhere else. From this solution for $f(\eta) and \theta(\eta)$ we can plot the 'analytic' solution in figure 3.2 Note that we have defined $\gamma = \frac{g\beta K}{\nu}$.

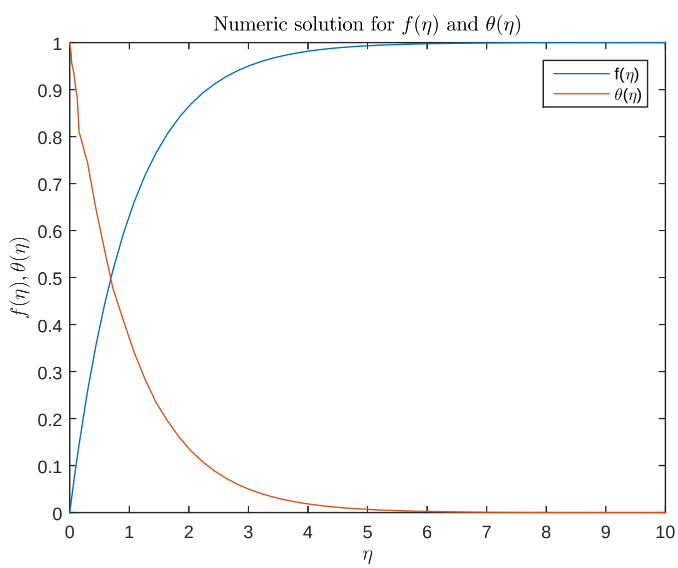

Figure 3:

3.3 Domain

From the solutions obtained for $f(\eta)$ and $\theta(\eta)$, we find that $\eta_{max} = 15$ is a good approximation to the behaviour at infinity. Subsituting this into (37) gives the extent of the grid in terms of y.

The grid size in the x dimension is set to the same value for no good reason.

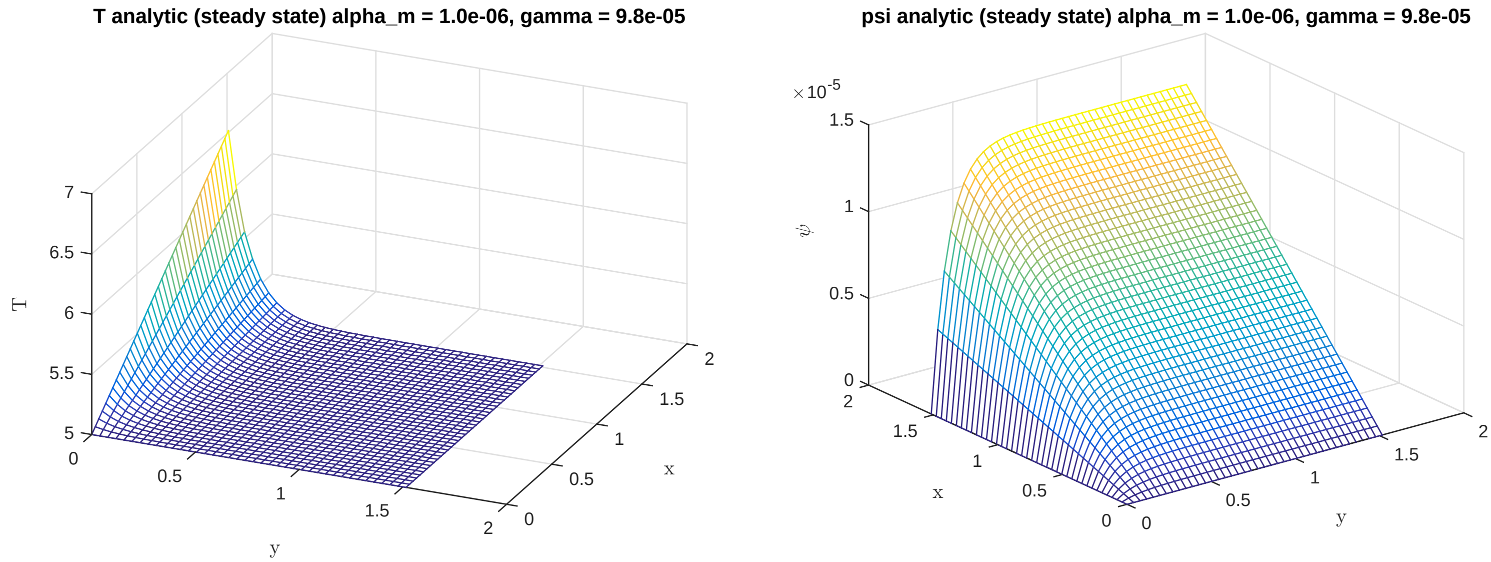

Figure 4: Analytic solutions for T and ψ. Note the axes ares rotated differently in the two graphs.

3.4 Testing the code

Firstly, we fix T at the analytic steady state value and calculate ψ, the error between this and the analytic value of ψ is small and decreases with the size of a grid square, confirming that the code correctly solves the poisson equation.

Next, we fix ψ to the analytic steady state solution and calculate the velocities from it, comparing these two those derived from the analytic solution for ψ

$$\psi = \sqrt{\frac{g\beta K}{\nu}\alpha_m} x f(\eta) \tag{53}$$

$$u = \frac{\partial \psi}{\partial y} = x\sqrt{\frac{g\beta K}{\nu}\alpha_m}\frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}f(\eta) \tag{54}$$

$$= x\sqrt{\frac{g\beta K}{\nu}\alpha_m}\sqrt{\frac{g\beta K}{\nu\alpha_m}}\theta(\eta) \tag{55}$$

$$= x\frac{g\beta K}{\nu}\theta(\eta) \tag{56}$$

$$v = -\frac{\partial \psi}{\partial x} = -\sqrt{\frac{g\beta K}{\nu}\alpha_m} f(\eta) \tag{57}$$

These are all calculated correctly, with the exception of the values at the boundary where the finite difference method is only first order. However these values aren't used, so this isn't an issue.

Finally we calculate T from analytic values of ψ and velocity, and this works correctly too.

Putting everything together,

4 The Horton-Rogers-Lapwood problem

As a final test of the cartesian code, I will calculate the nusselt number for a flow in a square domain $(0 < x < L, 0 < z < 1)$ satisfying

$$\frac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \frac{1}{R_m}\nabla^2\theta \tag{58}$$

$$\frac{\partial \theta}{\partial x} = -\nabla^2\psi \tag{59}$$

with boundary conditions

$$u = \psi_z = 0 \text{ at } x = 0, L \tag{60}$$

$$w = -\psi_x = 0 \text{ at } z = 0, 1 \tag{61}$$

$$\theta = \begin{cases} 0 & : z = 0 \\ 1 & : z = 1 \end{cases} \tag{62}$$

$$\theta_x = 0 \text{ at } x = 0, L \tag{63}$$

The first two are satisfied by setting $\psi = 0$ along all the boundaries. The third is easily introduced by fixing the values at the edge of the grid. The last is satisfied by writing

$$\theta_{1,j} - \theta_{2,j} = 0 \implies \theta_{1,j} = \theta_{2,j} \tag{64}$$

$$\theta_{N_x,j} - \theta_{N_x-1,j} = 0 \implies \theta_{N_x,j} = \theta_{N_x-1,j} \tag{65}$$

and substituting this into the discretization of the heat equation.

Linear stability analysis predicts that there should be an onset of convection at $4\pi^2$ for $L = 2$, and this is observed in the model.