1 Eliminating the chimney boundary

The momentum equation is

$$(\psi_r/r)_r = \begin{cases} \frac{rR_m}{2D_a}(p_z + \Theta^*) & 0 < r < a \\ -R_m\theta_r & a < r < b \end{cases} \tag{1}$$

And the jump conditions at $r = a$ are

$$[\psi(a)] = 0 \qquad [\psi_r(a)] = 0 \tag{2}$$

I have already integrated this for $0 < r < a$, and found

$$\psi(a) = \frac{a^4}{16Da}\left(\frac{\psi_r}{a} + R_m(\theta - \Theta^*)\right) + \frac{1}{2}a\psi_r(a) \tag{3}$$

Integrating for $a < r < 2$ gives

$$\psi_r = -rR_m(\theta - A) \tag{4}$$

Where A is an integration constant. To integrate again we make the 0th order approximation that $\theta(r) \approx \theta(b)$ for $a < r < b$ and find

$$\begin{aligned}
\psi_r(b) &= -bR_m(\theta(b) - A) \implies A = \frac{1}{bR_m}\psi_r(b) + \theta(b) && (5)\\
\psi_r(r) &= \frac{r}{b}\psi_r(b) && (6)\\
\psi(r) &= \frac{r^2}{2b}\psi_r(b) + B && (7)\\
&= \frac{1}{2}r\psi_r(r) + B && (8)
\end{aligned}$$

Applying the jump conditions, and using $\theta(r) \approx \theta(b) \implies \theta(a) \approx \theta(b)$ gives

$$\begin{aligned}
\psi^+(a) &= \psi^-(a) && (9)\\
\frac{1}{2}a\psi_r^+(a) + B &= \frac{a^4}{16Da}\left(\frac{\psi_r^-(a)}{a} + R_m(\theta(a) - \Theta^*)\right) + \frac{1}{2}a\psi_r^-(a) && (10)\\
&= \frac{a^4}{16Da}\left(\frac{\psi_r^+(a)}{a} + R_m(\theta(b) - \Theta^*)\right) + \frac{1}{2}a\psi_r^+(a) && (11)\\
B &= \frac{a^4}{16Da}\left(\frac{\psi_r(b)}{b} + R_m(\theta(b) - \Theta^*)\right) && (12)\\
\psi^+(r) &= \frac{1}{2}r\psi_r(r) + \frac{a^4}{16Da}\left(\frac{\psi_r(b)}{b} + R_m(\theta(b) - \Theta^*)\right) && (13)\\
\psi(b) &= \frac{1}{2}b\psi_r(b) + \frac{a^4}{16Da}\left(\frac{\psi_r(b)}{b} + R_m(\theta(b) - \Theta^*)\right) && (14)
\end{aligned}$$

If we had instead made the 1st order approximation $\theta(r) = \theta(b) - (b - r)\theta_r(b)$ we would have found

$$\begin{aligned}
\psi_r(r) &= -rR_m\left[\theta(b) - (b - r)\theta_r(b) - A\right] && (15)\\
\psi_r(b) &= -bR_m(\theta(b) - A) \implies A = \theta(b) + \frac{\psi_r(b)}{R_m b} && (16)\\
\psi_r(r) &= rR_m\left[(b - r)\theta_r(b) + \frac{\psi_r(b)}{R_m b}\right] && (17)\\
\psi_r(r) &= r\left[R_m b\theta_r(b) + \frac{\psi_r(b)}{b}\right] - r^2R_m\theta_r(b) && (18)\\
\psi(r) &= \frac{1}{2}r^2\left[R_m b\theta_r(b) + \frac{\psi_r(b)}{b}\right] - \frac{1}{3}r^3R_m\theta_r(b) + B && (19)\\
&= \frac{1}{2}r\psi_r(r) + \frac{1}{6}r^3R_m\theta_r(b) + B && (20)
\end{aligned}$$

And then $\theta(a) \approx \theta(b) - (b-a)\theta_r(b)$ so

$$\psi^+(a) = \psi^-(a) \tag{21}$$

$$\frac{1}{2}a\psi_r^+(a) + \frac{1}{6}a^3 R_m \theta_r(b) + B = \frac{a^4}{16Da}\left(\frac{\psi_r^-(a)}{a} + R_m(\theta(a) - \Theta^*)\right) + \frac{1}{2}a\psi_r^-(a) \tag{22}$$

$$\frac{1}{6}a^3 R_m \theta_r(b) + B = \frac{a^4}{16Da}\left(\frac{\psi_r^-(a)}{a} + R_m(\theta(a) - \Theta^*)\right) \tag{23}$$

$$B = \frac{a^4}{16Da}\left(R_m\left[(b-a)\theta_r(b) + \frac{\psi_r(b)}{R_m b}\right] + R_m\left[\theta(b) - (b-a)\theta_r(b) - \Theta^*\right]\right) - \frac{1}{6}a^3 R_m \theta_r(b) \tag{24}$$

$$= \frac{a^4 R_m}{16Da}\left(\frac{\psi_r(b)}{R_m b} + \theta(b) - \Theta^*\right) - \frac{1}{6}a^3 R_m \theta_r(b) \tag{25}$$

$$\psi(r) = \frac{1}{2}r\psi_r(r) + \frac{1}{6}r^3 R_m \theta_r(b) + \frac{a^4 R_m}{16Da}\left(\frac{\psi_r(b)}{R_m b} + \theta(b) - \Theta^*\right) - \frac{1}{6}a^3 R_m \theta_r(b) \tag{26}$$

$$\psi(b) = \frac{1}{2}b\psi_r(b) + \frac{a^4}{16Da}\left(\frac{\psi_r(b)}{b} + R_m\left[\theta(b) - \Theta^*\right]\right) + \frac{1}{6}(b^3 - a^3)R_m \theta_r(b) \tag{27}$$