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Morse Functions and Birth-Death Bifurcations

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eingereicht von:	Johannes Radu Hübers
geboren am:	13.08.2000
geboren in:	Berlin
Gutachter:	Prof. Chris Wendl
	Prof. Thomas Walpuski

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Contents

1 Introduction

Morse theory is the study of Morse functions and how they mirror a manifolds topological properties. A Morse function is a differentiable real-valued function on a manifold which only has non-degenerate critical points. The simplest Morse functions are smooth functions $f : M \to \mathbf{R}$ defined on a differentiable submanifold of $\mathbf{R}^n$ that assign each point $x \in M$ a "height" $f(x)$ above some plane in $\mathbf{R}^n$. It is unsurprising that such height functions encode much of the shape of the manifold, and then so do general Morse functions. They can for example be used to find CW structures, handle decompositions and define a homology theory on the underlying manifold. Despite their usefulness, C^k Morse functions are generic in the sense that they form an open and dense subset of $C^k(M)$. However, it is not always possible to find a homotopy $\{f_s\}_{s\in[0,1]}$ connecting two C^k Morse functions f_0 and f_1 such that each intermediary function f_t is C^k Morse itself.

In [Cer70] Jean Cerf proved a theorem that partially rectifies this defect. He showed that on a closed manifold two Morse functions can always be connected by a C^k homotopy in such a way that, except for finitely many times $t_1, \ldots, t_N$, all intermediary functions f_t are Morse themselves. In fact, there doesn't just exist one such homotopy but rather, it is the case that generic C^k homotopies have this property. Moreover, the non-Morse intermediary functions f_{t_i} each have just one degenerate critical point and at this point the homotopy has a birth-death bifurcation. At a birth bifurcation, as t increases, a critical point appears out of nothing, is "born", splitting into two non-degenerate critical points. At a death bifurcation two non-degenerate critical points collide, momentarily becoming one degenerate critical point and then disappearing, or "dying". Finally, the two born or dying critical points have Morse indices k and $k+1$. This has applications in Kirby calculus

The main goal of this thesis is to explain a proof of Cerf's theorem.

In chapter 2, we introduce the basics of Morse theory and everything else that needs to be known to understand the statement of Cerf's theorem. This includes the definition of birth-death bifurcations and a topology on a space of differentiable functions.

Chapter 3 gives basic definitions of concepts of differentiability on Banach spaces and Banach manifolds as the proof of Cerf's theorem requires these. Especially the implicit function theorem for Banach spaces will be useful. In the end we discuss Stephen Smale's generalization of Sard's theorem, which will be the most important tool in proving genericity.

Chapter 4 is the central part of this thesis. It is dedicated entirely to the proof of Cerf's theorem, using the concepts introduced in the preceding chapters. First it will be proved that the universal spaces of critical points $\mathcal{M}(\mathcal{P}^m)$ and $\mathcal{M}(\mathcal{P}^m; k)$ are Banach submanifolds of a space $[0,1] \times M \times \mathcal{P}^m$, where $\mathcal{P}^m$ is the set of all C^m homotopies between two Morse functions f_0 and f_1. These spaces contain all critical points for all intermediary functions for all possible homotopies. After proving this, we will show that for generic homotopies the space of critical points of all intermediary functions is a one-dimensional manifold, and the space of degenerated critical points is a zero-dimensional manifold. In particular, as $[0, 1] \times M$ is compact, this means that there are only finitely many non-Morse intermediary functions. Then it will be proved that for generic homotopies there is, at any time, at most one degenerate critical points. Finally, it is proved that these degeneracies are all birth-death bifurcations that the Morse indices of pairs of born or dying critical points are k and $k + 1$.

2 Morse functions, birth-death bifurcations and Cerf's theorem

This chapter introduces everything that's needed to understand the main theorem of this thesis. This includes the basics of Morse theory. Throughout, M is a smooth manifold of dimension n.

2.1 The natural linearization at a zero and the Hessian of a function

Let $E \to M$ be a vector bundle and $\sigma : M \to E$ a differentiable section of it.

Lemma 1. *At each zero $x \in M$ of σ, there exists a natural linearization*

$$D\sigma(x) : T_xM \to E_x, \quad v \mapsto D\sigma(x)v$$

of σ independent of any choices. It is the same as the tangent map $T_x\sigma$ when trivializing E in a neighborhood U around x and thus replacing σ by a map $U \to E_x$.

Proof. Recall that, for two covariant derivatives $D^{(i)}$, $D^{(ii)}$,

$$\Gamma(E) \times \Gamma(TM) \to \Gamma(E), \quad (\sigma, v) \mapsto D^{(i)}_v\sigma - D^{(ii)}_v\sigma$$

is tensorial not only in the vector field v but also in the section σ. This means that, when $\sigma(x) = 0$,

$$D^{(i)}_v\sigma(v) = D^{(ii)}_v\sigma(v). \tag{1}$$

Let $x \in M$ with $\sigma(x) = 0$ and suppose there is a trivialization of E on U around x, identifying σ with a map $\sigma : U \to E_x$. The tangent map is tensorial in the vector field and linear in the section. That is, for functions f, g, scalars a, b and functions $\sigma, \alpha, \beta : U \to E_x$, it holds that

$$T\sigma(fv + gw) = fT\sigma(v) + gT\sigma(w), \qquad T(a\alpha + b\beta)(v) = aT\alpha(v) + bT\beta(v).$$

This makes it a covariant derivative on the trivial vector bundle $E_x \times U \to U$ and, because the trivialization $E|_U \to E_x \times U$ is linear on fibers, it also defines a covariant derivative on $E|_U$. As, by 1, the linearization $D\sigma(x)$ is the same for all covariant derivatives, it follows that the natural linearization is equal to the tangent map in the trivialization. □

We now use the natural linearization to define the Hessian of a function. Let M be a Riemannian manifold and $f : M \to \mathbf{R}$ be a C^2 function on M. At a critical point $x \in M$, i.e. a point with $\nabla f(x) = 0$, the Hessian of f is defined as the natural linearization of the gradient vector field, $D(\nabla f)(x) : T_xM \to T_xM$. One useful property of the Hessian is that it is symmetric:

$$g(D(\nabla f)(x)v, w) = g(v, D(\nabla f)(x)w) \tag{2}$$

The Hessian is also often defined differently, for example as the second covariant derivative in which case it is a (outside critical points non-symmetric) bilinear form instead of an endomorphism (a $(0,2)$-tensor instead of a $(1,1)$-tensor). In this thesis however, it is regarded as an endomorphism.

2.2 Basic Morse theory

This section is a glimpse into Morse theory, the study of Morse functions on manifolds, and the topological information they encode. Proofs of most statements are omitted but can be found in [Mil73]. Throughout, M is a smooth manifold equipped with a Riemannian metric (although Morse functions can also be defined without the metric so it is just for convenience).

Definition 1. A Morse function on a smooth manifold M is a C^2 function $f : M \to \mathbf{R}$ for which every critical point is non-degenerate This means that at each $x \in M$ for which $\nabla f(x) = 0$, the Hessian $D(\nabla f)(x)$ is invertible.

Now, the Hessian being invertible might not be a very comprehensible criterion. The Morse lemma helps illustrate what's so special about Morse functions: At every critical point there exist coordinates in which they take a simple polynomial form:

Lemma 2 (Morse lemma)**.** *Suppose that $f : M \to \mathbf{R}$ has a non-degenerate critical point at $x_0 \in M$. Then there exist coordinates $\{x_k\}_{k=1}^n$ on a neighborhood U of x_0 in which f takes the form*

$$f(x) = f(x_0) - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2.$$

Here i is the number of negative eigenvalues of the Hessian at x_0, called the index of f at x_0. It varies from critical point to critical point.

Equivalently, the index at x_0 is the dimension of largest subspace of $T_{x_0}M$ on which $D(\nabla f_t)$ is negative definite.

It is easy to see that the given definition of Morse functions is equivalent to saying that a Morse function is a function whose derivative df is transverse to the zero-section in T^*M. This also makes it clear that the set $\mathrm{Crit}(f)$ of critical points of f is a codimension-n, i.e. 0-dimensional, submanifold of M. In other words, the critical points of a Morse function are isolated. Thus, Morse functions on compact manifolds have only finitely many critical points.

One interesting thing about Morse functions is that they define CW structures and handle decompositions for closed manifolds, which is useful for investigating its homotopy type. The connection between Morse functions and homotopy type is made clear in the following two basic theorems of Morse theory, whose proof can be found in [Mil73] §3. For every "height" $h \in \mathbf{R}$ the cell or handle decomposition of the part of M below this height, $M^h = \{x \in M \mid f(x) \leq h\}$ changes when h traverses a critical value. Precisely, an i-cell or i-handle is attached to $M^{h-\epsilon}$ when the critical point has index i.

Theorem 3. *Let $f : M \to \mathbf{R}$ be a Morse function on a manifold M. Suppose that $[a, b]$ contains no critical values of f and $f^{-1}[a, b]$ is compact. Then M^a and M^b are diffeomorphic.*

Theorem 4. *Let $f : M \to \mathbf{R}$ be Morse and $x \in M$ a (non-degenerate) critical point of f with index i. If, for $h = f(x)$ and some $\epsilon > 0$, $f^{-1}[h - \epsilon, h + \epsilon]$ is compact and contains no other critical points, then $M^{h+\epsilon}$ is homotopy equivalent to $M^{h-\epsilon}$ with an i-cell attached.*

On a closed manifold, a Morse function with all its critical points on different levels then generates a CW structure on M.

2.3 Homotopies of Morse functions and birth-death bifurcations

This thesis is concerned mainly with homotopies between Morse functions. For functions $f_0 : M \to \mathbf{R}$ and $f_1 : M \to \mathbf{R}$, a homotopy is a continuous map $H : I \times M \to \mathbf{R}$ ($I = [0, 1]$) such that $H(0, \cdot) = f_0$ and $H(1, \cdot) = f_1$. H is called a C^k homotopy if it is C^k as a map $I \times M \to \mathbf{R}$. We will mostly denote homotopies as families of functions,

$$H = \{f_s\}_{s \in I} = \{f_s\}, \quad \text{where for all } t \in I : \quad f_t = H(t, \cdot) \in C(M).$$

An important fact of Morse theory ([Hir76], p. 147) is that C^k Morse functions form an open and dense set in the vector space $C^k(M)$ when it is equipped with the right topology, meaning that Morse functions are the norm rather than the exception. One may then hope that one can find, for any pair of Morse functions f_0, f_1, a C^k homotopy $\{f_s\}$ connecting them that is itself Morse in the sense that

$$\forall t \in I : f_t \in C^k(M) \text{ is Morse.}$$

The following lemma implies that this is not generally the case:

Lemma 5. *Let $f_0, f_1 \in C^k(M)$ be Morse functions defined on a closed manifold M. If $\{f_s\}$ is a C^k Morse homotopy between f_0 and f_1, then f_0 and f_1 have the same number of critical points.*

The problem now is that Morse functions defined on the same manifold may have different amounts of critical points. Consider for example

$$f_k : S^1 = \mathbf{R}/2\pi\mathbf{Z} \to \mathbf{R}, \quad [\theta] \mapsto \sin(k\theta), \quad k = 1, 2, 3, \ldots$$

f_1 has two critical points, f_2 has four critical points, and so on. So no two different Morse functions in this family can be connected by a C^k Morse homotopy.

The lemma's proof use the evolution space of critical points

$$\mathcal{M}(\{f_s\}) := \{(t, x) \in I \times M \mid \nabla f_t(x) = 0\}.$$

This space encodes the evolution of critical points of f_t as t varies and it will be one of the primary objects of chapter 4. An illustration of this space for a particular homotopy can be found a few pages ahead in this section.

Throughout the proof, open intervals are to be understood as their intersection with I.

Proof of lemma 5. Pull back the vector bundle $TM \to M$ by $\pi_M : I \times M \to M$ to get a vector bundle $E \to I \times M$. Define the section

$$\sigma : I \times M \to E, \quad \sigma(t,x) = \nabla f_t(x).$$

Note that its zero set is exactly $\mathcal{M}(\{f_s\})$. In a local trivialization of E around (t,x), σ may be identified with a function with values in $E_{(t,x)} = T_x M$. Its derivative is then

$$D\sigma(t,x) : \mathbf{R} \oplus T_x M \to T_x M, \quad D\sigma(t,x) = D\sigma(t,x)|_{\mathbf{R}} + D\sigma(t,x)|_{T_x M}$$

where $D\sigma(t,x)|_{T_x M} = D(\nabla f_t)(x)$ is invertible because f_t is Morse. By the implicit function theorem, there exist neighborhoods $(t-\epsilon, t+\epsilon) \subset I$ of t and $U \subset M$ of x such that $\mathcal{M}(\{f_s\}) \cap (-\epsilon, \epsilon) \times U$ can be C^k parametrized by t. (This also proves that $\mathcal{M}(\{f_s\})$ is a submanifold of $I \times M$.)

Now, let $t_0 \in I$ and set $c = \#\mathrm{Crit}(f_{t_0})$. Our goal is to show that $\{t \in I : \#\mathrm{Crit}(f_t) = c\} = I$. Let $x_1(t_0), \ldots, x_c(t_0)$ be the critical points of f_{t_0} and $x_i : (t_0 - \epsilon_i, t_0 + \epsilon_i) \to U_i$ be local parametrizations of $\mathcal{M}(\{f_s\})$ around them, as per the implicit function theorem. Let

$$N = \left(\mathcal{M}(\{f_s\}) \setminus \bigcup_{i=1}^{c} (-\epsilon_i, \epsilon_i) \times U_i\right)^c.$$

This is a neighborhood of $\{t_0\} \times M$ in $I \times M$. As $\{t_0\} \times M$ is compact, there exists an open tube $T = (t_0 - \delta, t_0 + \delta) \times M$ around it that is contained in N. By definition of N, for all $t \in (t_0 - \delta, t_0 + \delta)$, the critical points $x \in M$ of f_t, i.e. points for which $(t,x) \in \mathcal{M}(\{f_s\})$, are contained in one of the neighborhoods $(t_0 - \epsilon_i, t_0 + \epsilon_i) \times U_i$, where they are parametrized as $x_i(t)$. So, the critical points of f_t are exactly $x_i(t)$, $i = 1, \ldots, c$, which is just as many as at t_0, so the set $\{t \in I : f_t \text{ has exactly } c \text{ critical points}\}$ is open. On the other hand, its complement

$$\bigcup_{d \in \mathbf{N}_0 : d \neq c} \{t \in I : f_t \text{ has exactly } d \text{ critical points}\}$$

is, as the union of sets which are open by the same argument, also open, and thus it is closed. As I is connected, we are done. □

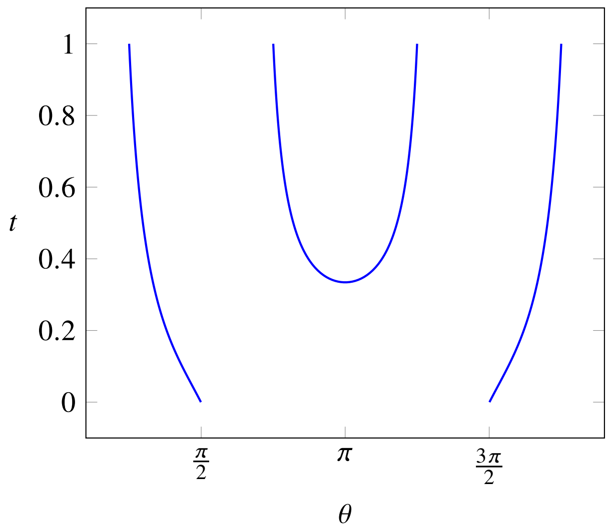
In the proof, the obstruction to Morse functions with different numbers of critical points having a C^k Morse homotopy was that the projection $\mathcal{M}(\{f_s\}) \to I$, $(t,x) \mapsto t$ is a submersion for all C^k Morse homotopies $\{f_s\}$. So, for a C^k homotopy between two Morse functions with differently many critical points, this projection will invariably fail to be submersive. As an example, consider the Morse functions

$$f_0, f_1 : S^1 \to \mathbf{R}, \quad f_0([\theta]) = \sin(\theta),\ f_1([\theta]) = \sin(2\theta)$$

and the C^k homotopy $\{f_s\}$, where

$$f_t([\theta]) = (1-t)f_0([\theta]) + t f_1([\theta]).$$

Its evolution space of critical points $\mathcal{M}(\{f_s\})$ is graphed below.



We see that f_0 has only two critical points, that then move smoothly as t grows. At a time $t_0 \approx 0.35$, there suddenly appears (is "born") a third critical point x_0, splitting into two critical points right away. The four critical points that exist now continue moving smoothly with t. What happened at (t_0, x_0) is called a birth bifurcation. Looking at the projection $\mathcal{M}(\{f_s\}) \to \mathbf{R}$, $(t, x) \mapsto t$, while (t_0, x_0) is a critical point of this projection, its second derivative is likely non-zero.

Definition 2. A C^k homotopy $\{f_s\} : I \times M \to \mathbf{R}$ has a birth-death bifurcation at an event $(t_0, x_0) \in I \times M$ if

$$(t_0, x_0) \in \mathcal{M}(\{f_s\}) = \{(t, x) \in \mathbf{R} \times M \mid \nabla f_t(x) = 0\}$$

and (t_0, x_0) has a neighborhood U in $\mathcal{M}(\{f_s\})$ that is a 1-dimensional submanifold of $I \times M$ and for which the projection $U \to \mathbf{R}$, $(t, x) \mapsto t$ has first derivative zero and second derivative non-zero at (t_0, x_0). In case the second derivative is positive, the bifurcation is a birth, if it is negative, it is a death.

The main theorem of this thesis, first proved by Jean Cerf in [Cer70], says that for any two C^m Morse functions f_0, f_1 there exists a comeager set of C^m homotopies $\{f_s\}$ which

– by the preceding discussion – may not be Morse but are at least regular in the sense that their only failures to be Morse are birth-death bifurcations.

Theorem 6 (Main theorem, [Cer70])**.** *Let M be a closed smooth manifold and f_0, f_1 : $M \to \mathbf{R}$ be two C^m, $m = 3, 4, \ldots, \infty$, Morse functions. Then there exists a comeager subset $\mathcal{P}^m_{\text{reg}} \subset \mathcal{P}^m$ of C^m homotopies $\{f_s\}$ for which all critical points are non-degenerate with the exception of finitely many birth-death bifurcations $(t_1, x_1), \ldots, (t_N, x_N)$ satisfying $0 < t_1 < \cdots < t_N < 1$. At each birth-death bifurcation a pair of critical points with Morse indices k and $k+1$ ($k \in \{0, \ldots, n-1\}$) is either created or annihilated.*

Some applications of Cerf's theorem Theorem 6 can be used to show that for every manifold M^n the number

$$\sum_{k=0}^{n} (-1)^k \# \operatorname{Crit}_k(f) \tag{3}$$

defined for a Morse function $f : M \to \mathbf{R}$, where $\# \operatorname{Crit}_k(f)$ is the number of critical points with Morse index k of f, does not depend on the particular Morse function. In other words, it is an invariant for the manifold. The idea of the proof is very simple. For all Morse functions f_0 and f_1 there exists a C^k homotopy $\{f_s\} \in \mathcal{P}^m_{\text{reg}}$. This homotopy is Morse everywhere except for finitely many t. When it is Morse, the non-degeneracy of the critical points implies with the implicit function theorem that the critical points x_i can be parametrized differentiably by t. The index of a t-parametrized critical point is constant. At degenerate times, there is exactly one birth-death bifurcation. There, two critical points either die or are born. Their indices are k and $k+1$, so in either case, as the sum is alternating, it will not change.

Arguably the most important application of Cerf's theorem, and more generally Cerf theory, is in the Kirby calculus. There it is used to prove that any two handle decompositions for a smooth closed manifold are related by finitely many cancellations of two handles, creations of two handles, and isotopies of the attaching maps (see e.g. [Kir89]). The reason is that every smooth handle decomposition corresponds to a Morse function. Then, given a generic homotopy connecting the Morse functions, a birth bifurcation of $\{f_s\}$ corresponds to the creation of two handles and a death bifurcation to a handle cancellation.

2.4 The topology on the space of homotopies $\mathcal{P}^m$

The main theorem (6) is a statement about the topological properties (comeagerness) of a subset $\mathcal{P}^m_{\text{reg}}$ of the infinite-dimensional space of functions

$$\mathcal{P}^m = \{\{f_s\} \in C^m(I \times M) \text{ matching } f_0, f_1\} \subset C^m(I \times M).$$

In order to make sense of this, it first needs to be clear what topology $\mathcal{P}^m$ is equipped with. The topology in question is the subspace topology induced by the C^m topology on $C^m(I \times M)$.

There are multiple ways of defining this topology, however, as $I \times M$ is compact, one of them is particularly useful: At first, let $m < \infty$. Covering $I \times M$ by finitely many coordinate charts $\phi_i : U_i \to V_i \subset \mathbf{R}^{n+1}$, we can define for $H \in C^m(I \times M)$,

$$\|H\|_{C^m(I\times M)} = \sum_i \|H \circ \phi_i^{-1}\|_{C^m(V_i)},$$

where $\|\cdot\|_{C^m(V_i)}$ is the usual C^m-norm on $C^m(V_i)$. One can show that all norms defined this way are equivalent and make $C^m(I\times M)$ a Banach space. This is, however, only a norm when the underlying manifold is compact. In the non-compact case similar topologies can be defined using subbases or jets. For this, see [Hir76].

We always consider $C^m(I \times M)$ equipped with the C^m topology and $\mathcal{P}^m$ with the subspace topology. Now, $\mathcal{P}^m$ is no vector space, however it is an affine space over the vector space

$$\mathcal{P}_0^m := \{\{g_s\} \in C^m(I \times M) \mid g_0 = g_1 = 0\}.$$

In particular, it is an infinite-dimensional Banach manifold as will be defined in section 3.2.

The inclusions $C^m(I \times M) \hookrightarrow C^k(I \times M)$ are continuous for all $k \le m < \infty$. This also holds for $m = \infty$, as we endow $C^\infty(I \times M)$ with the topology induced by the inclusions

$$C^\infty(I \times M) \hookrightarrow C^k(I \times M).$$

Unlike in the case $m < \infty$, $C^\infty(I \times M)$ is no Banach space, and neither is $\mathcal{P}_0^\infty$, so $\mathcal{P}^\infty$ is also no Banach manifold. This will be a problem in the proof in chapter 4, where the Sard–Smale theorem can then only be applied in the case $m < \infty$. We will need to extend the statements to the smooth case by using the following lemma whose proof depends on the fact that $\mathcal{P}^\infty$ is dense in $\mathcal{P}^k$ for all k.

Lemma 7. *For every $k \in \mathbf{N}_0$, if $O^k \subset \mathcal{P}^k$ is open and dense in $\mathcal{P}^k$, then $O^\infty = O^k \cap \mathcal{P}^\infty$ is open and dense in $\mathcal{P}^\infty$.*

Proof. For each l, the inclusion $i_l : \mathcal{P}^l \hookrightarrow \mathcal{P}^\infty$ induces the subspace topology $\mathcal{T}^l$ on $\mathcal{P}^\infty$. Observe that

$$\mathcal{T}^0 \subset \mathcal{T}^1 \subset \mathcal{T}^2 \subset \dots$$

The topology on $\mathcal{P}^\infty$ is defined as the smallest for which all of these inclusions are continuous. This means that $O^k \cap \mathcal{P}^\infty = i_k^{-1}(O^k)$ is open in $\mathcal{P}^\infty$. To check that it is also dense, note that the topology on $\mathcal{P}^\infty$ is induced by the basis $\mathcal{B} = \bigcup_{l=k}^\infty \mathcal{T}^l$. To check that O^∞ is dense in $\mathcal{P}^\infty$, it is then enough to show that O^∞ intersects every basis set. Hence, let $U^\infty \in \mathcal{T}^l$, $l \ge k$. By the definition of $\mathcal{T}^l$ there exists an open $U^l \subset \mathcal{P}^l$ for which $U^\infty = U^l \cap \mathcal{P}^\infty$. The inclusion $\mathcal{P}^k \hookrightarrow \mathcal{P}^l$ is continuous because $k \le l$, so $O^k \cap U^l$ is open in $\mathcal{P}^l$. As $\mathcal{P}^\infty$ is dense in $\mathcal{P}^l$, it intersects every open subset of $\mathcal{P}^l$, in particular $O^k \cap U^l$. Thus

$$O^\infty \cap U^\infty = (O^k \cap \mathcal{P}^\infty) \cap (U^l \cap \mathcal{P}^\infty) = \mathcal{P}^\infty \cap (O^k \cap U^l) \neq \emptyset,$$

showing that O^∞ intersects every set in $\mathcal{B}$, and is thus dense in $\mathcal{P}^\infty$. $\square$

A proof of the density of $C^\infty(I \times M)$ in $C^k(I \times M)$ can be found in [Hir76], theorem 3.3. Only little modification is needed to apply the proof to $\mathcal{P}^\infty \subset \mathcal{P}^k$ as well.

3 Differential calculus on Banach spaces, Banach manifolds and the Sard–Smale theorem

This chapter introduces concepts of differentiability for maps between Banach spaces and Banach manifolds. The chapter follows [Lan93] and [Lan01] closely. The last section is concerned with the Sard–Smale theorem, our number one tool in proving that something is comeager.

3.1 Differentiable maps between Banach spaces

In typical analysis courses the concepts of differentiability and derivative are only defined for functions between finite-dimensional real or complex vector spaces. However the definition of total derivative, that is, the derivative as a linear map approximating the function near a point, can be extended to infinite-dimensional Banach spaces without much effort. Recall that a Banach space is a complete normed vector space over the field of the real or complex numbers. In this thesis we will disregard the complex case, the field of scalars will always be $\mathbf{R}$.

Throughout this section, E, F and G will be Banach spaces. The set of bounded (continuous) linear maps $E \to F$ will be denoted $L(E, F)$. Equipped with the operator norm, it is itself a Banach space.

Definition 3. Let U be an open subset of E. A map $f : U \to F$ is differentiable at a point $x_0 \in U$ if there exists a bounded linear map $Df(x_0) : E \to F$ such that

$$\lim_{v \to 0} \frac{f(x_0 + v) - f(x_0) - Df(x_0)v}{\|v\|} = 0.$$

Then $Df(x_0)$ is called the derivative of f at x_0. A differentiable map $f : U \to F$ is a map which is differentiable at every point in U.

Because many Banach manifolds we encounter in the proof have boundary it is also necessary to introduce Banach half-spaces: A Banach half-space in E is a set $E^+_\lambda = \lambda^{-1}[0, \infty)$ where $\lambda : E \to \mathbf{R}$ is a non-zero bounded linear functional. Differentiable maps can also be defined between half-spaces:

Definition 4. Let U be an open subset of a half-space E^+. A map $f : U \to F$ to a Banach space is differentiable if, for every $x \in U$, there exist an open neighborhood N of x in the full Banach space E and a differentiable map

$$g : N \to F$$

such that f and g coincide on $U \cap N$. The derivative of f at x is then defined as the derivative of g at x

$$Df(x) := Dg(x) \in L(E, F).$$

This definition of the derivative is independent of the choice of extension g. Note that the derivative $Df(x)$ is itself an element of a Banach space $L(E, F)$. Thus, for differentiable f, consider the map

$$Df : U \to L(E, F), \quad x \mapsto Df(x).$$

If Df is continuous, f is said to be continuously differentiable. Of course, Df might itself be differentiable. Its derivative is then a map

$$D^2 f : U \to L(E, L(E, F)),$$

the second derivative of f. Higher order derivatives are defined inductively by $D^k f = D(D^{k-1} f)$. A map $f : U \to F$ for which the derivatives $Df, D^2 f, \ldots, D^{k-1} f, D^k f$ all exist and are continuous is said to be k times continuously differentiable, and the set of these maps is denoted by $C^k(U, F)$. Observe that the space $L(E, L(E, F))$ is naturally isomorphic to the space of bilinear maps $E \times E \to F$, so the second derivative $D^2 f(x_0)$ may also be regarded an element of this space, $L^2(E, F)$. Analogously, one can consider $D^k f$ to take its values in $L^k(E, F)$ the Banach space of k-linear maps $E \times \cdots \times E \to F$.

Conveniently, most of the usual properties of derivatives between finite-dimensional spaces also hold true with Banach spaces. In particular, for C^k maps $f : F \supset V \to G$ and $g : E \supset U \to F$, the composition $f \circ g : U \to G$ is also of class C^k and satisfies the chain rule

$$D(f \circ g)(x) = Df(g(x)) \circ Dg(x) \in L(E, G).$$

From this one immediately deduces that (C^k) diffeomorphisms, i.e. bijective differentiable (C^k) functions $U \to V$ with differentiable (C^k) inverse, have as derivatives Banach space isomorphisms.

3.2 Banach manifolds

Using the differential calculus on Banach spaces we can now define – potentially infinite-dimensional – Banach manifolds, manifolds modeled on Banach spaces instead of $\mathbf{R}^n$. The proof of the main theorem (6) will make heavy use of infinite-dimensional manifolds, in particular the affine space of all $\mathcal{P}^m = \{\{f_s\} \in C^m(I \times M) \mid \{f_s\} \text{ matches } f_0 \text{ and } f_1\}$ of all homotopies between f_0 and f_1, and the universal space of critical points

$$\mathcal{M}(\mathcal{P}^m) := \{(t, x, \{f_s\}) \in I \times M \times \mathcal{P}^m \mid \nabla f_t(x) = 0\},$$

the "disjoint union" of all the evolution spaces $\mathcal{M}(\{f_s\})$ of critical points. This section introduces some basic definitions. They are all analogous to those from finite-dimensional differential geometry.

The first definitions are those of the C^k atlas, a set of coordinate charts for a space, and of the C^k manifold. As before, a half-space of a Banach space E is a set $E^+ = \lambda^{-1}[0, \infty)$ where $\lambda : E \to \mathbf{R}$ is a non-zero bounded linear functional.

Definition 5. Let X be a topological space and E^+ be a Banach half-space. An atlas of class C^k, on X, $k \geq 0$, is a cover of X by charts $\phi_\alpha : U_\alpha \to E^+$, $\alpha \in A$, with C^k overlap. This is to say that

1. each U_α is an open subset of X and $\bigcup_{\alpha \in A} U_\alpha = X$,
2. each map $\phi_\alpha : U_\alpha \to E^+$ is a homeomorphism onto an open subset $\phi_\alpha(U_\alpha)$ of the half-space,
3. for all $\alpha, \beta \in I$ the transition map $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is of class C^k as a map between open subsets of the half-space.

Definition 6. A C^k Banach manifold modeled on E is a topological space X together with a C^k structure. A C^k structure is an equivalence class of C^k atlases on X, where atlases $\{\phi_\alpha : U_\alpha \to E^+\}_{\alpha \in A}$ and $\{\psi_\beta : U_\beta \to E^+\}_{\beta \in B}$ are equivalent if all their charts are compatible, meaning that, for all $\alpha \in A$ and $\beta \in B$, the transition map

$$\psi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta)$$

is of class C^k as a map between open subset of Banach half-spaces.

Of course, finite-dimensional manifolds are also Banach manifolds in this sense, modeled on the Banach space $\mathbf{R}^n$. An open subset of a Banach manifold is obviously itself a Banach manifold modeled on the same space, and so are products $X_1 \times X_2$ of Banach manifolds, then modeled on the direct sum $E_1 \oplus E_2$ of the model spaces.

Submanifolds are defined as in the finite-dimensional case, by means of slice charts:

Definition 7. Let X be a C^k Banach manifold where $k \geq 0$. A subset $Y \subset X$ for which each point $y \in Y$ has an open neighborhood U such that there exists a C^k-diffeomorphism $\varphi : U \to V_1 \times V_2$, where V_1 is an open subset of a Banach half-space E_1^+ and, if $y \in \partial X$, V_2 is an open subset of E_2^+, and if $y \notin \partial X$, V_2 is an open subset of E_2, satisfying

$$\varphi(x) \in V_1 \times \{0\} \iff x \in Y$$

is called a C^k submanifold of X. Its codimension is defined to be $\operatorname{codim} Y = \dim E_2$.

If $V_2 \subset E_2^+$, then the definition of differentiability has to be extended to products of Banach half-spaces, just like it has been extended from full spaces to half-spaces.

Every submanifold inherits a C^k structure from its ambient manifold, making it a manifold in its own right, modeled on the Banach half-space E_1^+.

Differentiable maps are also defined as for finite-dimensional manifolds:

Definition 8. A map $f : X \to Y$ between Banach manifolds modeled on E and F is C^k if for all charts $\phi : U \to E$ and all charts $\psi : V \to F$ the map

$$\psi \circ f \circ \phi^{-1} : \phi(U \cap V) \to F$$

is of class C^k.

For differentiable Banach manifolds, *tangent vectors* at a point $x \in X$ can be defined as equivalence classes of tuples (ϕ, v) where $\phi : U_\phi \to E^+$ is a chart at x and $v \in E$ is a vector. Two such tuples (ϕ, v) and (ψ, w) are equivalent, if $D(\phi \circ \psi^{-1})(\phi(x))v = w$. The *tangent space* to x, T_xX is then the set of all tangent vectors to x. Each chart ϕ then determines the structure of a vector space on T_xX by declaring $E \to T_xX$, $v \mapsto [(\phi, v)]$ an isomorphism. As the transition maps $\phi \circ \psi^{-1}$ are diffeomorphisms and thus their derivatives at x isomorphisms of E, this structure is independent of the choice of chart.

For a differentiable map $f : X \to Y$ between Banach manifolds, its *tangent map* at x is the linear map $T_xf : T_xX \to T_{f(x)}Y$ defined by

$$T_xf([\phi, v]) = [\psi, D(\psi \circ f \circ \phi^{-1})(x)v],$$

where ϕ is a chart around x and ψ is a chart around $f(x)$.

We'll also frequently make use of vector bundles, vector spaces at each point of the manifold that vary differentiably with the point. The definition is not very different from the one for finite-dimensional manifolds. The most important example of a vector bundle is the tangent bundle $TX = \bigsqcup_{x \in X} T_xX$ of a Banach manifold X. The tangent map $Tf : TX \to TY$ of a C^k map $f : X \to Y$, defined by $Tf(x, v) = (f(x), T_xf(v))$ for $(x, v) \in TX$, is itself a map of class C^{k-1} between Banach manifolds.

Definition 9. Let X be a C^k Banach manifold. A C^k vector bundle over X is a C^k Banach manifold V together with a C^k projection $p : V \to X$ such that for all $x \in X$

$$V_x := p^{-1}(x)$$

is equipped with the structure of a Banach space, and a set of compatible local trivializations whose domains cover X. Compatible local trivializations over open U_1, $U_2 \subset X$ are C^k-diffeomorphisms

$$\tau_i : p^{-1}(U_i) \to U_i \times V_0, \quad i = 1, 2,$$

where V_0 is a Banach space, such that

1. for the projection $\pi_{U_i} : U_i \times V_0 \to U_i$ it holds that $\pi_2 \circ \tau = p$,
2. at every $x \in U_i$, $\pi_{V_0} \circ \tau|_{p^{-1}(x)}$ is a Banach space isomorphism $V_x \to V_0$,
3. the map $U_1 \cap U_2 \to L(V_0, V_0), x \mapsto \tau_2 \circ \tau_1^{-1}(x, \cdot)$ is C^k.

Just like finite-dimensional manifolds in section 2.1, differentiable sections of vector bundles over Banach spaces have a unique natural linearization at each critical point which is identical to the tangent map in any trivialization.

Finally, it should be mentioned that the Banach manifolds that occur in this thesis, $\mathcal{P}^m$ and various submanifolds of $I \times M \times \mathcal{P}^m$, $3 \leq m < \infty$, will all be second-countable. This is important for applying the Sard–Smale theorem.

3.3 Inverse and implicit function theorems for Banach spaces

These theorems' finite-dimensional versions are introduced in typical analysis lectures and can be generalized to Banach manifolds. The inverse function theorem says that if the derivative of a map is surjective, then the map is a diffeomorphism locally. A proof can be found in [Lan93], pp. 361–363.

Theorem 8 (Inverse function theorem)**.** *Let E and F be Banach spaces and $f : U \to F$ a C^k, $k \geq 1$, function on an open subset U of E. If, at some $x \in E$, its derivative $Df(x) : E \to F$ is a Banach space isomorphism, then there exist open neighborhoods U of x and V of $f(x)$ such that $f|_U : U \to V$ is a C^k-diffeomorphism.*

The implicit function theorem for Banach spaces is used throughout chapter 4. Its proof, which is not discussed here but can be found in [Lan93], pp. 364–365, builds upon the inverse function theorem.

Theorem 9 (Implicit function theorem)**.** *Let U and V be open subsets of Banach spaces E and F respectively. If for a C^k map $f : U \times V \to G$*

$$Df(x_0, y_0)|_F \in L(F, G)$$

is an isomorphism at a zero $f(x_0, y_0) = 0$, then there exist open neighborhoods U_0 of x_0 and V_0 of y_0 and a C^k function $g : U_0 \to V_0$ satisfying for $(x, y) \in U_0 \times V_0$ that

$$f(x, y) = 0 \iff y = g(x).$$

Throughout the proof in chapter 4 we will often need to show that certain spaces are (Banach) manifolds. This will most often be done by defining a finite-rank vector bundle $V \to X$ over some ambient Banach manifold X such that the set in question is exactly the zero set of a C^k section σ of V. Then the following version of the implicit function theorem can prove that $\sigma^{-1}(0)$ is a submanifold of X:

Corollary 10. *Let X be a Banach manifold (with boundary) modeled on the Banach space E. Let $\pi : V \to X$ be a finite-rank vector bundle and $\sigma : X \to V$ be a C^k-smooth section. Suppose that the zero set $Z := \sigma^{-1}(0)$ satisfies that for each point $z \in Z$ the natural linearization $D\sigma(z) : T_z X \to V_z$ is surjective. Then Z is a C^k submanifold of X. Furthermore, its tangent space at z is*

$$T_z Z = \ker D\sigma(z)$$

and its codimension is $\operatorname{codim} Z = \operatorname{rk} V$.

Proof. As the submanifold property is local it suffices to prove this for the special case where X is an open subset of E. Then we may also assume that $V \to X$ is trivial, replacing the pullback bundle V by a vector space V and the section σ by a C^k function $\sigma : X \to V$. We know that the natural linearization of a vector bundle section is the same as the ordinary derivative of the corresponding function in a trivialization, so we know that the derivative $D\sigma(z_0) : E \to V$ is surjective for $z_0 \in Z = \sigma^{-1}(0)$. Then $D\sigma(z_0)$ descends to an isomorphism $E/E_1 \cong V$ where $E_1 = \ker D\sigma(z_0)$. As V is finite-dimensional, E_1 has finite codimension. Because, as the kernel of a continuous map, E_1 is also closed, it is complemented by some subspace $E_2 \subset E$. From now on we regard E as the sum $E_1 \oplus E_2$, denoting vectors as pairs (x, y). Then

$$D\sigma(x_0, y_0)|_{E_2} : E_2 \to V$$

is an isomorphism, so the implicit function theorem can be applied to find open neighborhoods $U_0 \ni x_0$, $V_0 \ni y_0$ and a C^k map $g : U_0 \to V_0$ that satisfies

$$\forall (x, y) \in U_0 \times V_0 : \quad f(x, y) = 0 \iff y = g(x).$$

Out of this map we can construct the slice chart

$$\phi : U_0 \times V_0 \to E_1 \oplus E_2, \quad \phi(x, y) = \begin{pmatrix} x \\ y - g(x) \end{pmatrix}.$$

In matrix form (where the ij-th entry is a linear map $E_j \to E_i$), its derivative is

$$D\phi(x_0, y_0) = \begin{pmatrix} \frac{D}{Dx} x & \frac{D}{Dy} x \\ \frac{D}{Dx}(y - g(x)) & \frac{D}{Dy}(y - g(x)) \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_{E_1} & 0 \\ -\frac{D}{Dx} g(x) & \mathrm{Id}_{E_2} \end{pmatrix}$$

which is clearly an isomorphism, hence, by the inverse function theorem, ϕ is a C^k-diffeomorphism. Now observe that for $(x, y) \in U_0 \times V_0$

$$\phi(x, y) \in E_1 \times \{0\} \iff y = g(x) \iff (x, y) \in Z,$$

which proves that Z is a C^k-submanifold. Its codimension must be $\operatorname{codim} Z = \dim E_2 = \dim V$ and its tangent space is

$$T_{(x_0,y_0)} Z = D\phi(x_0, y_0)^{-1}(E_1 \oplus \{0\}).$$

If for $(v, w) \in E_1 \oplus E_2$, $D\phi(x_0, y_0)(v + w) \in E_1 \oplus \{0\}$, then it is clear from the matrix representation above that $Dg(x_0)v = w$. Now, applying the chain rule with the function $(\mathrm{id}_{E_1}, g) : E_1 \to E_1 \times E_2$, $x \mapsto (x, g(x))$ which satisfies $\sigma \circ (\mathrm{id}_{E_1}, g) = 0$, we get

$$\begin{aligned} D\sigma(x_0, y_0) \begin{pmatrix} v \\ w \end{pmatrix} &= D\sigma(x_0, y_0) D(\mathrm{id}_{E_1}, g)(x_0) v \\ &= D(\sigma \circ (\mathrm{id}|_{E_1}, g)) v = 0, \end{aligned}$$

meaning that $T_{(x_0,y_0)} Z \subset \ker D\sigma(x_0, y_0)$. As $\operatorname{codim} T_{(x_0,y_0)} Z = \operatorname{codim} \ker D\sigma(x_0, y_0) < \infty$, it follows that these are actually identical. □

3.4 Sard's theorem for Banach manifolds

Theorem 6 is a genericity result. It proposes that homotopies $\{f_s\} \in \mathcal{P}^m$ with certain desirable properties are typical in a sense. Concretely, in chapter 4 the set of these homotopies will be proved comeager in $\mathcal{P}^m$. This will be accomplished by (repeatedly) applying the Sard–Smale theorem, a generalization of Sard's theorem to Banach manifolds. This section introduces concepts of genericity from general topology and the Sard–Smale theorem.

Definition 10.

- A subset Y of a topological space X is said to be *nowhere dense* if its closure has empty interior.
- The subset Y is *meager* if it is the union of countably many nowhere dense sets.
- A *comeager* subset of X is the complement to a subset, or equivalently, it contains the intersection of countably many open and dense sets.

Sard's theorem says that for certain functions $f : \mathbf{R}^n \to \mathbf{R}^m$ the critical values form a set of measure zero. Specifically the theorem is concerned with C^k functions for which the regularity k is greater than $n - m$ and 0. Smale's generalization of Sard's theorem is a similar statement about maps between Banach manifolds. However, these may be infinite-dimensional, in which case $n - m$ will no longer be well-defined. Conveniently,

$$n - m = \dim\ker L + \dim\operatorname{im} L - m = \dim\ker L - \dim\operatorname{coker} L$$

holds for all linear $L : \mathbf{R}^n \to \mathbf{R}^m$ and $\dim\ker L - \dim\operatorname{coker} L$ can be a well-defined expression even when L is a map between infinite-dimensional Banach spaces. This *Fredholm index* replaces $n - m$ in the Sard–Smale theorem.

Definition 11. A *Fredholm map* is a differentiable function $f : X \to Y$ between two Banach manifolds whose tangent map $T_x f$ at each point is a Fredholm operator between the Banach spaces $T_x X$ and $T_{f(x)} Y$, which is to say that $\ker T_x f$ and $\operatorname{coker} T_x f$ are finite-dimensional. The *index* of f at x is then defined

$$\operatorname{ind}_x f := \operatorname{ind} T_x f := \dim\ker T_x f - \dim\operatorname{coker} T_x f \in \mathbf{Z}.$$

If the manifold X is connected, the index is independent of x. Now we can give the statement of the Sard–Smale theorem. It was first proved in 1965 by Stephen Smale ([Sma65]).

Theorem 11 (Sard–Smale theorem, [Sma65]). *Let X and Y be second-countable Banach manifolds. Let $f : X \to Y$ be a C^k Fredholm map with $k > \max(\operatorname{ind} f, 0)$. Then the set of regular values of f is comeager in Y.*

In chapter 4, the theorem will be applied to the projection

$$\pi : \mathcal{M}(\mathcal{P}^m) \to \mathcal{P}^m$$

after the universal space of critical points $\mathcal{M}(\mathcal{P}^m) = \{t, x, \{f_s\}) \in I \times M \times \mathcal{P}^m \mid \nabla f_t(x) = 0\}$ has been proven a Banach manifold. The set of regular values of π is thus comeager. The Sard–Smale theorem will be applied to other such projections more often, each time producing a new comeager subset of $\mathcal{P}^m$. Taking the intersection of all these sets – which is also comeager – we then obtain the set $\mathcal{P}^m_{\text{reg}}$ that the main theorem 6 proposes exists.

4 Proof of the main theorem

4.1 Overview

Throughout this chapter, M will be a closed smooth n-dimensional manifold equipped with a Riemannian metric. f_0 and f_1 will be C^m Morse functions. Our objective is to prove theorem 6, the main theorem of this thesis. Let us first recall its statement:

Theorem 6 (Main theorem, [Cer70])**.** *Let M be a closed smooth manifold and f_0, f_1 : $M \to \mathbf{R}$ be two C^m, $m = 3, 4, \ldots, \infty$, Morse functions. Then there exists a comeager subset $\mathcal{P}^m_{\mathrm{reg}} \subset \mathcal{P}^m$ of C^m homotopies $\{f_s\}$ for which all critical points are non-degenerate with the exception of finitely many birth-death bifurcations $(t_1, x_1), \ldots, (t_N, x_N)$ satisfying $0 < t_1 < \cdots < t_N < 1$. At each birth-death bifurcation a pair of critical points with Morse indices k and $k+1$ ($k \in \{0, \ldots, n-1\}$) is either created or annihilated.*

For convenience, we list the main spaces that will be used in the proof.

- The space $\mathcal{P}^m$ of C^m-smooth homotopies between f_0 and f_1,

  $$\mathcal{P}^m := \{\{h_s\} \in C^m([0,1] \times M) \mid h_0 = f_0 \text{ and } h_1 = f_1\}.$$

  For $m < \infty$ this is a Banach manifold as it is an affine space over the Banach space

  $$\mathcal{P}^m_0 := \{\{g_s\} \in C^m(I \times M) \mid g_0 = g_1 = 0\}.$$

- For homotopies $\{f_s\} \in \mathcal{P}^m$ the evolution space

  $$\mathcal{M}(\{f_s\}) = \{(t, x) \in I \times M : \nabla f_t(x) = 0\}$$

  of critical points and

  $$\mathcal{M}(\{f_s\}; k) = \{(t, x) \in I \times M : \nabla f_t(x) = 0\},$$

  the space of critical points of of degeneracy $\dim\ker D(\nabla f_t)(x) = k$.

- The "universal spaces" of critical points (of degeneracy $\dim\ker D(\nabla f_t)(x) = k$), i.e. the spaces of all critical points (of index k) for all homotopies:

  $$\mathcal{M}(\mathcal{P}^m) = \{(t, x, \{f_s\}) \in I \times M \times \mathcal{P}^m : \nabla f_t(x) = 0\},$$
  $$\mathcal{M}(\mathcal{P}^m; k) = \{(t, x, \{f_s\} \in \mathcal{M}(\mathcal{P}^m) \mid \dim\ker D(\nabla f_t)(x) = k\}.$$

The proof will happen in multiple steps, each contained in one section of the chapter:

First, in lemmas 12 and 13, we will prove that $\mathcal{M}(\mathcal{P}^m)$ and $\mathcal{M}(\mathcal{P}^m; k)$ are C^{m-1} and respectively C^{m-2} Banach submanifolds of $I \times M \times \mathcal{P}^m$.

In the proof of proposition 16 the Sard–Smale theorem will be applied to the total spaces $\mathcal{M}(\mathcal{P}^m)$ and $\mathcal{M}(\mathcal{P}^m; k)$, finding that their sets of regular values are comeager in the space of homotopies $\mathcal{P}^m$. It follows then that for such regular values $\{f_s\} \in \mathcal{P}^m$, the evolution space of critical points, $\mathcal{M}(\{f_s\})$, forms a 1-dimensional submanifold of spacetime $I \times M$. Similarly, $\mathcal{M}(\{f_s\}; 1)$ is a 0-dimensional submanifold, i.e. a finite set, and $\mathcal{M}(\{f_s\}; k)$ is empty for $k \geq 2$, so the only failures of generic homotopies to be Morse at all times are in $\mathcal{M}(\{f_s\}; 1)$. $\mathcal{M}(\{f_s\}; 1)$ being 0-dimensional means, as $I \times M$ is compact, that there are only finitely many points in this space and hence also only finitely many times at which $\{f_s\}$ fails to be Morse.

One can restrict the set of regular values $\{f_s\}$ more such that it remains comeager but additionally the events in $\mathcal{M}(\{f_s\}; 1)$ (x is a degenerate point of $D(\nabla f_t)(x)$) occur at different times, i.e. at any time t there is at most one point that keeps f_t from being Morse. We will later deduce from this that there are never more than one birth-death bifurcations at the same time. This proposition (16) is again proved by showing some universal space is a Banach manifold and applying the Sard–Smale theorem to its projection onto $\mathcal{P}^m$.

The last two sections, 4.5 and 4.6, connect everything to the birth-death bifurcations introduced in section 2.3: First, in proposition 21 we prove that events in $\mathcal{M}(\{f_s\}; 1)$ are exactly birth-death bifurcations. And last, we prove in proposition 22 that the two born or dying critical points at this birth-death bifurcation have Morse indices k and $k+1$ for some $k = 0, \ldots, n-1$.

4.2 The Banach manifolds $\mathcal{M}(\mathcal{P}^m)$ and $\mathcal{M}(\mathcal{P}^m; k)$

Lemma 12. *For all $2 \leq m < \infty$, the space*

$$\mathcal{M}(\mathcal{P}^m) := \{(t, x, \{f_s\}) \in I \times M \times \mathcal{P}^m \mid \nabla f_t(x) = 0\}$$

is a codimension-n, C^{m-1} Banach submanifold of $I \times M \times \mathcal{P}^m$.

The proof of this lemma is simply an application of the implicit function theorem for Banach spaces. Note that we restricted to the case $m < \infty$. This is necessary because $\mathcal{P}^m_0$ is not a Banach space for $m = \infty$, so $I \times M \times \mathcal{P}^m$ is not a Banach manifold.

Proof. As a product of Banach manifolds (one with boundary, two without), $I \times M \times \mathcal{P}^m$ is a Banach manifold with tangent space

$$T_{(t,x,\{f_s\})}(I \times M \times \mathcal{P}^m) \cong \mathbf{R} \oplus T_x M \oplus \mathcal{P}^m_0.$$

We pull back $TM \to M$ by the projection $I \times M \times \mathcal{P}^m \to M$ to obtain a vector bundle $E \to I \times M \times \mathcal{P}^m$ with fibers $E_{(t,x,\{f_s\})} = T_x M$ of rank n. Then $\mathcal{M}(\mathcal{P}^m)$ is precisely the zero set $\sigma^{-1}(0)$ of the C^{m-1}-smooth section

$$\sigma : I \times M \times \mathcal{P}^m \to E, \quad (t, x, \{f_s\}) \mapsto \nabla f_t(x).$$

One now only needs to show that the linearization $D\sigma$ is surjective at any point in $\sigma^{-1}(0)$, then the implicit function theorem (version 10) implies that $\mathcal{M}(\mathcal{P}^m)$ is a submanifold of smoothness C^{m-1} and codimension n.

To this end, let $(t^0, x^0, \{f_s^0\}) \in \sigma^{-1}(0)$. As f_0 and f_1 are Morse, for $t^0 = 0$ and $t^0 = 1$, the Hessian $D(\nabla f_{t^0})(x) : T_{x^0}M \to T_{x^0}M$ is invertible. It is a restriction of the natural linearization $D\sigma(t^0, x^0, \{f_s^0\}) : \mathbf{R} \oplus T_{x^0}M \to T_{x^0}M$, so this linearization is also surjective. Thus, we may now assume that $0 \neq t_0 \neq 1$. Let $v \in T_xM$ be arbitrary. By choosing coordinates around x^0 in which we define $g(x) = v \cdot x$ and extending g to the rest of M (for example by multiplying with a smooth bump function that is 1 in a neighborhood of x^0 and supported in the coordinate chart) we find a function

$$g \in C^m(M) \quad \text{with} \quad \nabla g(x^0) = v.$$

Let $b : [0,1] \to [0,1]$ be a smooth bump function with $b(0) = b(1) = 0$ and $b(t^0) = 1$. Set $g_s(x) := b(s)g(x)$. Then $\{g_s\} \in \mathcal{P}_0^m$ and consequently $r \mapsto (t^0, x^0, \{f_s^0\} + r\{g_s\}) \in I \times M \times \mathcal{P}^m$ is a smooth path. Clearly,

$$\frac{d}{dr}\Big|_{r=0} \sigma(x^0, t^0, \{f_s^0 + rg_s\}) = \frac{d}{dr}\Big|_{r=0} \nabla(f_t^0 + rg_t)(x) = \nabla g_t(x) = v$$

which shows that the linearization $D\sigma(t^0, x^0, \{f_s^0\})$ is surjective. □

The next lemma illuminates the structure of $\mathcal{M}(\mathcal{P}^m)$: It shows that $\mathcal{M}(\mathcal{P}^m)$ can be partitioned into disjoint spaces $\mathcal{M}(\mathcal{P}^m; k)$, $k = 0, \dots, n$, each a submanifold of it. For $k > 0$, elements $(t, x, \{f_s\}) \in \mathcal{M}(\mathcal{P}^m; k)$ have that x is a degenerate critical point of f_t, meaning that f_t is not Morse. The positive codimension of $\mathcal{M}(\mathcal{P}^m; k)$ shows that such degenerate elements are unusual in $\mathcal{M}(\mathcal{P}^m)$.

Lemma 13. *For each $k = 1, 2, 3, \dots$, the space*

$$\mathcal{M}(\mathcal{P}^m; k) := \{(t, x, \{f_s\}) \in \mathcal{M}(\mathcal{P}^m) \mid \dim\ker D(\nabla f_t)(x) = k\}$$

is a Banach submanifold of $\mathcal{M}(\mathcal{P}^m)$ of smoothness C^{m-1}. Its codimension is $\frac{k(k+1)}{2}$.

The subsequent proof will use a lemma that shows that the spaces of linear maps of the same rank are submanifolds of $L(V, W)$ for V and W finite-dimensional. To prepare, let V and W be finite-dimensional real vector spaces and $A \in L(V, W)$ a linear map between them. Setting

$$V_2 := \ker A \quad \text{and} \quad W_1 := \operatorname{im} A$$

we can define splittings of V and W: $V = V_1 \oplus V_2$, $W = W_1 \oplus W_2$. In these splittings A can be written in matrix form

$$\begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad A_{11} \in L(V_1, W_1) \text{ an isomorphism.}$$

As the map $\pi : L(V, W) \to L(V_1, W_1)$, defined in matrix form by

$$\pi : \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \mapsto B_{11}, \quad \text{where} \quad B_{ij} \in L(V_j, W_i),$$

is continuous and the set of isomorphisms $V_1 \to W_1$ is open in $L(V_1, W_1)$, there exists a neighborhood $U \subset L(V, W)$ of A such that, with respect to the splittings, all $B \in U$ have the matrix form

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad B_{ij} \in L(V_j, W_i),$$

with B_{11} bijective. In particular, A has a neighborhood of linear maps B which all have $\operatorname{rk} B \geq \operatorname{rk} A$.

Lemma 14. *The map*

$$\Phi : U \to L(V_2, W_2), \quad B \mapsto B_{22} - B_{21} B_{11}^{-1} B_{12}$$

satisfies $\ker B \cong \ker \Phi(B)$ *and* $\operatorname{coker} B \cong \operatorname{coker} \Phi(B)$. *From this it follows that*

$$\dim \ker B = \dim \ker A \iff \Phi(B) = 0 \iff \dim \operatorname{coker} B = \dim \operatorname{coker} A.$$

Proof. V being the direct sum of V_1 and V_2 means that for any $v \in V$ there is a unique decomposition $v = v_1 + v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$. We will prove that the projection $\pi_2 : v \mapsto v_2$ defines an isomorphism $\ker B \cong \ker \Phi(B)$. Suppose $v \in \ker B$. Then

$$0 = Bv = (B_{11} v_1 + B_{12} v_2) + (B_{21} v_1 + B_{22} v_2)$$

from which we deduce

$$B_{12} v_2 = -B_{11} v_1 \quad \text{and} \quad B_{22} v_2 = -B_{21} v_1. \tag{4}$$

It follows that $v_2 \in \ker \Phi(B)$:

$$\Phi(B) v_2 = B_{22} v_2 - B_{21} B_{11}^{-1} B_{12} v_2 = -B_{21} v_1 + B_{21} B_{11}^{-1} B_{11} v_1 = 0$$

Injectivity: $\pi_2(v) = 0$ just means $v_2 = 0$. Equation 4 implies that

$$-B_{11} v_1 = B_{12} v_2 = B_{12} 0 = 0.$$

It follows from the invertibility of B_{11} that $v_1 = 0$, whence the projection $\pi_2|_{\ker B}$ is injective. To see that $\pi_2|_{\ker B} : \ker B \to \ker \Phi(B)$ is surjective, suppose $v_2 \in \ker \Phi(B) \subset V_2$. Then set

$$v_1 := -B_{11}^{-1} B_{12} v_2 \in V_1, \quad v := v_1 + v_2$$

and notice that $Bv = -B_{21}B_{11}^{-1}B_{12}v_2 + B_{22}v_2 = \Phi(v_2) = 0$. Thus the projection $V \to V_2$ defines an isomorphism $\ker B \cong \ker \Phi(B)$. By the rank-nullity theorem the isomorphism $\operatorname{coker} B \cong \operatorname{coker} \Phi(B)$ holds as well:

$$\begin{aligned}
\dim \operatorname{coker} B &= \dim W - \dim \operatorname{im} B \\
&= \dim W - \dim V + \dim \ker B \\
&= \dim W - \dim \operatorname{im} A - \dim \ker A + \dim \ker \Phi(B) \\
&= (\dim W - \dim W_1) - (\dim V_2 - \dim \ker \Phi(B)) \\
&= \dim W_2 - \dim \operatorname{im} \Phi(B).
\end{aligned}$$

It then follows that

$$\dim \ker B = \dim \ker A \iff \dim \ker \Phi(B) = \dim \ker A \iff \Phi(B) = 0$$

as $\ker A$ is the domain of $\Phi(B)$. By the rank-nullity theorem, this is clearly also equivalent to $\dim \operatorname{coker} B = \dim \operatorname{coker} A$. □

Remark 15. In this thesis, V will usually be the tangent space to the Riemannian manifold M, which is an inner product space. A will usually be the Hessian of a function, a symmetric endomorphism. In this case, Φ can be defined instead on an open neighborhood U of A in the (smaller) space of symmetric endomorphisms $\mathrm{End}_{\mathrm{sym}}(V)$. Then you also have that $V_2 = \ker A$ and $W_1 = \operatorname{im} A$ are orthogonal which means that we can do the above construction with identical splittings

$$V = V_1 \oplus V_2 = \operatorname{im} A \oplus \ker A, \quad W = W_1 \oplus W_2 = \operatorname{im} A \oplus \ker A,$$

meaning that Φ takes its values in $L(V_2, W_2) = \mathrm{End}(\ker A)$. Observe that, when in an orthonormal basis, B is represented by a symmetric matrix and thus

$$(\Phi(B))^T = (B_{22} - B_{21}B_{11}^{-1}B_{12})^T = B_{22}^T - B_{12}^T B_{11}^{-T} B_{21}^T = B_{22} - B_{21}B_{11}B_{12} = \Phi(B),$$

i.e. $\Phi(B)$ is also symmetric for all $B \in U$. Thus Φ is, in fact, a map

$$\Phi : \mathrm{End}_{\mathrm{sym}}(V) \supset U \to \mathrm{End}_{\mathrm{sym}}(\ker A)$$

still satisfying that $\dim \ker B = \dim \ker A \iff \Phi(B) = 0$.

Using all this, we can now prove lemma 13.

Proof. Just like for lemma 12, the proof is an application of the implicit function theorem. Let $(t_0, x_0, \{f_s\}_0) \in \mathcal{M}(\mathcal{P}^m; k)$. This time, pull back the endomorphism bundle

$\mathrm{End}(TM) \to M$ by the projection $\mathcal{M}(\mathcal{P}^m) \to M$ will be denoted E. Define the section

$$A : \mathcal{M}(\mathcal{P}^m) \to E, \quad A(t, x, \{f_s\}) := D(\nabla f_t)(x) \in \mathrm{End}(T_x M).$$

As $D(\nabla f_t)(x)$ is a Hessian at a critical point, it is symmetric and hence we may consider E with fibers $\mathrm{End}_{\mathrm{sym}}(T_x M)$ instead of $\mathrm{End}(T_x M)$. Trivializing E in a neighborhood $\mathcal{O} \subset \mathcal{M}(\mathcal{P}^m)$ of $(t_0, x_0, \{f_s\}_0)$ allows us to identify A with a map

$$A : \mathcal{O} \to \mathrm{End}_{\mathrm{sym}}(T_{x_0} M).$$

By remark 15, there is a neighborhood $U \subset \mathrm{End}_{\mathrm{sym}}(T_{x_0} M)$ of $A(t_0, x_0, \{f_s\}_0)$ and a map $\Phi : U \to \mathrm{End}_{\mathrm{sym}}(\ker A(t_0, x_0, \{f_s\}_0))$ satisfying $\dim\ker B = k$ if and only if $\Phi(B) = 0$. Thus

$$\Phi \circ A : \mathcal{M}(\mathcal{P}^m) \supset \mathcal{O} \to \mathrm{End}_{\mathrm{sym}}(\ker A(t_0, x_0, \{f_s\}_0)$$

satisfies

$$(t, x, \{f_s\}) \in \mathcal{M}(\mathcal{P}^m; k) \iff \dim\ker D(\nabla f_t)(x) = k \iff \Phi \circ A(t, x, \{f_s\}) = 0.$$

To apply the implicit function theorem we need to check that 0 is a regular value of $\Phi \circ A$. Recall the definition of Φ: $\Phi(B) = B_{22} - B_{21} B_{11}^{-1} B_{12}$ Because Φ is linear in B_{22}, it is a submersion. By the chain rule, it is enough now to verify that the derivative of $A : \mathcal{O} \to \mathrm{End}_{\mathrm{sym}}(T_{x_0} M)$ is surjective. To this end, let $H \in \mathrm{End}_{\mathrm{sym}}(T_{x_0} M)$ be arbitrary. We'll want to find a differentiable path

$$(-\epsilon, \epsilon) \to \mathcal{M}(\mathcal{P}^m), \quad r \mapsto (t(r), x(r), \{f_s\}(r))$$

such that $(t(0), x(0), \{f_s\}(0)) = (t_0, x_0, \{f_s\}_0)$ and

$$\left.\frac{d}{dr}\right|_{r=0} A(t(r), x(r), \{f_s\}(r)) = H. \tag{5}$$

Choose normal coordinates around x_0. These induce a basis of $T_{x_0} M$ and the endomorphism H can be regarded a symmetric matrix in this basis. Then define

$$g(x) := \frac{1}{2}(x - x_0) \cdot H(x - x_0).$$

for all x in the chart and extend this map to the rest of M. The purpose of this definition is that $\nabla g(x_0) = 0$ and $D(\nabla g)(x_0) = H$. As the function f_0 is Morse, all of its critical points satisfy $\dim\ker D(\nabla f_0) = 0$. The same holds for f_1. From this we deduce that $0 < t_0 < 1$. Then g can be used to define a smooth homotopy

$$g_s(x) := b(s) g(x), \quad s \in I,\ x \in M$$

where $b : I \to \mathbf{R}$ is some smooth bump function with $b(t_0) = 1$ and $b(0) = b(1) = 0$. Now we construct a path that satisfies 5. The time and space functions will be constant: $t(r) = t_0$, $x(r) = x_0$. Meanwhile, the homotopy $\{f_s\}$ will vary by

$$\{f_s\}(r) := \{f_s\}_0 + r\{g_s\}.$$

First, we must check that the path $(t, x, \{f_s\})$ really takes its values in $\mathcal{M}(\mathcal{P}^m)$:

$$\nabla(f_{t(r)}(r))(x(r)) = \nabla(f_{t_0}(0))(x_0) + r\nabla g_{t_0}(x_0) = 0 + r0 = 0$$

verifies this. Then, checking 5,

$$\frac{d}{dr}\bigg|_{r=0} D(\nabla(f_{t(r)}(r)))(x(r)) = \frac{d}{dr}\bigg|_{r=0} D(\nabla(f_{t_0}(0)))(x_0) + rD(\nabla g_{t_0})(x_0) = 0 + H,$$

which, as $H \in \mathrm{End}_{\mathrm{sym}}(T_{x_0}M)$ was arbitrary, proves that the derivative of A is surjective and hence that 0 is a regular value of $\Phi \circ A$.

Then we can apply the implicit function theorem to find that $\mathcal{M}(\mathcal{P}^m; k) \cap \mathcal{O}$ is a submanifold of $\mathcal{O}$. As the kernel of $A(t_0, x_0, \{f_s\}_0)$ has dimension k, the codimension is

$$\operatorname{codim}\{\Phi \circ A = 0\} = \dim \mathrm{End}_{\mathrm{sym}}(\ker A(t_0, x_0, \{f_s\}_0)) = \frac{k(k+1)}{2}.$$

Because $(t_0, x_0, \{f_s\}_0) \in \mathcal{M}(\mathcal{P}^m; k)$ was arbitrary, this proves lemma 13. $\square$

4.3 $\mathcal{M}(\{f_s\})$ and $\mathcal{M}(\{f_s\}; 1)$

Now that we proved that the "universal" spaces $\mathcal{M}(\mathcal{P}^m)$ and $\mathcal{M}(\mathcal{P}^m; k)$ are C^{m-1} manifolds, we can use this to show that for generic homotopies $\{f_s\} \in \mathcal{P}^m$ the evolution space of critical points $\mathcal{M}(\{f_s\})$ is a one-dimensional submanifold, and that $\mathcal{M}(\{f_s\}; 1)$, the space of those critical points where the homotopy fails to be Morse, is a zero-dimensional manifold, i.e. a finite set. All other $\mathcal{M}(\{f_s\}; k)$, $k \geq 2$, will be empty, so the only degeneracies keeping $\{f_s\}$ from being Morse are those in $\mathcal{M}(\{f_s\}; 1)$.

Proposition 16. *For all $3 \leq m \leq \infty$ there exists an open and dense set $\mathcal{P}^m_{\mathrm{reg}} \subset \mathcal{P}^m$ such that for all $\{f_s\} \in \mathcal{P}^m_{\mathrm{reg}}$ the spaces $\mathcal{M}(\{f_s\})$ and $\mathcal{M}(\{f_s\}; 1)$ are C^{m-1} submanifolds of $I \times M$ of dimensions 1 and 0, while for $k \geq 2$, $\mathcal{M}(\{f_s\}; k)$ is empty.*

The proposition's proof will proceed in multiple steps. We first restrict to the case $m < \infty$ because lemmas 12 and 13 are false for $m = \infty$ as $\mathcal{P}^\infty$ is no Banach manifold. The smooth case will be recovered in the end.

1. The projection $\pi : \mathcal{M}(\mathcal{P}^m) \to \mathcal{P}^m$ will be proven a Fredholm map of index 1. This will allow us to apply the Sard–Smale theorem to find that the regular values of π are comeager (i.e. generic) in $\mathcal{P}^m$.

2. Lemma 18 will show that for all regular values of π, the linearization $D\sigma|_{I\times M\times\{\{f_s\}\}}(t,x)$ is surjective for all $(t,x) \in \mathcal{M}(\{f_s\})$, hence the implicit function theorem can be applied to show that $\mathcal{M}(\{f_s\})$ is a submanifold.

3. We show that $\pi|_{\mathcal{M}(\mathcal{P}^m;k)} : \mathcal{M}(\mathcal{P}^m;k) \to \mathcal{P}^m$ is a Fredholm map of index $1 - k(k+1)/2$. We apply the Sard–Smale theorem again to find a comeager subset of $\mathcal{P}^m$.

4. Using lemma 18 we show that for all $\{f_s\}$ in the comeager subset, the map $\Phi \circ A$, defined locally around points of $\mathcal{M}(\mathcal{P}^m;k)$ on $\mathcal{M}(\mathcal{P}^m)$ with values in $\mathbf{R}$ has regular value zero and apply the implicit function theorem.

5. We prove that a set $\mathcal{P}^m_{\text{reg}}$ satisfying proposition 16 is open.

6. The density of $\mathcal{P}^\infty$ in $\mathcal{P}^m$ is used to extend the theorem to the case $m = \infty$.

First we must prove some algebraic lemmas the proof relies on. The first lemma will be applied to show that the projections $\mathcal{M}(\mathcal{P}^m) \to \mathcal{P}^m$ and $\mathcal{M}(\mathcal{P}^m;k) \to \mathcal{P}^m$ are Fredholm maps.

Lemma 17. *Suppose V is a finite-dimensional vector space and that E is a Banach space. Consider the projection $\pi : V \oplus E \to E$. This is a Fredholm operator of index* $\dim V$. *Furthermore, for every closed subspace $S \subset V \oplus E$ of finite codimension* $\operatorname{codim} S = k$, *the restriction of the projection $\pi|_S$ is a Fredholm operator $S \to E$ and*

$$\operatorname{ind}(\pi|_S) = \dim V - \operatorname{codim} S.$$

Proof. Clearly, $\ker \pi = V$ and thus $\dim\ker\pi = \dim V < \infty$. As π is surjective, it follows that it is Fredholm. Its index is

$$\operatorname{ind}\pi = \dim\ker\pi - \operatorname{codim}\operatorname{im}\pi = \dim V.$$

It is a basic fact that for two Fredholm operators f and g their composition $f \circ g$ is also Fredholm and has index $\operatorname{ind}(f \circ g) = \operatorname{ind}(f) + \operatorname{ind}(g)$. The inclusion $\iota : S \hookrightarrow V \oplus E$ is a Fredholm operator:

$$\dim\ker\iota = 0, \quad \operatorname{codim}\operatorname{im}\iota = \operatorname{codim} S, \quad \operatorname{ind}\iota = -\operatorname{codim} S.$$

Observe that $\pi|_S = \pi \circ \iota$. It is thus Fredholm with

$$\operatorname{ind}\pi|_S = \operatorname{ind}(\pi \circ \iota) = \operatorname{ind}\pi + \operatorname{ind}\iota = \dim V - \operatorname{codim} S. \qquad \square$$

The second lemma (which can be found with a simple proof in [Wen], p. 210) will be used to deduce the surjectivity of $D\sigma$ from the surjectivity of $T\pi$, which will then allow us to apply the implicit function theorem, proving that $\mathcal{M}(\{f_s\})$ is a submanifold. It will be applied similarly for $\mathcal{M}(\{f_s\};k)$, $k \geq 1$.

Lemma 18. *Let X_1, X_2 and Y be vector spaces and $L_1 : X_1 \to Y$, $L_2 : X_2 \to Y$ be linear maps. Then, if $L : X_1 \oplus X_2 \to Y$, $(x_1, x_2) \mapsto L_1 x_1 + L_2 x_2$ is surjective, the projection $\Pi : \ker L \to X_2$, $(x_1, x_2) \mapsto x_2$ satisfies*

$$\ker \Pi \cong \ker L_1 \quad \textit{and} \quad \operatorname{coker} \Pi \cong \operatorname{coker} L_1.$$

The next lemma is a consequence of the open mapping theorem from functional analysis. The proof is omitted but may be found in [Lan93], pp. 396–397.

Lemma 19. *Let E and F be Banach spaces. In the Banach space of bounded linear operators $L(E, F)$, the subset of surjective linear maps is open.*

Now we get to the proof of proposition 16.

Proof. For now we restrict to the case $3 \leq m < \infty$. The smooth case will be recovered in the end of the proof. Consider the projection

$$\pi : \mathcal{M}(\mathcal{P}^m) \to \mathcal{P}^m, \quad (t, x, \{f_s\}) \mapsto \{f_s\}.$$

As $\mathcal{M}(\mathcal{P}^m)$ is just a regular level set of the section σ defined in the proof of lemma 12, the tangent space to $\mathcal{M}(\mathcal{P}^m)$ is exactly the kernel of the linearization of σ:

$$T_{(t,x,\{f_s\})}\mathcal{M}(\mathcal{P}^m) = \ker D\sigma(t, x, \{f_s\}) \subset \mathbf{R} \oplus T_x M \oplus \mathcal{P}_0^m$$

Then the derivative of π is the restriction of the projection $\mathbf{R} \oplus T_x M \oplus \mathcal{P}_0^m \to \mathcal{P}_0^m$ to the tangent space of $\mathcal{M}(\mathcal{P}^m)$. According to lemma 12, $\mathcal{M}(\mathcal{P}^m)$ has codimension n. As $\mathbf{R} \oplus T_x M$ is of finite dimension $n+1$, we can then apply lemma 17 to find that $T_{(t,x,\{f_s\})}\pi$ is a Fredholm operator with $\operatorname{ind}(T_{(t,x,\{f_s\})}\pi) = (n+1) - n = 1$ for all $(t, x, \{f_s\}) \in \mathcal{M}(\mathcal{P}^m)$. This means that π is a Fredholm map of index 1.

From the assumption $m \geq 3$, it follows that $\mathcal{M}(\mathcal{P}^m)$ is a C^2 manifold and then $\pi : \mathcal{M}(\mathcal{P}^m) \to \mathcal{P}^m$ is also C^2. As $2 > 1 = \max(0, \operatorname{ind}\pi)$, we may apply the Sard–Smale theorem (11) finding that the set of regular values of π, from now on referred to by $\mathcal{P}^m_{\mathrm{reg};\pi}$, is comeager in $\mathcal{P}^m$. These regular values $\{f_s\} \in \mathcal{P}^m_{\mathrm{reg};\pi}$ satisfy that for all $(t, x) \in \mathcal{M}(\{f_s\})$ the tangent map $T_{(t,x,\{f_s\})}\pi$ is surjective. Now we apply lemma 18: The tangent space $T_{(t,x,\{f_s\})}(I \times M \times \mathcal{P}^m)$ decomposes as the direct sum $T_{(t,x)}(I \times M) \oplus T_{\{f_s\}}\mathcal{P}^m$. The linearization of the section σ then decomposes as

$$L := D\sigma(t, x, \{f_s\}) = D\sigma|_{T(I\times M)}(t, x, \{f_s\}) \oplus D\sigma|_{T\mathcal{P}^m}(t, x, \{f_s\}) =: L_1 \oplus L_2.$$

In the proof of lemma 12, we showed that $L = D\sigma(t, x, \{f_s\})$ is surjective. Thus, we may apply lemma 18 with L, L_1 and L_2 as defined above, and with

$$\Pi := T_{(t,x,\{f_s\})}\pi \quad \text{defined on} \quad T_{(t,x,\{f_s\})}\mathcal{M}(\mathcal{P}^m) = \ker D\sigma(t, x, \{f_s\}) = \ker L,$$

the restriction of the projection onto $T_{\{f_s\}}\mathcal{P}^m$. We obtain

$$\operatorname{coker} D\sigma|_{I\times M\times\{f_s\}}(t,x,\{f_s\}) \cong \operatorname{coker} T_{(t,x,\{f_s\})}\pi = 0,$$

so the linearization of $\sigma|_{I\times M\times\{f_s\}}$ is surjective. As $\mathcal{M}(\{f_s\})$ is non-empty (on a closed manifold, every function has a critical point) the implicit function theorem shows that

$$\mathcal{M}(\{f_s\}) = \{\sigma|_{I\times M\times\{f_s\}} = 0\}$$

is a codimension-n, C^{m-1} submanifold of $I \times M$.

Now we go on to similarly prove that for generic $\{f_s\}$, $\mathcal{M}(\{f_s\}; 1)$ is a submanifold of $\mathcal{M}(\{f_s\})$ and $\mathcal{M}(\{f_s\}; k)$ is empty for $k \geq 2$. The way it was shown in section 4.2 that $\mathcal{M}(\mathcal{P}^m; k)$ is a manifold was by using that, locally around each $(t, x, \{f_s\}) \in \mathcal{M}(\mathcal{P}^m; k)$, it is the regular zero set of a function

$$\Phi A : \mathcal{M}(\mathcal{P}^m) \supset \mathcal{O} \to \operatorname{End}_{\mathrm{sym}}(\ker D(\nabla f_t)(x)).$$

The kernel of $D(\nabla f_t)(x)$ is k-dimensional, hence we may regard ΦA as a map with codomain $\mathbf{R}^{\frac{k(k+1)}{2}}$. The derivative

$$D(\Phi A)(t, x, \{f_s\}) : T_{(t,x,\{f_s\})}\mathcal{M}(\mathcal{P}^m) \to \mathbf{R}^{\frac{k(k+1)}{2}}$$

is surjective and will later on, when we apply lemma 18, play the role of L in the lemma.

$\mathcal{M}(\mathcal{P}^m; k)$ is of codimension $n+\frac{k(k+1)}{2}$ in $I\times M\times\mathcal{P}^m$, so, by lemma 17, π_k is Fredholm with index $(n + 1) - (n + \frac{k(k+1)}{2}) = 1 - \frac{k(k+1)}{2}$. As the Banach manifold $\mathcal{M}(\mathcal{P}^m; k)$ is at least C^1 (meaning that π_k is too), we can apply the Sard–Smale theorem and find out that the set $\mathcal{P}^m_{\mathrm{reg};k}$ of regular values of π_k is comeager in $\mathcal{P}^m$. As the intersection of countably many comeager sets,

$$\mathcal{P}^m_{\mathrm{reg}} := \mathcal{P}^m_{\mathrm{reg};\pi} \cap \bigcap_{k=1}^{\infty} \mathcal{P}^m_{\mathrm{reg};k},$$

the set of all $\{f_s\} \in \mathcal{P}^m$ which are regular values for all projections $\pi, \pi_1, \pi_2, \dots$, is comeager itself.

Now, for $\{f_s\} \in \mathcal{P}^m_{\mathrm{reg}}$, as $\{f_s\} \in \mathcal{P}^m_{\mathrm{reg};\pi}$ and thus $\mathcal{M}(\{f_s\})$ is a 1-dimensional submanifold of $\mathcal{M}(\mathcal{P}^m)$, the 1-dimensional closed linear subspace $X_1 := T_{(t,x)}\mathcal{M}(\{f_s\})$ gives rise to a splitting

$$T_{(t,x,\{f_s\})}\mathcal{M}(\mathcal{P}^m) = X_1 \oplus X_2,$$

where X_2 is some complementary space. We decompose

$$D(\Phi A)(t, x, \{f_s\}) = D(\Phi A)(t, x, \{f_s\})|_{X_1} \oplus D(\Phi A)(t, x, \{f_s\})|_{X_2} =: L_1 \oplus L_2$$
$$X_1 \oplus X_2 \to \mathbf{R}^{\frac{k(k+1)}{2}}.$$

Defining the projection $\hat{\Pi} : X_1 \oplus X_2 \to X_2$, we want to prove that $\Pi = \hat{\Pi}|_{\ker D(\Phi A)}$ is surjective: First note that, as

$$T_{(t,x,\{f_s\})}\pi : X_1 \oplus X_2 \to T_{\{f_s\}}\mathcal{P}^m$$

is surjective and $\ker T_{(t,x,\{f_s\})}\pi = X_1$, $T_{(t,x,\{f_s\})}\pi|_{X_2} : X_2 \to T\mathcal{P}^m$ is an isomorphism. Then, as $T_{(t,x,\{f_s\})}\pi_k$ is also surjective and $\Pi = T\pi|_{X_2}^{-1} \circ T\pi_k$, Π is surjective. Lemma 18 implies

$$\operatorname{coker} D(\Phi A)(t,x,\{f_s\})|_{X_1} \cong \operatorname{coker} \Pi = 0, \tag{6}$$

meaning that $D(\Phi A)$ is surjective when restricted to $X_1 = T\mathcal{M}(\{f_s\})$. The implicit function theorem then implies that $\mathcal{M}(\{f_s\};k)$ is a submanifold of $\mathcal{M}(\{f_s\})$ of codimension $\frac{k(k+1)}{2}$.

Claim. *The comeager set $\mathcal{P}^m_{\text{reg}}$ of all $\{f_s\}$ is open.*

Because $D(\nabla f_t)(x)$ is defined on the n-dimensional T_xM, $\dim\ker D(\nabla f_t)(x)$ can never exceed n. Thus for all $k > n$, the set $\mathcal{M}(\mathcal{P}^m;k)$ is empty and then the set of regular values of $\pi_k : \mathcal{M}(\mathcal{P}^m;k) \to \mathcal{P}^m$ is all of $\mathcal{P}^m$. Hence

$$\mathcal{P}^m_{\text{reg}} = \mathcal{P}^m_{\text{reg};\pi} \cap \bigcap_{k=1}^{\infty} \mathcal{P}^m_{\text{reg};k} = \mathcal{P}^m_{\text{reg}} \cap \mathcal{P}^m_{\text{reg};1} \cap \dots \cap \mathcal{P}^m_{\text{reg};n}. \tag{7}$$

Knowing this, suppose $\mathcal{P}^m_{\text{reg}}$ is not open. Then there exists an $\{f_s\} \in \mathcal{P}^m_{\text{reg}}$ and a sequence $\{f_s^n\} \in (\mathcal{P}^m_{\text{reg}})^c$ converging to $\{f_s\}$. By 7, each $\{f_s^n\}$ lies outside one of the sets $\mathcal{P}^m_{\text{reg};\pi}, \mathcal{P}^m_{\text{reg};1}, \dots, \mathcal{P}^m_{\text{reg};n}$, meaning that there is a projection $p^n \in \{\pi, \pi_1, \dots, \pi_n\}$ for which $\{f_s^n\}$ is a critical value. As this set of projections is finite, we may choose a subsequence, still called $\{f_s^n\}$, such that every $\{f_s^n\}$ is a critical value of the same projection $p \in \{\pi, \pi_1, \dots, \pi_n\}$. Then there must exist a sequence $(t^n, x^n) \in \mathcal{M}(\{f_s\}^n)$, or in case $p = \pi_k$, $(t^n, x^n) \in \mathcal{M}(\{f_s^n\};k)$ for some $k = 1, \dots, n$, such that $T_{(t^n,x^n,\{f_s^n\})}p$ is not surjective. By compactness of $I \times M$, this sequence converges to a limit (t,x). As each $(t^n,x^n) \in \mathcal{M}(\{f_s^n\})$, we have in a trivialization of the pullback of TM by the projection $(t,x,\{f_s\}) \to x$ around $(t,x,\{f_s\})$, that

$$\nabla f_t(x) = \lim \nabla f^n_{t^n}(x^n) = 0$$

and thus $(t,x) \in \mathcal{M}(\{f_s\})$. Now we must distinguish between the cases:

Case $p = \pi$. We trivialize the tangent bundle $T\mathcal{M}(\mathcal{P}^m) \to \mathcal{M}(\mathcal{P}^m)$ around $(t,x,\{f_s\})$ and $T\mathcal{P}^m \to \mathcal{P}^m$ around $\{f_s\}$. This way, $T\pi$ may be identified with a map $(\tau, \chi, \{\varphi_s\}) \mapsto T_{(\tau,\chi,\{\varphi_s\})}\pi \in L(T\mathcal{M}(\mathcal{P}^m), T_{\{f_s\}}\mathcal{P}^m)$, which is, as π is C^1, continuous. By lemma 19, the set of surjective linear operators is open in $L(T_{(t,x,\{f_s\})}\mathcal{M}(\mathcal{P}^m), T_{\{f_s\}}\mathcal{P}^m)$. Thus, as $T_{(t^n,x^n,\{f_s^n\})}\pi$ is non-surjective, so must be $T_{(t,x,\{f_s\})}$, a contradiction to $\{f_s\}$ being a regular value of π.

Case $p = \pi_1$. Choosing a trivialization of the pull-back endomorphism bundle over $I \times M \times \mathcal{P}^m$ at $(t,x,\{f_s\})$, we see that $D(\nabla f_t)(x) = \lim D(\nabla f^n_{t^n})(x^n)$. Recall that every endomorphism has a neighborhood in which all other endomorphisms have rank at least as large. This means that in $\mathrm{End}(T_x M)$, the subset of endomorphisms of nullity ≥ 1 is closed. Then, as $\dim\ker D(\nabla f_{t^n}(x^n)) = 1$ for all n, $\dim\ker D(\nabla f_t)(x) \geq 1$. However, by assumption $\{f_s\} \in \mathcal{P}^m_{\mathrm{reg}}$ for all $k \geq 2$, $\mathcal{M}(\{f_s\}; k)$ is empty for all those k. Thus $(t, x) \in \mathcal{M}(\{f_s\}; 1)$. Now, $T_{(t,x,\{f_s\})}\pi_1$ is surjective but $T_{(t^n,x^n,\{f^n_s\})}$, thus the same argument as in case $p = \pi$ yields a contradiction.

Case $p = \pi_k$ **for some** $k \geq 2$. As in the case $p = \pi_1$, the set of endomorphisms of nullity at least k being closed implies that $\dim\ker D(\nabla f_t)(x) \geq k$, meaning that (t, x) lies in one of the sets $\mathcal{M}(\{f_s\}; l)$, $k \leq l \leq n$, all of which are empty by the assumption that $\{f_s\} \in \mathcal{P}^m_{\mathrm{reg}}$.

We see that all cases lead to contradictions, thus $\mathcal{P}^m_{\mathrm{reg}}$ is open. As it is comeager, the Baire category theorem implies that it is also dense.

The reason we had to restrict to $m < \infty$ so far is that the proof relied on $\mathcal{P}^m$ being a Banach manifold. This is not the case when $m = \infty$, as then the model space $\mathcal{P}^\infty_0 \subset C^\infty(I \times M)$ is no Banach space. However, extending the proposition to this case is not difficult.

Claim. *Proposition 16 also holds true in the case $m = \infty$.*

Let us set

$$\mathcal{P}^\infty_{\mathrm{reg}} := \mathcal{P}^3_{\mathrm{reg}} \cap \mathcal{P}^\infty$$

and check that this satisfies the proposition. We know then that for all $\{f_s\} \in \mathcal{P}^\infty_{\mathrm{reg}}$ the spaces $\mathcal{M}(\{f_s\})$ and $\mathcal{M}(\{f_s\}; 1)$ are C^2, submanifolds of the right codimensions. For $\mathcal{P}^\infty_{\mathrm{reg}}$ to satisfy proposition 16, they must however be C^∞ submanifolds. To see that they are, recall that we proved for $\{f_s\} \in \mathcal{P}^3_{\mathrm{reg}}$ that $D\sigma|_{I\times M\times\{f_s\}}(t, x, \{f_s\})$ and $D(\Phi A)|_{\mathcal{M}(\{f_s\})}(t, x, \{f_s\})$ are surjective. Unlike before, now that $\{f_s\}$ is additionally smooth, σ and ΦA are also smooth, so applying the implicit function theorem shows that $\mathcal{M}(\{f_s\})$ and $\mathcal{M}(\{f_s\}; 1)$ really are submanifolds of class C^∞. Of course, we already proved that $\mathcal{M}(\{f_s\}; k) = \emptyset$ for $k \geq 2$, and no additional smoothness needs to be verified here.

Showing that $\mathcal{P}^\infty_{\mathrm{reg}}$ is open and dense in $\mathcal{P}^\infty$ is merely an application of lemma 7.

□

4.4 Events in $\mathcal{M}(\{f_s\}; 1)$ occur at different times

We know now that generic homotopies, that is homotopies $\{f_s\}$ in the open and dense set $\mathcal{P}^m_{\mathrm{reg}}$, are Morse at all except finitely many times. These are the times of the events in $\mathcal{M}(\{f_s\}; 1)$. The next step on the way to theorem 6 is to show that, at these degenerate times t, there is always only one degenerate critical point x of f_t. This is not true for all $\{f_s\} \in \mathcal{P}^m_{\mathrm{reg}}$, but we can prove that the $\{f_s\}$ in $\mathcal{P}^m_{\mathrm{reg}}$ for which it is true still form a comeager set in $\mathcal{P}^m$.

Proposition 20. *One may further restrict the set $\mathcal{P}^m_{\mathrm{reg}}$ from proposition 16 so that it remains comeager but additionally satisfies that events $(t, x) \in \mathcal{M}(\{f_s\}; 1)$, i.e. events where f_t*

fails to be Morse, occur at different times. In other words, for all $\{f_s\} \in \mathcal{P}^m_{\text{reg}}$ *and all* $(t_1, x_1), (t_2, x_2) \in \mathcal{M}(\{f_s\}; 1)$ *with* $t_1 = t_2$ *it then holds that* $x_1 = x_2$.

For the first three steps, we must restrict to the case $m < \infty$ again. In the last step, the smooth case is recovered.

1. First we show that

$$\mathcal{U} := \{(t_1, x_1, t_2, x_2, \{f_s\}) \in (I \times M)^2 \times \mathcal{P}^m \mid (t_1, x_1), (t_2, x_2) \in \mathcal{M}(\{f_s\}; 1), \\ (t_1, x_1) \neq (t_2, x_2)\}$$

   is a codimension-$(2n + 2)$ submanifold of $(\mathring{I} \times M)^2 \times \mathcal{P}^m$ (where $\mathring{I} = (0, 1)$). The proof is similar to those of lemmas 12 and 13 combined.

2. Next, we prove that $p : \mathcal{U} \to I^2$, $(t_1, x_1, t_2, x_2, \{f_s\}) \mapsto (t_1, t_2)$ is a submersion and thus $p^{-1}(\Delta)$ is a codimension-$(2n + 3)$ submanifold of $(\mathring{I} \times M)^2 \times \mathcal{P}^m$.

3. Notice that $(t_1, x_1, t_2, x_2, \{f_s\}) \in p^{-1}(\Delta)$ means that $t_1 = t_2$ but $x_1 \neq x_2$. Thus, for the proposition to hold, $\{f_s\}$ must be removed from $\mathcal{P}^m_{\text{reg}}$. This tells us that the goal of the proof is to show that, for the projection

$$\pi : (\mathring{I} \times M)^2 \times \mathcal{P}^m \to \mathcal{P}^m,$$

   the set $\mathcal{P}^m_{\text{reg}} \setminus \pi(p^{-1}(\Delta))$ is comeager in $\mathcal{P}^m$. To see this, we apply the Sard–Smale theorem to $\pi|_{p^{-1}(\Delta)}$, showing that the regular values of $\pi|_{p^{-1}(\Delta)}$ are comeager in $\mathcal{P}^m$. However, for dimensional reasons, $\pi|_{p^{-1}(\Delta)}$ can have no regular points and thus all regular values are outside $\pi(p^{-1}(\Delta))$, which must then be meager.

4. In the end, we extend to the case $m = \infty$.

Proof. **Step 1.** We show that $\mathcal{U}$ is a C^{m-2} submanifold of $(\mathring{I} \times M)^2 \times \mathcal{P}^m$ of codimension $2n + 2$. Let

$$\mathcal{U}_0 := \{(t_1, x_1, t_2, x_2, \{f_s\}) \in (\mathring{I} \times M)^2 \times \mathcal{P}^m \mid (t_1, x_1) \neq (t_2, x_2)\}.$$

The diagonal $\{(t_1, x_1) = (t_2, x_2)\} \subset (\mathring{I} \times M)^2$ is closed and the projection $(\mathring{I} \times M)^2 \times \mathcal{P}^m \to (\mathring{I} \times M)^2$ continuous, hence $\mathcal{U}_0$ is an open submanifold of $(\mathring{I} \times M)^2 \times \mathcal{P}^m$. To see that $\mathcal{U} \subset \mathcal{U}_0$, observe that for all $(t_1, x_1, t_2, x_2, \{f_s\}) \in \mathcal{U}$ the times t_1 and t_2 can't be 0 or 1. This is because, as f_0 and f_1 are Morse, they have no degenerate critical points and hence there are no events $(0, x)$ or $(1, x)$ in $\mathcal{M}(\{f_s\}; 1)$.

The proof that $\mathcal{U}$ is a submanifold is similar to the proof of lemmas 12 and 13. First we pull back the vector bundle

$$TM \oplus TM \to M \times M$$

to $\mathcal{U}_0$ by the projection $(t_1, x_1, t_2, x_2, \{f_s\}) \mapsto (x_1, x_2) \in M \times M$. This bundle will be called E. Its fibers are $E_{(t_1,x_1,t_2,x_2,\{f_s\})} = T_{x_1}M \oplus T_{x_2}M$. Now define the C^{m-1} section

$$\sigma : \mathcal{U}_0 \to E, \quad \sigma(t_1, x_1, t_2, x_2, \{f_s\}) = (\nabla f_{t_1}(x_1), \nabla f_{t_2}(x_2)).$$

To apply the implicit function theorem, we show that at every zero the linearization of σ is surjective: For this, we use a smooth the bump function

$$b : I \times M \to \mathbf{R} \quad \text{satisfying} \quad b(t,x) = \begin{cases} 0 & \text{for } t = 0, 1 \\ 0 & \text{in a neighborhood of } (t_2, x_2) \\ 1 & \text{in a neighborhood of } (t_1, x_1). \end{cases} \tag{8}$$

Let $(v, 0) \in T_{x_1}M \oplus T_{x_2}M = E_{(t_1,x_1,t_2,x_2,\{f_s\})}$ be arbitrary. As in the proof of lemma 12 one can construct a function

$$g \in C^\infty(M), \quad \nabla g(x_1) = v.$$

Extending g to $I \times M$ by

$$g_s(x) := b(s,x)g(x)$$

yields a homotopy $\{g_s\} \in \mathcal{P}_0^m$ with $g_{t_2}(x_2) = 0$, and then

$$\frac{d}{dr}\bigg|_{r=0} \sigma(t_1, x_1, t_2, x_2, \{f_s\} + r\{g_s\}) = \frac{d}{dr}\bigg|_{r=0} (r\nabla g_{t_1}(x_1), r\nabla g_{t_2}(x_2)) = (v, 0).$$

By exchanging the roles of (t_1, x_1) and (t_2, x_2) in the foregoing constructions one can just as well show this for any vector $(0, w)$, hence $D\sigma(t_1, x_1, t_2, x_2, \{f_s\})$ is surjective. Applying the implicit function theorem then shows that $\sigma^{-1}(0)$ is a codimension-$2n$ C^{m-1}-smooth submanifold of $\mathcal{U}_0$.

Next, pull back the vector bundle

$$\mathrm{End}_{\mathrm{sym}}(TM) \oplus \mathrm{End}_{\mathrm{sym}}(TM) \to M \times M$$

to $\sigma^{-1}(0)$ and, again, call it E. The fibers are $E_{(t_1,x_1,t_2,x_2,\{f_s\})} = \mathrm{End}_{\mathrm{sym}}(T_{x_1}M) \oplus \mathrm{End}_{\mathrm{sym}}(T_{x_2}M)$. Define the C^{m-2} section

$$A : \sigma^{-1}(0) \to E, \quad A(t_1, x_1, t_2, x_2, \{f_s\}) = (D(\nabla f_{t_1})(x_1), D(\nabla f_{t_2})(x_2)).$$

Around every $(t_1, x_1, t_2, x_2, \{f_s\}) \in \sigma^{-1}(0)$ with $\dim\ker D(\nabla f_{t_1})(x_1) = \dim\ker D(\nabla f_{t_2})(x_2) = 1$, we trivialize E, thus identifying A with a locally defined function

$$A : \sigma^{-1}(0) \supset \mathcal{O} \to \mathrm{End}_{\mathrm{sym}}(T_{x_1}M) \oplus \mathrm{End}_{\mathrm{sym}}(T_{x_2}M).$$

By lemma 14, there exist neighborhoods $U_1 \subset \mathrm{End}_{\mathrm{sym}}(T_{x_1}M)$, $U_2 \subset \mathrm{End}_{\mathrm{sym}}(T_{x_2}M)$ and submersions $\Phi_1 : U_1 \to \mathrm{End}(\ker A(t_1, x_1, t_2, x_2, \{f_s\})) \cong \mathbf{R}$, $\Phi_2 : U_2 \to \mathbf{R}$ for which

$$(\tau_1, \chi_1, \tau_2, \chi_2, \{\phi_s\}) \in \mathcal{U} \iff ((\Phi_1, \Phi_2) \circ A)(\tau_1, \chi_1, \tau_2, \chi_2, \{\phi_s\}) = 0.$$

As in the proof of lemma 13, for every $H \in \mathrm{End}_{\mathrm{sym}}(T_{x_1}M)$ exists a function $g \in C^\infty(M)$ which satisfies

$$\nabla g(x_1) = 0 \quad \text{and} \quad D(\nabla g)(x_1) = H.$$

Multiplying with the bump function 8, $g_s(x) := b(s,t)g(x)$ is a homotopy $\{g_s\} \in \mathcal{P}_0^m$ with

$$\nabla g_{t_1}(x_1) = 0,\ D(\nabla g_{t_1}(x_1)) = H \quad \text{and} \quad \nabla g_{t_2}(x_2) = 0,\ D(\nabla g_{t_2}(x_2)) = 0.$$

Then

$$\begin{aligned}\frac{d}{dr}\Big|_{r=0} A(t_1, x_1, t_2, x_2, \{f_s\} + r\{g_s\}) &= \frac{d}{dr}\Big|_{r=0} (D(\nabla(f_{t_1} + rg_{t_1})(x_1), D(\nabla(f_{t_2} + rg_{t_2}))(x_2))) \\ &= \frac{d}{dr}\Big|_{r=0} r(D(\nabla g_{t_1})(x_1), D(\nabla g_{t_2})(x_2)) = (H, 0).\end{aligned}$$

Results $(0, H)$ are clearly possible by exchanging (t_1, x_1) and (t_2, x_2) in the above constructions. Thus, as $r \mapsto (t_1, x_1, t_2, x_2, \{f_s\} + r\{g_s\})$ is a path through $\sigma^{-1}(0)$, A has surjective differential. Together with the fact that Φ_1 and Φ_2 are submersions (c.f. their definition in lemma 14), it becomes apparent that $(\Phi_1, \Phi_2) \circ A : \mathcal{O} \to \mathbf{R}^2$ is a submersion and thus the implicit function theorem can be used to show that $\mathcal{O} \cap \mathcal{U}$ is a C^{m-2}, codimension-2 submanifold of $\mathcal{O} \subset \sigma^{-1}(0)$. It follows that $\mathcal{U}$ really is a C^{m-2}, codimension-$(2n+2)$ submanifold of $(\mathring{I} \times M)^2 \times \mathcal{P}^m$.

Step 2. We prove now that $p : \mathcal{U} \to \mathring{I}^2$, $(t_1, x_1, t_2, x_2, \{f_s\}) \mapsto (t_1, t_2)$ is a submersion, and thus $p^{-1}(\Delta)$ is a codimension-$(2n+3)$ submanifold of $(\mathring{I} \times M)^2 \times \mathcal{P}^m$. Let $(t_1(0), x_1(0), t_2(0), x_2(0), \{f_s^{(0)}\}) \in \mathcal{U}$ be arbitrary. We must show that the derivative

$$T_{(t_1(0), x_1(0), t_2(0), x_2(0), \{f_s^{(0)}\})}p : \mathbf{R} \oplus T_{x_1}M \oplus \mathbf{R} \oplus T_{x_2}M \oplus \mathcal{P}_0^m \to \mathbf{R} \oplus \mathbf{R}$$

is surjective. We do this by finding, for the vectors $(1, 0)$ and $(0, 1) \in T_{(t_1, t_2)}(I \times I)$, paths

$$(t_1, x_1, t_2, x_2, \{f_s\}) : (-\epsilon, \epsilon) \to \mathcal{U}, \quad r \mapsto (t_1(r), x_1(r), t_2(r), x_2(r), \{f_s^{(r)}\})$$

through $\mathcal{U}$ that match the initial value $(t_1(0), x_1(0), t_2(0), x_2(0), \{f_s^{(0)}\})$ at $r = 0$. Construct such a path by choosing a smooth bump function $b : I \times M \to \mathbf{R}$ that satisfies

$$b(t, x) = \begin{cases} 0 & \text{for t in neighborhoods of 0 and 1} \\ 1 & \text{for (t,x) in a neighborhood of } (t_1(0), x_1(0)) \\ 0 & \text{for (t,x) in a neighborhood of } (t_2(0), x_2(0)). \end{cases} \tag{9}$$

Then define $\{f_s^{(r)}\}$ by

$$f_s^{(r)}(x) = f_{s-rb(s,x)}^{(0)}(x).$$

Observe that this is defined: As $b(s,x) = 0$ for s sufficiently close to 0 or 1, $s - rb(s,x)$ is contained in $[0,1]$ for all $s \in [0,1]$ and r in an interval $(-\epsilon, \epsilon)$, so $f_{s-rb(s,x)}^{(0)}$ is defined. Next, one defines the path $t_1(r) = t_1(0) + r$. By the definition of the bump function in 9, for small r and for x close to $x_1(0)$, it holds that $b(t(r), x) = 1$. Then, for these r and x,

$$f_{t_1(r)}^{(r)}(x) = f_{t_1(r)-rb(t_1(r))}^{(0)}(x) = f_{t_1(0)}^{(0)}(x).$$

Setting $x_1(r) = x_1(0)$ constant, observe that

$$\begin{gathered}\nabla(f_{t_1(r)}^{(r)})(x_1(r)) = \nabla(f_{t_1(0)}^{(0)})(x_1(0)) = 0,\\ \dim\ker D(\nabla f_{t_1(r)}^{(r)}(x_1(r))) = \dim\ker D(\nabla f_{t_1(0)}^{(0)})(x_1(0)) = 1.\end{gathered} \tag{10}$$

Equally, setting $t_2(r) = t_2(0)$ and $x_2(r) = x_2(0)$ constantly and using the fact that, in a neighborhood of $(t_2(0), x_2(0))$, the bump function b is zero, we find

$$\begin{gathered}\nabla(f_{t_2(r)}^{(r)})(x_2(r)) = \nabla(f_{t_2(0)}^{(0)})(x_2(0)) = 0,\\ \dim\ker D(\nabla f_{t_2(r)}^{(r)}(x_2(r))) = \dim\ker D(\nabla f_{t_2(0)}^{(0)})(x_2(0)) = 1.\end{gathered} \tag{11}$$

10 and 11 together mean that $(t_1, x_1, t_2, x_2, \{f_s\})$ really is a path in $\mathcal{U}$. Clearly, the derivative along the path is

$$\frac{d}{dr}\Big|_{r=0} p(t_1(r), x_1(r), t_2(r), x_2(r), \{f_s^{(r)}\}) = \begin{pmatrix}\dot{t}_1(0)\\ \dot{t}_2(0)\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}.$$

An analogous construction, in which the roles of t_1 and t_2 are swapped, yields another path $(t_1, x_1, t_2, x_2, \{f_s\})$ through $\mathcal{U}$ along which the derivative of p at $r = 0$ is $(0,1)$. This then shows that $T_{(t_1(0), x_1(0), t_2(0), x_2(0), \{f_s^{(0)}\})}p$ is surjective, hence p is a submersion.

We deduce that as p is transverse to the diagonal

$$\Delta = \{(t_1, t_2) \in \mathbf{R} \times \mathbf{R} \mid t_1 = t_2\},$$

implying that the preimage $p^{-1}(\Delta)$ is a C^{m-2} submanifold of $\mathcal{U}$. Its codimension in $(\mathring{I} \times M)^2 \times \mathcal{P}^m$ is $\operatorname{codim} \mathcal{U} + \dim \Delta = 2n + 3$.

Step 3. We apply the Sard–Smale theorem to find that $\pi(p^{-1}(\Delta))$ is meager in $\mathcal{P}^m$. The projection

$$\pi : (\mathring{I} \times M)^2 \times \mathcal{P}^m \to \mathcal{P}^m, \quad (t_1, x_1, t_2, x_2, \{f_s\}) \mapsto \{f_s\}$$

is clearly a Fredholm map with index $\dim(\mathring{I} \times M)^2 = 2n + 2$. Lemma 17 shows that $\pi|_{p^{-1}(\Delta)}$

is also Fredholm and has index

$$\operatorname{ind}(\pi|_{p^{-1}(\Delta)}) = \operatorname{ind}(\pi) - \operatorname{codim}(p^{-1}(\Delta)) = (2n+2) - (2n+3) = -1.$$

As $\pi_{|}p^{-1}$ is C^{m-2} and $m \geq 3$, we may then apply the Sard–Smale theorem and thus find that that the set of regular values of $\pi|_{p^{-1}(\Delta)}$ is comeager in $\mathcal{P}^m$.

However, as

$$\dim\ker T_{(t_1,x_1,t_2,x_2,\{f_s\})}\pi = 2n+2 < \operatorname{codim} p^{-1}(\Delta)$$

for all $(t_1, x_1, t_2, x_2, \{f_s\}) \in (\mathring{I} \times M)^2 \times \mathcal{P}^m$, the restriction $\pi|_{p^{-1}(\Delta)}$ can't have any regular points. This means that the set of regular values of $\pi|_{p^{-1}(\Delta)}$ is just the complement of the image, i.e. $\pi(p^{-1}(\Delta))^c$ is comeager in $\mathcal{P}^m$. By definition, $\pi(p^{-1}(\Delta))$ is then meager in $\mathcal{P}^m$. Removing a meager set from a comeager set leaves the set comeager, thus if we replace $\mathcal{P}^m_{\text{reg}}$ by $\mathcal{P}^m_{\text{reg}} \setminus \pi(p^{-1}(\Delta))$ we are left with a comeager set of homotopies $\{f_s\}$ that satisfy both propositions 16 and 20.

Step 4. We now extend to the case $m = \infty$. Set $\mathcal{P}^\infty_{\text{reg}} := \mathcal{P}^3_{\text{reg}} \cap \mathcal{P}^\infty$. We want to show that $\mathcal{P}^\infty_{\text{reg}}$ is comeager in $\mathcal{P}^\infty$. As $\mathcal{P}^3_{\text{reg}}$ is comeager in $\mathcal{P}^3$, it contains an intersection

$$\bigcap_{k=1}^\infty O^3_k \subset \mathcal{P}^3_{\text{reg}}, \quad \text{where } O^3_k \text{ is open and dense in } \mathcal{P}^3.$$

Then, clearly, $\bigcap_{k=1}^\infty (O^3_k \cap \mathcal{P}^\infty) \subset \mathcal{P}^\infty_{\text{reg}}$. By lemma 7, each set $O^3_k \cap \mathcal{P}^\infty$ is open and dense in $\mathcal{P}^\infty$. Thus $\mathcal{P}^\infty_{\text{reg}}$ is comeager. □

4.5 $\mathcal{M}(\{f_s\}; 1)$ corresponds to birth-death bifurcations

Proposition 21. *For $\{f_t\} \in \mathcal{P}^m_{\text{reg}}$, a point $(t_0, x_0) \in \mathcal{M}(\{f_t\})$ is a birth-death bifurcation if and only if it lies in $\mathcal{M}(\{f_t\}; 1)$.*

Proof. For the " $\Longrightarrow$ " direction, let $(t_0, x_0) \in I \times M$ be a birth-death bifurcation. As $\{f_t\} \in \mathcal{P}^m_{\text{reg}}$, $\mathcal{M}(\{f_t\})$ is a 1-dimensional of $I \times M$. Choose a local parametrization $\gamma : (-\epsilon, \epsilon) \to I \times U$, $s \mapsto \gamma(s) = (t(s), x(s))$, $\gamma(0) = (t_0, x_0)$ of $\mathcal{M}(\{f_s\})$, where U is an open neighborhood of x_0 such that TM is trivial over U. This means that $\{\nabla f_t\}$ can be seen as a function $I \times U \to \mathbf{R}^n$ and is constantly zero when composed with the parametrization: $\nabla f_{t(s)}(x(s)) = 0$. Then, using that $\dot{t}(0) = 0$, as (t_0, x_0) is a birth-death bifurcation,

$$\begin{aligned}
0 &= \frac{d}{ds}\Big|_{s=0} \nabla f_{t(s)}(x(s)) \\
&= T_{\gamma(0)}\{\nabla f_t\}(\dot{\gamma}(0)) = T_{(t_0,x_0)}\{\nabla f_t\}(\dot{t}(0), \dot{x}(0)) \\
&= \frac{d}{ds}\Big|_{s=0} \{\nabla f_t\}(t_0 + s \cdot 0, x_0 + s \cdot \dot{x}(0)) = \frac{d}{ds}\Big|_{s=0} \nabla f_{t_0}(x_0 + s \cdot \dot{x}(0)) \\
&= D(\nabla f_{t_0})(x_0)\dot{x}(0).
\end{aligned}$$

As (t,x) is a parametrization, i.e. a diffeomorphism, it has non-zero derivative. As $\dot{t}(0) = 0$, this implies that $\dot{x}(0) \neq 0$ so $\ker D(\nabla f_{t_0})(x_0)$ contains a non-zero vector. It follows that $\dim\ker D(\nabla f_{t_0})(x_0) \geq 1$, so $(t_0, x_0) \in \mathcal{M}(\{f_t\}; k)$ for some $k \geq 1$. As $\mathcal{M}(\{f_t\}; k) = \emptyset$ for $k \geq 2$, we obtain $(t_0, x_0) \in \mathcal{M}(\{f_t\}; 1)$.

For the other direction: Suppose $(t_0, x_0) \in \mathcal{M}(\{f_s\}; 1)$. As $\mathcal{M}(\{f_s\})$ is a submanifold of $I \times M$, there exists a C^2 parametrization $(t,x) : (-\epsilon, \epsilon) \to \mathcal{M}(\{f_s\})$ with $(t(0), x(0)) = (t_0, x_0)$. Now one trivializes the vector bundle $I \times TM \to I \times M$ around (t_0, x_0). This local trivialization induces an obvious local trivialization of $I \times \mathrm{End}_{\mathrm{sym}}(TM) \to I \times M$. With the help of these trivializations, the sections $\nabla f : I \times M \to I \times TM$, $(t,x) \mapsto (t, \nabla f_t(x))$ and $D(\nabla f) : I \times M \to I \times \mathrm{End}_{\mathrm{sym}}(TM)$, $(t,x) \mapsto (t, D(\nabla f_t)(x))$ can be identified with maps taking values in the vector spaces $T_{x_0}M$ and $\mathrm{End}_{\mathrm{sym}}(T_{x_0}M)$, respectively. Then one clearly has $\nabla f \circ (t,x) \equiv 0$ and thus

$$0 = \frac{d}{ds}\Big|_{s=0} \nabla f(t(s), x(s)) = D(\nabla f)(t_0, x_0) \begin{pmatrix} \dot{t}(0) \\ \dot{x}(0) \end{pmatrix}. \tag{12}$$

Using the Sard–Smale theorem, we proved in section 4.3 that, for all $(t,x) \in \mathcal{M}(\{f_s\})$, the linear map $D(\nabla f)(t,x) : \mathbf{R} \times T_xM \to T_xM$ has full rank, i.e. nullity 1. Furthermore, as $(t_0, x_0) \in \mathcal{M}(\{f_s\}; 1)$, the linearization $D(\nabla f_{t_0})(x_0) : T_{x_0}M \to T_{x_0}M$ also has nullity 1. Putting these two facts together it follows that $\ker D(\nabla f)(t_0, x_0) \subset \{0\} \oplus T_{x_0}M$. Equation 12 implies that $(\dot{t}(0), \dot{x}(0)) \in \ker D(\nabla f)(t_0, x_0)$, and thus we find that $\dot{t}(0) = 0$.

It remains to show that $\ddot{t}(0) \neq 0$. One has

$$\begin{aligned} 0 &= \frac{d^2}{ds^2}\Big|_{s=0} \nabla f(t(s), x(s)) = \frac{d}{ds}\Big|_{s=0} D(\nabla f)(t(s), x(s))(\dot{t}(s), \dot{x}(s)) \\ &= \Big(\frac{d}{ds}\Big|_{s=0} D(\nabla f)(t(s), x(s))\Big)(\dot{t}(0), \dot{x}(0)) + D(\nabla f)(t_0, x_0)(\ddot{t}(0), \ddot{x}(0)). \end{aligned}$$

If one now plugs in $\dot{t}(0) = 0$ and, to provoke a contradiction, supposes $\ddot{t}(0) = 0$, then one obtains

$$\begin{aligned} 0 &= \Big(\frac{d}{ds}\Big|_{s=0} D(\nabla f)(t(s), x(s))\Big)(0, \dot{x}(0)) + D(\nabla f)(t_0, x_0)(0, \ddot{x}(0)) \\ &= \frac{d}{ds}\Big|_{s=0} D(\nabla f_{t(s)})(x(s))\dot{x}(0) + D(\nabla f_{t_0})(x_0)\ddot{x}(0) \\ &= \dot{A}(0)\dot{x}(0) + A(0)\ddot{x}(0) \end{aligned}$$

where $A(s) := D(\nabla f_{t(s)}(x(s))$. We will prove later that

$$\dot{A}(0)(\ker A(0)) \pitchfork \operatorname{im} A(0). \tag{13}$$

As $\ker A(0)$ is 1-dimensional and $\operatorname{im} A(0)$ is $(n-1)$-dimensional, we deduce that $\dot{A}(0)$ is injective when restricted to $\ker A(0)$. Then

$$\begin{aligned}
0 &= \frac{d}{ds}\Big|_{s=0} \nabla f(t(s), x(s)) \\
&= D(\nabla f)(t_0, x_0)(\dot{t}(0), \dot{x}(0)) = D(\nabla f)(t_0, x_0)(0, \dot{x}(0)) \\
&= D(\nabla f_{t_0})(x_0)\dot{x}(0) = A(0)\dot{x}(0),
\end{aligned}$$

which implies that $\dot{x}(0) \in \ker A(0)$. Then it follows from 13 that

$$\dot{A}(0)\dot{x}(0) = 0 \quad \text{and} \quad A(0)\ddot{x}(0) = 0. \tag{14}$$

Now, as (t, x) is a parametrization, i.e. a diffeomorphism, it has non-zero derivative. As $\dot{t}(0) = 0$, this means that $\dot{x}(0) \neq 0$. However, as $\dot{A}(0)$ is injective on $\ker A(0)$, and $\dot{x}(0) \in \ker A(0) \setminus \{0\}$, it must be that $\dot{A}(0)\dot{x}(0) \neq 0$, contradicting 14.

To conclude the proof we still need to check the transversality in 13: As $A(0)$ is a Hessian at a critical point, it is symmetric. This means that there exists an orthonormal basis $\{v_i\}_{i=1}^n$ of $T_{x_0}M$ with respect to which $A(0)$ takes diagonal form

$$A(0) = \begin{pmatrix} a_{11} & & & \\ & \ddots & & \\ & & a_{n-1,n-1} & \\ & & & 0 \end{pmatrix}.$$

Next we need that the matrix-valued path $A(s)$ intersects the manifold (see lemma 14) $N := \{B \in \mathbf{R}^{n\times n} \mid \dim\ker B = 1\}$ transversely at time $s = 0$. To see this, recall that, locally, $N = \Phi^{-1}(0)$ where Φ is defined in lemma 14. Equation 6 in the proof of proposition 16 we showed that, as $\{f_s\} \in \mathcal{P}^m_{\text{reg}}$, it holds that $\frac{d}{ds}\big|_{s=0}\Phi(A(s)) \neq 0$. Hence

$$T_{A(0)}\Phi(\dot{A}(0)) = \frac{d}{ds}\Big|_{s=0}\Phi(A(s)) \neq 0, \tag{15}$$

from which we infer that $\dot{A}(0) \notin \ker T_{A(0)}\Phi = T_{A(0)}N$. It is easy to see that the tangent space to N is

$$T_{A(0)}N = \{(b_{ij})_{i,j=1,\dots,n} \in \mathbf{R}^{n\times n} \mid b_{nn} = 0\}.$$

In the basis $\{v_i\}$ every non-zero vector w in the kernel of $A(0)$ has the form

$$w = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ w^n \end{pmatrix}, \quad w^n \in \mathbf{R} \setminus \{0\}.$$

Now, $\dot{A}(0) \pitchfork T_{A(0)}N$, so as a matrix $\dot{A}(0) = (\dot{a}_{ij})$ the component $\dot{a}_{nn}$ must be non-zero.

It follows that $(\dot{A}(0)w)^n = \dot{a}_{nn} w^n \neq 0$ from which we deduce that $\dot{A}(0)w \notin \operatorname{im} A(0)$. As $\dim \operatorname{im} A(0) = n-1$, this proves that $\dot{A}(0)(\ker A(0)) \pitchfork \operatorname{im} A(0)$. ◻

4.6 Morse indices at birth-death bifurcations

Finally, we will prove

Proposition 22. *For every $\{f_s\} \in \mathcal{P}^m_{\mathrm{reg}}$ it holds that, at each birth-death bifurcation $(t_0, x_0) \in \mathcal{M}(\{f_s\}; 1)$, if $(t,x) : (-\epsilon, \epsilon) \to \mathcal{M}(\{f_s\}; 1)$ is a parametrization of $\mathcal{M}(\{f_s\})$ with $(t(0), x(0)) = (t_0, x_0)$, then for all $s < 0$ the Morse index of $f_{t(s)}$ is k and it is $k+1$ for $s > 0$, or vice-versa.*

In the proof, all paths $s \mapsto \text{something}(s)$, or $r \mapsto \text{something}(r)$, are defined on an interval containing 0. Throughout the proof, this interval keeps getting smaller.

Proof. Let $(t_0, x_0) \in \mathcal{M}(\{f_s\}; 1)$. We start again by pulling back the bundle $\mathrm{End}_{\mathrm{sym}}(TM) \to M$ by the projection $I \times M \to M$, $(t,x) \mapsto x$ and then trivializing the pull-back bundle around (t_0, x_0). For a parametrization $(t(s), x(s)) \in \mathcal{M}(\{f_s\})$ with $(t(0), x(0)) = (t_0, x_0)$ the map

$$s \mapsto D(\nabla f_{t(s)})(x(s)) \in \mathrm{End}_{\mathrm{sym}}(T_{x(s)}M)$$

can be regarded as a path

$$A : s \mapsto A(s) \in \mathrm{End}_{\mathrm{sym}}(T_{x_0}M)$$

in the vector space of symmetric endomorphisms. Consider again the map $\Phi : U \to \mathbf{R}$ defined on an open neighborhood $U \subset \mathrm{End}_{\mathrm{sym}}(T_{x_0}M)$ of $A(0)$, as in remark 15. We showed in equation 15 that, as $\{f_s\} \in \mathcal{P}^m_{\mathrm{reg}}$, the derivative $T_{A(0)}\Phi(\dot{A}(0))$ is non-zero.

Consider the map

$$(s, \lambda) \mapsto \Phi(A(s) - \lambda) \in \mathbf{R}$$

which is defined for (s, λ) in some neighborhood $O \subset \mathbf{R} \times \mathbf{R}$ of $(0,0)$. It satisfies

$$\frac{\partial}{\partial s}\Big|_{s=0,\lambda=0} \Phi(A(s) - \lambda) = T_{A(0)}\Phi(\dot{A}(0)) \neq 0$$

and then, by continuity, this is also true for other values of (s, λ) in a neighborhood of $(0,0)$, again denoted by O. One can thus apply the implicit function theorem to the map to obtain that

$$\Lambda := \{(s, \lambda) \in O \mid \dim\ker(A(s) - \lambda) = 1\}$$

is a C^{m-2} submanifold of $\mathbf{R}^2$ of codimension 1. Its tangent space at points $(s, 0)$ is

$$T_{(s,0)}\Lambda = \ker\left(\tfrac{\partial}{\partial s}\big|_{\lambda=0}\Phi(A(s) - \lambda) \quad \tfrac{\partial}{\partial \lambda}\big|_{\lambda=0}\Phi(A(s) - \lambda)\right)$$

which contains the vector $(-\partial_\lambda|_{\lambda=0}\Phi(A(s) - \lambda), \partial_s\Phi(A(s)))$ with second component non-zero, making Λ transverse to the submanifold $\{(s, \lambda) \in \mathbf{R} \times \mathbf{R} \mid \lambda = 0\}$. Thus, any parametrization of Λ,

$$r \mapsto (s(r), \lambda(r)),$$

satisfies $\dot{\lambda}(0) \neq 0$. By the definition of Λ, the parametrization satisfies

$$\dim \ker(A(s(r)) - \lambda(r)) = 1,$$

i.e. $\lambda(r)$ is a simple eigenvalue of $A(r) := A(s(r))$.

Now we want to use our C^{m-2} eigenvalue function λ which, because $\lambda(0) = 0$ and $\dot{\lambda}(0) \neq 0$, changes sign at $r = 0$, to prove that the index of $A(r)$ changes by one at $r = 0$. For each r, set

$$L(r) := \ker(A(r) - \lambda(r)), \quad V(r) := \operatorname{im}(A(r) - \lambda(r)).$$

As $A(r) - \lambda(r)$ is symmetric, these spaces are orthogonal. At $r = 0$ we may choose an orthonormal eigenbasis basis $\{v_1(0), \ldots, v_n(0)\}$ of $A(0)$ for $T_{x_0}M$ such that $v_1(0), \ldots, v_k(0)$ are negative eigenvectors, $v_{k+1}(0), \ldots, v_{n-1}(0)$ are positive eigenvectors, and $v_n(0)$ is a null eigenvector. This means that

$$v_1(0), \ldots, v_{n-1}(0) \in V(0), \quad v_n(0) \in L(0).$$

Now we define the vectors

$$w_i(r) := (A(r) - \lambda(r))v_i(0), \quad i = 1, \ldots, n-1$$
$$w_n(r) := v_n(0).$$

Choosing some volume form $\omega \in \Lambda^n T^*_{x_0}M$ and using that, because $w_1(0) = v_1(0), \ldots, w_n(0) = v_n(0)$ are linearly independent, $\omega(w_1(0), \ldots, w_n(0)) \neq 0$, it follows by continuity that

$$\omega(w_1(r), \ldots, w_n(r)) \neq 0 \quad \text{for all } r \text{ sufficiently close to } 0.$$

This means that the $w_1(r), \ldots, w_n(r)$ are also linearly independent. Also, note that

$$\langle w_1(r), \ldots, w_{n-1}(r)\rangle = \operatorname{im}(A(r) - \lambda(r))$$

as $w_1(r), \ldots, w_{n-1}(r)$ are linearly independent elements of the $(n-1)$-dimensional (because $\dim \ker(A(r)-\lambda(r)) = 1$) space $\operatorname{im}(A(r)-\lambda(r))$. Then, applying the Gram–Schmidt process to w_i, $i = 1, \ldots, n-1$ like

$$v_i(r) := w_i(r) - \sum_{j=1}^{i-1} \frac{\langle v_j(r), w_i(r)\rangle}{\langle v_j(r), v_j(r)\rangle} v_j(r), \quad i = 1, \ldots, n-1,$$

the v_i, $i \leq n-1$ are C^{m-2} pointwise orthonormal functions $(-\epsilon, \epsilon) \to T_{x_0}M$ spanning

$\mathrm{im}(A(r) - \lambda(r))$. When one then does the same to $w_n(r) = v_n(0)$,

$$v_n(r) := w_n(r) - \sum_{j=1}^{n-1} \frac{\langle v_j(r), w_n(r)\rangle}{\langle v_j(r), v_j(r)\rangle} v_j(r),$$

the resulting function $v_n : (-\epsilon, \epsilon) \to T_{x_0}M$ takes values

$$v_n(r) \in \mathrm{im}(A(r) - \lambda(r))^\perp = \ker(A(r) - \lambda(r)).$$

In other words, it is a C^{m-1}-smooth varying eigenvector of $A(r)$ for eigenvalue $\lambda(r)$. The functions v_i, $i \leq n-1$ may be regarded as sections of the trivial vector bundle $\overline{T_{x_0}M} \to (-\epsilon, \epsilon)$. They are an orthonormal frame of $V = \bigsqcup_r V(r)$, showing that the latter is a subbundle. Similarly, v_n shows that $L = \bigsqcup_r L(r)$ is a rank 1 vector bundle over $(-\epsilon, \epsilon)$.

As for each r, the basis $\{v_1(r), \ldots, v_n(r)\}$ is orthonormal, the corresponding matrix of $A(r)$ in the basis then has diagonal entries

$$a_{ii}(r) = \langle v_i(r), A(r)v_i(r)\rangle .$$

At $r = 0$, the diagonal entries $a_{ii}(0)$ are negative for $i = 1, \ldots, k$ and positive for $i = k+1, \ldots, n-1$. By continuity, this holds for all r in a neighborhood of 0. However, the last diagonal entry is

$$a_{nn}(r) = \langle v_i(r), A(r)v_i(r)\rangle = \langle v_i(r), \lambda(r)v_i(r)\rangle = \lambda(r).$$

As $\lambda(0) = 0$ and $\dot\lambda(0) \neq 0$, $a_{nn}(r)$ then changes sign at $r = 0$. By Sylvester's law of inertia the number of negative diagonal entries is exactly the index of $A(r)$. This means that there really is a change in the index of $A(r)$ from $k+1$ to k or vice-versa at $r = 0$.

□

Wrapping up the proof of Cerf's theorem (6) Let us restate Cerf's theorem and check that we have really proved it.

Theorem 6 (Main theorem, [Cer70])**.** *Let M be a closed smooth manifold and f_0, f_1 : $M \to \mathbf{R}$ be two C^m, $m = 3, 4, \ldots, \infty$, Morse functions. Then there exists a comeager subset $\mathcal{P}^m_{\mathrm{reg}} \subset \mathcal{P}^m$ of C^m homotopies $\{f_s\}$ for which all critical points are non-degenerate with the exception of finitely many birth-death bifurcations $(t_1, x_1), \ldots, (t_N, x_N)$ satisfying $0 < t_1 < \cdots < t_N < 1$. At each birth-death bifurcation a pair of critical points with Morse indices k and $k+1$ ($k \in \{0, \ldots, n-1\}$) is either created or annihilated.*

The setting of the theorem was a smooth closed Riemannian manifold M and two C^m Morse functions f_0 and f_1. Proposition 16 showed that there exists an open and dense subset $\mathcal{P}^m_{\mathrm{reg}}$ of $\mathcal{P}^m$ such that for each homotopy $\{f_s\} \in \mathcal{P}^m_{\mathrm{reg}}$, the set $\mathcal{M}(\{f_s\})$ is a 1-dimensional C^{m-1} submanifold of $I \times M$, the set $\mathcal{M}(\{f_s\}; 1)$ is a 0-dimensional submanifold, and the sets

$\mathcal{M}(\{f_s\}; k)$ are empty for $k \geq 2$. The reason why the theorem is stated only for $m \geq 3$, lies therein that the manifold $\mathcal{M}(\mathcal{P}^m)$ is only a C^{m-1} submanifold and thus, to apply the Sard–Smale theorem to the equally C^{m-1} projection $\pi : \mathcal{M}(\mathcal{P}^m) \to \mathcal{P}^m$ we need that $m \geq 3$ because its Fredholm index is 1.

Thus we have a finite ($\mathcal{M}(\{f_s\}; 1)$ is a 0-dimensional submanifold of a compact space) sequence $t_1, \ldots, t_N \in I$ such that, for all other t, f_t is Morse. Furthermore, we now know that the only failures of $\{f_s\}$ to be Morse are in $\mathcal{M}(\{f_s\}; 1)$.

Proposition 20 stated that we may further shrink $\mathcal{P}^m_{\text{reg}}$ such that it remains comeager and that now two degeneracies in $\mathcal{M}(\{f_s\}; 1)$ occur at the same time. This means that for each t_i there is exactly one degenerate critical point x_i of f_{t_i} and we may order the sequence $0 < t_1 < \cdots < t_N < 1$.

Proposition 21 implies that, as each $(t_i, x_i) \in \mathcal{M}(\{f_s\}; 1)$, at all these events there is a birth-death bifurcation and these are all birth-death bifurcations of $\{f_s\}$. Finally, proposition 22 shows that the two critical points that are born or die at (t_i, x_i) have index k and $k + 1$, i.e. indices are just one apart.

We see that really all parts of the main theorem have been proven.

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Berlin, den 05. September 2022,

Johannes Radu Hübers

Johannes Radu Hübers