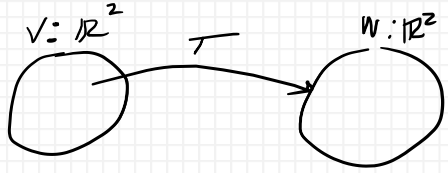


$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}\frac{1}{2}x\\2y\end{bmatrix}$$

la rep. matricial de T es

$$\begin{bmatrix}\frac{1}{2} & 0\\0 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{1}{2}x\\2y\end{bmatrix}$$

$(2\times2)(2\times1) = 2\times1$

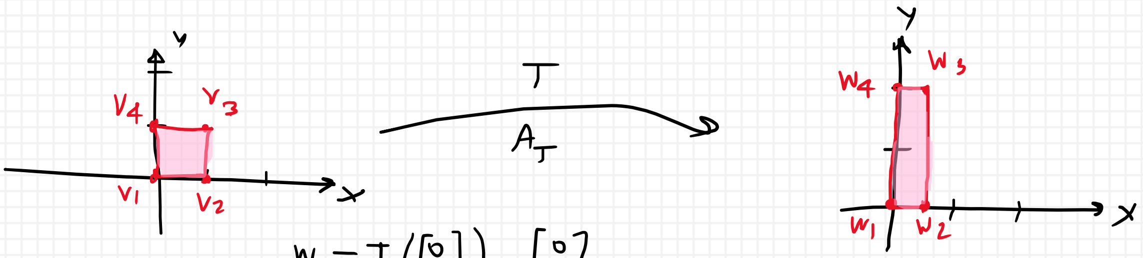
Graphical Intuition in Two Dimensions

Let us gain some intuition for determinants, eigenvectors, and eigenvalues using different linear mappings. Figure 4.4 depicts five transformation matrices $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_5$ and their impact on a square grid of points, centered at the origin:

In geometry, the area-preserving properties of this type of shearing parallel to an axis is also known as Cavalieri's principle of equal areas for parallelograms

- $\boldsymbol{A}_1 = \begin{bmatrix}\frac{1}{2} & 0\\0 & 2\end{bmatrix}$. The direction of the two eigenvectors correspond to the canonical basis vectors in $\mathbb{R}^2$, i.e., to two cardinal axes. The vertical axis is extended by a factor of 2 (eigenvalue $\lambda_1 = 2$), and the horizontal axis is compressed by factor $\frac{1}{2}$ (eigenvalue $\lambda_2 = \frac{1}{2}$). The mapping is area preserving ($\det(\boldsymbol{A}_1) = 1 = 2 \cdot \frac{1}{2}$).

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}\frac{1}{2}x\\2y\end{bmatrix}$$



$$w_1 = T\left(\begin{bmatrix}0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\0\end{bmatrix}$$

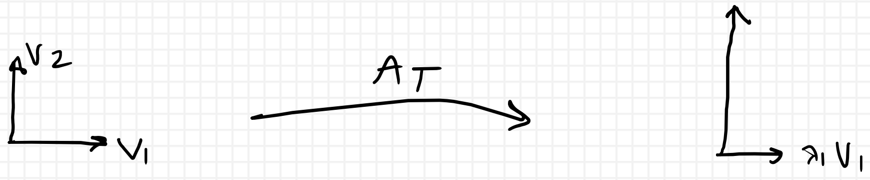
$$w_2 = T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}$$

$$w_3 = T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}\frac{1}{2}\\2\end{bmatrix}$$

$$w_4 = T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\2\end{bmatrix}$$

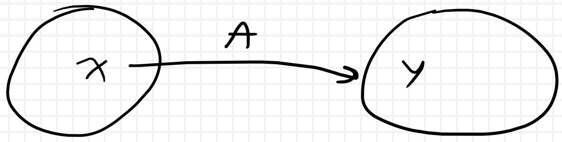
$$|A_T| = \begin{vmatrix}\frac{1}{2} & 0\\0 & 2\end{vmatrix} = 1$$

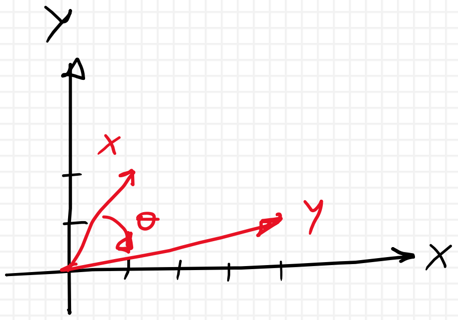
$$A_T = \begin{bmatrix}\frac{1}{2} & 0\\0 & 2\end{bmatrix} \qquad \lambda_1 = \frac{1}{2} \qquad \lambda_2 = 2$$



SINGULAR VALUE DECOMPOSITION

$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$

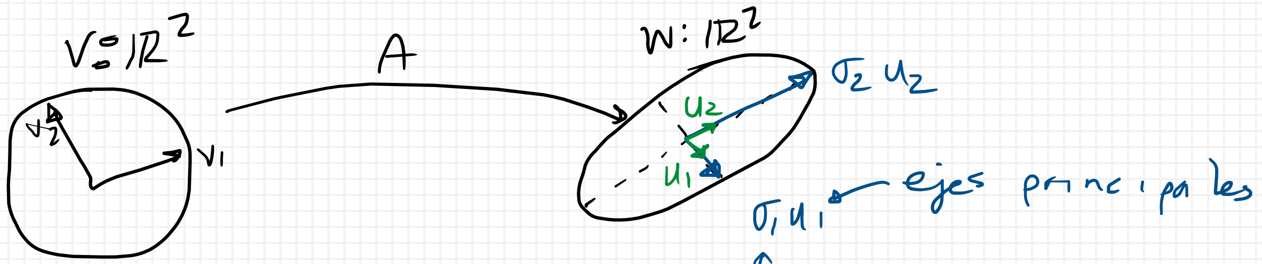




$y = Ax = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{bmatrix}$, $\quad A = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$

rotación en un ángulo θ $\qquad$ alargamiento



$\mathcal{B}_V = \{v_1, v_2\}$

círculo $\qquad$ elipse

* para $\mathbb{R}^n$

esfera de dimensión n $\xrightarrow{A}$ elipsoide de dimensión n

$v_1, v_2, \ldots, v_n$ ← vectores ortogonales

$u_1, u_2, \ldots, u_n$ ← vectores ortonormales

$\sigma_1, \sigma_2, \ldots, \sigma_n$ ← alargamiento

$Av_1 = \sigma_1 u_1 \qquad Av = \lambda v$

$Av_j = \sigma_j u_j, \quad j = 1, 2, \ldots, n$

$$\underbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}}_{A \in \mathbb{C}^{n\times n}} \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}}_{V \in \mathbb{C}^{n\times n}} = \underbrace{\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}}_{U \in \mathbb{C}^{n\times n}} \underbrace{\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}}_{\Sigma \in \mathbb{C}^{n\times n}}$$

$AV = U\Sigma$, notar que U y V son matrices ortogonales

entonces $U^{-1} = U^*$ y $V^{-1} = V^*$

Teorema de SVD: cada matriz $A \in \mathbb{C}^{n\times n}$ tiene una descomposición en valores singulares

* valores singulares $\{\sigma_j\}$ son determinados de manera unívoca y si A es cuadrada σ_j son distintos
* $\{u_j\}$ y $\{v_j\}$ son únicos.

$$AV = U\Sigma$$

Desp. A y nos queda

$$A = U\Sigma V^*$$ → Podimos descomponer a la matriz A como la multiplicación de 3 matrices.

Desconocemos U, V, Σ

$$A^T(A = U\Sigma V^*)$$
$$A^TA = A^T U\Sigma V^*$$
$$A^TA = (U\Sigma V^*)^T U\Sigma V^*$$
$$A^TA = V\Sigma U^TU\Sigma V^*$$
$$A^TA = V\Sigma I\Sigma V^*$$
$$A^TA = V\Sigma^2V^*$$
$$A^TAV = V\Sigma^2V^*V$$
$$A^TAV = V\Sigma^2 I$$
$$\boxed{A^TAV = V\Sigma^2}$$

$$Av = \lambda v$$

obtenemos V y Σ

$$(A = U\Sigma V^*)A^T$$
$$AA^T = U\Sigma V^*(U\Sigma V^*)^T$$
$$AA^T = U\Sigma V^* V\Sigma U^T$$
$$AA^T = U\Sigma^2U^T$$
$$AA^TU = U\Sigma^2U^TU$$
$$\boxed{AA^TU = U\Sigma^2}$$

$$Av = \lambda v$$

obtenemos U y Σ

$$\lambda_j = \sigma_j^2 \rightarrow \sigma_j = \sqrt{\lambda_j}$$

valores y vectores propios de $A^TA \rightarrow V$ y Σ

$$V = \left[\begin{array}{c|c|c|c} v_1 & v_2 & \cdots & v_n \end{array}\right] , \quad \Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}$$

$$\sigma_1 = \sqrt{\lambda_1}, \ \sigma_2 = \sqrt{\lambda_2}, \ldots, \ \sigma_n = \sqrt{\lambda_n}$$

valores y vectores propios de $AA^T \rightarrow U$ y Σ

APLICACIÓN DE SVD EN PROCESAMIENTO DE IMÁGENES

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Partimos de la función $z(x,y) = x^2y - x^2 - y^2$, definida en un intervalo de $-5 \le x \le 5$ y $-5 \le y \le 5$, igualmente espaciadas con enteros

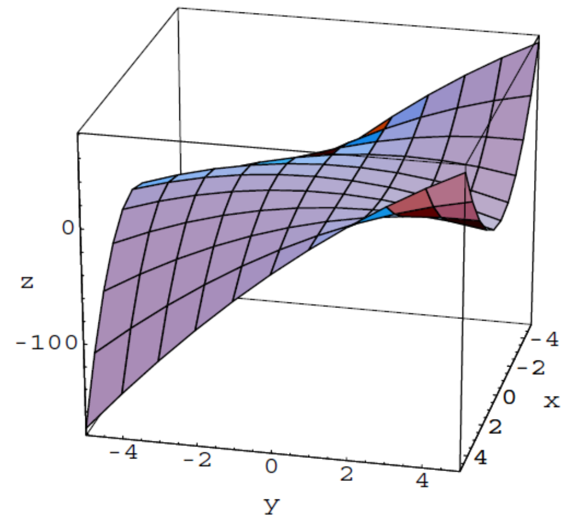

Figure 1: $z = x^2y - x^2 - y^2$ using 11×11 grid

$11 \times 11 = 121$ datos

Si nosotros formamos la sig. matriz:

$$A = \begin{bmatrix} -175 & -141 & -109 & -79 & -51 & -25 & -1 & 21 & 41 & 59 & 75 \\ -121 & -96 & -73 & -52 & -33 & -16 & -1 & 12 & 23 & 32 & 39 \\ -79 & -61 & -45 & -31 & -19 & -9 & -1 & 5 & 9 & 11 & 11 \\ -49 & -36 & -25 & -16 & -9 & -4 & -1 & 0 & -1 & -4 & -9 \\ -31 & -21 & -13 & -7 & -3 & -1 & -1 & -3 & -7 & -13 & -21 \\ -25 & -16 & -9 & -4 & -1 & 0 & -1 & -4 & -9 & -16 & -25 \\ -31 & -21 & -13 & -7 & -3 & -1 & -1 & -3 & -7 & -13 & -21 \\ -49 & -36 & -25 & -16 & -9 & -4 & -1 & 0 & -1 & -4 & -9 \\ -79 & -61 & -45 & -31 & -19 & -9 & -1 & 5 & 9 & 11 & 11 \\ -121 & -96 & -73 & -52 & -33 & -16 & -1 & 12 & 23 & 32 & 39 \\ -175 & -141 & -109 & -79 & -51 & -25 & -1 & 21 & 41 & 59 & 75 \end{bmatrix}$$

$z(0,0) = x^2y - x^2 - y^2 = 0$

$z(1,0) = 1^2(0) - 1^2 - 0^2 = -1$

$z(5,5) = 5^2 5 - 5^2 - 5^2 = 75$

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La matriz A la vamos a descomponer aplicando SVD.

```
%% SVD Proyecto 1
close;
clear;
clc;

A = [ -175  -141  -109   -79   -51   -25    -1    21    41    59    75
  -121   -96   -73   -52   -33   -16    -1    12    23    32    39
   -79   -61   -45   -31   -19    -9    -1     5     9    11    11
   -49   -36   -25   -16    -9    -4    -1     0    -1    -4    -9
   -31   -21   -13    -7    -3    -1    -1    -3    -7   -13   -21
   -25   -16    -9    -4    -1     0    -1    -4    -9   -16   -25
   -31   -21   -13    -7    -3    -1    -1    -3    -7   -13   -21
   -49   -36   -25   -16    -9    -4    -1     0    -1    -4    -9
   -79   -61   -45   -31   -19    -9    -1     5     9    11    11
  -121   -96   -73   -52   -33   -16    -1    12    23    32    39
  -175  -141  -109   -79   -51   -25    -1    21    41    59    75
];

[U,S,V] = svd(A)
[V1,D1,W1] = eig(A*A');
[V2,D2,W2] = eig(A'*A);
```

La matriz A se puede escribir como

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T$$

Podemos observar que si consideramos

$$A_1 = \sigma_1 u_1 v_1^T \qquad \dim(u_1) = 11 \times 1$$

$$A_2 = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T \qquad \dim(v_1) = 11 \times 1$$

A_1 requiere $11 + 11 + 1 = 23$ (19 %)

A_2 requiere $(11)4 + 2 = 46$ (38 %)