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16.5 1-8 Determine a) el rotacional y b) la divergencia del campo vectorial.

$$F(x,y,z) = \underbrace{(x+yz)}_{x}\,i + \underbrace{(y+xz)}_{y}\,j + \underbrace{(z+xy)}_{z}\,k$$

• Formula: Divergencia

$$\operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot F$$

$$\operatorname{rot} F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) i + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) k$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Divergencia:

$$\frac{\partial P}{\partial x} = 1 \qquad \frac{\partial Q}{\partial y} = 1 \qquad \frac{\partial R}{\partial z} = 1 \Rightarrow 1+1+1 = \underline{\underline{3}}$$

Rotacional:

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \underset{P}{x+yz} & \underset{Q}{y+xz} & \underset{R}{z+xy} \end{vmatrix}$$

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = (x) - (x) = 0 \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (z) - (z) = 0$$

$$\frac{\partial R}{\partial y} - \frac{\partial P}{\partial z} = (y) - (y) = 0 \Rightarrow 0 + 0 + 0 = \underline{\underline{0}}$$

2.-

$$\mathbf{F}(x,y,z) = \underbrace{\overset{P}{xy^2z^3}}_{x}\,\mathbf{i} + \underbrace{\overset{Q}{x^3yz^2}}_{y}\,\mathbf{j} + \underbrace{\overset{R}{x^2y^3z}}_{z}\,\mathbf{K}$$

• Divergencia $\nabla \cdot F =$

$$\frac{\partial P}{\partial x} = y^2z^3 + \frac{\partial Q}{\partial y} = x^3z^2 + \frac{\partial R}{\partial z} = x^2y^3$$

$$\Rightarrow y^2z^3 + x^3z^2 + x^2y^3$$

• Rotacional $\nabla \times F =$

$$\begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \underbrace{xy^2z^3}_{P} & \underbrace{x^3yz^2}_{Q} & \underbrace{x^2y^3z}_{R} \end{vmatrix} = \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = 3x^2y^2z - 2x^3yz$$

$$\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} = 2xy^3z - 3xy^2z^2 \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^2yz^2 - 2xyz^3$$

$$\Rightarrow (3x^2y^2z - 2x^3yz)\,\mathbf{i} - (2xy^3z - 3xy^2z^2)\,\mathbf{j} + (3x^2yz^2 - 2xyz^3)\,\mathbf{K}$$

$$= x^2yz(3y - 2x)\,\mathbf{i} - xy^2z(2y - 3z)\,\mathbf{j} + xyz^2(3x - 2z)\,\mathbf{K}$$

3.- $\mathbf{F}(x,y,z) = xye^z\,\mathbf{i} + yze^x\,\mathbf{K}$

• Divergencia

$$\frac{\partial P}{\partial x} = ye^z \qquad \frac{\partial Q}{\partial y} = 0 \qquad \frac{\partial R}{\partial z} = ye^x$$

$$\Rightarrow ye^z + ye^x = \underline{\underline{y(e^z + e^x)}}$$

• Rotacional

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \underset{P}{xye^z} & \underset{Q}{0} & \underset{R}{yze^x} \end{vmatrix}$$

$\frac{\partial R}{\partial y} - \cancel{\frac{\partial Q}{\partial z}} = ze^x$ $\quad$ $\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} = yze^x - xye^z$

$\cancel{\frac{\partial Q}{\partial x}} - \frac{\partial P}{\partial y} = -xe^z \Rightarrow ze^x i - y(ze^x - xe^z)j + (-xe^z)k$

8.-

Divergencia $\nabla \cdot F$ $\quad F(x,y,z) = \left\langle \underset{P}{\frac{x}{y}}, \underset{Q}{\frac{y}{z}}, \underset{R}{\frac{z}{x}} \right\rangle$

$\frac{\partial P}{\partial x} = \frac{x}{y} = xy^{-1} = \frac{1}{y}$ $\quad$ $\frac{\partial Q}{\partial y} = \frac{1}{z}$ $\quad$ $\frac{\partial R}{\partial z} = \frac{1}{x}$

$\Rightarrow \frac{1}{y} + \frac{1}{z} + \frac{1}{x}$

Rotacional:

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \underset{P}{\frac{x}{y}} & \underset{Q}{\frac{y}{z}} & \underset{R}{\frac{z}{x}} \end{vmatrix} = \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = 0 - \left(-\frac{y}{z^2}\right)$$

$\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} = \left(-\frac{z}{x^2}\right) - 0$ $\quad$ $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - \left(-\frac{x}{y^2}\right)$

$\Rightarrow \frac{y}{z^2} i + \frac{z}{x^2} + \frac{x}{y^2}$

30.- Sea $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ y $r = |\mathbf{r}|$

a) $\nabla \cdot \mathbf{r} = 3$ = Divergencia

$$\frac{\partial p}{\partial x} = 1 \quad \frac{\partial Q}{\partial y} = 1 \quad \frac{\partial R}{\partial z} = 1 = 1+1+1 = 3$$

B) $\nabla \cdot (r\mathbf{r}) = 4r$

$$\nabla \cdot (r\mathbf{r}) = (\nabla r) \cdot \mathbf{r} + r\nabla \cdot \mathbf{r}$$

$$\nabla r = \frac{\partial r}{\partial x}\mathbf{i} + \frac{\partial r}{\partial y}\mathbf{j} + \frac{\partial r}{\partial z}\mathbf{k}$$

$$\nabla \mathbf{r} = \frac{\partial r_x}{\partial x} + \frac{\partial r_y}{\partial r} + \frac{\partial r_z}{\partial z}$$

$$\mathbf{r} = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}$$

$$\nabla \cdot (r\mathbf{r}) = \left(\frac{\partial r}{\partial x}\mathbf{i} + \frac{\partial r}{\partial y}\mathbf{j} + \frac{\partial r}{\partial z}\mathbf{k}\right) \cdot (r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}) + r\left(\frac{\partial r_x}{\partial x} + \frac{\partial r_y}{\partial y} + \frac{\partial r_z}{\partial z}\right)$$

$$\Rightarrow \left[r_x\left(\frac{\partial r}{\partial x}\right) + r_y\left(\frac{\partial r}{\partial y}\right) + r_z\left(\frac{\partial r}{\partial z}\right)\right] + \left(r\frac{\partial r_x}{\partial x} + r\frac{\partial r_y}{\partial y} + r\frac{\partial r_z}{\partial z}\right)$$

$$\Rightarrow r(x,y,z)\left(\frac{\partial r}{\partial x} + \frac{\partial r}{\partial y} + \frac{\partial r}{\partial z}\right) \Rightarrow r + r + r + r = \underline{\underline{4r}}$$

C) $\nabla^2 r^3$ $\quad r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2} \therefore r^3 \Rightarrow (x^2+y^2+z^2)^{3/2}$

$$\nabla^2 r^3 = \frac{\partial^2 r^3}{\partial x} \Rightarrow \frac{\partial'}{\partial x}(x^2+y^2+z^2)^{3/2} = 3x\underbrace{(x^2+y^2+z^2)^{1/2}}_{r}$$

$$\frac{\partial''}{\partial x} = 3\frac{\partial}{\partial x}(rx) = \underline{\underline{\frac{3\,\partial r_x}{\partial x}}}$$

$$\nabla^2 r^3 = \frac{\partial^2 r^3}{\partial y} = \frac{\partial'}{\partial y}(x^2+y^2+z^2)^{3/2} = 3y(x^2+y^2+z^2)^{1/2}$$

$$\frac{\partial''}{\partial y} \quad 3\frac{\partial}{\partial y}(ry) = \frac{3\partial r_y}{\partial y}$$

$$\nabla^2 r^3 = \frac{\partial^2 r^3}{\partial z} = \frac{\partial^2}{\partial z}(x^2+y^2+z^2)^{3/2} = 3z(x^2+y^2+z^2)^{1/2} = 0$$

$$\frac{\partial''}{\partial z} = \frac{3\partial(rz)}{\partial z} = \frac{3\partial rz}{\partial z}$$

Entonces: $3\frac{\partial rx}{\partial x} + \frac{3\partial ry}{\partial y} + \frac{3\partial rz}{\partial z} = 3\left(\frac{\partial rx}{\partial x} + \frac{\partial ry}{\partial y} + \frac{\partial rz}{\partial z}\right)$

Si $\nabla\cdot(r\mathbf{r}) = \frac{\partial rx}{\partial x} + \frac{\partial ry}{\partial y} + \frac{\partial rz}{\partial z} = 4r$

sustituyendo

$3(4r) = \underline{\underline{12r}}$

31.- Verifique cada una de las entidades.

$\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad r = \|\mathbf{r}\|$

a) $\nabla r = \mathbf{r}/r$

$\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \qquad r = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}}\cancel{2}x(x^2+y^2+z^2)^{-1/2} = \frac{x}{(x^2+y^2+z^2)^{1/2}} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{1}{\cancel{2}}\cancel{2}y(x^2+y^2+z^2)^{-1/2} = \frac{y}{(x^2+y^2+z^2)^{1/2}} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{1}{2}2z(x^2+y^2+z^2)^{-1/2} = \frac{z}{(x^2+y^2+z^2)^{1/2}} = \frac{z}{r}$$

$$\nabla r = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k} = \frac{(x\hat{i}+y\hat{j}+z\hat{k})}{r} \Rightarrow \frac{\mathbf{r}}{r}$$

b) $\vec{\nabla}\times\mathbf{r} = 0$

$$\nabla\times r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) = 1 - 1 = 0$$

$$= \frac{\partial z}{\partial x} - \frac{\partial x}{\partial z} = 1 - 1 = 0$$

$$\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} = 1 - 1 = 0 \qquad 0+0+0 = 0$$

$\nabla\times\mathbf{r} = 0$

c) $\nabla(1/r) = -\mathbf{r}/r^3$ $\quad = r = (x^2+y^2+z^2)^{1/2} = \frac{1}{r} \quad \frac{1}{(x^2+y^2+z^2)}$

$$\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

$$\frac{\partial r}{\partial x} (x^2+y^2+z^2)^{-1/2} = -\frac{1}{\cancel{2}} \cancel{2}x (x^2+y^2+z^2)^{-3/2} = -x(x^2+y^2+z^2)^{-3/2}$$

$$= \frac{x}{(x^2+y^2+z^2)^{3/2}} = \frac{x}{\sqrt[3]{x^2+y^2+z^2}} = -\frac{x}{r^3} \therefore \frac{\partial r}{\partial y} = \frac{-y}{r^3} \quad \frac{\partial r}{\partial z} = \frac{-z}{r^3}$$

$$-\overbrace{\left(\frac{x}{r^3} i + \frac{y}{r^3} j + \frac{z}{r^3} k\right)}^{r} = \boxed{\frac{-\mathbf{r}}{r^3}}$$

d) $\nabla \ln(r) = \mathbf{r}/r^2$ $\quad r = (x^2+y^2+z^2)^{1/2}$

$$\nabla \ln\left[(x^2+y^2+z^2)^{1/2}\right] = \frac{1}{(x^2+y^2+z^2)^{1/2}} \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2} =$$

$$\frac{1}{(x^2+y^2+z^2)^{1/2}} \frac{1}{\cancel{2}} \cancel{2}x (x^2+y^2+z^2)^{-1/2} = \underbrace{\frac{1}{(x^2+y^2+z^2)^{1/2}}}_{r} \cdot \frac{x}{\underbrace{(x^2+y^2+z^2)^{1/2}}_{r}}$$

$$= \frac{\partial x}{\partial x} \frac{x}{r^2} \therefore \frac{\partial}{\partial y} \frac{y}{r^2} \therefore$$

$$\frac{\partial}{\partial z} \frac{z}{r^2} = \frac{\overbrace{x i + y j + z k}^{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^2}$$