$$\begin{array}{ccccccc}
\mathbf{c}_1 = \mathbf{0} & \mathbf{c}_2 & \mathbf{c}_3 & \cdots & \mathbf{c}_i & \cdots & \mathbf{c}_{2^k} \\
\mathbf{e}_2 & \mathbf{c}_2 + \mathbf{e}_2 & \mathbf{c}_3 + \mathbf{e}_2 & \cdots & \mathbf{c}_i + \mathbf{e}_2 & \cdots & \mathbf{c}_{2^k} + \mathbf{e}_2 \\
\mathbf{e}_3 & \mathbf{c}_2 + \mathbf{e}_3 & \mathbf{c}_3 + \mathbf{e}_3 & \cdots & \mathbf{c}_i + \mathbf{e}_3 & \cdots & \mathbf{c}_{2^k} + \mathbf{e}_3 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
\mathbf{e}_j & \mathbf{c}_2 + \mathbf{e}_j & \mathbf{c}_3 + \mathbf{e}_j & \cdots & \mathbf{c}_i + \mathbf{e}_j & \cdots & \mathbf{c}_{2^k} + \mathbf{e}_j \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
\mathbf{e}_{2^{n-k}} & \mathbf{c}_2 + \mathbf{e}_{2^{n-k}} & \mathbf{c}_3 + \mathbf{e}_{2^{n-k}} & & \mathbf{c}_i + \mathbf{e}_{2^{n-k}} & & \mathbf{c}_{2^k} + \mathbf{e}_{2^{n-k}}
\end{array}$$

Figure 10.7 Standard array for an (n,k) block code.

the error pattern chosen as the first element in a row has not previously appeared in the standard array.

3. Step 2 is repeated until all the possible error patterns have been accounted for.

Figure 10.7 illustrates the structure of the standard array so constructed. The 2^k columns of this array represent the disjoint subsets $D_1, D_2, \ldots, D_{2^k}$. The 2^{n-k} rows of the array represent the cosets of the code, and their first elements $\mathbf{e}_2, \ldots, \mathbf{e}_{2^{n-k}}$ are called *coset leaders*.

For a given channel, the probability of decoding error is minimized when the most likely error patterns (i.e., those with the largest probability of occurrence) are chosen as the coset leaders. In the case of a binary symmetric channel, the smaller we make the Hamming weight of an error pattern, the more likely it is for an error to occur. Accordingly, the standard array should be constructed with each coset leader having the minimum Hamming weight in its coset.

We are now ready to describe a decoding procedure for linear block codes:

1. For the received vector $\mathbf{r}$, compute the syndrome $\mathbf{s} = \mathbf{r}\mathrm{H}^{\mathrm{T}}$.
2. Within the coset characterized by the syndrome $\mathbf{s}$, identify the coset leader (i.e., the error pattern with the largest probability of occurrence); call it $\mathbf{e}_0$.
3. Compute the code vector

$$\mathbf{c} = \mathbf{r} + \mathbf{e}_0 \tag{10.26}$$

as the decoded version of the received vector $\mathbf{r}$.

This procedure is called *syndrome decoding*.

EXAMPLE 1

Hamming Codes

For any positive integer $m \geq 3$, there exists a linear block code with the following parameters:

code length	$n = 2^m - 1$
number of message bits	$k = 2^m - m - 1$
number of parity-check bits	$n - k = m$

Such a linear block code for which the error-correcting capability $t = 1$ is called a Hamming code.[3] To be specific, consider the example of $m = 3$, yielding the $(7,4)$ Hamming code with $n = 7$ and $k = 4$. The generator of this code is defined by

$$\mathbf{G} = \left[\begin{array}{ccc:cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$
$$\qquad\qquad \underbrace{}_{\mathbf{P}} \qquad \underbrace{}_{\mathbf{I}_k}$$

which conforms to the systematic structure of (10.12).

The corresponding parity-check matrix is given by

$$\mathbf{H} = \left[\begin{array}{ccc:cccc} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right]$$
$$\qquad\qquad \underbrace{}_{\mathbf{I}_{n-k}} \qquad \underbrace{}_{\mathbf{P}^{\mathrm{T}}}$$

The operative property embodied in this equation is that the columns of the parity-check matrix **P** consist of all the nonzero m-tuples, where $m = 3$.

With $k = 4$, there are $2^k = 16$ distinct message words, which are listed in Table 10.1. For a given message word, the corresponding codeword is obtained by using (10.13). Thus, the application of this equation results in the 16 codewords listed in Table 10.1.

In Table 10.1, we have also listed the Hamming weights of the individual codewords in the (7,4) Hamming code. Since the smallest of the Hamming weights for the nonzero codewords is 3, it follows that the minimum distance of the code is 3, which is what it should be by definition. Indeed, all Hamming codes have the property that the minimum distance $d_{\min} = 3$, independent of the value assigned to the number of parity bits m.

To illustrate the relation between the minimum distance $d_{\min}$ and the structure of the parity-check matrix **H**, consider the codeword 0110100. In matrix multiplication, defined

Table 10.1 Codewords of a (7,4) Hamming code

Message word	Codeword	Weight of codeword	Message word	Codeword	Weight of codeword
0000	0000000	0	1000	1101000	3
0001	1010001	3	1001	0111001	4
0010	1110010	4	1010	0011010	3
0011	0100011	3	1011	1001011	3
0100	0110100	3	1100	1011100	4
0101	1100101	4	1101	0001101	3
0110	1000110	3	1110	0101110	4
0111	0010111	4	1111	1111111	7

by (10.16), the nonzero elements of this codeword "sift" out the second, third, and fifth columns of the matrix **H**, yielding

$$\begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

We may perform similar calculations for the remaining 14 nonzero codewords. We thus find that the smallest number of columns in **H** that sums to zero is 3, reconfirming the defining condition $d_{\min} = 3$.

An important property of binary Hamming codes is that they satisfy the condition of (10.25) with the equality sign, assuming that $t = 1$. Thus, assuming single-error patterns, we may formulate the error patterns listed in the right-hand column of Table 10.2. The corresponding eight syndromes, listed in the left-hand column, are calculated in accordance with (10.20). The zero syndrome signifies no transmission errors.

Suppose, for example, the code vector [1110010] is sent and the received vector is [1100010] with an error in the third bit. Using (10.19), the syndrome is calculated to be

$$\mathbf{s} = [1100010]\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\1&1&0\\0&1&1\\1&1&1\\1&0&1\end{bmatrix}$$

$$= \begin{bmatrix}0&0&1\end{bmatrix}$$

From Table 10.2 the corresponding coset leader (i.e., error pattern with the highest probability of occurrence) is found to be [0010000], indicating correctly that the third bit of the received vector is erroneous. Thus, adding this error pattern to the received vector, in accordance with (10.26), yields the correct code vector actually sent.

Table 10.2 Decoding table for the (7,4) Hamming code defined in Table 10.1

Syndrome	Error pattern
000	0000000
100	1000000
010	0100000
001	0010000
110	0001000
011	0000100
111	0000010
101	0000001