Ma 221 - Course Overview

Definitions and Terminology

- Classifications
  - Ordinary or partial
  - Order
  - Linear or non-linear
- Solutions
  - Explicit or implicit
  - Interval of validity
- Initial value problem
  - Existence of unique solution

First Order Differential Equations

Autonomous D.E.

$$\frac{dy}{dx} = f(y)$$

- Critical points & constant solutions
  - Phase portrait
  - Classification
    - Asymptotically stable (Attractor)
    - Unstable (Repeller)
    - Semi-stable (Neither)

Separable Equations

$$\frac{dy}{dx} = g(x)p(y)$$
$$h(y)dy = g(x)dx$$
$$\int h(y)dy = \int g(x)dx$$

Linear Equations

$$\frac{dy}{dx} + p(x)y = q(x)$$

Integrating Factor

$$IF = e^{\int p(x)dx}$$

$$e^{\int p(x)dx}\left(\frac{dy}{dx} + p(x)y\right) = e^{\int p(x)dx}[q(x)]$$
$$\frac{d}{dx}\left(e^{\int p(x)dx}y\right) = e^{\int p(x)dx}[q(x)]$$

Exact Equations

$$M(x,y)dx + N(x,y)dy = 0$$

Test for exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

When test is passed, there is $F(x,y)$ such that

$$M = \frac{\partial F}{\partial x} \qquad and \quad N = \frac{\partial F}{\partial y}$$

Find $F(x,y)$.
Solution is

$$F(x,y) = c$$

Bernoulli D.E.

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = q(x)$$

Substitution

$$z = y^{1-n}$$

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

yields a d.e. in z.

$$\frac{1}{1-n}\frac{dz}{dx} + p(x)z = q(x)$$

$$\frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x)$$

which is a linear d.e. in z. After solving for z, don't forget to go back to y,

In all cases, the arbitrary constant resulting from integration is used to satisfy any initial condition.

Second Order Linear Differential Equations

Form of equation

$$L[y] = a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

When $g(x) = 0$, the d.e. is homogeneous, otherwise non-homogeneous.

Form of general solution

Homogeneous d.e.

$$y_c = c_1y_1 + c_2y_2$$

where y_1 and y_2 are linearly independent solutions of the homogeneous equation.

Non-homogeneous d.e

$$y = y_c + y_p$$

where y_p is a [particular] solution of the non-homogeneous equation.

Wronskian

A test for linear independence of solutions

$$W[y_1,y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

$W[y_1,y_2] \neq 0$ for linearly independent solutions of the homogeneous d.e.

Superposition

Homogeneous d.e.

If $L[y_1] = 0$ and $L[y_2] = 0$, then $L[c_1y_1 + c_2y_2] = 0$, for any constants c_1 and c_2.

Nonhomogeneous d.e

If $L[y_1] = g_1(x)$ and $L[y_2] = g_2(x)$, then $L[y_1 + y_2] = g_1(x) + g_2(x)$.

Homogeneous D.E.

Constant coefficients -

$$ay'' + by' + cy = 0$$

Solve auxiliary (characteristic) equation -

$$p(m) = am^2 + bm + c = 0$$

2 real roots

$$m = m_1, m_2$$
$$y_c = c_1e^{m_1x} + c_2e^{m_2x}$$

repeated real roots

$$m = m_1$$
$$y_c = (c_1 + c_2x)e^{m_1x}$$

2 complex roots

$$m = \alpha \pm i\beta$$
$$y_c = (c_1\cos\beta x + c_2\sin\beta x)e^{\alpha x}$$

Cauchy-Euler D.E.

$$ax^2y'' + bxy' + cy = 0$$

Solve auxiliary (indicial) equation -

$$am^2 + (b-a)m + c = 0$$

2 real roots

$$m = m_1, m_2$$

$$y_c = c_1x^{m_1} + c_2x^{m_2}$$

repeated real roots

$$m = m_1$$

$$y_c = (c_1 + c_2 \ln x)x^{m_1}$$

2 complex roots

$$m = \alpha \pm i\beta$$

$$y_c = [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]x^{\alpha}$$

Non-homogeneous D.E.

Undetermined coefficients

Constant coefficient d.e.

$$ay'' + by' + cy = f(x)$$

$$f(x) = ce^{ax}$$

$$f(x) = (A\cos\beta x + B\sin\beta x)e^{\alpha x}$$

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

products of above

Be careful if $f(x)$ is a solution of the homogeneous equation.

Variation of parameters

$$y'' + p(x)y' + q(x)y = f(x)$$

$$y = v_1y_1 + v_2y_2$$

$$y_1v_1' + y_2v_2' = 0$$

$$y_1'v_1' + y_2'v_2' = f(x)$$

The solution of these linear equations is

$$v_1' = \frac{-f(x)y_2}{[y_1y_2' - y_2y_1']} = \frac{-f(x)y_2}{W[y_1,y_2]}$$

$$v_2' = \frac{f(x)y_1}{[y_1y_2' - y_2y_1']} = \frac{f(x)y_1}{W[y_1,y_2]}$$

Integration completes the solution.

Note: These formulae assume that the coefficient of y” is 1.

Mathematical Modeling

Spring/Mass System

Free damped motion

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0$$

β is the damping coefficient and k is the spring rate. Since they represent physical quantities, both are positive. We rewrite the equation.:

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

The auxiliary equation is $m^2 + 2\lambda m + \omega^2 = 0$. The discriminant, $\lambda^2 - \omega^2$, which determines the nature of the solution has physical meaning.

$\lambda^2 - \omega^2 > 0$: Overdamped system. Two exponential solutions with negative exponents.
$\lambda^2 - \omega^2 = 0$: Critically damped system. The dividing case. Solution appears similar to the above case.
$\lambda^2 - \omega^2 < 0$: Underdamped system. Complex conjugate roots with negative real parts. Some oscillation with decreasing amplitude.

Forced damped motion

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(x)$$

As usual, the solution is the sum of the general solution to the homogeneous equation and a particular solution.

$$y = y_c + y_p$$

In all cases described above the complementary function contains exponential functions with negative exponents and hence goes to zero with time. Such functions are called transients and the steady state solution comes entirely from the particular solution

Laplace Transforms

Definition

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$
$$= F(s) = \hat{f}(s)$$

Calculate Laplace Transform from definition

Properties

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$
$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y\} - y(0)$$
$$\mathcal{L}\{y''(t)\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$$
$$\mathcal{L}\{f(t-a)U(t-a)\} = e^{-as}F(s)$$

Note: $U(t-a)$ in the last line is the unit step function. $U(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a. \end{cases}$

Inverse Laplace Transform

Partial Fractions

Use of Laplace transform for a solution of initial value problems

Transform the differential equation $[y \rightarrow Y = \mathcal{L}\{y\}]$

Solve for the transform of the solution

Apply the inverse transform to obtain the solution $[Y \rightarrow y = \mathcal{L}^{-1}\{y\}]$

Partial Differential Equations

Separation of Variables

$u(x,t) = X(x) \cdot T(t)$

Obtain ordinary differential equations for $X(x)$ and $T(t)$.

Boundary Value Problems

Eigenvalues and eigenfunctions

$$DE: \quad L[y] + \lambda y = 0$$
$$BC: \quad \alpha_1 y(a) + \beta_1 y'(a) = 0$$
$$BC: \quad \alpha_2 y(b) + \beta_2 y'(b) = 0$$

Three cases to be examined (discriminant positive, zero or negative)

Fourier Series

Full Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}\right)x + b_n \sin\left(\frac{n\pi}{L}\right)x\right]$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left[\left(\frac{n\pi}{L}\right)x\right]dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left[\left(\frac{n\pi}{L}\right)x\right]dx$$

Convergence

$f(x)$ when f is continuous at x and $-L < x < L$

Average value, $\frac{f(x-)+f(x+)}{2}$, at jumps

Periodic extension with period $2L$

Special cases

Even function $f(-x) = f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}\right)x\right]$$

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left[\left(\frac{n\pi}{L}\right)x\right]dx$$

Odd function $f(-x) = -f(x)$

$$f(x) = \sum_{n=1}^{\infty}\left[b_n \sin\left(\frac{n\pi}{L}\right)x\right]$$

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left[\left(\frac{n\pi}{L}\right)x\right]dx$$

Half-range Fourier Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left[\left(\frac{n\pi}{L}\right)x\right]dx$$

Half-range Fourier Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left[\left(\frac{n\pi}{L}\right)x\right]dx$$

Convergence

$f(x)$ when f is continuous at x and $0 < x < L$

Average value, $\frac{f(x-)+f(x+)}{2}$, at jumps

Extension (odd for sine series, even for cosine series) $-L < x < L$

Periodic extension with period $2L$

Partial Differential Equations

Initial Boundary Values Problems

Heat Equation

Wave Equation

Procedure

Separation of Variables $u(x,t) = X(x)T(t)$

Eigenvalue problem from boundary conditions to obtain $X(x)$

Ordinary differential equation for $T(t)$

Combine and sum to obtain formal solution

Fourier expansion of initial conditions using eigenfunctions