

Let's talk about Thurstone & Co.: An information-theoretical model for comparative judgments, and its statistical translation

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Abstract

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Keywords: Probability, Directed Acyclic Graphs, Bayesian methods, Thurstonian model, Comparative judgement, Structural Causal Models, Statistical modeling

1. Introduction

In *comparative judgment* (CJ) studies, judges assess a specific trait or attribute across various stimuli by performing pairwise comparisons (Thurstone, 1927b,a). Each comparison produces a dichotomous outcome, indicating which stimulus is perceived to exhibit a higher trait level. For example, when assessing text quality, judges compare pairs of written texts (the stimuli) to determine the relative quality each text exhibit (the trait) (Laming, 2004; Pollitt, 2012; Whitehouse, 2012; van Daal et al., 2016; Lesterhuis, 2018; Coertjens et al., 2017; Goossens and De Maeyer, 2018; Bouwer et al., 2023).

Numerous studies have documented the effectiveness of CJ in assessing traits and competencies over the past decade. These studies have emphasized three aspects of the method's effectiveness: its reliability, validity, and practical applicability. Research on reliability indicates that CJ requires a relatively small number of pairwise comparisons (Verhavert et al., 2019; Crompvoets et al., 2022) to produce trait scores that are as precise and consistent as those generated by other assessment methods (Coertjens et al., 2017; Goossens and De Maeyer, 2018; Bouwer et al., 2023). Furthermore, evidence suggests that the reliability and time efficiency of CJ are comparable, if not superior, to those of other assessment methods when employing adaptive comparison algorithms (Pollitt, 2012; Verhavert et al., 2022; Mikhailiuk et al., 2021). On the other hand, research on

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Preprint submitted to Psychometrika

November 15, 2024

validity suggests that scores generated by CJ can accurately represent the traits under measurement (Whitehouse, 2012; van Daal et al., 2016; Lesterhuis, 2018; Bartholomew et al., 2018; Bouwer et al., 2023). Finally, research on practical applicability highlights the method’s versatility across both educational and non-educational contexts (Jones, 2015; Bartholomew et al., 2018; Jones et al., 2019; Marshall et al., 2020; Bartholomew and Williams, 2020; Boonen et al., 2020).

Nevertheless, despite the increasing number of CJ studies, unsystematic and fragmented research approaches have left several critical issues unaddressed. This research primarily focuses on three: the over-reliance on Thurstone’s Case V assumptions in the statistical analysis of CJ data, the apparent disconnect between CJ’s trait measurement and hypothesis testing, and the unclear role of comparison algorithms on the method’s reliability and validity. The following sections will discuss each of these issues in detail, followed by the introduction of a theoretical model and its statistical translation, which aims to address all three concerns simultaneously.

2. Three critical issues in CJ literature

In its most general form, Thurstone’s theory (1927a) posits that the dichotomous outcome resulting from comparing two stimuli is determined by two factors: the discriminial process of each stimulus and their discriminial difference. The *discriminal process* refers to the psychological effect each stimulus has on the judges, or more simply stated, the judges’ perception of the trait level of each stimulus. Thurstone assumes that the discriminial process for each stimulus follows a Normal distribution. In this distribution, the mode (mean), known as the *modal discriminial process*, represents the position of the stimulus on the trait continuum, while the dispersion, known as the *discriminal dispersion*, reflects variability in the perceived trait level of the stimulus. Figure 1 shows example distributions of discriminial process for two stimuli (objects).

However, since the discriminial mode and dispersion of a single stimulus are not directly observable without comparing it to other stimuli, the *law of comparative judgment* becomes essential. This law asserts that when assessing a specific trait by comparing two stimuli, the stimulus positioned further along the continuum is perceived as having a higher level of that trait. Thus, the observed dichotomous outcome is determined by the distribution of the difference between the discriminial processes of the two stimuli, called the *discriminal difference*. Figure 2 shows an example distribution of the discriminial difference for two stimuli (objects).

Importantly, the theory’s general form primarily addresses pairwise comparisons of stimuli made by a single judge (Thurstone, 1927a, pp. 267). Consequently, to enhance its practical applicability, Thurstone introduced five distinct cases, each defined by progressively simplifying assumptions. Figure 3 summarizes these cases, focusing on key assumptions such as the distribution of discriminial processes, the similarity of discriminial dispersion across stimuli, the correlation between stimuli, and which judges perform the comparisons. For a comprehensive discussion of this progression, refer to Thurstone (1927a) and Bramley (2008, pp. 248-253).

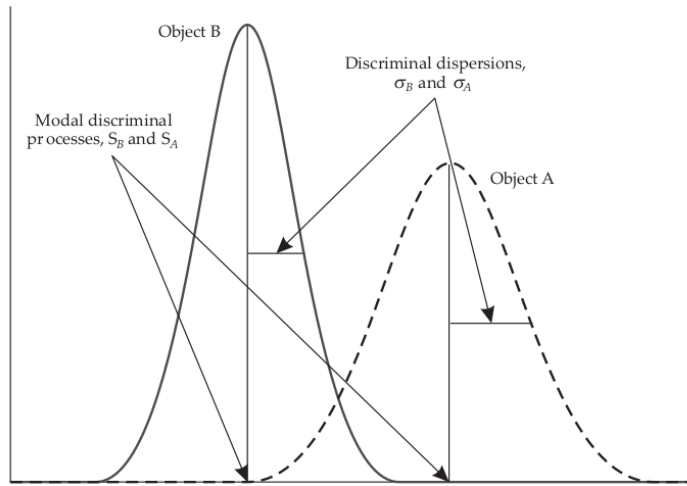


Figure 1: Example distributions of discriminative processes for two stimuli (objects). Extracted from [Bramley \(2008, pp. 249\)](#).

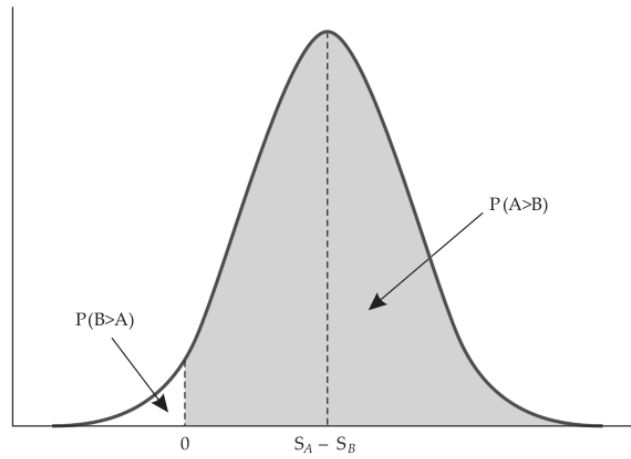


Figure 2: Distribution of the discriminative difference between two stimuli (objects). Extracted from [Bramley \(2008, pp. 251\)](#).

Assumption	Thurstone's					BTL model
	Case I	Case II	Case III	Case IV	Case V	
Discriminal process (distribution)	Normal	Normal	Normal	Normal	Normal	Logistic
Discriminal dispersion (between stimuli)	Different	Different	Different	Similar	Equal	Equal
Correlation (between stimuli)	Constant	Constant	Zero	Zero	Zero	Zero
Which judges compare?	Single	Multiple	Multiple	Multiple	Multiple	Multiple

Figure 3: Thurstones cases and assumptions

2.1. The Case V and the statistical analysis of CJ data

Surprisingly, despite its reliance on the largest number of simplifying assumptions (Bramley, 2008, pp. 253), Case V remains the most widely used case in the CJ literature. This popularity is largely due to its simplified statistical representation in the Bradley-Terry-Luce (BTL) model (Bradley and Terry, 1952; Luce, 1959). The BTL model mirrors Case V's assumptions, with one key difference: while Case V assumes a Normal distribution for the discriminational processes, the BTL model uses the more mathematically tractable logistic distribution (Bramley, 2008, pp. 254) (see Figure 3). This substitution has little effect on the model's estimation or interpretation, as the Normal and logistic distributions differ by a scaling factor of approximately 1.7 (van der Linden, 2017, pp. 16) (refer to Figure 4).

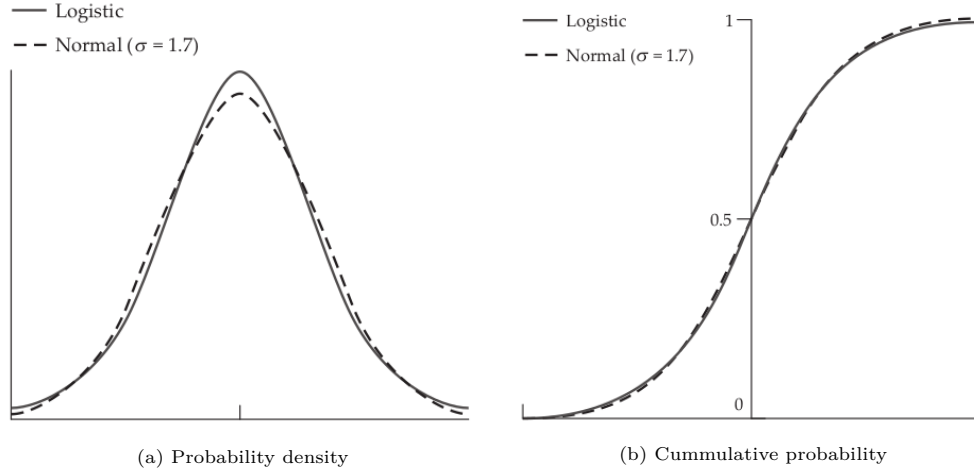


Figure 4: Probability density and cumulative probability of the logistic and Normal distributions. Extracted from Bramley (2008, pp. 254-255).

However, Case V (and its BTL model counterpart) was originally developed to provide a “rather coarse scaling” of traits (Thurstone, 1927a, pp. 269), prioritizing statistical simplicity over precision in trait measurement. As a result, its assumptions may not be suitable for applications beyond the psychophysical contexts, which typically involve

simple, homogeneous stimuli. Thurstone himself cautioned that its use “should not be made without (an) experimental test” (Thurstone, 1927a, pp. 270), further acknowledging that some assumptions could become problematic with less homogeneous stimuli, such as handwriting or English compositions (Thurstone, 1927b, pp. 374). Consequently, given that current CJ applications often involve more complex, less homogeneous stimuli, two assumptions of Case V may not always hold theoretical or in practice: the assumption of zero correlation and equal dispersion between stimuli.

Regarding the assumption of *zero correlation between stimuli* ...

2.2. The disconnect between trait measurement and hypothesis testing

Building on the previous section, it is evident that the BTL model typically functions as the measurement model for the trait under assessment (Andrich, 1978; Bramley, 2008). A measurement model specifies how manifest variables contribute to the estimation of latent variables (Everitt and Skrondal, 2010). For example, when evaluating text quality, researchers use the BTL model to process the dichotomous outcomes resulting from the pairwise comparisons (the manifest variables) to estimate scores that reflect the underlying quality level of the texts (the latent variable) (Laming, 2004; Pollitt, 2012; Whitehouse, 2012; van Daal et al., 2016; Lesterhuis, 2018; Coertjens et al., 2017; Goossens and De Maeyer, 2018; Bouwer et al., 2023).

Researchers then typically use the estimated BTL scores, or their transformations, to conduct additional analyses or hypothesis tests. For example, these scores have been used to identify ‘misfit’ judges and stimuli (Pollitt, 2012; van Daal et al., 2017; Goossens and De Maeyer, 2018), detect biases in judges’ ratings (Pollitt and Elliott, 2003; Pollitt, 2012), calculate correlations with other assessment methods (Goossens and De Maeyer, 2018; Bouwer et al., 2023), or test hypotheses related to the underlying trait of interest (Bramley and Vitello, 2019; Boonen et al., 2020; Bouwer et al., 2023; van Daal et al., 2017; Jones et al., 2019; Gijzen et al., 2021).

However, the statistical literature advises caution when using estimated scores to conduct additional analyses or hypotheses tests. A key consideration is that BTL scores are parameter estimates that inherently carry uncertainty. Ignoring this uncertainty can introduce bias into the analysis and reduce the precision of hypothesis tests. Notably, the direction and magnitude of the bias are often unpredictable; results may be attenuated, exaggerated, or remain unaffected, depending on the amount of uncertainty present in the scores and the actual effects being tested (Kline, 2023, pp. 25; Hoyle, 2023, pp. 137). Furthermore, reduced precision in hypothesis tests weakens their statistical power, ultimately increasing the likelihood of committing type-I or type-II errors (McElreath, 2020).

To mitigate these risks, principles from Structural Equation Modeling (SEM) (Hoyle, 2023, pp. 138) and Item Response Theory (IRT) (Fox, 2010, chap. 6; van der Linden, 2017, chap. 24) recommend conducting these analyses and tests within a structural model. A structural model specifies how different manifest or latent variables influence the latent variable of interest (Everitt and Skrondal, 2010). This approach allows analyses that can account for both the BTL scores and their uncertainties simultaneously, rather than treating them as separate elements. Therefore, an integrated approach that combines CJ’s measurement and structural models can offer significant advantages.

2.3. The role and impact of comparison algorithms

3. Theory

3.1. A theoretical model for CJ

3.2. From theory to statistics

4. Discussion

4.1. Findings

4.2. Limitations and further research

5. Conclusion

Declarations

Funding: The project was founded through the Research Fund of the University of Antwerp (BOF).

Financial interests: The authors have no relevant financial interest to disclose.

Non-financial interests: Author XX serve on advisory board of Company Y but receives no compensation this role.

Ethics approval: The University of Antwerp Research Ethics Committee has confirmed that no ethical approval is required.

Consent to participate: Not applicable

Consent for publication: All authors have read and agreed to the published version of the manuscript.

Availability of data and materials: No data was utilized in this study.

Code availability: All the code utilized in this research is available in the digital document located at: https://jriverspejo.github.io/paper2_manuscript/.

AI-assisted technologies in the writing process: The authors used ChatGPT, an AI language model, during the preparation of this work. They occasionally employed the tool to refine phrasing and optimize wording, ensuring appropriate language use and enhancing the manuscript's clarity and coherence. The authors take full responsibility for the final content of the publication.

CRediT authorship contribution statement: *Conceptualization:* S.G., S.D.M., T.vD., and J.M.R.E.; *Methodology:* S.D.M., T.vD., and J.M.R.E.; *Software:* J.M.R.E.; *Validation:* J.M.R.E.; *Formal Analysis:* J.M.R.E.; *Investigation:* J.M.R.E.; *Resources:* S.G., S.D.M., and T.vD.; *Data curation:* J.M.R.E.; *Writing - original draft:* J.M.R.E.; *Writing - review & editing:* S.G., S.D.M., and T.vD.; *Visualization:* J.M.R.E.; *Supervision:* S.G. and S.D.M.; *Project administration:* S.G. and S.D.M.; *Funding acquisition:* S.G. and S.D.M.

6. Appendix

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