Let's talk about Thurstone & Co.: An information-theoretical model for comparative judgments, and its statistical translation

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Abstract

(to do)

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1. Introduction

In *comparative judgment* (CJ) studies, judges assess a specific trait or attribute across various stimuli by performing pairwise comparisons (Thurstone, 1927; Pollitt, 2004, 2012a). Each comparison produces a dichotomous outcome, indicating which stimulus is perceived to exhibit a higher trait level. For example, when assessing text quality, judges compare pairs of written texts (the stimuli) to determine the relative quality each text exhibit (the trait) (Laming, 2004; Pollitt, 2012b; Whitehouse, 2012; van Daal et al., 2016; Lesterhuis, 2018; Coertjens et al., 2017; Goossens and De Maeyer, 2018; Bouwer et al., 2023).

Numerous studies have documented the effectiveness of CJ in assessing traits and competencies over the past decade. These studies have emphasized three aspects of the method's effectiveness: its reliability, validity, and practical applicability. Research on reliability indicates that CJ requires a relatively small number of pairwise comparisons (Verhavert et al., 2019; Crompvoets et al., 2022) to produce trait scores that are as precise and consistent as those generated by other assessment methods (Coertjens et al., 2017; Goossens and De Maeyer, 2018; Bouwer et al., 2023). Furthermore, evidence suggests that the reliability and time efficiency of CJ are comparable, if not superior, to those of other assessment methods when employing adaptive comparison algorithms (Pollitt,

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2012b; Verhavert et al., 2022; Mikhailiuk et al., 2021). On the other hand, research on validity suggests that scores generated by CJ can accurately represent the traits under measurement (Whitehouse, 2012; van Daal et al., 2016; Lesterhuis, 2018; Bartholomew et al., 2018; Bouwer et al., 2023). Finally, research on practical applicability highlights the method's versatility across both educational and non-educational contexts (Jones, 2015; Bartholomew et al., 2018; Jones et al., 2019; Marshall et al., 2020; Bartholomew and Williams, 2020; Boonen et al., 2020).

Nevertheless, despite the increasing number of CJ studies, unsystematic and fragmented research approaches have left several critical issues unaddressed. This research primarily focuses on three: the over-reliance on Thurstone's Case V assumptions in the statistical analysis of CJ data, the apparent disconnect between CJ's trait measurement and hypothesis testing, and the unclear role and influence of comparison algorithms on the method's reliability and validity. The following sections will discuss each of these issues in detail, followed by the introduction of a theoretical model and its statistical translation, which aims to address all three concerns simultaneously.

2. Three critical issues in CJ literature

2.1. The Case V and the statistical analysis of CJ data

In its most general form, Thurstone's theory (1927) posits that when a judge compares two stimuli, the resulting dichotomous outcome depends on two main factors: the discriminal process and the law of comparative judgment. The *discriminal process* refers to the psychological effect each stimulus has on the judges, or more simply stated, the judges' perception of the trait level of each stimulus. Thurstone assumes that the discriminal process for each stimulus follows a Normal distribution. In this distribution, the mode (mean), known as the *modal discriminal process*, represents the position of the stimulus on the trait continuum, while the dispersion, known as the *discriminal dispersion*, reflects variability in the perceived trait level of the stimulus. Figure 1 shows a visual depiction of the discriminal process of two stimuli (objects).

However, since the discriminal mode and dispersion of a single stimulus are not directly observable without comparing it to other stimuli, the *law of comparative judgment* becomes essential. This law asserts that when assessing a specific trait by comparing two stimuli, the stimulus positioned further along the continuum is perceived as having a higher level of that trait. Thus, the observed dichotomous outcome is determined by the distribution of the difference between the discriminal processes of the two stimuli, called the *discriminal difference*. Figure 2 shows a visual depiction of the discriminal difference of two stimuli (objects).

Thurstone's general form of the theory primarily applies to pairwise comparisons of stimuli made by a single judge (Thurstone, 1927, pp. 267). Therefore, to enhance the theory's practical applicability, he developed five distinct cases, each incorporating a series of increasingly simplifying assumptions. Figure 3 outlines each case along with their key assumptions. For a detailed explanation of this theoretical progression, see Thurstone (1927) and Bramley (2008, pp. 248-253).

Surprisingly, Case V is the most commonly used case in the CJ literature, despite incorporating the largest number of simplifying assumptions (Bramley, 2008, pp. 253). This

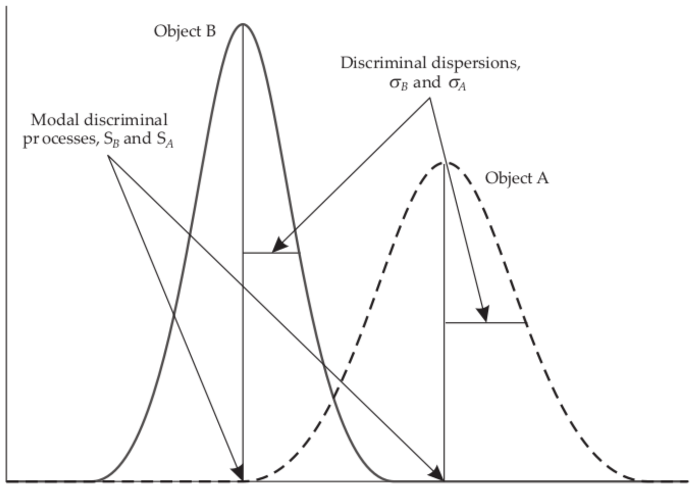

Figure 1: Discriminal processes of two stimuli (objects). Extracted from Bramley (2008, pp. 249)

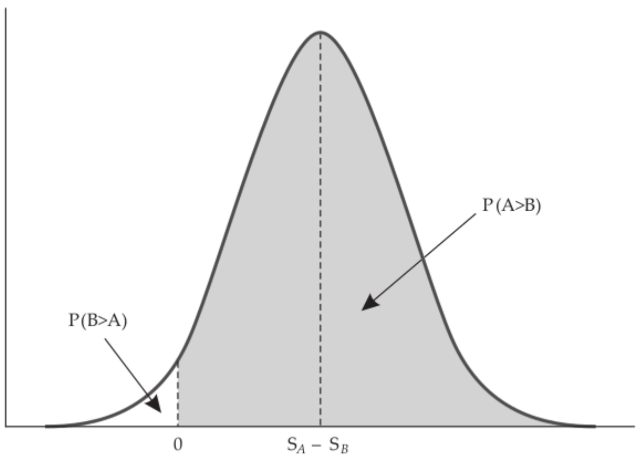

Figure 2: Discriminal difference between two stimuli (objects). Extracted from Bramley (2008, pp. 251)

	Thurstone's					BTL
Assumption	**Case I**	**Case II**	**Case III**	**Case IV**	**Case V**	**model**
Discriminal process (distribution)	Normal	Normal	Normal	Normal	Normal	Logistic
Discriminal dispersion (between stimuli)	Different	Different	Different	Similar	Equal	Equal
Correlation (between stimuli)	Constant	Constant	Zero	Zero	Zero	Zero
How many judges compare?	Single	Multiple	Multiple	Multiple	Multiple	Multiple

Figure 3: Thurstones cases and asumptions

widespread adoption is primarily due to the Bradley-Terry-Luce (BTL) model (Bradley and Terry, 1952; Luce, 1959), which offers a simpler statistical representation of the case. The BTL model closely aligns with Case V's assumptions, with one key difference: the distribution of the discriminal process. While Case V assumes a Normal distribution, the BTL model uses the more mathematically tractable logistic distribution (Bramley, 2008, pp. 254) (see Figure 3). However, this adjustment does not significantly affect the model's estimation or interpretation, as the logistic and normal distributions are related to each other by a factor of approximately 1.7 (van der Linden, 2017, pp. 16).

However, because Case V was originally developed to provide a "rather coarse scaling" of traits (Thurstone, 1927, pp. 269), prioritizing statistical ease of use over precision in trait measurement, two of its assumptions may be causing more problems than expected: the assumption of zero correlation between the stimuli, and the assumption of equal dispersion between stimuli.

2.2. The disconnect between trait measurement and hypothesis testing

As outlined in the previous section, the BTL model typically functions as the measurement model in a CJ study (Andrich, 1978; Bramley, 2008). A measurement model specifies how manifest variables contribute to the estimation of latent variables (Everitt and Skrondal, 2010). For example, when evaluating text quality, researchers use the BTL model to process the dichotomous outcomes resulting from the pairwise comparisons (the manifest variables) to estimate scores that reflect the underlying quality level of the texts (the latent variable) (Laming, 2004; Pollitt, 2012b; Whitehouse, 2012; van Daal et al., 2016; Lesterhuis, 2018; Coertjens et al., 2017; Goossens and De Maeyer, 2018; Bouwer et al., 2023).

Researchers then typically use the estimated BTL scores, or their transformations, to conduct additional analyses or hypothesis tests. For example, these scores have been used to identify 'misfit' judges and stimuli (Pollitt, 2012b; van Daal et al., 2017; Goossens and De Maeyer, 2018), detect biases in judges' ratings (Pollitt and Elliott, 2003; Pollitt, 2012b), calculate correlations with other assessment methods (Goossens and De Maeyer, 2018; Bouwer et al., 2023), or test hypotheses related to the underlying trait of interest (Bramley and Vitello, 2019; Boonen et al., 2020; Bouwer et al., 2023; van Daal et al., 2017; Jones et al., 2019; Gijsen et al., 2021).

However, the statistical literature advises caution when using estimated scores to conduct additional analyses or hypotheses tests. A key consideration is that BTL scores are parameter estimates that inherently carry uncertainty. Ignoring this uncertainty can introduce bias into the analysis and reduce the precision of hypothesis tests. Notably, the direction and magnitude of the bias are often unpredictable; results may be attenuated, exaggerated, or remain unaffected, depending on the amount of uncertainty present in the scores and the actual effects being tested (Kline, 2023, pp. 25; Hoyle, 2023, pp. 137). Furthermore, reduced precision in hypothesis tests weakens their statistical power, ultimately increasing the likelihood of committing type-I or type-II errors (McElreath, 2020).

To mitigate these risks, principles from Structural Equation Modeling (SEM) (Hoyle, 2023, pp. 138) and Item Response Theory (IRT) (Fox, 2010, chap. 6; van der Linden, 2017, chap. 24) recommend conducting these analyses and tests within a structural model. A structural model specifies how different manifest or latent variables influence the latent variable of interest (Everitt and Skrondal, 2010). This approach allows analyses that can account for both the BTL scores and their uncertainties simultaneously, rather than treating them as separate elements. Therefore, an integrated approach that combines CJ's measurement and structural models can offer significant advantages.

2.3. The role and impact of comparison algorithms

3. Theory

3.1. A theoretical model for CJ

3.2. From theory to statistics

4. Discussion

4.1. Findings

4.2. Limitations and further research

5. Conclusion

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6. Appendix

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