Let’s talk about Thurstone & Co.: An information-theoretical model for comparative judgments, and its statistical translation

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Abstract

(to do)

# Introduction

In *comparative judgment* (CJ) studies, judges assess a specific trait or attribute across various stimuli by performing pairwise comparisons (Thurstone 1927b, 1927a). Each comparison produces a dichotomous outcome, indicating which stimulus is perceived to exhibit a higher trait level. For example, when assessing text quality, judges compare pairs of written texts (the stimuli) to determine the relative quality each text exhibit (the trait) (Laming 2004; Pollitt 2012; Whitehouse 2012; van Daal et al. 2016; Lesterhuis 2018a; Coertjens et al. 2017; Goossens and De Maeyer 2018; Bouwer et al. 2023).

Numerous studies have documented the effectiveness of CJ in assessing traits and competencies over the past decade. These studies have emphasized three aspects of the method’s effectiveness: its reliability, validity, and practical applicability. Research on reliability indicates that CJ requires a relatively small number of pairwise comparisons (Verhavert et al. 2019; Crompvoets, Béguin, and Sijtsma 2022) to produce trait scores that are as precise and consistent as those generated by other assessment methods (Coertjens et al. 2017; Goossens and De Maeyer 2018; Bouwer et al. 2023). Furthermore, evidence suggests that the reliability and time efficiency of CJ are comparable, if not superior, to those of other assessment methods when employing adaptive comparison algorithms (Pollitt 2012; Verhavert, Furlong, and Bouwer 2022; Mikhailiuk et al. 2021). Meanwhile, research on validity suggests that scores generated by CJ can accurately represent the traits under measurement (Whitehouse 2012; van Daal et al. 2016; Lesterhuis 2018a; Bartholomew et al. 2018; Bouwer et al. 2023), while research on practical applicability highlights the method’s versatility across both educational and non-educational contexts (Kimbell 2012; Jones and Inglis 2015; Bartholomew et al. 2018; Jones et al. 2019; Marshall et al. 2020; Bartholomew and Williams 2020; Boonen, Kloots, and Gillis 2020).

Nevertheless, despite the increasing number of CJ studies, unsystematic and fragmented research approaches have left several critical issues unaddressed. The present study primarily focuses on three: the over-reliance on the assumptions of Thurstone’s Case V in the statistical analysis of CJ data, the apparent disconnect between CJ’s trait measurement and hypothesis testing, and the unclear role of the diverse assessment design features on CJ’s reliability and validity. The following sections begin with a brief overview of Thurstone’s theory and a detailed discussion of these issues. Subsequently, the study introduces a theoretical model for CJ that builds upon Thurstone’s theory, alongside its statistical translation, designed to address all three concerns simultaneously.

# Thurstone’s theory

In its most general form, Thurstone’s theory (1927a) suggests that two factors determine the dichotomous outcome of pairwise comparisons: the discriminal process of each stimulus and their discriminal difference. The *discriminal process* refers to the psychological effect each stimulus exerts on the judges, or more simply, the underlying perception of the stimulus’ trait level. According to the theory, the discriminal process for each stimulus follows a Normal distribution. The mode (mean) of this distribution, referred to as the *modal discriminal process*, represents the stimulus’ position on the trait continuum. Meanwhile, the dispersion of the distribution, referred to as the *discriminal dispersion*, reflects the variability in the perceived trait level of the stimulus.

However, since the discriminal process of a single stimulus is not directly observable, the *law of comparative judgment* becomes essential. This law states that in pairwise comparisons, the stimulus positioned further along the trait continuum is perceived as having a higher level of that trait. Thus, the theory assumes the observed dichotomous outcome is determined by the distribution of the difference between the underlying discriminal processes of the stimuli, referred to as the *discriminal difference*. This suggests that the outcome depends on the relative distance between stimuli, rather than their absolute positions on the trait continuum.

These concepts can be illustrated through an example. For instance, in the context of evaluating text quality, [Figure 1 (a)](#fig-discriminal_process) could depict the underlying discriminal process distributions for two written texts, highlighting differences in their discriminal dispersions and modal discriminal processes along the quality trait continuum. Furthermore, [Figure 1 (b)](#fig-discriminal_difference) could display the discriminal difference distribution for these texts, showing that text A is perceived to exhibit significantly higher quality than text B, as indicated by the shaded gray area. Consequently, the dichotomous outcome of this comparison would favor text A.

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| --- | --- | --- |
| |  | | --- | | (a) Discriminal processes | |  |

|  |  |
| --- | --- |
| |  | | --- | | (b) Discriminal difference | |

Figure 1: Example distribution of discriminal processes and their discriminal difference for two written texts (stimuli or objects). Extracted from Bramley (2008, 249–51).

Importantly, the general form of Thurstone’s theory primarily addressed pairwise comparisons of stimuli made by a single judge (Thurstone 1927a, 267). Thus, for practical application, Thurstone introduced five distinct cases derived from this general form, each defined by progressively simplifying assumptions. [Table 1](#tbl-thurstone_cases) summarizes these cases, highlighting key assumptions such as the distribution of discriminal processes, the similarity of discriminal dispersions across stimuli, the correlation between stimuli, and the number of judges performing the comparisons. For a comprehensive discussion of this progression, refer to Thurstone (1927a) and Bramley (2008, 248–53).

|  |
| --- |
| Table 1: Thurstones cases and asumptions |

# Three critical issues in CJ literature

## The Case V and the statistical analysis of CJ data

Despite its reliance on the largest number of simplifying assumptions (Bramley 2008, 253; Kelly, Richardson, and Isaacs 2022, 677), Case V remains the most widely used case in the CJ literature. This popularity is largely due to its simplified statistical representation in the Bradley-Terry-Luce (BTL) model (Bradley and Terry 1952; Luce 1959). The BTL model mirrors the assumptions of Case V, with one key difference: while Case V assumes a Normal distribution for the discriminal processes of stimuli, the BTL model uses the more mathematically tractable Logistic distribution (Andrich 1978; Bramley 2008, 254) (see [Table 1](#tbl-thurstone_cases)). This substitution has little impact on the model’s estimation or interpretation, as the Normal and Logistic distributions differ by a scaling factor of approximately (van der Linden 2017, 1:16) (see [Figure 2](#fig-logistic_vs_normal)).

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| --- | --- | --- |
| |  | | --- | | (a) Probability density | |  |

|  |  |
| --- | --- |
| |  | | --- | | (b) Cummulative probability | |

Figure 2: Probability density and cumulative probability of the logistic and Normal distributions. Extracted from Bramley (2008, 254–55).

However, Case V was originally developed to provide a “rather coarse scaling” of traits (Thurstone 1927a, 269), prioritizing statistical simplicity over precision in trait measurement (Kelly, Richardson, and Isaacs 2022, 677). As a result, its assumptions may not be suitable for applications beyond the psycho-physical contexts for which it was created. Thurstone himself cautioned that its use “should not be made without (an) experimental test” (Thurstone 1927a, 270), acknowledging that some assumptions could prove problematic in the presence of complex traits or heterogeneous stimuli, such as handwriting or English compositions (Thurstone 1927b, 374). Consequently, given that modern CJ applications frequently involve these types of traits and stimuli, two main assumptions of Case V may not consistently hold in theory or practice: the zero correlation and equal dispersion between stimuli.

The assumption of *zero correlation between stimuli* is more easily understood through an example. For instance, when using pairwise comparisons to evaluate text quality, the assumption suggests that a judge’s perception of the trait in one text does not influence his perception of the same trait on another. Thurstone attributed this independence to the cancellation of potential judges’ biases, driven by two opposing and equally weighted effects occurring during the pairwise comparisons (Thurstone 1927a, 268). This cancellation was mathematically demonstrated by Andrich (1978), assuming discriminal processes with additive biases. However, it is easy to imagine two scenarios where the zero correlation assumption almost certainly does not hold: when the pairwise comparison involves multidimensional, complex traits with heterogeneous stimuli, and when an additional hierarchical structure is relevant to the stimuli.

In the first scenario, evidence suggests that when judges are confronted with multi-dimensional, complex traits and heterogeneous stimuli, such as when evaluating written text quality, their assessments are influenced by multiple, intricate aspects of the stimuli (van Daal et al. 2016; Lesterhuis 2018b; Chambers and Cunningham 2022). In such cases, it is not inconceivable that these various aspects of the stimuli unevenly influence the judges’ perception of one text over another, leading to biases that resist cancellation, as the effects of the factors involved in the comparison are neither equally weighted nor opposing. For example, this may occur when a judge, while assessing the argumentative quality of a text, places disproportionate emphasis on grammatical errors, ultimately favoring texts with fewer errors but weaker arguments. Although the literature does not provide direct evidence for this specific argument, studies such as Pollitt and Elliott (2003) do offer evidence of judges’ biases, supporting the idea that the factors involved in pairwise comparisons may not always cancel out.

## The disconnect between trait measurement and hypothesis testing

Building on the previous section, it is evident that the BTL model commonly functions as the measurement model for the trait of interest in CJ experiments (Andrich 1978; Bramley 2008). A measurement model specifies how manifest variables contribute to the estimation of latent variables (Everitt and Skrondal 2010). For example, when evaluating text quality, researchers use the BTL model to process the dichotomous outcomes resulting from the pairwise comparisons (the manifest variables) to estimate scores that reflect the underlying quality level of texts (the latent variable) (Laming 2004; Pollitt 2012; Whitehouse 2012; van Daal et al. 2016; Lesterhuis 2018a; Coertjens et al. 2017; Goossens and De Maeyer 2018; Bouwer et al. 2023).

Researchers then typically use the estimated BTL scores, or their transformations, to conduct additional analyses and tests, or to make decisions regarding the exclusion of certain data in these analyses and tests. The literature shows that these scores have been employed to calculate correlations with other assessment methods (Goossens and De Maeyer 2018; Bouwer et al. 2023) or to test hypotheses related to the underlying traits of interest (Bramley and Vitello 2019; Boonen, Kloots, and Gillis 2020; Bouwer et al. 2023; van Daal et al. 2017; Jones et al. 2019; Gijsen et al. 2021). Additionally, the BTL scores have been used to detect biases in judges’ ratings (Pollitt and Elliott 2003; Pollitt 2012), as well as to identify “misfit” judges and stimuli (Pollitt 2012; van Daal et al. 2017; Goossens and De Maeyer 2018), with considerations for their possible exclusion.

However, the statistical literature advises caution when using estimated scores to conduct additional analyses and tests while warning against eliminating data through ad-hoc univariate procedures. A key consideration is that BTL scores are parameter estimates that inherently carry uncertainty. Ignoring this uncertainty can bias the analysis and reduce the precision of hypothesis tests. Notably, the direction and magnitude of such biases are often unpredictable. Results may be attenuated, exaggerated, or remain unaffected depending on the degree of uncertainty in the scores and the actual effects being tested (Kline 2023, 25; Hoyle 2023, 137). Moreover, eliminating data using ad-hoc univariate procedures using these scores compounds this issue by excluding possible valuable information from the analyses (McElreath 2020), further introducing bias. Finally, the reduced precision in hypothesis tests weakens their statistical power, increasing the likelihood of type-I or type-II errors (McElreath 2020).

To mitigate these risks, principles from Structural Equation Modeling (SEM) (Hoyle 2023, 138) and Item Response Theory (IRT) (Fox 2010, chap. 6; van der Linden 2017, vol. 1, chap. 24) recommend conducting these analyses and tests within a structural model. A structural model specifies how different manifest or latent variables influence the latent variable of interest (Everitt and Skrondal 2010). This approach allows analyses that can account for both the BTL scores and their uncertainties simultaneously, rather than treating them as separate elements. Therefore, an integrated approach that combines CJ’s measurement and structural models can offer significant advantages.

## Assessment design features and their role on reliability and validity

# An updated theoretical and statistical model for CJ

## The theoretical model

## From theory to statistics

# Discussion

## Findings

## Limitations and further research

# Conclusion

# Declarations

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# Appendix

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