Let's talk about Thurstone & Co.: An information-theoretical model for comparative judgments, and its statistical translation

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Abstract

This study revisits Thurstone's law of comparative judgment (CJ), focusing on two prominent issues of traditional approaches. First, it critiques the heavy reliance on Thurstone's Case V assumptions and, by extension, the Bradley-Terry-Luce (BTL) model when analyzing CJ data. Specifically, the study raises concerns about the assumptions of equal discriminal dispersions and zero correlation between the stimuli. While these assumptions simplify the trait measurement model, they may fail to capture the complexity of CJ data, potentially leading to unreliable and inaccurate trait estimates. Second, the study highlights the apparent disconnect between CJ's trait measurement and hypothesis testing processes. Although separating these processes simplifies the analysis of CJ data, it may also undermine the reliability of various statistical results derived from these processes.

To address these issues, the study extends Thurstone's general form using a systematic and integrated approach based on Causal and Bayesian inference methods. This extension integrates core theoretical principles alongside key CJ assessment design features, such as the selection of judges, stimuli, and comparisons. It then translates these elements into a probabilistic statistical model for analyzing dichotomous CJ data, overcoming the rigid assumptions of Case V and the BTL model.

Finally, the study emphasizes the relevance of this extension for contemporary empirical CJ research, particularly stressing the need for bespoke CJ models tailored to the experiments and data assumptions. It also lays the foundation for broader applications, encouraging researchers across the social sciences to adopt more robust and interpretable methodologies.
Keywords: causal inference, directed acyclic graphs, structural causal models, bayesian statistical methods, thurstonian model, comparative judgement, probability, statistical modeling

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1. Introduction

In *comparative judgment* (CJ) studies, judges assess a specific trait or attribute across different stimuli by performing pairwise comparisons (Thurstone, 1927b,a). Each comparison produces a dichotomous outcome, indicating which stimulus is perceived to have a higher trait level. For example, when assessing writing quality, judges compare pairs of written texts (the stimuli) to determine the relative writing quality each text exhibit (the trait) (Pollitt, 2012b; van Daal et al., 2016; Lesterhuis, 2018; Coertjens et al., 2017; Goossens and De Maeyer, 2018; Bouwer et al., 2023).

Numerous studies have documented the effectiveness of CJ in assessing traits and competencies over the past decade. These studies have highlighted three aspects of the method's effectiveness: its reliability, validity, and practical applicability. Research on reliability suggests that CJ requires a relatively modest number of pairwise comparisons (Verhavert et al., 2019; Crompvoets et al., 2022) to generate trait scores that are as precise and consistent as those generated by other assessment methods (Coertjens et al., 2017; Goossens and De Maeyer, 2018; Bouwer et al., 2023). In addition, the evidence suggests that the reliability and time efficiency of CJ are comparable, if not superior, to those of other assessment methods when employing adaptive comparison algorithms (Pollitt, 2012b; Verhavert et al., 2022; Mikhailiuk et al., 2021). Meanwhile, research on validity indicates the capacity of CJ scores to represent accurately the traits under measurement (Whitehouse, 2012; van Daal et al., 2016; Lesterhuis, 2018; Bartholomew et al., 2018; Bouwer et al., 2023). Lastly, research on its practical applicability highlights CJ's versatility across both educational and non-educational contexts (Kimbell, 2012; Jones and Inglis, 2015; Bartholomew et al., 2018; Jones et al., 2019; Marshall et al., 2020; Bartholomew and Williams, 2020; Boonen et al., 2020).

Nevertheless, despite the increasing number of CJ studies, research in this domain remains unsystematic and fragmented, leaving several critical issues unresolved. This study identifies and discusses two prominent issues of traditional approaches that can undermine the reliability and validity of CJ's trait estimates. First, it critiques the heavy reliance on Thurstone's Case V assumptions (Thurstone, 1927a) and, by extension, the Bradley-Terry-Luce (BTL) model (Bradley and Terry, 1952; Luce, 1959) when analyzing CJ data. Specifically, the study raises concerns about the assumptions of equal discriminal dispersions and zero correlation between the stimuli. While these assumptions simplify the trait measurement model, they may fail to capture the complexity

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of some traits or account for heterogeneous stimuli, potentially leading to unreliable and inaccurate trait estimates. Second, the study highlights the disconnect between CJ's trait measurement and hypothesis testing processes. Although separating these processes simplifies the analysis of CJ data, it may also undermine the reliability of various statistical inferences derived from these processes (McElreath, 2020; Kline, 2023; Hoyle, 2023).

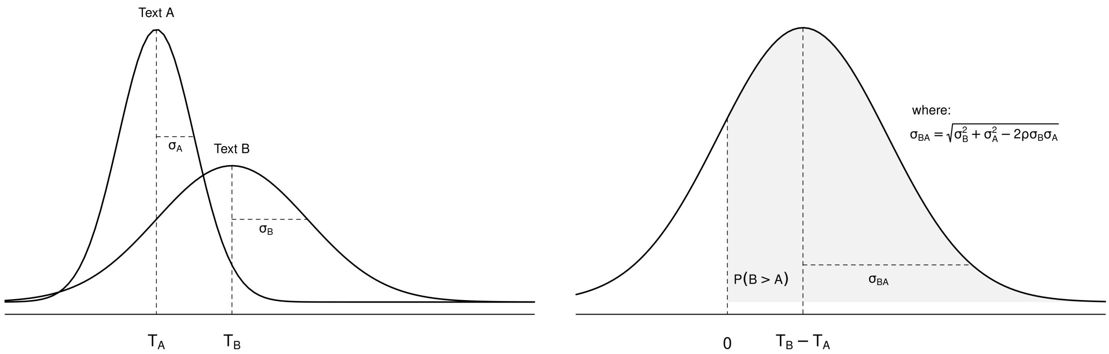
To address these issues, this study extends Thurstone's general form through a systematic and integrated approach that combines causal and Bayesian inference methods. In addition to potentially enhancing measurement reliability and validity, and improving statistical accuracy in hypothesis testing, this approach offers two key advantages. First, it clarifies the interactions among all actors and processes involved in CJ assessments. Second, it shifts the current comparative data analysis paradigm from passively accepting Case V and the BTL model assumptions to actively testing whether those assumptions fit the data under analysis.

The study divides its content into six main sections. Section 2 provides an overview of Thurstone's theory. Section 3 discusses the identified issues in detail. Section 4 extends Thurstone's general form to address these challenges. The extension integrates core theoretical principles alongside key CJ assessment design features, such as the selection of judges, stimuli, and comparisons. Section 5 translates these theoretical and practical elements into a probabilistic statistical model to analyze dichotomous pairwise comparison data. Finally, Section 6 discusses the findings, explores avenues for future research, and detail the challenges for future researchers.

2. Thurstone's theory

In its most general form, Thurstone's theory addresses pairwise comparisons wherein a single judge evaluates multiple stimuli (Thurstone, 1927a). The theory posits that two key factors determine the dichotomous outcome of these comparisons: the discriminal process of each stimulus and their discriminal difference. The *discriminal process* captures the psychological impact each stimulus exerts on the judge or, more simply, his perception of the stimulus trait. The theory assumes that the discriminal process for any given stimulus forms a Normal distribution along the trait continuum (Thurstone, 1927a). The mode (mean) of this distribution, known as the *modal discriminal process*, indicates the stimulus position on this continuum, while its dispersion, referred to as the *discriminal dispersion*, reflects variability in the perceived trait of the stimulus.

Figure 1a illustrates the hypothetical discriminal processes along a quality trait continuum for two written texts. The figure indicates that the modal discriminal process for Text B is positioned further along the continuum than that of Text A ($T_B > T_A$), suggesting that Text B exhibits higher quality. Additionally, the figure highlights that Text B has a broader distribution compared to Text A, which arises from its larger discriminal dispersion ($\sigma_B > \sigma_A$).



(a) Discriminal processes (b) Discriminal difference

Figure 1: Hypothetical discriminal processes and discriminant difference along a quality trait continuum for two written texts.

However, since the individual discriminal processes of the stimuli are not directly observable, the theory introduces the *law of comparative judgment.* This law posits that in pairwise comparisons, a judge perceives the stimulus with a discriminal process positioned further along the trait continuum as possessing more of the trait (Bramley, 2008). This suggests that pairwise comparison outcomes depend on the relative distance between stimuli, not their absolute positions on the continuum. Indeed, the theory assumes that the difference between the underlying discriminal processes of the stimuli, referred to as the *discriminal difference*, determines the observed dichotomous outcome. Furthermore, the theory assumes that because the individual discriminal processes form a Normal distribution on the continuum, the discriminal difference will also conform to a Normal distribution (Andrich, 1978). In this distribution, the mode (mean) represents the average relative separation between the stimuli, and its dispersion indicates the variability of that separation.

Figure 1b illustrates the distribution of the discriminal difference for the two hypothetical texts. The figure indicates that the judge perceives Text B as having significantly higher quality than Text A. Two key observations support this conclusion: the positive difference between their modal discriminal processes ($T_B - T_A > 0$) and the probability area where the discriminal difference distinctly favors Text B over Text A, represented by the shaded gray area denoted as $P(B > A)$. As a result, the dichotomous outcome of this comparison is more likely to favor Text B over A.

3. Two Prominent Issues in Traditional CJ Practice

Thurstone noted from the outset that his general formulation, described in Section 2, led to a *trait scaling problem*. Specifically, the model required estimating more "unknown" parameters than the number of available pairwise comparisons (Thurstone, 1927a). For instance, in a CJ assessment with five texts, the general form would require estimating 20 parameters: five modal discriminal processes, five discriminal dispersions, and 10 correlations–one per comparison (see Table 1). However, a single judge could only provide $\binom{5}{2} = 10$ unique comparisons, an insufficient data set to estimate the required parameters.

Table 1: Thurstones cases and their asumptions

Assumption	General form	Thurstone's Case I	Case II	Case III	Case IV	Case V	BTL model
Discriminal process (distribution)	Normal	Normal	Normal	Normal	Normal	Normal	Logistic
Discriminal dispersion (between stimuli)	Different	Different	Different	Different	Similar	Equal	Equal
Correlation (between stimuli)	One per pair	Constant	Constant	Zero	Zero	Zero	Zero
How many judges compare?	Single	Single	Multiple	Multiple	Multiple	Multiple	Multiple

To address this issue and facilitate the practical implementation of the theory, Thurstone developed five cases derived from this general form, each progressively incorporating additional simplifying assumptions (Thurstone, 1927a). In Case I, Thurstone postulated that pairs of stimuli would maintain a constant correlation across all comparisons. In Case II, he allowed multiple judges to undertake comparisons instead of confining evaluations to a single judge. In Case III, he posited that there was no correlation between stimuli. In Case IV, he assumed that the stimuli exhibited similar dispersions. Finally, in Case V, he replaced this assumption with the condition that stimuli had equal discriminal dispersions. Table 1 summarizes the assumptions of the general form and the five cases. For a detailed discussion of these cases and their progression, refer to Thurstone (1927a) and Bramley (2008).

However, Thurstone developed Case V prioritizing statistical simplicity over precise trait measurement and offering no guidance on how to use its trait estimates for statistical inference or hypothesis testing. Specifically, Thurstone cautioned that its use "should not be made without (an) experimental test" (Thurstone, 1927a, pp. 270), as it imposes the most extensive set of simplifying assumptions (Bramley, 2008; Kelly et al., 2022) (see Table 1). Moreover, because Thurstone's primary goal was to produce a "rather coarse scaling" of traits and "allocate the compared stimuli on this continuum" (Thurstone, 1927a, pp. 269), his theory did not support formal statistical inference. Despite these limitations, CJ research has predominantly relied on Case V to measure different traits, which raises significant concerns about the reliability and validity of such measurements in contexts where the case's assumptions may not hold (Kelly et al., 2022; Andrich, 1978). Furthermore, although the CJ tradition has attempted to address the gap in hypothesis testing by relying on the point estimates of traits–or their transformations–the statistical literature cautions against using these estimates as the sole basis for statistical inference, as such practices introduce bias in the analysis and reduce the precision of hypothesis tests (McElreath, 2020; Kline, 2023; Hoyle, 2023). Next, both issues are discussed in more depth.

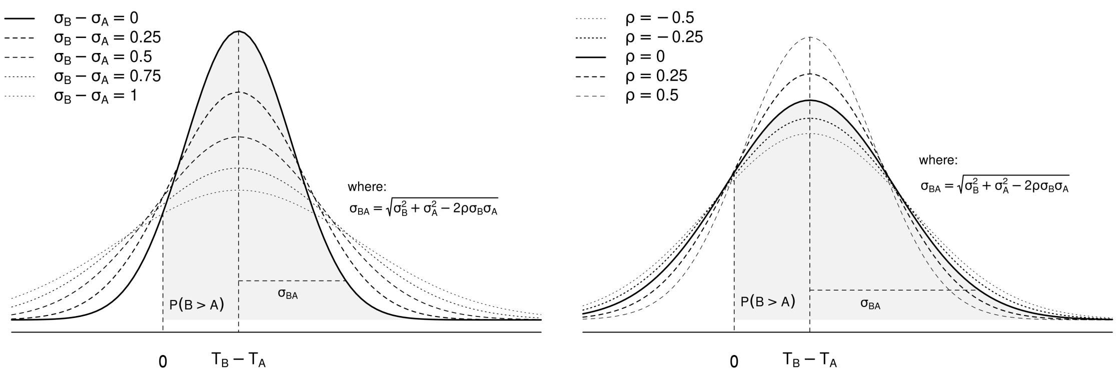
3.1. The Case V and the statistical analysis of CJ data

As previously discussed, Case V remains the most widely used model in CJ literature. This preference largely stems from the widespread adoption of the BTL model, which provides a simplified statistical representation of the case. The BTL model mirrors most of Case V's assumptions, with one notable distinction. While Case V assumes a Normal distribution for the stimuli' discriminal processes, the BTL model uses the more mathematically tractable Logistic distribution (Andrich, 1978; Bramley, 2008) (see Table 1). However, this substitution has minimal impact on trait estimation or model interpretation because the scale of the discriminal process (i.e., the latent trait) is arbitrary up to a non-monotonic transformation (van der Linden, 2017a; McElreath, 2021). That is, as long as the substitution (transformation) preserves the data rank order, the choice of distribution for the discriminal processes is inconsequential. This condition is satisfied in this case, as the Normal and Logistic distributions exhibit analogous statistical properties, differing only by a scaling factor of approximately 1.7 (van der Linden, 2017a).

However, Thurstone acknowledged that some assumptions of Case V could be problematic when researchers assess complex traits or heterogeneous stimuli (Thurstone, 1927b). Thus, given that modern CJ applications often involve such traits and stimuli, two key assumptions of Case V, and by extension, the BTL model, may not always hold in theory or practice. These assumptions are the equal dispersion and zero correlation between stimuli.

3.1.1. The assumption of equal dispersions between stimuli

According to the theory, discrepancies in the discriminal dispersions of stimuli shape the distribution of the discriminal difference, directly influencing the outcome of pairwise comparisons. A thought experiment can help illustrate this idea. In it, researchers observe the discriminal processes for two texts, A and B, assuming that the dispersion for Text A remains constant and that the two texts are uncorrelated ($\rho = 0$). Figure 2a demonstrates that an increase in the uncertainty associated with the perception of Text B relative to Text A ($\sigma_B - \sigma_A$), broadens the distribution of their discriminal difference. This broadening affects the probability area where the discriminal difference distinctly favors Text B over Text A, expressed as $P(B > A)$, ultimately influencing the comparison outcome. Additionally, the figure reveals that when the discriminal dispersions of the texts are equal, as in the BTL model ($\sigma_B - \sigma_A = 0$), the discriminal difference distribution is more narrow than when the dispersions differ. As a result, the discriminal difference is more likely to favor Text B over Text A, as it is represented by the shaded gray area.



(a) Discriminal Difference distribution under varying discrepancies in stimuli dispersions

(b) Discriminal Difference distribution under varying levels of correlation between stimuli

Figure 2: The effect of dispersion discrepancies and stimuli correlation on the distribution of the discriminal difference.

In experimental practice, however, the thought experiment occurs in reverse. Researchers first observe the comparison outcome and then use the BTL model to infer the discriminal difference between stimuli and their respective discriminal processes (Thurstone, 1927b). Consequently, the outcome's ability to reflect *true* differences between stimuli largely depends on the validity of the model's assumptions (Kohler et al., 2019), in this case, the assumption of equal dispersions. When the assumption accurately captures the complexity of the data, the BTL model estimates a discriminal difference distribution that accurately represents the *true* discriminal difference between the texts. This scenario is illustrated in Figure 2a, when the model's discriminal difference distribution aligns with the *true* discriminal difference distribution, represented by the thick continuous line corresponding to $\sigma_B - \sigma_A = 0$. The accuracy of this discriminal difference then ensures reliable estimates for the texts' discriminal processes.

Notably, while assuming equal dispersions simplifies the trait measurement model, evidence from the CJ literature suggests that this assumption may fail to accommodate for heterogeneous stimuli, such as handwritten texts or English compositions (Thurstone, 1927b; Andrich, 1978; Bramley, 2008; Kelly et al., 2022). The presence of the so-called *misfit texts*–texts that elicit more judgment discrepancies than others (Pollitt, 2004, 2012b,a; Goossens and De Maeyer, 2018)–may signal this limitation, as such discrepancies can arise from larger discriminal dispersions due to stimulus heterogeneity or because the texts are genuine outliers–that is, texts with distinctive characteristics that deviate markedly from the rest of the sample (Grubbs, 1969). In either case, the BTL model's assumptions prevent it from adequately accounting for or addressing these anomalies, leaving exclusion of the "problematic" texts as the primary remedy (Pollitt, 2012a,b).

Significant statistical and measurement issues can arise when the assumption of equal dispersions between stimuli does not hold. Specifically, the BTL model may overestimate the trait's reliability, that is, the degree to which the outcome accurately reflects the *true* discriminal differences between stimuli. This overestimation, in turn, results in spurious conclusions about these differences (McElreath, 2020; Wu et al., 2022) and, by extension, about the underlying discriminal processes of stimuli. Figure 2a also illustrates this scenario when the model's discriminal difference distribution aligns with the thick continuous line for $\sigma_B - \sigma_A = 0$, while the *true* discriminal difference follows any discontinuous line where $\sigma_B - \sigma_A \neq 0$. Furthermore, if researchers acknowledge that *misfit statistics* may represent texts with different dispersions or outlying observations, the common CJ practice of excluding stimuli based on these statistics may unintentionally discard valuable information (Miller, 2023), and introduce bias into the trait estimates (Zimmerman, 1994; McElreath, 2020). The direction and magnitude of these biases remain unpredictable, as they depend on which stimuli researchers exclude from the analysis.

3.1.2. The assumption of zero correlation between stimuli

The correlation between two stimuli ρ measures how much the judges' perception of a specific trait in one stimulus depends on their perception of the same trait in other stimulus. Similar to the discriminal dispersions, this correlation shapes the distribution of the discriminal difference, directly impacting the outcomes of pairwise comparisons. Assuming that the discriminal dispersions for a couple of texts remain constant, Figure 2b shows that as the correlation between the two texts increases, the distribution of their discriminal difference becomes narrower. This narrowing, in turn, affects the probability that the discriminal difference distinctly favors Text B over Text A–denoted as $P(B > A)$–and thus directly influences the comparison outcome. Furthermore, the figure shows that when two texts are independent or uncorrelated, as assumed in the BTL model ($\rho = 0$), the distribution of their discriminal difference is less narrow than in scenarios where the texts are positively correlated. As a result, it becomes less likely for the comparison to favor Text B over Text A, as indicated by the larger shaded area.

Despite these notable differences in the distribution of the discriminal difference under various correlational assumptions, in practice, assessment designs often adopt the assumption of no correlation between stimuli based on an old theoretical justification. Specifically, Thurstone (1927a) argued that stimuli could be treated as uncorrelated because judges' biases–arising from two opposing and equally weighted effects occurring during the pairwise comparisons–would cancel each other out. This idea was later formalized by Andrich (1978), who provided a mathematical demonstration of this cancellation using the BTL model under the assumption of discriminal processes with additive biases. However, evidence from the CJ literature indicates that the assumption of zero correlation does not hold in practice in at least two cases: when intricate aspects of multidimensional, complex traits or heterogeneous stimuli influence judges' perceptions or when additional hierarchical structures are relevant to the stimuli.

Research on text quality assessments suggests that when judges evaluate complex, multidimensional traits or heterogeneous stimuli, they often rely on a variety of intricate stimulus characteristics to inform their judgments (van Daal et al., 2016; Lesterhuis et al., 2018; Chambers and Cunningham, 2022). Regardless of their relevance, these characteristics may not receive equal weight or consistently oppose one another across comparisons. As a result, they may exert a disproportionate influence on judges' perceptions, generating biases that persist rather than cancel out. For example, this could occur when a judge assessing the argumentative quality of a text may place disproportionate emphasis on handwriting clarity, thereby favoring neatly written texts despite their weaker arguments. Moreover, because the discriminal process of stimuli becomes an observable outcome only through the judges' perceptions, these biases could introduce dependencies between the stimuli (van der Linden, 2017b). While direct evidence for this exact scenario is limited, existing studies document the presence of judge bias in CJ (Pollitt and Elliott, 2003; van Daal et al., 2016; Bartholomew et al., 2020), reinforcing the argument that the factors influencing pairwise comparisons do not always cancel each other out.

In the second case, the shared context or inherent connections introduced by additional hierarchical structures may create dependencies between stimuli–a statistical phenomenon known as clustering

(Everitt and Skrondal, 2010). For instance, when the same individual produces multiple texts, those texts often share several features such as writing style or overall quality. Although the CJ literature acknowledges the existence of such hierarchical structures (e.g., Boonen et al., 2020), the statistical approaches to account for this additional source of dependence have been insufficient. For instance, when CJ data incorporates multiple samples of stimuli from the same individuals, researchers frequently rely on (averaged) point estimates of the BTL scores to conduct subsequent analyses and tests at the individual level (Bramley and Vitello, 2019; Boonen et al., 2020; Bouwer et al., 2023; van Daal et al., 2017; Jones et al., 2019; Gijsen et al., 2021).

Thus, erroneously assuming zero correlation between stimuli can also lead to significant statistical and measurement issues. In particular, neglecting judges' biases or relevant hierarchical structures can create dimensional mismatches in the model, leading to the over- or underestimation of trait reliability (Ackerman, 1989; Hoyle, 2023) and even introduce statistical biases (Wu et al., 2022). These inaccuracies can result in spurious conclusions about the discriminal differences (McElreath, 2020) and, by extension, the underlying discriminal processes of the stimuli. One such spurious conclusion could be the incorrect classification of stimuli or judges as *misfits*. Figure 2b illustrates how assuming zero correlation can undermine trait reliability: the discriminal difference distribution of the BTL scores follows the thick continuous line ($\rho = 0$), while the *true* discriminal difference may correspond to any discontinuous line where $\rho \neq 0$.

Finally, removing *misfit* judges–that is, judges whose assessments deviate markedly from the shared consensus (Pollitt, 2012a,b; van Daal et al., 2016; Goossens and De Maeyer, 2018; Wu et al., 2022), and may appear as outliers under the BTL model (Wu et al., 2022)–risks discarding valuable information and introducing bias into trait estimates (Miller, 2023). The direction and magnitude of these biases remain unpredictable, as they depend on which judges researchers exclude from the analysis (Zimmerman, 1994; O'Hagan, 2018; McElreath, 2020).

3.2. The disconnect between trait measurement and hypothesis testing

Researchers in CJ studies typically use the BTL model to measure traits and position the compared stimuli along a latent continuum (Thurstone, 1927a). The CJ literature shows that research frequently relies on point estimates of these traits–typically the BTL scores or its transformations–to conduct statistical inference or hypothesis testing. For example, researchers have used these scores to identify 'misfit' judges and stimuli (Pollitt, 2012b; van Daal et al., 2016; Goossens and De Maeyer, 2018), detect biases in judges' ratings (Pollitt and Elliott, 2003; Pollitt, 2012b), calculate correlations with other assessment methods (Goossens and De Maeyer, 2018; Bouwer et al., 2023), or test hypotheses related to the underlying trait of interest (Casalicchio et al., 2015; Bramley and Vitello, 2019; Boonen et al., 2020; Bouwer et al., 2023; van Daal et al., 2017; Jones et al., 2019; Gijsen et al., 2021).

Nevertheless, while separating the trait measurement and hypothesis testing processes simplifies the analysis of CJ data, the statistical literature cautions against relying solely on the point estimates of BTL scores to conduct statistical inference or hypothesis tests, as this practice can undermine the resulting statistical conclusions. A key consideration is that BTL scores are parameter estimates that inherently carry uncertainty (measurement error). Ignoring this uncertainty can bias the analysis and reduce the precision of hypothesis tests. The direction and magnitude of such biases are often unpredictable. Results may be attenuated, exaggerated, or remain unaffected depending on the degree of uncertainty in the scores and the actual effects being tested (McElreath, 2020; Kline, 2023; Hoyle, 2023). Furthermore, the reduced precision in hypothesis tests diminishes their statistical power, increasing the likelihood of committing type-I or type-II errors (McElreath, 2020).

In aggregate, the heavy reliance on Thurstone's Case V assumptions in the statistical analysis of comparative data can compromise the reliability of trait estimates. This overreliance may also undermine their validity (Perron and Gillespie, 2015), particularly when coupled with the disconnect between the trait measurement and hypothesis testing processes. The structural approach to causal inference can address these issues by offering a systematic and integrated framework to extend Thurstone's general form. This approach can also strengthen measurement reliability and validity while enhancing the statistical accuracy of hypothesis tests.

4. Extending Thurstone's general form

The *structural approach* to causal inference provides a formal framework for identifying causes and estimating their effects using data. The approach uses structural causal models (SCMs) and directed acyclic graphs (DAGs) (Pearl, 2009; Pearl et al., 2016; Gross et al., 2018; Neal, 2020) to formally and graphically represent the assumed causal structure of a system, such as the one found in CJ assessments. Essentially, SCMs and DAGs function as *conceptual models* on which identification analysis rests. *Identification analysis* determines whether an estimator can accurately compute an estimand based solely on its (causal) assumptions, regardless of random variability (Schuessler and Selb, 2023). Here, *estimands* represent the specific quantities researchers aim to determine (i.e., a parameter) (Everitt and Skrondal, 2010). *Estimators* denote the methods or functions that transform data into an estimate (e.g., a statistical model), while *estimates* are the numerical values approximating the estimand (Neal, 2020; Everitt and Skrondal, 2010).

A motivating example that will appear throughout this study clarifies these concepts. In this example, researchers aim to answer the question: "To what extent do different teaching methods influence students' ability to produce high-quality written texts?" To investigate this, a researcher designs a CJ assessment by randomly assigning students (individuals) to two groups, each receiving a different teaching method. Judges then compare pairs of students' written texts (stimuli) to produce a dichotomous outcome reflecting the relative quality of each text (trait). Based on this setup, researchers can reformulate the research question as the estimand: "*On average*, is there a difference in the ability to produce high-quality written texts between the two groups of students?".

Following standard CJ practices, researchers would typically use estimates from the BTL model–or its transformations–to approximate this estimand. However, as discussed in Section 3, Thurstone's Case V and the BTL model exhibit several statistical and measurement limitations. These limitations hinder the model's ability to identify various estimands relevant to CJ inquiries, including the one described in the example.

Fortunately, SCMs and DAGs support identification analysis through two key advantages[1]. First, regardless of complexity, they can represent various causal structures using only five fundamental building blocks (Neal, 2020; McElreath, 2024). This feature allows researchers to decompose complex structures into manageable components, facilitating their analysis. Second, they depict causal relationships in a non-parametric way. This flexibility enables feasible identification strategies without requiring specification of the types of variables, the functional forms relating them, or the parameters of those functional forms (Pearl et al., 2016).

Thus, this section addresses the issues identified in Section 3 by extending Thurstone's general form using the structural approach to causal inference. Specifically, it combines the core theoretical principles outlined in Section 2 with key CJ assessment design features, such as the selection of

[1]In depth explanation of these topics is beyond the scope of this study, thus, readers seeking a more profound understanding can refer to introductory papers such as Pearl (2010), Rohrer (2018), Pearl (2019), and Cinelli et al. (2020), and introductory books like Pearl and Mackenzie (2018), Neal (2020), and McElreath (2020) are useful. For more advanced study, seminal papers such as Neyman (1923), Rubin (1974), Spirtes et al. (1991), and Sekhon (2009), along with books such as Pearl (2009), Morgan and Winship (2014), and Hernán and Robins (2025), are recommended.

judges, stimuli, and comparisons. Section 4.1 introduces the *conceptual-population model*, which incorporates these theoretical principles and assumes an idealized setting where researchers observe a *conceptual population* of comparative judgment data–that is, data representing all repeated judgments made by every available judge for each pair of stimuli produced by each pair of individuals in the population. Conversely, Section 4.2 presents the *sample-comparison model*, which integrates the assessment design features and reflects a more realistic setting where researchers access only a sample of judges, individuals, stimuli, and comparisons from the conceptual population.

4.1. The conceptual-population model

In the conceptual-population model, the idealized scenario of a *conceptual population* of comparative data enables the integration of Thurstone's theoretical principles and provides a foundation for proposing innovations aimed at addressing some of the issues discussed in Section 3.

4.1.1. Integrating the first theoretical principles

Before incorporating the first theoretical principles of Thurstone's theory, it is essential to further define SCMs. SCMs are formal mathematical models characterized by a set of *endogenous* variables V, a set of *exogenous* variables E, and a set of functions F (Pearl, 2009; Pearl et al., 2016; Cinelli et al., 2020). Endogenous variables are those whose causal mechanisms a researcher chooses to model (Neal, 2020). In contrast, exogenous variables represent *errors* or *disturbances* arising from omitted factors that the investigator chooses not to model explicitly (Pearl, 2009). Lastly, the functions, referred to as *structural equations*, express the endogenous variables as non-parametric functions of other endogenous and exogenous variables. These functions use the symbol '$:=$' to denote the asymmetrical causal dependence between variables and the symbol '$\perp$' to represent *d-separation*, a concept akin to statistical (conditional) independence.

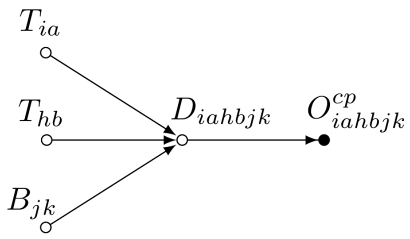
SCM 3a presents the first theoretical principles embedded in the conceptual-population model, which evaluates the impact of different teaching methods on students' writing ability. This SCM outlines the relationship between the conceptual-population outcome (O^{cp}_{iahbjk}) and several related variables. The subscripts i and h identify the students who authored the texts (i.e., the individuals). The indices a and b represent the texts under comparison (i.e., the stimuli). The index j indicates the judge conducting the comparison, while the index k accounts for assessment conditions where a judge compares the same pair of stimuli multiple times, i.e., a *repeated measures designs* (Lawson, 2015, pp. 366-376). Thus, the indexing system supports comparisons between different texts written by the same student ($i = h$; $a \neq b$) and between texts written by distinct students ($i \neq h$; where

$a = b$ is permitted), each compared once or repeatedly by all judges ($j = 1, \dots, n_J$; $k = 1, \dots, n_K$; where $n_J > 1$ and $n_K \geq 1$). However, it excludes cases where a judge compares a student's text to itself, whether once or multiple times ($i = h$; $a = b$; $j = 1, \dots, n_J$; $k = 1, \dots, n_K$; where $n_J > 1$ and $n_K \geq 1$), as such comparison lacks practical relevance within the CJ framework. Here, n_J indicates the total number of judges, and n_K denotes the number of repeated judgments each judge performs.

$$O^{cp}_{iahbjk} := f_O(D_{iahbjk})$$

$$D_{iahbjk} := f_D(T_{ia}, T_{hb}, B_{jk})$$

(a) SCM



(b) DAG

Figure 3: Conceptual-population model, scalar form.

In line with Thurstone's theory, SCM 3a depicts the texts' discriminal processes (T_{ia}, T_{hb}) and their discriminal difference (D_{iahbjk}) (see Section 2). Additionally, the SCM incorporates a key CJ design feature: the judges' biases (B_{kj}). This extension builds on the arguments presented in Section 3.1.2, contending that the discriminal difference becomes an observable outcome only through judges' perceptions. Given that such perceptions may be imperfect–and that each judge may carry some degree of bias (see Pollitt and Elliott, 2003; van Daal et al., 2016)–it is reasonable that judges' perceptions (bias) should be treated as an integral component of the CJ system from the outset, as this leads to a more accurate representation of the data-generating process underlying the pairwise comparisons. This model defines the preliminary set of endogenous variables, $V = \{O_{iahbjk}, D_{iahbjk}, T_{ia}, T_{hb}, B_{kj}\}$, and the preliminary set of structural equations, $F = f_O, f_D$, which capture the non-parametric dependencies among these variables.

Notably, every SCM has an associated DAG (Pearl et al., 2016; Cinelli et al., 2020). A DAG is a *graph* consisting of nodes connected by edges, where nodes represent random variables. The term *directed* indicates that edges or arrows extend from one node to another, indicating the direction of causal influence. The absence of an edge implies no direct relationship between the nodes. The term *acyclic* means that the causal influences do not form loops, ensuring the influences do not cycle back on themselves (McElreath, 2020). DAGs conventionally depict observed variables as solid black circles and unobserved (latent) variables as open circles (Morgan and Winship, 2014).

Although DAGs conventionally omit exogenous variables for simplicity, the DAGs presented in this section includes exogenous variables to improve clarity and reveal potential issues related to conditioning and confounding (Cinelli et al., 2020).

Figure 3b displays the DAG corresponding to SCM 3a, illustrating the expected causal relationships outlined in Thurstone's theory. The graph shows that the discriminal processes of the texts (T_{ia}, T_{hb}) influence their discriminal difference (D_{iahbjk}), which in turn determines the outcome (O^{cp}_{iahbjk}). It also highlights the influence of judges' biases (B_{kj}) on the discriminal difference. Additionally, the DAG differentiates between observed endogenous variables, such as the outcome (solid black circle), and latent endogenous variables, including the texts' discriminal processes, their discriminal difference, and the judges' biases (open circles).

4.1.2. The conceptual-population *data structure*

Although specifying a data structure is not mandatory when using SCMs and DAGs, defining one improves clarity and facilitates the description of the system. Thus, to re-express the scalar form of the CJ system shown in Figure 3 into an equivalent vectorized form, we first define the vectors I and J, along with the matrices IA and JK, as in Equation (1). Here, each element of I represents a unique individual i or h, where n_I denotes the total number of individuals. Similarly, each element of J corresponds to a unique judge j, with n_J indicating the total number of judges. Moreover, each row of IA represents a unique pairing of individuals i, h with stimuli a, b. As a result, the matrix IA contains $n_I \cdot n_A$ rows and 2 columns, where n_A specifies the number of stimuli available per individual. Likewise, each row of JK associates a judge j with a (repeated) judgment index k. Consequently, the matrix JK has $n_J \cdot n_K$ rows and 2 columns, where n_K indicates the number of repeated judgments each judge makes.

Additionally, we construct the matrix R to map each row of the IA matrix with a corresponding row from the JK matrix. This matrix has n rows and 6 columns, where $n = \binom{n_I \cdot n_A}{2} \cdot n_J \cdot n_K$. Here, the term $\binom{n_I \cdot n_A}{2}$ represents the binomial coefficient, which quantifies the total number of unique comparisons possible between every pair of stimuli generated by each pair of individuals in the population. Thus, we define the matrix as in Equation (1).

$$I = \begin{bmatrix} 1 \\ \vdots \\ i \\ \vdots \\ h \\ \vdots \\ n_I \end{bmatrix}; J = \begin{bmatrix} 1 \\ \vdots \\ j \\ \vdots \\ n_J \end{bmatrix}; IA = \begin{bmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & n_A \\ \vdots & \vdots \\ i & a \\ \vdots & \vdots \\ h & b \\ \vdots & \vdots \\ n_I & 1 \\ \vdots & \vdots \\ n_I & n_A \end{bmatrix}; JK = \begin{bmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & n_K \\ \vdots & \vdots \\ j & k \\ \vdots & \vdots \\ n_J & 1 \\ \vdots & \vdots \\ n_J & n_K \end{bmatrix}; R = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 2 & 1 & n_K \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ i & a & h & b & j & k \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n_I & n_A - 1 & n_I & n_A & n_J & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n_I & n_A - 1 & n_I & n_A & n_J & n_K \end{bmatrix} \tag{1}$$

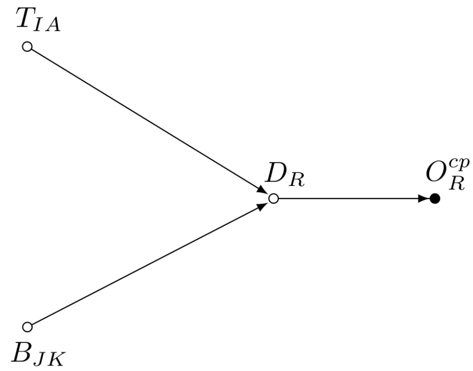
It is easier to visualize the structure of the previously defined vectors and matrices by considering an example. Assuming $n_I = 5$, $n_A = 2$, $n_J = 3$, and $n_K = 3$, the vectors and matrices described in Equation (1) take the form as in Equation (2).

$$I = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}; \; J = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \; IA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \\ 2 & 2 \\ 3 & 1 \\ 3 & 2 \\ 4 & 1 \\ 4 & 2 \\ 5 & 1 \\ 5 & 2 \end{bmatrix}; \; JK = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 2 & 1 \\ 2 & 2 \\ 2 & 3 \\ 3 & 1 \\ 3 & 2 \\ 3 & 3 \end{bmatrix}; \; R = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 2 & 1 & 3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 5 & 2 & 1 & 1 \\ 1 & 1 & 5 & 2 & 1 & 2 \\ 1 & 1 & 5 & 2 & 1 & 3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 4 & 2 & 5 & 2 & 3 & 1 \\ 4 & 2 & 5 & 2 & 3 & 2 \\ 4 & 2 & 5 & 2 & 3 & 3 \\ 5 & 1 & 5 & 2 & 3 & 1 \\ 5 & 1 & 5 & 2 & 3 & 2 \\ 5 & 1 & 5 & 2 & 3 & 3 \end{bmatrix} \tag{2}$$

Now, using Equation (1), we can re-express SCM 3a and DAG 3b in an equivalent vectorized form, as shown in Figure 4. In this depiction, the outcome O_R^{cp}, the texts' discriminal difference D_R, their discriminal processes T_{IA}, and the judges' biases B_{JK} are represented as vectors rather than scalar values. These vectors capture all the observations from the conceptual population. Specifically, O_R^{cp} and D_R are observed and latent vectors of length n, respectively. Moreover, T_{IA} and B_{JK} are latent vectors of lengths $n_I \cdot n_A$ and $n_J \cdot n_K$, respectively.

$$O_R^{cp} := f_O(D_R)$$
$$D_R := f_D(T_{IA}, B_{JK})$$

(a) SCM



(b) DAG

Figure 4: Conceptual-population model, initial vectorized form.

4.1.3. Integrating hierarchical structural components

Building on the principles of Structural Equation Modeling (SEM) (Hoyle, 2023) and Item Response Theory (IRT) (Fox, 2010; van der Linden, 2017a), the conceptual-population model integrates two *hierarchical structural components* to examine how different *relevant* [2] variables–whether observed or latent–affect the primary latent variable of interest (Everitt and Skrondal, 2010). This hierarchical design enables researchers to formulate and test hypotheses that account for both the nested structure of stimuli and the uncertainties inherent in trait estimation (see Section 3.1.2 and Section 3.2 for a discussion of these considerations).

The top branch of DAG 5b illustrates the first component, where *relevant* [3] student-related variables X_I, such as teaching method, and students' idiosyncratic errors e_I causally influence the latent variable representing students' writing-quality trait T_I. The error term e_I captures variations in students' traits unexplained by X_I. Here, X_I is an observed matrix with n_I rows and q_I independent columns (variables), and both e_I and T_I are latent vectors of length n_I. Additionally, this

[2] *Relevant variables* are those that satisfy the *backdoor criterion* (Neal, 2020, pp 37), that is, they belong to a *sufficient adjustment set* (Pearl, 2009; Pearl et al., 2016; Morgan and Winship, 2014). A *sufficient* set (potentially empty) blocks all non-causal paths between a predictor and an outcome without opening new ones (Pearl, 2009). Refer also to footnote 1.

[3] refer to footnote 2.

branch shows how T_I, along with *relevant* [4] text-related variables X_{IA} (e.g., text length), and texts' idiosyncratic errors e_{IA} causally influence the texts' written-quality trait T_{IA}, the first primary latent variable of interest. The error term e_{IA} captures variations in the texts' traits that remain unexplained by T_I or X_{IA}. Here, X_{IA} is an observed matrix with dimensions $n_I \cdot n_A$ rows and q_{IA} independent columns (variables), while e_{IA} and T_{IA} are latent matrices with n_I rows and n_A columns.

Similarly, the bottom branch of DAG 5b depicts the second component, where *relevant* [5] judge-related variables Z_J, such as judgment expertise, and judges' idiosyncratic errors e_J causally influence the latent variable representing judges' bias B_J. The error e_J captures variations in judges' bias unexplained by Z_J. Here, Z_J is an observed matrix with n_J rows and q_J independent columns (variables), and both e_J and B_J are latent vectors of length n_J. Furthermore, the branch shows how B_J, along with *relevant* [6] judgment-related variables Z_{JK} (e.g., the number of judgments a judge makes), and judgments' idiosyncratic errors e_{JK} causally influence the judges' biases associated with each text B_{JK}, the second primary latent variable of interest. The error e_{JK} captures variations in judgments unexplained by B_J or Z_{JK}. Here, Z_{JK} is an observed matrix with dimension $n_J \cdot n_K$ rows and q_{JK} independent columns (variables), while e_{JK} and B_{JK} are latent latent matrices with n_J rows and n_K columns

Notably, all variables and functions shown in SCM 5a and DAG 5b are part of the set of endogenous variables V, structural equations F, and exogenous variables E for the conceptual-population model. Additionally, the figures demonstrate that all exogenous variables are independent of one another, as indicated by the relationships $e_{IA} \perp \{e_I, e_{JK}, e_J\}$, $e_I \perp \{e_{JK}, e_J\}$ and $e_{JK} \perp e_J$ and the absence of connecting arrows.

Overall, the conceptual-population model extends Thurstone's general form by introducing key innovations to address the limitations discussed in Section 3.1.2 and Section 3.2. These enhancements include accounting for judges' biases and integrating hierarchical structural components. Nevertheless, despite its promise of enhancing measurement accuracy and precision, the model still depends on the unrealistic assumption that researchers have access to data from the *conceptual population*. Since researchers rarely meet this assumption in practice, they must consider a more realistic scenario.

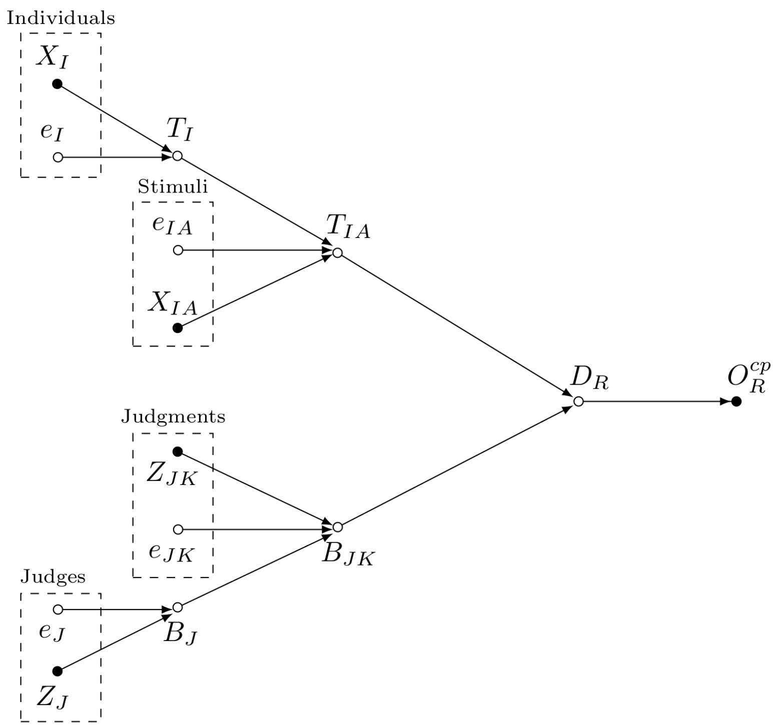
[4] refer to footnote 2.

[5] refer to footnote 2.

[6] refer to footnote 2.

$$
\begin{aligned}
O_R^{cp} &:= f_O(D_R) \\
D_R &:= f_D(T_{IA}, B_{JK}) \\
T_{IA} &:= f_T(T_I, X_{IA}, e_{IA}) \\
T_I &:= f_T(X_I, e_I) \\
B_{JK} &:= f_B(B_J, Z_{JK}, e_{JK}) \\
B_J &:= f_B(Z_J, e_J) \\
e_I &\perp \{e_J, e_{IA}, e_{JK}\} \\
e_J &\perp \{e_{IA}, e_{JK}\} \\
e_{IA} &\perp e_{JK}
\end{aligned}
$$

(a) SCM



(b) DAG

Figure 5: Conceptual-population model, final vectorized form.

4.2. The sample-comparison model

The sample-comparison model presents a more realistic scenario than the conceptual-population model. First, it explicitly assumes researchers work with a data sample consisting of a limited number of repeated judgments (n_K^s) from a sample of judges (n_J^s) and a specific number of texts (n_A^s) from a sample of students (n_I^s), all drawn from the conceptual population (Section 4.2.1). Second, the model assumes that judges do not perform *all repeated judgments* within the data sample (Section 4.2.2). Instead, they conduct a sufficient number of stimuli comparisons, n_C, to ensure an accurate estimation of the proportion $P(B > A)$, as proposed by Thurstone (1927a).

4.2.1. The sample mechanism

To incorporate the sampling mechanism and facilitate the interpretation of the sample-comparison model, we first define the *data sampling process* using the binary vector variables S_I, S_J, S_{IA}, and S_{JK} as follows:

$$S_I = \begin{bmatrix} i_{(1)} \\ \vdots \\ i_{(i)} \\ \vdots \\ i_{(h)} \\ \vdots \\ i_{(nI)} \end{bmatrix} ; \; S_J = \begin{bmatrix} j_{(1)} \\ \vdots \\ j_{(j)} \\ \vdots \\ j_{(nJ)} \end{bmatrix} ; \; S_{IA} = \begin{bmatrix} ia_{(1,1)} \\ \vdots \\ ia_{(1,n_A)} \\ \vdots \\ ia_{(i,a)} \\ \vdots \\ ia_{(h,b)} \\ \vdots \\ ia_{(nI,1)} \\ \vdots \\ ia_{(nI,nA)} \end{bmatrix} ; \; S_{JK} = \begin{bmatrix} jk_{(1,1)} \\ \vdots \\ jk_{(1,n_K)} \\ \vdots \\ jk_{(j,k)} \\ \vdots \\ jk_{(nJ,1)} \\ \vdots \\ jk_{(nJ,nK)} \end{bmatrix} \tag{3}$$

Where each element of S_I is a binary value indicating the presence or absence of corresponding elements in the vector I, as in Equation (4). We apply the same logic to S_J using vector J (not shown). Thus, the vectors S_I and S_J contains n_I and n_J elements, respectively.

$$i_{(i)} = \begin{cases} 1 & \text{if data element } i \text{ from } I \text{ is sampled} \\ 0 & \text{if data element } i \text{ from } I \text{ is missing} \end{cases} \tag{4}$$

Similarly, each element of S_{IA} is a binary value indicating the presence or absence of data rows in the matrices IA, as defined in Equation (5). We apply the same logic to S_{JK} using the matrix JK (not shown). Thus, the vectors S_{IA} and S_{JK} contains $n_I \cdot n_A$ and $n_J \cdot n_K$ elements, respectively.

$$ia_{(i,a)} = \begin{cases} 1 & \text{if data elements } i, a \text{ from } IA \text{ are sampled} \\ 0 & \text{if data elements } i, a \text{ from } IA \text{ are missing} \end{cases} \tag{5}$$

We can illustrate the structure of these vectors more clearly with an example. Suppose researchers exclude the second student, the second text from each student, and the third judge from the setup shown in Equation (2). Given $n_I = 5$, $n_A = 2$, $n_J = 3$, and $n_K = 3$, the resulting vectors would have the following structure:

$$S_I = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} ; \; S_J = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} ; \; S_{IA} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} ; \; S_{JK} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

Notably, Equation (6) shows that missing observations in the vectors S_I and S_J–which represent unsampled students and judges–directly determine which observations are missing in S_{IA} and S_{JK}. In other words, researchers can only observe texts and judgments from students and judges initially included in the sample. The equation also shows that the sum of observed elements in S_I equals the number of sampled students (n^s_I) and that a similar sum in vector S_J equals the sampled judges (n^s_J). Conversely, the sum of observed elements in S_{IA} represents the total sampled texts across all sampled students ($n^s_I \cdot n^s_A$), while a similar sum in vector S_{JK} represents the total sampled repeated judgments across all sampled judges ($n^s_J \cdot n^s_K$). Notice that in this example, because the design systematically excludes every third repeated judgment, researchers can also express S_{JK} using $n_K = n^s_K = 2$.

Finally, we define the *sample mechanism* S in Equation (8), which maps each element of S_{IA} to every element of S_{JK}. Each element $s_{(i,a,h,b,j,k)}$ is a binary value indicating the presence or absence of data rows in the matrix R resulting from the sample mechanism, as in Equation (7). Thus, the vector contains n elements, matching the number of rows in R, and the sum of its elements represents the total data sample: $n^s = \binom{n^s_I \cdot n^s_A}{2} \cdot n^s_J \cdot n^s_K$. Here, the term $\binom{n^s_I \cdot n^s_A}{2}$ represents the binomial coefficient, which quantifies the total number of unique comparisons possible between every pair of sampled stimuli generated by each pair of sampled individuals.

$$s_{(i,a,h,b,j,k)} = \begin{cases} 1 & \text{if data elements } i,a,h,b,j,k \text{ from } R \text{ are sampled} \\ 0 & \text{if data elements } h,i,a,b,j,k \text{ from } R \text{ are missing} \end{cases} \tag{7}$$

$$S = \begin{bmatrix} s_{(1,1,1,2,1,1)} \\ \vdots \\ s_{(1,1,1,2,1,n_K)} \\ \vdots \\ s_{(i,a,h,b,j,k)} \\ \vdots \\ s_{(n_I,n_A-1,n_I,n_A,n_J,1)} \\ \vdots \\ s_{(n_I,n_A-1,n_I,n_A,n_J,1)} \end{bmatrix} \tag{8}$$

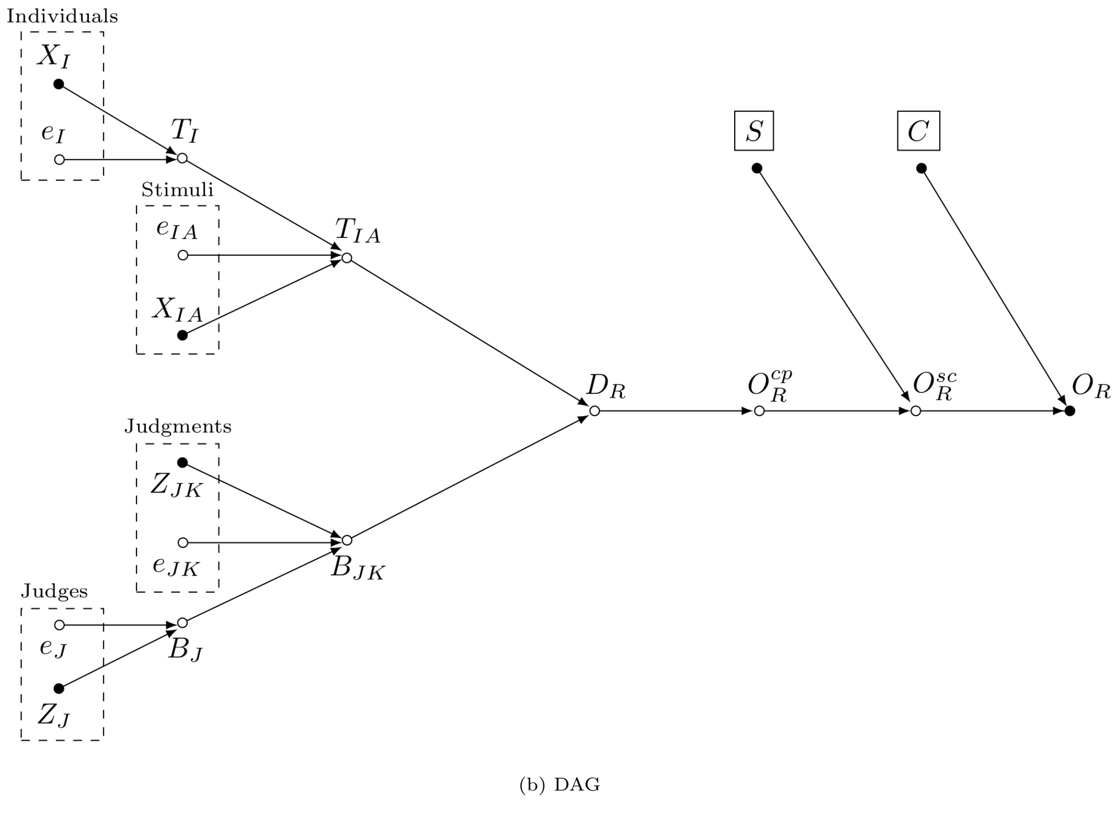
With the definition of S, we incorporate the sample mechanism into the conceptual-population model. Following the convention of McElreath (2020) and Deffner et al. (2022), DAG 6b represents the conceptual-population outcome O_R^{cp} as unobserved, emphasizing that researchers cannot directly access this outcome due to the sampling mechanism. The DAG also depicts the *sample design* vector S as a causal factor influencing the sample-comparison outcome O_R^{sc}. A square encloses S, indicating that it is a conditioned variable. In this context, *conditioning* means that researchers restrict their focus to the elements of O_R^{cp} that satisfy $s_{(i,a,h,b,j,k)} = 1$ (Neal, 2020; McElreath, 2020). In essence, S is a vector that selects *all repeated judgments made by a subset of judges for a subset of stimuli produced by the sampled individuals.*

Notably, the DAG shows that S is independent of all other variables in the model. This implies that DAG 6b applies exclusively to Simple Random Sampling (SRSg) designs. In these designs, each repeated judgment, judge, stimulus, and individual has the same probability of being included in the sample as any other observation within their respective groups (Lawson, 2015).

However, due to concerns about the practical feasibility of the comparison task (Boonen et al., 2020), CJ assessments rarely implement an exhaustive pairings of sampled judges, stimuli, and individuals. Thus, a realistic scenario must account for the fact that judges typically compare only a subset of stimuli authored by a sample of individuals.

$$
\begin{aligned}
O_R &\coloneqq f_C(O_R^{sc}, C) \\
O_R^{sc} &\coloneqq f_S(O_R^{cp}, S) \\
O_R^{cp} &\coloneqq f_O(D_R) \\
D_R &\coloneqq f_D(T_{IA}, B_{JK}) \\
T_{IA} &\coloneqq f_T(T_I, X_{IA}, e_{IA}) \\
T_I &\coloneqq f_T(X_I, e_I) \\
B_{JK} &\coloneqq f_B(B_J, Z_{JK}, e_{JK}) \\
B_J &\coloneqq f_B(Z_J, e_J) \\
e_I &\perp \{e_J, e_{IA}, e_{JK}\} \\
e_J &\perp \{e_{IA}, e_{JK}\} \\
e_{IA} &\perp e_{JK}
\end{aligned}
$$

(a) SCM



(b) DAG

Figure 6: Sample-comparison model, final vectorized form

4.2.2. The comparison mechanism

As in the previous section, we begin defining the *comparison mechanism* using the binary vector variable C to facilitate the interpretation of the sample-comparison model. Equation (9) shows that C contains n elements corresponding to the number of rows in the R matrix, with each element $c_{(i,a,h,b,j,k)}$ being a binary value indicating the presence or absence of data rows in R, a definition similar to that of $s_{(i,a,h,b,j,k)}$ in Equation (7).

$$C = \begin{bmatrix} c_{(1,1,1,2,1,1)} \\ \vdots \\ c_{(1,1,1,2,1,n_K)} \\ \vdots \\ c_{(i,a,h,b,j,k)} \\ \vdots \\ c_{(n_I,n_A-1,n_I,n_A,n_J,1)} \\ \vdots \\ c_{(n_I,n_A-1,n_I,n_A,n_J,1)} \end{bmatrix} \tag{9}$$

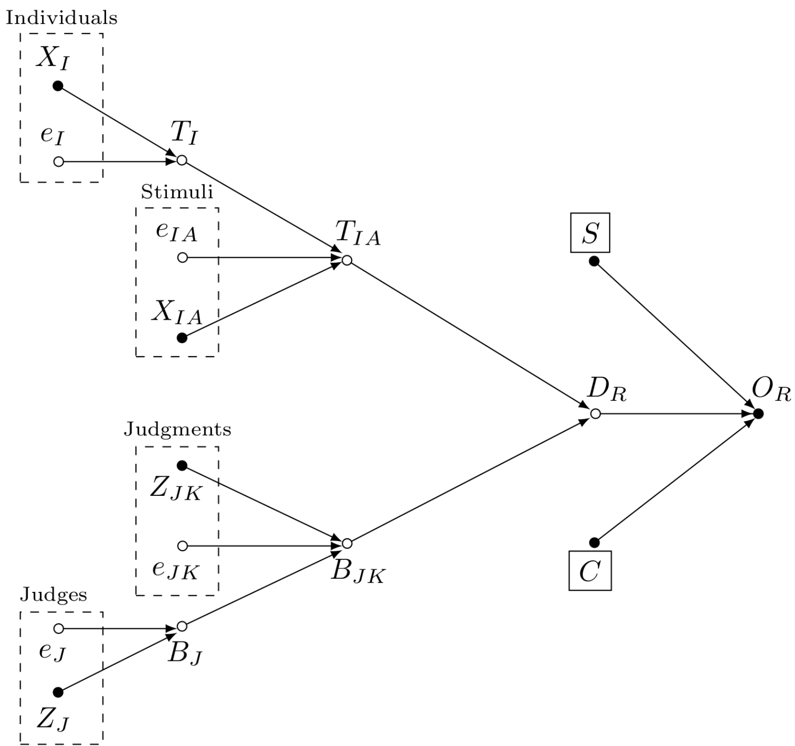
The DAG 6b also incorporates the *comparison mechanism* C into the conceptual-population model. It shows the sample-comparison outcome O_R^{sc} as unobserved, emphasizing that researchers cannot directly access this variable because of the comparison mechanism. The DAG further shows C as a conditioned variable (enclosed in a square) that causally influences the observed outcome O_R. This structure implies that C determines *which repeated judgments judges make for the stimuli produced by the individuals*. In essence, C reflects the assumption that judges *do not* perform all possible repeated judgments but instead complete a sufficient number, n_C, to enable the accurate estimation of the proportion $P(B > A)$ for each stimulus pair (Thurstone, 1927a, pp. 267).

Notably, DAG 6b also shows that C is independent of all other variables in the model. This independence implies that the conceptual model represented by the DAG applies exclusively to Random Allocation Comparative Designs (Bramley, 2015), or Incomplete Block Designs (Lawson, 2015), where every repeated judgment has an equal probability of being included in the sample.

Finally, since it is standard to assume that the distribution of the conceptual-population outcome O_R^{cp} also holds for O_R^{sc} and O_R, we can reformulate the sample-comparison model in Figure 6 into the equivalent form shown in Figure 7. This reformulation produces a model that applies directly to a sample of comparative data. In this version, the unobserved outcomes O_R^{cp} and O_R^{sc} are omitted, and O_R inherits the structural equation f_O that originally defined O_R^{cp}. Moreover, the definition of O_R now reflects its direct dependence on the discriminal difference D_R and the sample and comparison mechanisms, S and C.

$$
\begin{aligned}
O_R &:= f_O(D_R, S, C) \\
D_R &:= f_D(T_{IA}, B_{JK}) \\
T_{IA} &:= f_T(T_I, X_{IA}, e_{IA}) \\
T_I &:= f_T(X_I, e_I) \\
B_{JK} &:= f_B(B_J, Z_{JK}, e_{JK}) \\
B_J &:= f_B(Z_J, e_J) \\
e_I &\perp \{e_J, e_{IA}, e_{JK}\} \\
e_J &\perp \{e_{IA}, e_{JK}\} \\
e_{IA} &\perp e_{JK}
\end{aligned}
$$

(a) SCM



(b) DAG

Figure 7: Comparative judgment model

In summary, the SCM 7a and DAG 7b extend Thurstone's general form to address several limitations of the BTL model. These extensions account for judge biases (see Section 4.1.1), reflect the hierarchical structure of stimuli and incorporate measurement error in trait estimation and hypothesis testing (see Section 4.1.3), and even clarify the role of the sample and comparison mechanisms in CJ assessments (see Section 4.2). However, they do not resolve concerns about the assumption of equal dispersions among stimuli discussed in Section 3.1.1. Since this concern relates to the statistical assumption underlying the distribution of the discriminal process, we develop a formal statistical model to address it in the next section.

5. From SCM to statistical model

Using the structural causal model (SCM) 7a, we can derive a statistical model that addresses violations of the equal dispersion assumption (see Section 3.1.1). This derivation is possible because a fully specified SCM encodes functional and probabilistic information, which we can replace with suitable functions and probabilistic assumptions (Pearl et al., 2016). Specifically, SCM 7a allows us to express the joint distribution of our complex CJ system as a product of simpler conditional probability distributions (CPDs)[7], as shown in Equation (10). For clarity, we treat expressions such as $Y := f_Y(X)$, $P(Y \mid X)$, and $Y \sim f(Y \mid X)$ as equivalent, where $P(Y \mid X)$ and $f(Y \mid X)$ represent the CPD of Y given X.

$$
\begin{aligned}
P(O_R, &S, C, D_R, T_{IA}, X_{IA}, e_{IA}, T_I, X_I, e_I, B_{JK}, Z_{JK}, e_{JK}, B_J, Z_J, e_J) \\
&= P(O_R \mid D_R, S, C) \cdot P(S) \cdot P(C) \cdot P(D_R \mid T_{IA}, B_{JK}) \\
&\quad \cdot P(T_{IA} \mid T_I, X_{IA}, e_{IA}) \cdot P(T_I \mid X_I, e_I) \\
&\quad \cdot P(B_{JK} \mid B_J, Z_{JK}, e_{JK}) \cdot P(B_J \mid Z_J, e_J) \\
&\quad \cdot P(X_{IA}) \cdot P(X_I) \cdot P(Z_{JK}) \cdot P(Z_J) \\
&\quad \cdot P(e_{IA}) \cdot P(e_I) \cdot P(e_{JK}) \cdot P(e_J)
\end{aligned}
\tag{10}
$$

Each CPD in Equation (10) rests on specific assumptions, which we outline in the statistical model presented in Figure 8c. The model starts by assuming that O_R follows a Bernoulli distribution[8],

[7]This re-expression is possible because the *chain rule* of probability and the *Bayesian Network Factorization (BNF)* property. For further details, see Pearl et al. (2016) and Neal (2020).

[8]The binomial distribution–including its special case, the Bernoulli distribution–represent a maximum entropy distribution for binary events (McElreath, 2020, pp. 34). This means that the Bernoulli distribution is the most consistent alternative when only two un-ordered outcomes are possible and their expected frequencies are assumed to be constant (McElreath, 2020, pp. 310). For a detailed discussion of the binomial as a maximum entropy distribution, see McElreath (2020, sec. 10.1.2).

reflecting the binary nature of CJ outcomes. Furthermore, following the conventions of Generalized Linear Models (GLMs) (McCullagh and Nelder, 1983; Lee and Nelder, 1996; Agresti, 2015), the distribution links O_R to the latent discriminal difference vector D_R using an inverse-logit function: $\text{inv\_logit}(x) = 1/(1 + \exp(-x))$.

$$
\begin{aligned}
O_R &:= f_O(D_R, S, C) \\
D_R &:= f_D(T_{IA}, B_{JK}) \\
T_{IA} &:= f_T(T_I, X_{IA}, e_{IA}) \\
T_I &:= f_T(X_I, e_I) \\
B_{JK} &:= f_B(B_J, Z_{JK}, e_{JK}) \\
B_J &:= f_B(Z_J, e_J)
\end{aligned}
$$

$$
\begin{aligned}
& e_I \perp \{e_J, e_{IA}, e_{JK}\} \\
& e_J \perp \{e_{IA}, e_{JK}\} \\
& e_{IA} \perp e_{JK}
\end{aligned}
$$

(a) SCM

$$
\begin{aligned}
& P(O_R \mid D_R, S, C) \\
& P(D_R \mid T_{IA}, B_{JK}) \\
& P(T_{IA} \mid T_I, X_{IA}, e_{IA}) \\
& P(T_I \mid X_I, e_I) \\
& P(B_{JK} \mid B_J, Z_{JK}, e_{JK}) \\
& P(B_J \mid Z_J, e_J)
\end{aligned}
$$

$$
P(e_I)P(e_{IA})P(e_J)P(e_{JK})
$$

(b) Probabilistic model

$$
\begin{aligned}
O_R &\overset{iid}{\sim} \text{Bernoulli}\left[\text{inv\_logit}(D_R)\right] \\
D_R &= (T_{IA}[i, a] - T_{IA}[h, b]) + B_{JK}[j, k] \\
T_{IA} &= T_I + \beta_{XA} X_{IA} + e_{IA} \\
T_I &= \beta_{XI} X_I + e_I \\
B_{JK} &= B_J + \beta_{ZK} Z_{JK} + e_{JK} \\
B_J &= \beta_{ZJ} Z_J + e_J
\end{aligned}
$$

$$
\begin{aligned}
e &\sim \text{Multi-Normal}(\mu, \Sigma) \\
\Sigma &= VQV
\end{aligned}
$$

(c) Statistical model

Figure 8: Comparative judgment model, SCM, probabilistic and statistical model assuming different discriminal dispersions for the student's traits

While the joint distribution in Equation (10) includes the probability distributions of the sampling and comparison mechanisms, $P(S)$ and $P(C)$, as well as those of the predictor variables–$P(X_{IA})$, $P(X_I)$, $P(Z_{JK})$, and $P(Z_J)$–all of these probabilities are omitted from the statistical model 8c. This omission is justified because, while these distributions contribute to the overall joint distribution of the data, the variables S, C, X_{IA}, X_I, Z_{JK}, and Z_J are observed and independent of any other variable in the model. As observed variables, they do not require distributional assumptions in the same way the idiosyncratic errors do. Their independence follows from the underlying random selection procedures that govern the variables[9].

[9]Randomization ensures that data–and, by extension, an estimator–satisfies several key identification properties, such as common support, no interference, and consistency. The most critical property, however, is the elimination of confounding. *Confounding* occurs when an external variable, such as X_I, simultaneously influences both the outcome (e.g., O_R) and a variable of interest (e.g., S), resulting in spurious associations between the latter two (Everitt and Skrondal, 2010). Randomization ensure the absence of confounding by effectively decoupling the association between

Next D_R is defined as the difference between the discriminal processes $T_{IA}[i,a]$ and $T_{IA}[h,b]$, representing the underlying written-quality trait of the compared texts, plus the corresponding repeated judge bias $B_{JK}[j,k]$. Note that if it is assumed that $B_{JK}[j,k]$ reflects the difference in stimulus-specific biases, i.e., $B_{JK}[j,k] = B_{JK}[i,a,j,k] - B_{JK}[h,b,j,k]$, the discriminal difference can be re-written as:

$$\begin{aligned} D_R &= (T_{IA}[i,a] - T_{IA}[h,b]) + B_{JK}[j,k] \\ &= (T_{IA}[i,a] + B_{JK}[i,a,j,k]) - (T_{IA}[h,b] + B_{JK}[h,b,j,k]) \\ &= T^*_{IA}[i,a] - T^*_{IA}[h,b] \end{aligned} \tag{11}$$

This formulation reveals that the discriminal difference captures a *pure interaction effect*, in which neither the texts' discriminal processes nor the judges' biases alone determine the outcome, but their interaction does (Attia et al., 2022). Put simply, this mathematical description captures the idea that the stimuli' discriminal processes become an observable outcome only through the lens of judges' perceptions (i.e., their biases). For clarity, the square brackets in D_R indicate the relevant indices for each trait vector; they do not imply any subsetting of the data.

Now the functional forms for T_{IA}, T_I, B_{JK}, and B_J are specified. T_{IA} is modeled as a linear combination of the students' underlying writing-quality traits T_I, the effects of relevant text-related variables on quality assessment $\beta_{XA} X_{IA}$ (such as the influence of text length), and the text-specific idiosyncratic errors e_{IA}. Similarly, T_I is expressed as a linear combination of relevant student-related variables affecting the quality assessment $\beta_{XI} X_I$, and student-specific idiosyncratic errors e_I. For the judge-specific terms, B_{JK} is modeled as a linear combination of the judge's individual bias B_J, the influence of relevant judgment-related variables on quality assessment $\beta_{ZK} Z_{JK}$ (e.g., how the number of judgments affect the evaluation), and judgment-specific idiosyncratic errors e_{JK}. Finally, B_J is defined as a linear combination of relevant judge-level variables influencing the quality assessment $\beta_{ZJ} Z_J$ (such as judgment expertise) and judge-specific idiosyncratic errors e_J.

Next, the probabilistic assumptions for the idiosyncratic errors e_I, e_{IA}, e_J, and e_{JK} are specified. Unlike other variables in the model, these error terms exhibit indeterminacies in their *location*, *orientation*, and *scale* due to the lack of an inherent scale in the associated latent variables T_I, T_{IA}, B_J, and B_{JK}. Thus, to identify the latent variable model these indeterminacies must be

the variable of interest and any other variable, except for the outcome itself. For a more detailed discussion on the benefits of randomization, see Pearl (2009), Morgan and Winship (2014), Neal (2020), and Hernán and Robins (2025).

resolved (Depaoli, 2021; de Ayala, 2009). Drawing on principles from SEM (Hoyle, 2023), the vector of idiosyncratic errors $e = [e_I, e_{IA}, e_J, e_{JK}]^T$ are assumed to follow a Multivariate Normal distribution with mean vector μ and a covariance matrix $\Sigma = VQV$, with V denoting a diagonal matrix of standard deviations and Q a correlation matrix. To address the *location* indeterminacy, the errors' mean vector is set to zero:

$$\mu = [0, 0, 0, 0]^T \tag{12}$$

Following SCM 8a, the *orientation* indeterminacy is solved by assuming that the errors are uncorrelated. This assumption leads to the definition of the error's correlation matrix, Q, as the identity matrix:

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

To resolve the *scale* indeterminacy, the diagonal matrix V is defined as follows:

$$V = \begin{bmatrix} s_{XI} & 0 & 0 & 0 \\ 0 & p_{IA} & 0 & 0 \\ 0 & 0 & s_{ZJ} & 0 \\ 0 & 0 & 0 & p_{JK} \end{bmatrix} \tag{14}$$

Here, s_{XI} represents the standard deviation for the individuals, p_{IA} for the stimuli, s_{ZJ} for the judges, and p_{JK} for the judgments. It is assumed s_{XI} varies depending on the teaching method group to which each student belongs. Using the example from Section 4, where the teaching method $X_I = \{1, 2\}$, the model sets the constraint according to Equation (15). This constraint anchors the scale of the individuals' latent trait while relaxing the assumption of equal dispersion for the stimuli, thereby addressing the concerns raised in Section 3.1.1.

$$\sum_{g=1}^{2} s_{XI}[g]/2 = 1 \tag{15}$$

Because the error vector e follows an uncorrelated Multivariate Normal distribution, the marginal distribution of e_{IA} is a univariate Normal distribution with mean zero and standard deviation p_{IA}. Thus, p_{IA} is set as a proportion of 1 to establish the scale of the stimuli' latent trait relative to the scale of the individuals' trait. Note that as a result, T_{IA} is also normally distributed. This configuration effectively reinstates Thurstone's original assumption of Normal discriminal processes for the stimuli (see Table 1).

Similarly, it is assumed that s_{ZJ} varies depending on the groups to which each judge belongs. For instance, if $Z_J = \{1, 2, 3\}$ represents three groups of judges with varying expertise, the model sets the constraint according to Equation (16). This constraint anchors the scale of the judges' latent trait and relaxes the assumption of equal dispersion for the judgments.

$$\sum_{g=1}^{2} s_{ZJ}[g]/3 = 1 \tag{16}$$

Conversely, p_{JK} is defined as a proportion of 1 to establish the scale of the judgments' latent trait relative to the scale of the judges' trait.

Finally, we use *Bayesian inference methods* to convert the statistical model 8c into a practical statistical tool for analyzing paired comparison data. Bayesian inference offers three main advantages in this context. First, it handles complex and overparameterized models, where the number of parameters exceeds the number of observations (Baker, 1998; Kim and Cohen, 1999). This feature is essential for our implementation, as the proposed model is indeed overparameterized. Second, it incorporates prior information to constrain parameter estimates within plausible bounds, thereby mitigating estimation issues like non-convergence or improper solutions that often affect frequentist methods (Martin and McDonald, 1975; Seaman III et al., 2011). Prior distributions are used to define the error distribution and set the scale of latent variables (Depaoli, 2014). Third, Bayesian inference supports robust inferences from small samples, where the asymptotic properties underlying frequentist methods are less reliable (Baldwin and Fellingham, 2013; Lambert et al., 2006; Depaoli, 2014). This feature is particularly relevant in CJ assessments, as researchers often collect large volumes of paired comparisons but work with relatively small samples of judges, stimuli, and individuals to test hypotheses.

The **Declarations** section of this document provides a link to the model code, along with an alternative specification that assumes equal discriminal dispersions. We tested both versions of the model with success using `Stan` (Stan Development Team., 2021, version 2.26.1).

6. Discussion

Thurstone introduced the Law of Comparative Judgment to measure psychological traits of stimuli through pairwise comparisons (Thurstone, 1927b,a). In its general form, the theory models single-judge comparisons across multiple, potentially correlated stimuli. Each comparison produces a dichotomous outcome indicating which stimulus the judge perceives as having a higher trait level. However, Thurstone identified one key challenge in this general formulation: the measurement model required estimating more "unknown" parameters than the number of available pairwise comparisons (Thurstone, 1927a). To address this issue and facilitate the theory's practical applicability, he formulated five cases, each progressively incorporating several simplifying assumptions.

Among these, Case V remains the most widely used model in empirical CJ research, mainly due to the widespread adoption of the BTL model. The BTL model mirrors the core assumptions of Case V–namely, equal discriminal dispersions and zero correlation among stimuli' discriminal processes–but replaces the processes' normal distribution with the more mathematically tractable logistic distribution (Andrich, 1978; Bramley, 2008). Although this substitution has minimal impact on trait estimation or model interpretation (van der Linden, 2017a; McElreath, 2021), the simplifying assumptions of the BTL model–and by extension, of Case V–may fail to capture the complexity of some traits or account for heterogeneous stimuli (Thurstone, 1927b; Andrich, 1978; Bramley, 2008; Kelly et al., 2022), potentially leading to unreliable and inaccurate trait estimates (Ackerman, 1989; Zimmerman, 1994; McElreath, 2020; Hoyle, 2023).

Moreover, because Thurstone's original goal was to produce a "coarse scaling" of traits and allocate stimuli along this continuum (Thurstone, 1927b, pp. 269), his theory offered no guidance on how to use trait estimates for statistical inference. The CJ tradition has attempted to address this gap by separating trait estimation from hypothesis testing, relying on point estimates, such as BTL scores or their transformations, for inference. While this approach simplifies analysis, it can also introduce bias and compromise the reliability of the resulting inferences (McElreath, 2020; Kline, 2023; Hoyle, 2023).

To address the limitations of Thurstone's Case V and the BTL model, this study extends Thurstone's general form using a systematic, integrated approach that combines causal and Bayesian inference methods. The approach begins with the development of a conceptual model, represented as a Structural Causal Model (SCM) and a Directed Acyclic Graph (DAG) (Pearl, 2009; Pearl et al., 2016; Gross et al., 2018; Neal, 2020). This model integrates Thurstone's core theoretical principles, such as the discriminal processes of stimuli, alongside key CJ assessment design features, including judges' bias, sampling procedures, and comparison mechanisms, to disentangle the causal processes underlying the CJ system.

The approach then translates the SCM into a bespoke statistical model that allows researchers to analyze CJ data when violations to the assumptions of equal dispersion and zero correlation occur, and when statistical inference is necessary. In particular, this model accounts for judge biases, captures the hierarchical structure of stimuli, incorporates measurement error into the hypothesis testing process, and accommodates heterogeneity in discriminal dispersions. By addressing all these issues, these methodological innovations have the potential to enhance the reliability and validity of trait measurement in CJ (Perron and Gillespie, 2015), while also improving the accuracy of statistical inferences.

Beyond these potential benefits, the approach offers two additional advantages. First, it clarifies the roles and interactions of all actors and processes involved in CJ assessments. Second, it shifts the analytic paradigm from passively accepting the assumptions of Case V and the BTL model to actively testing their fit with observed data. Together, these advantages establish a principled framework for evaluating best practices in designing CJ assessments, one that better aligns with the demands of contemporary CJ contexts (Kelly et al., 2022), offering new insights into existing research and opening promising avenues for future inquiry.

6.1. Future research directions using our approach

Among the many potential directions for future research, three avenues deserve particular attention due to their direct impact on the reliability and validity of CJ trait estimates, as well as on the accuracy of statistical inferences. The following sections outline these avenues and explain how our approach facilitates their investigation.

6.1.1. The impact of sampling and comparison mechanisms

Although sampling and comparison mechanisms are central to modern CJ assessments, it is striking that most CJ literature has examined them within a limited scope. Researchers have primarily investigated the effects of adaptive comparative judgment (ACJ) designs on trait reliability (Pollitt, 2012a,b; Bramley, 2015; Verhavert et al., 2022; Mikhailiuk et al., 2021; Gray et al., 2024) or proposed practical guidelines for the number of comparisons judges should make (Verhavert et al., 2019; Crompvoets et al., 2022). While these studies offer valuable insights, they also overlook the broader role that these mechanisms play within the CJ system. As this oversight likely stems from a more fundamental lack of conceptual clarity about how these mechanisms function within the system, this study integrates these mechanisms into the conceptual model of CJ.

The explicit integration of the sampling and comparison mechanisms offers a new perspective on how these mechanisms shape the CJ process. Specifically, it clarifies their role as sources of missing data in CJ's data-generating process, that is, as mechanisms that determine which observations are missing from the final data sample. This new perspective encourages the application of Little and Rubin's principled missing data framework (2020), allowing a more rigorous evaluation of existing claims about these missing data mechanisms, their influence on CJ outcomes, and their implications for designing and evaluating more complex assessments setups.

This study circumvents the need to apply this missing data framework by deliberately structuring the sampling and comparison mechanisms to be independent of any observed or unobserved variables, including the outcome. In other words, these mechanisms are designed to produce data that are *missing completely at random* (MCAR) (Little and Rubin, 2020). This design offers one key advantage: it generates simple random samples that satisfy the condition of *ignorability*, allowing researchers to legitimately *ignore* missing data during analysis without introducing bias (Everitt and Skrondal, 2010; Kohler et al., 2019; Neal, 2020).

However, many modern CJ applications rely on more complex assessment designs, in which the sampling and comparison mechanisms introduce more intricate forms of missingness such as *missing at random* (MAR) or *missing not at random* (MNAR) (Little and Rubin, 2020). A prominent example is the previously discussed ACJ design, where prior judgment outcomes inform the selection of stimulus pairs for subsequent comparisons (Pollitt, 2012a,b; Bramley, 2015). This pair selection process suggests that ACJ's comparison mechanism is outcome-dependent, potentially classifying the method as a generator of MNAR data. If this classification holds, the mixed findings on ACJ's capabilities become more comprehensible: some studies find that the method improves trait reliability (Pollitt and Elliott, 2003; Pollitt, 2012a,b), while others contend that it artificially inflates these gains (Bramley, 2015; Bramley and Vitello, 2019; Crompvoets et al., 2020, 2022).

Regardless of the underlying missingness mechanisms, any CJ assessment design would benefit from explicitly defining its assumptions–a practice supported by our approach. This clarity enables researchers to evaluate how the sampling and comparison mechanisms affect trait estimation and statistical inference within each design. Such assessments are particularly relevant given the common misconception in the CJ literature that Thurstone's model can naturally handle even non-random missing data without compromising the reliability or validity of trait estimates (Bramley, 2008).

6.1.2. *The effects of judges' bias on the reliability of traits*

Despite the growing notion that various stimulus-related factors influence judges' perceptions (van Daal et al., 2016; Lesterhuis et al., 2018; Chambers and Cunningham, 2022) and that these influences may not always cancel each other out, few studies in the CJ literature provide empirical evidence for judges' biases (Pollitt and Elliott, 2003; van Daal et al., 2016; Bartholomew et al., 2020). This gap likely persists not due to a lack of interest or research but because researchers often rely on ad-hoc detection methods, such as 'misfit' statistics, that may not be well-suited for the task (Kelly et al., 2022). To address this limitation, this study treats judges' biases as an integral component of the CJ system from the outset. This approach offers one key advantage: it provides a more accurate representation of the data-generating process behind pairwise comparisons, one that acknowledges that the discriminal processes of stimuli become an observable outcome only through judges' perceptions, which may exhibit bias.

The explicit integration of judges' bias into CJ's conceptual model then paves the way for investigating several relevant research questions. One key question is whether researchers can validly analyze CJ data under the assumption of "sample-free" trait calibration, specifically under the hypothesis that judges exhibit no systematic bias. This question is particularly relevant because many researchers still regard "sample-freeness" as an inherent property of the BTL model (Bramley, 2008; Andrich, 1978) despite growing evidence of persistent biases. Another critical question is whether training or expertise can help judges avoid focusing on irrelevant stimulus features, such as handwriting, over more central criteria like argumentative quality in writing assessments (Kelly et al., 2022). Exploring these questions may also provide insights into what it truly means to be an "expert" within the CJ context (Kelly et al., 2022).

Moreover, since judges rely on these stimulus-related factors when evaluating complex, multidimensional traits (van Daal et al., 2016; Lesterhuis et al., 2018; Chambers and Cunningham, 2022), and these factors account for variation in judgment accuracy (Gill and Bramley, 2013; van Daal et al., 2017; van Daal, 2020; Gijsen et al., 2021), it is reasonable to expect that assessments also vary according to judge-specific attributes such as gender, age, culture, income, education, training, or expertise (Kelly et al., 2022). Prior studies support this view (Bartholomew et al., 2020; McMahon and Jones, 2015). Thus, building on the discussion in Section 6.1.1, researchers could further explore how judges' selection influences the formation of a "shared consensus" and whether these attributes introduce systematic biases or distortions in the observed trait distribution (Deffner et al., 2022). Furthermore, if such attributes indeed undermine the assumption of "sample-freeness," it becomes essential to explore strategies for mitigating their influence and to determine how many judges (and how many judgments per judge) are needed to produce reliable trait estimates under these conditions. In addition, it is worth considering whether *repeated measures designs*, in which judges evaluate the same stimulus pairs multiple times (Lawson, 2015), can improve judgment consistency and accuracy. As anticipated, the approach presented in this study provides the necessary structure to investigate rigorously these questions.

6.1.3. The identification of 'misfitting' judges and stimuli

Although the CJ literature clearly defines *misfit* judges and stimuli, CJ researchers have rarely examined how these observations relate to Thurstonian theory. In particular, they have not identified which elements of Thurstone's theory might account for the occurrence of misfits. This disconnect likely stems from the fact that CJ researchers derive misfit statistics from residual analysis and outlier detection methods rather than from Thurstonian principles. Specifically, *misfit judges* are typically defined as those whose assessments diverge significantly from the "shared consensus" (Pollitt, 2012a,b; van Daal et al., 2016; Goossens and De Maeyer, 2018; Wu et al., 2022), while *misfit stimuli* are those that elicit more judgment discrepancies than others (Pollitt, 2004, 2012a,b; Goossens and De Maeyer, 2018). Both definitions closely mirror the statistical concept of outliers, that is, observations that deviate markedly from the rest of the sample in which they occur (Grubbs, 1969). But this resemblance extends beyond the definitions themselves, as CJ researchers often identify misfits using conventional outlier detection procedures, such as transforming BTL model residuals into diagnostic statistics and comparing them against predefined thresholds (Pollitt, 2012a,b; Wu et al., 2022).

Nevertheless, it is not the classification of misfits as outliers that raises concerns for trait measurement and inference. Rather, the concern lies in the prevailing CJ practice of classify them through ad-hoc detection procedures and then excluding them from analysis (Pollitt, 2012a,b), often without providing empirical evidence for the various hypotheses proposed to explain their occurrence. In this regard, it is essential to recognize that outliers can only be defined relative to a specific model (McElreath, 2020). As such, detection procedures based on models like the BTL, which relies on strong assumptions, should be approached with caution, as they may not be well-suited for the task (Kelly et al., 2022). In particular, because the BTL model may not always accurately reflect the underlying data-generating process of the CJ system, misfit classification based on this model risks misidentifying observations. Furthermore, the exclusion of these observations carries additional risks. The statistical literature cautions that removing outliers can discard valuable information (Miller, 2023) and introduce bias into trait estimates. The direction and magnitude of this bias are often unpredictable, as they depend on which observations researchers exclude from the analysis (Zimmerman, 1994; O'Hagan, 2018; McElreath, 2020). Finally, precisely because of its rigid assumptions, the BTL model lacks the flexibility to adequately test many of the hypotheses proposed to explain the presence of misfits.

In contrast, the approach presented in this study provides a rigorous framework that enables the examination of several relevant hypotheses. For instance, researchers could investigate whether misfit judges are those who exhibit (an outlying degree of) systematic bias or greater variability in their judgments compared to their peers. Similarly, researchers might explore whether misfit stimuli exhibit more variable discriminal processes relative to other stimuli or if they are genuinely outlying cases. Moreover, since outliers are not necessarily "bad data" (McElreath, 2020), our approach also offers a principled alternative that retains misfits in the analysis without compromising trait estimation or inference. This alternative involves adapting the proposed model into robust measurement models (McElreath, 2020), a broad class of procedures designed to reduce the sensitivity of parameter estimates to mild or moderate departures from model assumptions (Everitt and Skrondal, 2010). This alternative also relates to a broader discussion in social science, which is: that when researchers face low predictive capacity from their models, they should not only search for "new" variables or alternative procedures but also consider employing more sophisticated measurement models (Wainer et al., 1978).

6.2. Study limitations and practical challenges for applied CJ researchers

The process of deriving conclusions from observed data always requires assumptions, whether the data are observational or experimental (Kohler et al., 2019; Deffner et al., 2022). The proposed approach is not an exception to this fundamental principle. As with all approaches grounded on causal inference, it relies on expert knowledge and assumptions about the variables' causal structure that are often untestable at the outset (Hernán and Robins, 2025). As such, the approach does not seek to yield automatic answers when applied to a given CJ dataset. Instead, it aims to encourage researchers to formulate precise questions and to make their assumptions explicit, fostering a generalizable understanding of the CJ system under study (Rohrer et al., 2022; Deffner et al., 2022; Sterner et al., 2024). This clarity is critical because a model's ability to produce accurate estimates and valid inferences depends heavily on how well the data and inferential goals align with its underlying assumptions (Kohler et al., 2019). Although this alignment remains empirically untested for the proposed models, the theory-driven nature of our approach provides a solid foundation for future empirical evaluation of its causal assumptions (Deffner et al., 2022), which are well supported by both theory and existing evidence.

Moreover, this theoretical commitment to causal inference also introduces several practical challenges that applied CJ researchers must navigate. These fall into two main categories: first, acquiring the foundational knowledge necessary to apply the approach effectively; and second, dedicating greater attention to both conceptual and statistical modeling.

6.2.1. Required foundational knowledge

To apply the approach effectively, CJ researchers require foundational knowledge in two areas. First, they need a solid understanding of causal inference principles. Second, they must learn to translate the functional and probabilistic content of conceptual models into bespoke statistical models. One example that illustrates the importance of a sound knowledge of causal inference is the recurrent assumption that predictor variables are "relevant" to the research context–interpreting this relevance as their inclusion in a *sufficient adjustment set* (Pearl, 2009; Pearl et al., 2016; Morgan and Winship, 2014). However, we do not fully explore what this assumption entails or its implications for model specification, estimation, and inference. As a result, CJ researchers must deeply engage with these and other complex ideas, including how SCMs encode functional and probabilistic information. To assist this effort, this study includes key references throughout the study to guide applied CJ researchers toward a deeper understanding of these concepts [10].

Developing the skills to translate conceptual models into bespoke statistical models also presents challenges. Although Bayesian inference methods offer a more accessible path for many applied CJ researchers to develop these skills–by reducing the need for specialized knowledge in areas like optimization theory, which frequentist methods often require–they still demand a working knowledge of other technical concepts such as probabilistic programming languages (PPLs), probability distributions, and convergence. These requirements can be non-trivial to master. To support CJ researchers in developing these skills, this study provides a link to the statistical model code and alternative model specifications in the **Declarations** section of this document [11].

[10] refer to footnote 1, 2, and the detailed online document referenced in the Declarations section

[11] Seminal texts on Bayesian inference methods, such as Gelman et al. (2014) and McElreath (2020), offer valuable support for developing a deeper understanding of these models.

6.2.2. *Attention to conceptual and statistical modeling*

Even after acquiring the necessary foundational knowledge, CJ researchers still face two additional challenges when applying our approach to their specific research context. First, they must verify whether the conceptual model provides a *faithful* [12] representation of the CJ system under study. Second, they need to assess whether the statistical translation of the model can accurately estimate the intended estimands (i.e., parameters) from empirical data. To ensure a faithful representation of the CJ system, we encourage researchers to treat our conceptual and statistical models as starting points rather than universal solutions for all CJ designs or datasets. While these models may prove adequate in some contexts, researchers cannot assume this apriori. Instead, they need to adapt these models to their specific contexts, paying close attention to assessment design features and causal assumptions. As discussed in Section 6.2, our approach facilitates this process by offering a transparent framework for articulating new assumptions and guiding the design of CJ assessments.

Conversely, to evaluate the estimation capabilities of the statistical model, researchers must tackle the challenge of performing identification analysis. As outlined in Section 4, *identification analysis* determines whether a statistical model can accurately compute a given estimand (e.g., a parameter) based solely on its (causal) assumptions, independent of random variability (Schuessler and Selb, 2023). Identification is crucial because it is a necessary condition for consistency. *Consistency* is the property of an estimator (e.g., a statistical model) whose estimates converge to the "true" value of an estimand as data size approaches infinity (Everitt and Skrondal, 2010). Without identification, consistency is impossible–even with infinite, error-free data–and thus, meaningful inference from finite samples cannot be achieved (Schuessler and Selb, 2023). Although performing identification analysis through formal derivations may seem like a natural next step, the complexity of the CJ system renders such an approach impractical at the outset. Instead, simulation-based methods, such as power analysis, provide a more practical and flexible alternative, enabling researchers to examine the consistency of estimates without relying on cumbersome mathematical proofs. Nonetheless, the development of formal identification proofs remains a significant goal for future CJ research. Notably, our approach supports both strategies by providing the probabilistic foundation for formal derivations and the necessary statistical structure for simulation-based methods.

[12] Avoid confusing this term with the *faithfulness assumption* (Neal, 2020; Hernán and Robins, 2025) described in the causal inference literature. For further details, see footnote 1.

7. Conclussion

This paper underscores the need to extend Thurstone's theory to meet the demands of contemporary empirical CJ research. It specifically advocates for the development of bespoke CJ models aligned with the underlying data-generating processes of specific CJ assessment designs, as these models have the potential to strengthen the robustness of trait estimates and the clarity of inferences. The work also outlines a clear trajectory for advancing CJ research in both theoretical and applied domains. Finally, the study also lays the groundwork for broader applications, encouraging researchers across the social sciences to adopt more robust and interpretable methodologies.

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