Let's talk about Thurstone & Co.: An information-theoretical model for comparative judgments, and its statistical translation

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Abstract

This study revisits Thurstone's law of comparative judgments (CJ) by addressing two key limitations in traditional approaches. Firstly, it addresses the overreliance on the assumptions of Thurstone's Case V in the statistical analysis of CJ data. Secondly, it addresses the apparent disconnect between CJ's approach to trait measurement and hypothesis testing. We put forward a systematic approach based on causal analysis and Bayesian statistical methods, which results in a model that facilitates a more comprehensive understanding of the factors influencing CJ experiments while offering a robust statistical translation. The new model accommodates unequal dispersions and correlations between stimuli, enhancing the reliability and validity of CJ's trait estimation, thereby ensuring the accurate measurement and interpretation of comparative data. The paper highlights the relevance of this updated framework for modern empirical research, particularly in education and social sciences. This contribution advances current research methodologies, providing a robust foundation for future applications in diverse fields.

Keywords: causal inference, directed acyclic graphs, structural causal models, bayesian statistical methods, thurstonian model, comparative judgement, probability, statistical modeling

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1. Introduction

In *comparative judgment* (CJ) studies, judges assess a specific trait or attribute across different stimuli by performing pairwise comparisons (Thurstone, 1927b,a). Each comparison produces a dichotomous outcome, indicating which stimulus is perceived to have a higher trait level. For example, when assessing writing quality, judges compare pairs of written texts (the stimuli) to

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determine the relative writing quality each text exhibit (the trait) (Laming, 2004; Pollitt, 2012b; Whitehouse, 2012; van Daal et al., 2016; Lesterhuis, 2018a; Coertjens et al., 2017; Goossens and De Maeyer, 2018; Bouwer et al., 2023).

Numerous studies have documented the effectiveness of CJ in assessing traits and competencies over the past decade. These studies have highlighted three aspects of the method's effectiveness: its reliability, validity, and practical applicability. Research on reliability suggests that CJ requires a relatively modest number of pairwise comparisons (Verhavert et al., 2019; Crompvoets et al., 2022) to generate trait scores that are as precise and consistent as those generated by other assessment methods (Coertjens et al., 2017; Goossens and De Maeyer, 2018; Bouwer et al., 2023). In addition, the evidence suggests that the reliability and time efficiency of CJ are comparable, if not superior, to those of other assessment methods when employing adaptive comparison algorithms (Pollitt, 2012b; Verhavert et al., 2022; Mikhailiuk et al., 2021). Meanwhile, research on the validity of CJ scores indicates their capacity to represent accurately the traits under measurement (Whitehouse, 2012; van Daal et al., 2016; Lesterhuis, 2018a; Bartholomew et al., 2018; Bouwer et al., 2023). Moreover, research on CJ's practical applicability highlights its versatility across both educational and non-educational contexts (Kimbell, 2012; Jones and Inglis, 2015; Bartholomew et al., 2018; Jones et al., 2019; Marshall et al., 2020; Bartholomew and Williams, 2020; Boonen et al., 2020).

Nevertheless, despite the increasing number of CJ studies, research in this domain remains unsystematic and fragmented, leaving several critical issues unresolved. This study identifies and discusses two prominent issues that can undermine the reliability and validity of CJ's trait estimates. The first issue arises from the excessive reliance on Thurstone's Case V assumptions in the statistical analysis of CJ data. The second stems from the apparent disconnect between CJ's approach to trait measurement and hypothesis testing. The study then addresses these issues by extending Thurstone's general form through a systematic and integrated approach based on causal and Bayesian inference methods.

As a result, the study divides its content into six main sections. Section 2 provides an overview of Thurstone's theory. Section 3 discusses the identified issues in detail. Section 4 extends Thurstone's theory to address these challenges. The extension integrates core theoretical principles alongside key assessment design features relevant to CJ experiments, such as the selection of judges, stimuli, and comparisons. Section 5 translates these theoretical and practical elements into a probabilistic statistical model for the analysis of dichotomous pairwise comparison data. Section 6 discusses the

implications of the findings and explores avenues for future research. Finally, Section 7 summarizes the study's key insights.

2. Thurstone's theory

In its most general form, Thurstone's theory addresses pairwise comparisons wherein a single judge evaluates multiple stimuli (Thurstone, 1927a, pp. 267). The theory posits that two key factors determine the dichotomous outcome of these comparisons: the discriminal process of each stimulus and their discriminal difference. The *discriminal process* captures the psychological impact each stimulus exerts on the judge or, more simply, his perception of the stimulus trait. The theory assumes that the discriminal process for any given stimulus forms a Normal distribution along the trait continuum (Thurstone, 1927a, pp. 266). The mode (mean) of this distribution, known as the *modal discriminal process*, indicates the stimulus position on this continuum, while its dispersion, referred to as the *discriminal dispersion*, reflects variability in the perceived trait of the stimulus.

Figure 1a illustrates the hypothetical discriminal processes along a quality trait continuum for two written texts. The figure indicates that the modal discriminal process for Text B is positioned further along the continuum than that of Text A ($T_B > T_A$), suggesting that Text B exhibits higher quality. Additionally, the figure highlights that Text B has a broader distribution compared to Text A, which arises from its larger discriminal dispersion ($\sigma_B > \sigma_A$).

(a) Discriminal processes   (b) Discriminal difference
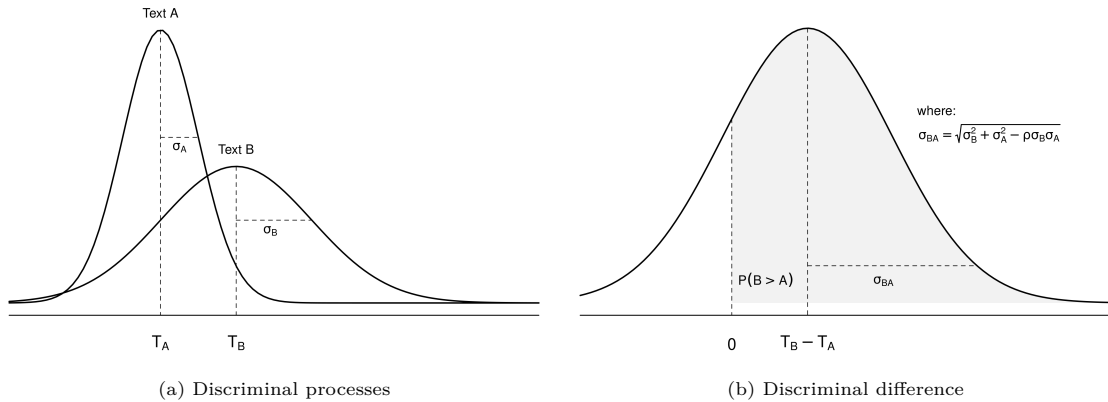
Figure 1: Hypothetical discriminal processes and discriminant difference along a quality trait continuum for two written texts.

However, since the individual discriminal processes of the stimuli are not directly observable, the theory introduces the *law of comparative judgment*. This law posits that in pairwise comparisons, a judge perceives the stimulus with a discriminal process positioned further along the trait continuum

as possessing more of the trait (Bramley, 2008, pp. 251). This suggests that the relative distance between stimuli, rather than their absolute positions on the continuum, likely defines the outcome of pairwise comparisons. Indeed, the theory assumes that the difference between the underlying discriminal processes of the stimuli, referred to as the *discriminal difference*, determines the observed dichotomous outcome. Furthermore, the theory assumes that because the individual discriminal processes form a Normal distribution on the continuum, the discriminal difference will also conform to a Normal distribution (Andrich, 1978). In this distribution, the mode (mean) represents the relative separation between the stimuli, and its dispersion indicates the variability of that separation.

Figure 1b illustrates the distribution of the discriminal difference for the hypothetical texts depicted in Figure 1a. The figure indicates that the judge perceives Text B as having significantly higher quality than Text A. This conclusion is supported by two key observations: the positive difference between their modal discriminal processes $(T_B - T_A > 0)$ and the probability area where the discriminal difference distinctly favors Text B over Text A, represented by the shaded gray area denoted as $P(B > A)$. As a result, the dichotomous outcome of this comparison is more likely to favor Text B over Text A.

3. Two prominent issues in CJ literature

This section identifies and discusses the two prominent issues that can undermine the reliability and validity of CJ's trait estimates. Section 3.1 examines the the excessive reliance on Thurstone's Case V assumptions in the statistical analysis of CJ data. Section 3.2 focuses on the apparent disconnect between CJ's approach to trait measurement and hypothesis testing.

3.1. The Case V and the statistical analysis of CJ data

Thurstone noted from the outset that the general form of the theory, as outlined in Section 2, gave rise to a trait scaling problem. Specifically, the model required estimating more "unknown" parameters than the available pairwise comparisons (Thurstone, 1927a, pp. 267). To address this issue and facilitate the practical implementation of the theory, he developed five cases derived from this general form, each progressively incorporating additional simplifying assumptions into the model.

In Case I, Thurstone postulated that pairs of stimuli would maintain a constant correlation across all comparisons. In Case II, he allowed multiple judges to undertake comparisons instead of confining evaluations to a single judge. In Case III, he posited that there was no correlation between stimuli.

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Table 1: Thurstones cases and their asumptions

Assumption	General form	Thurstone's					BTL model
		Case I	Case II	Case III	Case IV	Case V	
Discriminal process (distribution)	Normal	Normal	Normal	Normal	Normal	Normal	Logistic
Discriminal dispersion (between stimuli)	Different	Different	Different	Different	Similar	Equal	Equal
Correlation (between stimuli)	One per pair	Constant	Constant	Zero	Zero	Zero	Zero
How many judges compare?	Single	Single	Multiple	Multiple	Multiple	Multiple	Multiple

In Case IV, he assumed that the stimuli exhibited similar dispersions. Finally, in Case V, he replaced this assumption with the condition that stimuli had equal discriminal dispersions. Table 1 summarizes the assumptions of the general form and the five cases. For a detailed discussion of these cases and their progression, refer to Thurstone (1927a) and Bramley (2008, pp. 248–253).

Notably, despite relying on the most extensive set of simplifying assumptions (Bramley, 2008, pp. 253; Kelly et al., 2022, pp. 677), Case V remains the most widely used case in the CJ literature. This popularity stems mainly from its simplified statistical representation in the Bradley-Terry-Luce (BTL) model (Bradley and Terry, 1952; Luce, 1959). The BTL model mirrors the assumptions of Case V, with one notable distinction: whereas Case V assumes a Normal distribution for the stimuli's discriminal processes, the BTL model uses the more mathematically tractable Logistic distribution (Andrich, 1978; Bramley, 2008, pp. 254) (see Table 1). This substitution has little impact on the model's estimation or interpretation, as the Normal and Logistic distributions exhibit analogous statistical properties, differing only by a scaling factor of approximately 1.7 (van der Linden, 2017a, pp. 16).

However, Thurstone originally developed Case V to provide a "rather coarse scaling" of traits (Thurstone, 1927a, pp. 269), prioritizing statistical simplicity over precision in trait measurement (Kelly et al., 2022, pp. 677). He explicitly warned against its untested application, stating that its use "should not be made without (an) experimental test" (Thurstone, 1927a, pp. 270). Furthermore, he acknowledged that some assumptions could prove problematic when researchers assess complex traits or heterogeneous stimuli (Thurstone, 1927b, pp. 376). Consequently, given that modern CJ applications frequently involve such traits and stimuli, two main assumptions of Case V and, by extension, of the BTL model may not consistently hold in theory or practice, namely the assumption of equal dispersion and zero correlation between stimuli.

3.1.1. The assumption of equal dispersions between stimuli

According to the theory, discrepancies in the discriminal dispersions of stimuli shape the distribution of the discriminal difference, exerting a direct influence on the outcome of pairwise comparisons. A thought experiment can illustrate this concept. In this experiment, the researcher observes the discriminal processes for the texts depicted in Figure 1a. Furthermore, the experiment assumes that the discriminal dispersion for Text A remains constant and that the texts are uncorrelated ($\rho = 0$). Figure 2a reveals that an increase in the uncertainty associated with the perception of Text B in comparison to Text A, $(\sigma_B - \sigma_A)$, broadens the distribution of their discriminal difference. This broadening affects the probability area where the discriminal difference distinctly favors Text B over Text A, expressed as $P(B > A)$, ultimately influencing the comparison outcome. Additionally, the figure reveals that when the discriminal dispersions of the texts are equal $(\sigma_B - \sigma_A = 0)$, the discriminal difference is more narrow compared to situations where their dispersions differ. Consequently, the discriminal difference is more likely to favor Text B over Text A (shaded gray area)

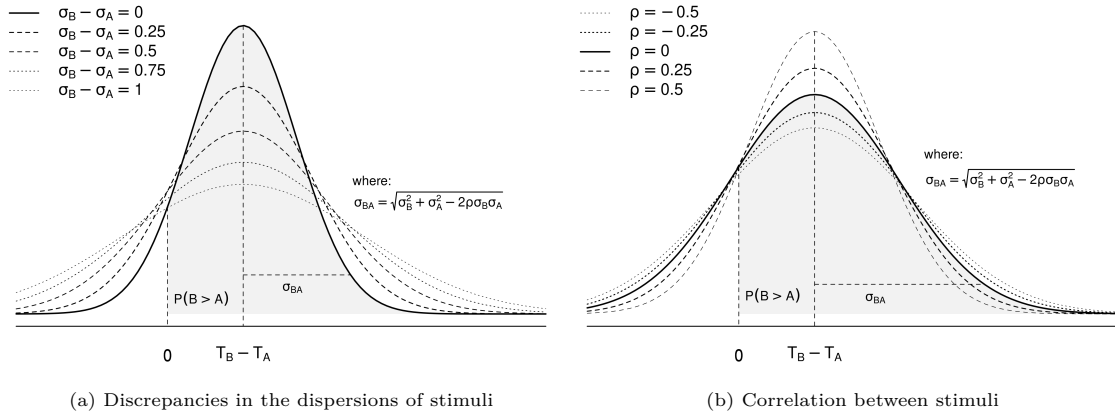(a) Discrepancies in the dispersions of stimuli (b) Correlation between stimuli

Figure 2: The effect of dispersion discrepancies and stimulus correlation on the distribution of the discriminal difference.

In experimental practice, however, the thought experiment occurs in reverse. Researchers first observe the comparison outcome and then use the BTL model to infer the discriminal difference between the stimuli and their respective discriminal processes (Thurstone, 1927b, pp. 373). Therefore, the outcome's ability to reflect the "true" differences between stimuli largely depends on the validity of the model's assumptions (Kohler et al., 2019, pp. 150), particularly the assumption of equal dispersions. For instance, when researchers observe a sample of outcomes favoring Text B over Text A and correctly assume equal dispersions between the texts, the BTL model estimates a discriminal difference distribution that accurately represents the "true" discriminal difference of the texts. This scenario is illustrated in Figure 2a, when the model's discriminal difference distri-

bution aligns with the "true" distribution, represented by the thick continuous line corresponding to $\sigma_B - \sigma_A = 0$. The accuracy of the discriminal difference then ensures reliable estimates for the texts' discriminal processes (citation needed?).

However, Thurstone argued that the assumption of equal dispersions may not be applicable when researchers assess complex traits or heterogeneous stimuli (Thurstone, 1927b, pp. 376), as these traits and stimuli can introduce judgment discrepancies due to their unique features (van Daal et al., 2016; Lesterhuis, 2018b; Chambers and Cunningham, 2022). Indeed, evidence of this violation may already be present in the CJ literature in the form of misfit statistics, which measure judgment discrepancies associated with specific stimuli (Pollitt, 2004, pp. 12; Goossens and De Maeyer, 2018, pp. 20). For example, labeling texts as "misfits" indicates that comparisons involving these texts result in more judgment discrepancies than those involving other texts (Pollitt, 2012a,b; van Daal et al., 2016; Goossens and De Maeyer, 2018). These discrepancies, in turn, suggest that the discriminal differences for "misfit" texts have broader distributions, indicating that their discriminal processes may also exhibit more variation than that of other texts. A similar line of reasoning applies to the concept of "misfit" judges, whose evaluations deviate substantially from the shared consensus due to the unique characteristics of the stimuli or the judges themselves. These "misfit" judges and their associated deviations can give rise to additional statistical and measurement issues, which we discuss in more detail in Section 3.1.2.

Thus, erroneously assuming equal dispersions between stimuli, can give rise to significant statistical and measurement issues. For instance, the model may overestimate the degree to which the outcome accurately reflects the "true" discriminal differences between stimuli. This overestimation can result in researchers drawing spurious conclusions about these differences (McElreath, 2020, pp. 370) and, by extension, about the underlying discriminal processes of stimuli. Figure 2a also illustrates this issue when the model's discriminal difference distribution aligns with the thick continuous line for $\sigma_B - \sigma_A = 0$, while the "true" discriminal difference follows any discontinuous line where $\sigma_B - \sigma_A \neq 0$. Additionally, if researchers recognize that misfit statistics highlight these critical differences in dispersions, the conventional CJ practice of excluding stimuli based on these statistics (Pollitt, 2012a,b; van Daal et al., 2016; Goossens and De Maeyer, 2018) can unintentionally discard valuable information, introducing bias into trait estimates (Zimmerman, 1994; McElreath, 2020, chap. 12). The direction and magnitude of these biases are often unpredictable, as they depend on which stimuli are excluded from the analysis.

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3.1.2. The assumption of zero correlation between stimuli

The correlation, represented by the symbol ρ, measures how much a judge's perception of a specific trait in one stimulus depends on their perception of the same trait in another. As with the discriminal dispersions, this correlation shapes the distribution of the discriminal difference, directly impacting the outcomes of pairwise comparisons. A similar thought experiment, as the one in Section 3.1.1, can illustrate this concept. The experiment now assumes that the discriminal dispersions for both texts remain constant. Figure 2b reveals that as the correlation between the texts increases, the distribution of their discriminal difference becomes narrower. This narrowing affects the area under the curve where the discriminal difference distinctly favors Text B over Text A, denoted as $P(B > A)$, thus influencing the comparison outcome. Furthermore, the figure shows that when two texts are independent or uncorrelated ($\rho = 0$), their discriminal difference is less narrow compared to scenarios where the texts are highly correlated. Consequently, the discriminal difference is less likely to favor Text B over Text A (shaded gray area).

Again, in experimental practice, researchers approach this process in reverse. They begin by observing the sample of outcomes favoring Text B over Text A and then use the BTL model to estimate the discriminal difference and the discriminal processes of the stimuli. Given that the BTL model assumes independent discriminal processes across comparisons, if this assumption holds, then the model estimates a discriminal difference distribution that accurately reflects the "true" discriminal difference of the texts. This scenario is also illustrated in Figure 2b when the discriminal difference distribution of the model aligns with the "true" distribution, represented by the thick continuous line corresponding to $\rho = 0$. Once more, the accuracy of the discriminal difference ensures reliable estimates for the discriminal processes of the texts (citation needed?).

Notably, Thurstone attributed the lack of correlation between stimuli to the cancellation of potential judges' biases. He argued that this cancellation resulted from two opposing and equally weighted effects occurring during pairwise comparisons (Thurstone, 1927a, pp. 268). Andrich (1978) provided a mathematical demonstration of this cancellation using the BTL model under the assumption of discriminal processes with additive biases. However, it is easy to imagine at least two scenarios in which the zero correlation assumption is almost certainly invalid: when the pairwise comparison involves multidimensional, complex traits with heterogeneous stimuli and when an additional hierarchical structure is relevant to the stimuli.

In the first scenario, the intricate aspects of multidimensional, complex traits may introduce

dependencies between the stimuli due to certain judges' biases that resist cancellation. Research on text quality suggests that when judges evaluate these traits, they often rely on various intricate characteristics of the stimuli to form their judgments (van Daal et al., 2016; Lesterhuis, 2018b; Chambers and Cunningham, 2022). These additional relevant characteristics are unlikely to be equally weighted or opposing. As a result, they may exert an uneven influence on judges' perceptions, creating biases in their judgments. Furthermore, since the discriminal difference of the stimuli becomes an observable outcome only through the judges' perceptions, these biases may ultimately introduce dependencies between the stimuli (van der Linden, 2017b, pp. 346). For example, this could occur when a judge assessing the argumentative quality of a text places more weight on its grammatical accuracy than other judges, thereby favoring texts with fewer errors but weaker arguments. While direct evidence for this particular scenario is lacking, studies such as Pollitt and Elliott (2003) demonstrate the presence of such biases, supporting the notion that the factors influencing pairwise comparisons may not always cancel out.

In the second scenario, the shared context or inherent connections created by additional hierarchical structures may further introduce dependencies between stimuli, creating a statistical phenomenon known as clustering (Everitt and Skrondal, 2010). Although the CJ literature acknowledges the existence of such hierarchical structures, the statistical handling of this additional source of dependence between stimuli has been inadequate. For instance, when CJ data incorporates multiple samples of stimuli from the same individuals, researchers frequently rely on (averaged) estimated BTL scores to conduct subsequent analyses and tests at the individual hierarchical level (Bramley and Vitello, 2019; Boonen et al., 2020; Bouwer et al., 2023; van Daal et al., 2017; Jones et al., 2019; Gijsen et al., 2021). This approach, however, can introduce additional statistical and measurement issues, which we discuss in greater detail in Section 3.2.

In any case, similar to Section 3.1.1, assuming zero correlation between stimuli by neglecting additional relevant traits, excluding judges based on misfit statistics, or ignoring hierarchical (grouping) structures can cause significant statistical and measurement issues. In general, the model may over- or underestimate how accurately the outcome reflects the "true" discriminal differences between stimuli. Such inaccuracies can result in spurious inferences about these differences and, by extension, about the stimuli's discriminal processes. This scenario is illustrated by Figure 2b, when the model's discriminal difference distribution aligns with the thick continuous line for $\rho = 0$, while the "true" discriminal difference follows any discontinuous line where $\rho \neq 0$.

In particular, neglecting relevant traits, such as judges' biases, can cause dimensional mismatches in the BTL model, artificially inflating the trait's reliability (Hoyle, 2023, pp. 341) or, worse, introducing bias into the trait's estimates (Ackerman, 1989). Excluding judges based on misfit statistics risks discarding valuable information, which may further bias the trait's estimates (Zimmerman, 1994; McElreath, 2020, chap. 12). Finally, ignoring hierarchical structures may reduce the precision of model parameter estimates, potentially overestimating the trait's reliability (Hoyle, 2023, pp. 482).

3.2. The disconnect between trait measurement and hypothesis testing

Building on the previous section, it is clear that, despite its limitations, the BTL model is commonly used as a measurement model in CJ assessments. A measurement model specifies how manifest variables contribute to the estimation of latent variables (Everitt and Skrondal, 2010). For example, when evaluating writing quality, researchers use the BTL model to process the dichotomous outcomes resulting from the pairwise comparisons (the manifest variables) to estimate scores that reflect the underlying level of writing quality (the latent variable) (Laming, 2004; Pollitt, 2012b; Whitehouse, 2012; van Daal et al., 2016; Lesterhuis, 2018a; Coertjens et al., 2017; Goossens and De Maeyer, 2018; Bouwer et al., 2023).

Researchers then typically use these estimated BTL scores, or their transformations, to conduct additional analyses or hypothesis tests. For example, these scores have been used to identify 'misfit' judges and stimuli (Pollitt, 2012b; van Daal et al., 2016; Goossens and De Maeyer, 2018), detect biases in judges' ratings (Pollitt and Elliott, 2003; Pollitt, 2012b), calculate correlations with other assessment methods (Goossens and De Maeyer, 2018; Bouwer et al., 2023), or test hypotheses related to the underlying trait of interest (Casalicchio et al., 2015; Bramley and Vitello, 2019; Boonen et al., 2020; Bouwer et al., 2023; van Daal et al., 2017; Jones et al., 2019; Gijsen et al., 2021).

However, the statistical literature advises caution when using estimated scores for additional analyses and tests. A key consideration is that BTL scores are parameter estimates that inherently carry uncertainty. Ignoring this uncertainty can bias the analysis and reduce the precision of hypothesis tests. Notably, the direction and magnitude of such biases are often unpredictable. Results may be attenuated, exaggerated, or remain unaffected depending on the degree of uncertainty in the scores and the actual effects being tested (Kline, 2023, pp. 25; Hoyle, 2023, pp. 137). Finally, the reduced precision in hypothesis tests diminishes their statistical power, increasing the likelihood of committing type-I or type-II errors (McElreath, 2020).

In aggregate, the excessive reliance on Thurstone's Case V assumptions in the statistical analysis of comparative data can compromise the reliability of the trait estimates. This overreliance may also undermine their validity (Perron and Gillespie, 2015, pp. 2), especially when combined with the disconnect between CJ's approach to trait measurement and hypothesis testing. However, the structural approach to causal inference can address these issues by providing a systematic and integrated framework that enhances statistical accuracy while strengthening measurement reliability and validity.

4. Extending Thurstone's theory

The *structural approach* to causal inference provides a formal framework for identifying causes and estimating their effects using data. The approach uses structural causal models (SCMs) and directed acyclic graphs (DAGs) (Pearl, 2009; Pearl et al., 2016; Gross et al., 2018; Neal, 2020) to formally and graphically represent the assumed causal structure of a system, such as the one found in CJ experiments.

Essentially, SCMs and DAGs function as *conceptual models* on which identification analysis rests (Schuessler and Selb, 2023, pp. 4). *Identification analysis* helps researchers to determine whether an estimator can accurately compute an estimand based solely on its (causal) assumptions, regardless of random variability (Schuessler and Selb, 2023, pp. 4). Here, *estimands* represent the specific quantities researchers aim to determine (Everitt and Skrondal, 2010). *Estimators* denote the methods or functions that transform data into an estimate, while *estimates* are the numerical values approximating the estimand (Neal, 2020; Everitt and Skrondal, 2010).

A motivating example helps to clarify these concepts. This example will appear throughout the document to illustrate various aspects of the proposed extension. In this example, researchers aim to determine: "To what extent do different teaching methods influence students' ability to produce high-quality written texts?" To investigate this, researcher design a CJ experiment by randomly assigning students (individuals) to two groups, each receiving a different teaching method. Judges then compare pairs of students' written texts (stimuli), to produce a dichotomous outcome that reflects the relative quality of each text (trait). Based on this setup, researchers reformulate the research question as the estimand: "*On average*, is there a difference in the ability to produce high-quality written texts between the two groups of students?". Finally, following current CJ practices, researchers then rely on estimates from the BTL model, or its transformations, to approximate this estimand.

However, Section 3 presents compelling evidence that Thurstone's Case V, and by extension the BTL model, suffers from several statistical and measurement limitations. These limitations hinder the model's ability to identify various estimands relevant to CJ inquiries, including the one described in the motivating example. Identification is crucial because it is a necessary condition for ensuring consistent estimators. *Consistency* refers to the property of an estimator whose estimates converge to the "true" value of the estimand as the data size approaches infinity (Everitt and Skrondal, 2010). Without identification, consistency cannot be achieved, even with "infinite" and error-free data. Thus, deriving meaningful insights from finite data becomes impossible (Schuessler and Selb, 2023, pp. 5).

Luckily, SCMs and DAGs support identification analysis through two key advantages. First, regardless of complexity, they can represent various causal structures using only five fundamental building blocks (Neal, 2020; McElreath, 2020). This feature allows researchers to decompose complex structures into manageable components, facilitating their analysis (McElreath, 2020). Second, they depict causal relationships in an interactive, non-parametric way. This flexibility enables feasible identification strategies without requiring specification of the types of variables, the functional forms relating them, or the parameters of those functional forms (Pearl et al., 2016, pp. 35).

Thus, this section addresses the issues identified in Section 3 by extending Thurstone's theory using the structural approach to causal inference. Specifically, it leverages the capabilities of this approach to formalize the combination of the core theoretical principles outlined in Section 2 with key assessment design features relevant to CJ experiments, such as the selection of judges, stimuli, and comparisons. In addition to enhancing statistical accuracy and strengthening measurement reliability and validity, the approach offers two key advantages. First, it clarifies the interactions among all actors and processes involved in CJ experiments. Second, it shifts the current comparative data analysis paradigm from passively accepting the model assumptions to actively testing whether those assumptions fit the data under analysis.

Accordingly, Section 4.1 incorporates the theoretical principles into what we call the *conceptual-population model*. This model assumes that researchers have access to comparison data from a conceptual population, that is, data representing all repeated judgments made by every available judge for each pair of stimuli produced by each pair of individuals in the population, hence its name. Conversely, Section 4.2 integrates the assessment design features into what we call the *sample-comparison model*. This model assumes a more realistic scenario where researchers only have access

to a sample of judges, individuals, stimuli, and comparisons from the conceptual population.

4.1. The conceptual-population model

Before incorporating the theoretical principles into the *conceptual-population model*, it is essential to define SCMs. SCMs are formal mathematical models characterized by a set of *endogenous* variables V, a set of *exogenous* variables E, and a set of functions F (Pearl, 2009; Cinelli et al., 2020). Endogenous variables are those whose causal mechanisms a researcher chooses to model (Neal, 2020). In contrast, exogenous variables represent *errors* or *disturbances* arising from omitted factors that the investigator chooses not to model explicitly (Pearl, 2009, pp. 27,68). Lastly, the functions, referred to as *structural equations*, express the endogenous variables as non-parametric functions of other endogenous and exogenous variables. These functions use the symbol ':=' to denote the asymmetrical causal dependence between variables and the symbol ' $\perp$ ' to represent *d-separation*, a concept akin to (conditional) independence.

SCM 3a illustrates the preliminary integration of theoretical principles into the conceptual-population model of the motivating example. This SCM maps the relationship between the conceptual-population outcome (O_{hiabjk}^{cp}) and several related variables. The subscripts h and i label the students who authored the texts (individuals), while the indices a and b represent the compared texts (stimuli). Moreover, the index j refers to the judge, and k specifies the judgment number, accounting for experimental conditions where judges perform the same judgment multiple times, i.e., a *repeated measures design* (Lawson, 2015, pp. 366-376). Notably, the indexing system allows comparisons between texts written by the same student ($h = i$, $a \neq b$) and between texts written by distinct students ($h \neq i$, where $a = b$ is possible). However, it excludes cases where judges compare a student's text to itself ($h = i$, $a = b$), as such comparison lacks practical meaning within the CJ framework.

In line with Thurstone's theory, SCM 3a depicts the texts' discriminal processes (T_{ha}, T_{ib}) and their discriminal difference (D_{hiabjk}) (see Section 2). Additionally, based on the arguments in Section 3.1.2 and the recommendations of Andrich (1978) and Wainer et al. (1978), the SCM incorporates the judges' biases (B_{kj}). Together with the outcome, these variables form the preliminary set of endogenous variables, $V = \{O_{hiabjk}, D_{hiabjk}, T_{ha}, T_{ib}, B_{kj}\}$. Finally, the SCM shows the preliminary set of structural equations, $F = \{f_O, f_D\}$, which define the non-parametric dependencies among these variables.

$$O^{cp}_{hiabjk} := f_O(D_{hiabjk})$$

$$D_{hiabjk} := f_D(T_{ha}, T_{ib}, B_{jk})$$

(a) SCM
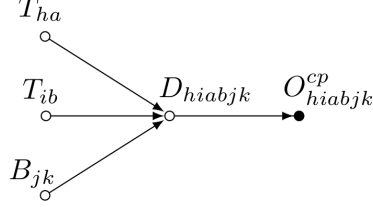
(b) DAG

Figure 3: Conceptual-population model, scalar form.

Notably, every SCM has an associated DAG (Pearl et al., 2016; Cinelli et al., 2020). A DAG is a graph consisting of nodes connected by edges, where nodes represent random variables. The term *directed* indicates that edges or arrows extend from one node to another, indicating the direction of causal influence, while the absence of an edge implies no relationship between the nodes. The term *acyclic* means that the causal influences do not form loops, ensuring the influences do not cycle back on themselves (McElreath, 2020). DAGs conventionally represent observed variables with solid black circles and unobserved (latent) variables with open circles (Morgan and Winship, 2014). They also display conditioned variables by surrounding them with a square. Here, *conditioning* refers to restricting the statistical analysis to a subset of the population defined by the values of the given variable (Neal, 2020, pp. 32). While DAGs often omit exogenous variables for simplicity, following a *standard representation*, this section adopts the *magnified representation* of DAGs, as including exogenous variables can improve clarity and help reveal potential issues related to conditioning and confounding (Cinelli et al., 2020).

Figure 3b displays the DAG corresponding to SCM 3a, illustrating the causal relationships among variables based on Thurstone's theory and the practical aspects of CJ experiments. Particularly, the graph shows how the texts' discriminal processes (T_{ha}, T_{ib}) influence their discriminal difference (D_{hiabjk}), which in turn determines the outcome. It also depicts the judges' biases (B_{kj}) as an additional factor influencing the texts' discriminal difference. Furthermore, the DAG distinguish between observed endogenous variables, such as the outcome, and latent endogenous variables, which include the texts' discriminal processes, their discriminal difference, and the judges' biases.

Although specifying a system's data structure is not mandatory for using SCMs and DAGs, defining one can sometimes enhance clarity and facilitate the description of the system. Thus, to simplify the description of a CJ experiment, we first define the vectors I and J, along with the matrices IA and JK, as follows:

$$
I_{n_I} = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ h \\ \vdots \\ i \\ \vdots \\ n_I \end{bmatrix} \; ; \; IA_{(n_I \cdot n_A) \times 2} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & n_A - 1 \\ 1 & n_A \\ \vdots & \vdots \\ h & a \\ \vdots & \vdots \\ i & b \\ \vdots & \vdots \\ n_I & n_A - 1 \\ n_I & n_A \end{bmatrix} \; ; \; J_{n_J} = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ j \\ \vdots \\ n_J \end{bmatrix} \; ; \; JK_{(n_J \cdot n_K) \times 2} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & n_K - 1 \\ 1 & n_K \\ \vdots & \vdots \\ j & k \\ \vdots & \vdots \\ n_J & n_K - 1 \\ n_J & n_K \end{bmatrix} \tag{1}
$$

where an element of I corresponds to a unique individual h or i, while each row of IA corresponds to a unique pairing of individuals h, i with stimuli a, b. Here, n_I represents the number of individuals, while n_A denotes the number of stimuli available per individual. Thus, vector I has a length of n_I, and matrix IA has a dimension of $(n_I \cdot n_A)$ rows and 2 columns. The vector and the matrix consist of scalars from the set of natural numbers $\mathbb{N}$.

Similarly, an element of J corresponds to a unique judge j. Moreover, each row of JK specifies a unique pairing of a judge j with a judgment number k. In this case, n_J indicates the total number of judges, and n_K represents the number of judgments each judge performs. As a result, vector J has a length of n_J, and matrix JK has a dimension of $(n_J \cdot n_K)$ rows and 2 columns. Like the previous structures, the vector and the matrix consist of scalars from the set of natural numbers $\mathbb{N}$.

Additionally, we define the matrix $RPUV$, where each row uniquely pairs the corresponding rows from the matrices IA and JK. This matrix reflects the fundamental idea that the discriminal difference of the stimuli becomes an observable outcome only through the judges' perceptions. Thus, the matrix is defined as follows:

$$RPUV_{n\times6} = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 2 & 1 & n_K-1 \\ 1 & 1 & 1 & 2 & 1 & n_K \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ h & i & a & b & j & k \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n_I & n_I & n_A-1 & n_A & n_J & 1 \\ n_I & n_I & n_A-1 & n_A & n_J & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n_I & n_I & n_A-1 & n_A & n_J & n_K-1 \\ n_I & n_I & n_A-1 & n_A & n_J & n_K \end{bmatrix} \in \mathbb{N} \tag{2}$$

This matrix also consists of scalars from the set of natural numbers $\mathbb{N}$ and has a dimension of $n = \binom{n_I \cdot n_A}{2} \cdot n_J \cdot n_K$ rows and 6 columns. The matrix's dimension results from the conceptual-population model's assumption that researchers have access to data capturing all repeated judgments made by every available judge for each pair of stimuli produced by each pair of individuals in the population. Here, the term $\binom{n_I \cdot n_A}{2}$ represents the Binomial coefficient, which quantifies the total number of individual-stimuli comparison pairs within the conceptual population. The coefficient is defined as follows:

$$\binom{n_I \cdot n_A}{2} = \frac{(n_I \cdot n_A)!}{2!(n_I \cdot n_A - 2)!}$$

Readers can better visualize the structure of these vectors and matrices by considering small values for the parameters corresponding to number of individuals, the number of available stimuli per individual, the total number of judges, and the number of judgments each judge performs. For instance, assuming $n_I = 5$, $n_A = 2$, $n_J = 3$, and $n_K = 3$, the matrices described in equations (1) and (2) take the following form:

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$$I_5 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \; ; \; IA_{10\times2} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \\ 2 & 2 \\ 3 & 1 \\ 3 & 2 \\ 4 & 1 \\ 4 & 2 \\ 5 & 1 \\ 5 & 2 \end{bmatrix} \; ; \; J_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \; ; \; JK_{9\times2} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 2 & 1 \\ 2 & 2 \\ 2 & 3 \\ 3 & 1 \\ 3 & 2 \\ 3 & 3 \end{bmatrix} \; ; \; RPUV_{405\times6} = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 2 & 1 & 3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 5 & 1 & 1 & 2 \\ 1 & 1 & 5 & 1 & 1 & 3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 5 & 1 & 3 & 1 \\ 1 & 1 & 5 & 1 & 3 & 2 \\ 1 & 1 & 5 & 1 & 3 & 3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 5 & 1 & 5 & 2 & 3 & 1 \\ 5 & 1 & 5 & 2 & 3 & 2 \\ 5 & 1 & 5 & 2 & 3 & 3 \end{bmatrix}$$

Thus, using the data structure defined by the matrices in equations (1) and (2), we can re-express SCM 3a and DAG 3b in an equivalent vectorized form, as shown in Figure 4. In this depiction, the outcome O^{cp}_{RPUV}, the texts' discriminal difference D_{RPUV}, their discriminal processes T_{IA}, and the judges' biases B_{JK} are represented as vectors rather than scalar values. These vectors encompass all values from the conceptual population. Specifically, O^{cp}_{RPUV} is an observed vector of length $\binom{nI \cdot n_A}{2} \cdot n_J \cdot n_K$, while D_{RPUV} is a latent vector of the same dimension. Additionally, T_{IA} and B_{JK} are latent vectors of lengths $(n_I \cdot n_A)$ and $(n_J \cdot n_K)$, respectively.

$$O^{cp}_{RPUV} := f_O(D_{RPUV})$$

$$D_{RPUV} := f_D(T_{IA}, B_{JK})$$

(a) SCM
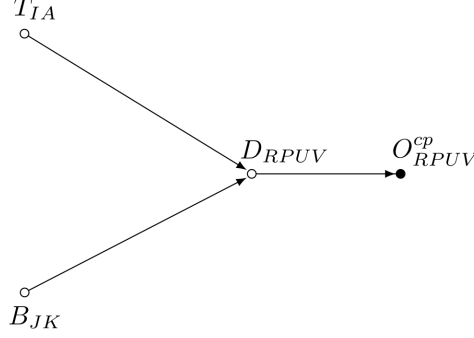
(b) DAG

Figure 4: Conceptual-population model, vectorized form.

Moreover, building on the arguments developed in Section 3.2 and the principles of Structural Equation Modeling (SEM) (Hoyle, 2023, pp. 138) and Item Response Theory (IRT) (Fox, 2010, chap. 6; van der Linden, 2017a, chap. 24), the conceptual-population model includes two hierarchical structural components. Each component defines how different observed or latent variables influence the primary latent variable of interest (Everitt and Skrondal, 2010). This addition facilitates analyses integrating the trait measurement and hypothesis testing processes, simultaneously accounting for the traits and their associated uncertainties, thereby reducing bias and enhancing measurement precision.

Figure 5 displays the two hierarchical structural components for the motivating example. Specifically, the top branch of DAG 5b depicts the first component. This component describes how the students' writing-quality trait (T_I) and a set of *relevant* text-related variables (X_{IA}) causally influence the texts' written-quality trait (T_{IA}). The DAG also highlights the idiosyncratic errors (e_{IA}), which capture variations in the texts' trait that neither the students' trait nor other relevant text-related variables can explain. Here, T_I and e_{IA} are latent vectors of length n_I and ($n_I \cdot n_A$), respectively. Additionally, X_{IA} is an observed matrix with a dimension of ($n_I \cdot n_A$) rows and q_{IA} columns.

Furthermore, the top branch of DAG 5b describes how a set of *relevant* student-related variables

(X_I) causally influence the students' writing-quality trait (T_I). The DAG also includes the idiosyncratic errors (e_I), which capture variations in the students' traits that the relevant student-related variables cannot explain. Here, e_I is a latent vector of length n_I, and X_I is an observed matrix with n_I rows and q_I columns.

$$O_{RPUV}^{cp} := f_O(D_{RPUV})$$

$$D_{RPUV} := f_D(T_{IA}, B_{JK})$$

$$T_{IA} := f_T(T_I, X_{IA}, e_{IA})$$

$$T_I := f_T(X_I, e_I)$$
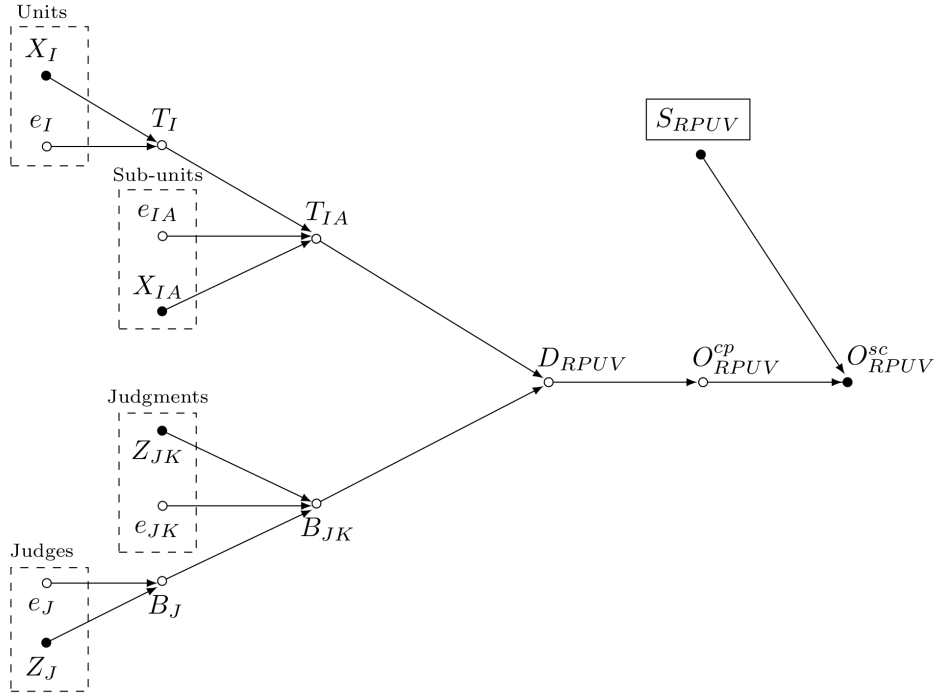
$$B_{JK} := f_B(B_J, Z_{JK}, e_{JK})$$

$$B_J := f_B(Z_J, e_J)$$

$$e_{IA} \perp \{e_I, e_{JK}, e_J\}$$

$$e_I \perp \{e_{JK}, e_J\}$$

$$e_{JK} \perp e_J$$

(a) SCM
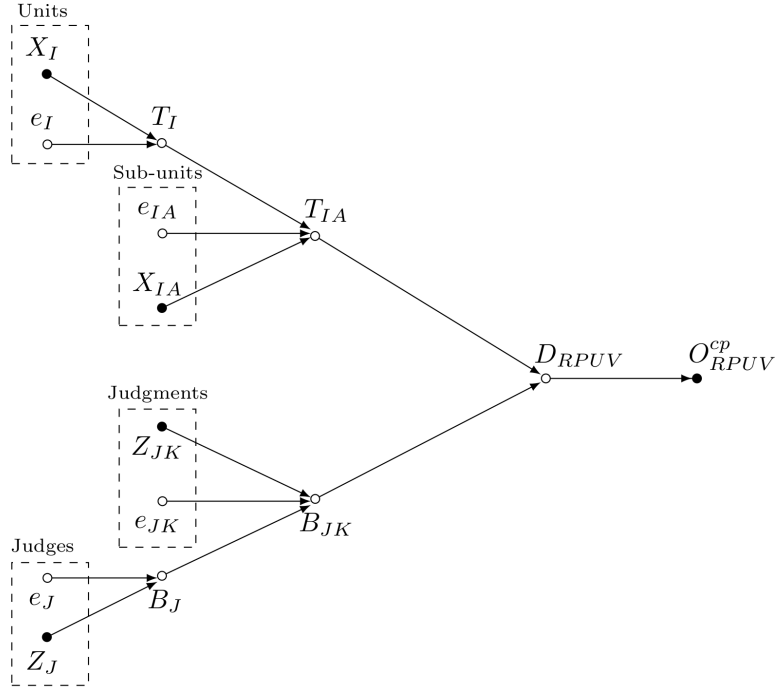
(b) DAG

Figure 5: Conceptual-population model, final vectorized form.

Similarly, the bottom branch of DAG 5b depicts the second component. This component describes how the judges' bias (B_J) and a set of *relevant* judgment-related variables (Z_{JK}) causally influence the biases associated with each text B_{JK}. The DAG also highlights the idiosyncratic errors (e_{JK}), which capture variations in the texts' bias that neither the judges' bias nor other relevant judgment-related variables can explain. In this description, B_J and e_{JK} are latent vectors of length n_J and $(n_J \cdot n_K)$, respectively. Additionally, Z_{JK} is an observed matrix with a dimension of $(n_J \cdot n_K)$ rows and q_{JK} columns.

Moreover, the bottom branch of DAG 5b describes how a set of *relevant* judge-related variables (Z_J) causally influence the judges' bias (B_J). The DAG also accounts for idiosyncratic errors (e_J), capturing the variations in the judges' bias that relevant judge-related variables cannot explain. Here, e_J is a latent vector of length n_J, and Z_J is an observed matrix with n_J rows and q_J columns.

Notably, all variables and functions shown in SCM 5a and DAG 5b define the set of endogenous variables V, structural equations F, and exogenous variables E for the conceptual-population model. Additionally, the figures demonstrate that all exogenous variables are independent of one another, as indicated by the relationships $e_{IA} \perp \{e_I, e_{JK}, e_J\}$, $e_I \perp \{e_{JK}, e_J\}$ and $e_{JK} \perp e_J$.

Overall, the conceptual-population model extends Thurstone's theory by introducing key innovations to address the limitations discussed in Section 3. These enhancements include accounting for judges' biases and integrating hierarchical structural components. Nevertheless, despite its promise of enhancing measurement accuracy and precision, the model still depends on the unrealistic assumption that researchers have access to data encompassing all repeated judgments from every judge for each pair of stimuli produced by every individual in the population. Since researchers rarely meet this assumption in practice, they must consider a more realistic scenario. The extensions to the conceptual-population model, derived from this realistic scenario, are concisely described in the sample-comparison model.

4.2. The sample-comparison model

In the sample-comparison model, researchers have access to a limited number of judgments ($n^s K$) from a sample of judges ($n^s J$), along with a specific number of texts ($n^s A$) from a sample of students ($n^s I$) drawn from the conceptual population.

Figure 6 illustrates the sample-comparison model, incorporating the sample mechanism. Following

the convention of McElreath (2020, pp. 499-516), the DAG 6b shows several conditioned variables $(S_R, S_P, S_U,$ and $S_V)$ that causally influence the sample-comparison outcome O_{RPUV}^{sc}, reflecting how the outcome is a sample drawn from the conceptual population. The DAG also depicts the conceptual-population outcome O_{RPUV}^{cp} is now an unobserved variable, emphasizing the researchers' inability to access such an outcome.

Concatenating the matrices

$$O^{sc}_{RPUV} := f_O(O^{cp}_{RPUV}, S_{RPUV})$$

$$O^{cp}_{RPUV} := f_O(D_{RPUV})$$

$$D_{RPUV} := f_D(T_{IA}, B_{JK})$$

$$T_{IA} := f_T(T_I, X_{IA}, e_{IA})$$

$$T_I := f_T(X_I, e_I)$$

$$B_{JK} := f_B(B_J, Z_{JK}, e_{JK})$$

$$B_J := f_B(Z_J, e_J)$$

$$e_{IA} \perp \{e_I, e_{JK}, e_J\}$$

$$e_I \perp \{e_{JK}, e_J\}$$

$$e_{JK} \perp e_J$$

(a) SCM

(b) DAG

Figure 6: Sample-comparison model including sampling mechanism

Considering comparison mechanisms

$$O_{RPUV} := f_C(O_{RPUV}^{sc}, C)$$

$$O_{RPUV}^{sc} := f_O(O_{RPUV}^{cp}, S_{RPUV})$$

$$O_{RPUV}^{cp} := f_O(D_{RPUV})$$

$$D_{RPUV} := f_D(T_{IA}, B_{JK})$$

$$T_{IA} := f_T(T_I, X_{IA}, e_{IA})$$

$$T_I := f_T(X_I, e_I)$$

$$B_{JK} := f_B(B_J, Z_{JK}, e_{JK})$$

$$B_J := f_B(Z_J, e_J)$$

$$e_{IA} \perp \{e_I, e_{JK}, e_J\}$$

$$e_I \perp \{e_{JK}, e_J\}$$

$$e_{JK} \perp e_J$$

(a) SCM
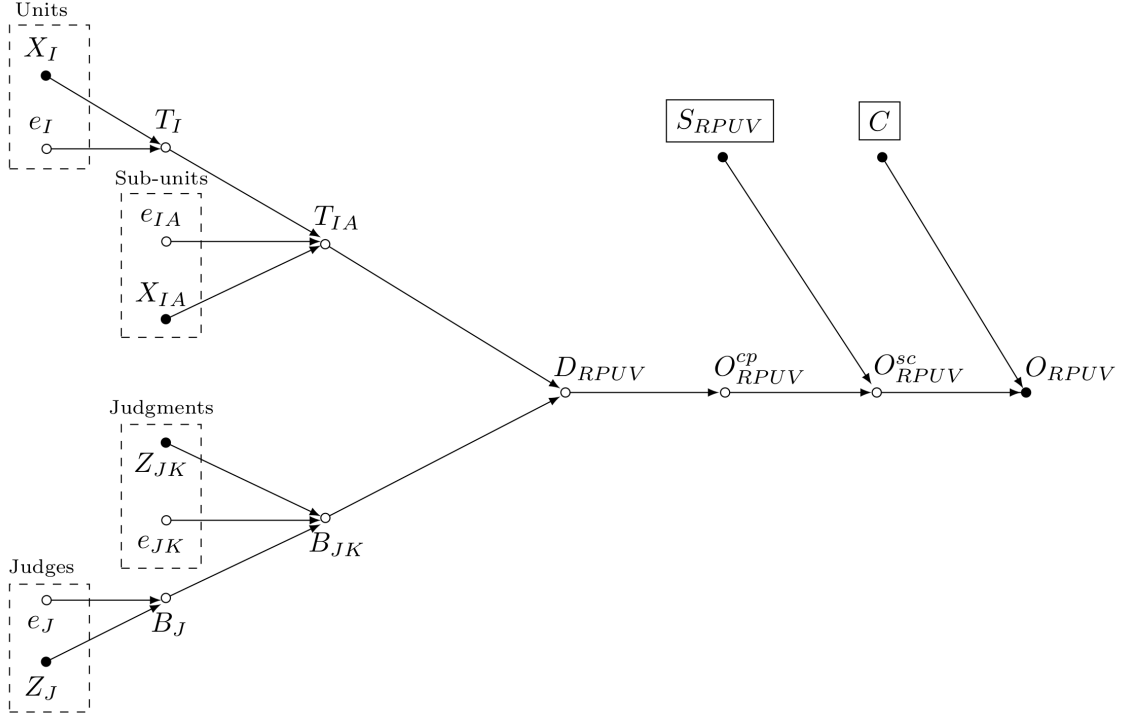
(b) DAG

Figure 7: Sample-comparison model including sample and comparison mechanisms

5. Abandoning the BTL model

$$O_{RPUV} = O^{sc}_{RPUV} \odot C$$

$$O^{sc}_{RPUV} = O^{cp}_{RPUV} \odot S_{RPUV}$$

$$O^{cp}_{RPUV} \sim \text{Bernoulli} \left[\text{inv\_logit}(D_{RPUV}) \right]$$

$$D_{RPUV} = (T_{IA}[R_1, P_1] - T_{IA}[R_2, P_2]) + B_{JK}[U, V]$$

$$T_{IA} = T_I + \beta_{XA} X_{IA} + e_{IA}$$

$$T_I = \beta_{XI} X_I + e_I$$

$$B_{JK} = B_J + \beta_{ZK} Z_{JK} + e_{JK}$$

$$B_J = \beta_{ZJ} Z_J + e_J$$

$$e_{IA} \sim \text{Normal}(0, p_{IA})$$

$$e_I \sim \text{Normal}(0, 1)$$

$$e_{JK} \sim \text{Normal}(0, p_{JK})$$

$$e_J \sim \text{Normal}(0, 1)$$

(a) Statistical model

$$O_{RPUV} := f_C(O^{sc}_{RPUV}, C)$$

$$O^{sc}_{RPUV} := f_O(O^{cp}_{RPUV}, S_R, S_P, S_U, S_V)$$

$$O^{cp}_{RPUV} := f_O(D_{RPUV})$$

$$D_{RPUV} := f_D(T_{IA}, B_{JK})$$

$$T_{IA} := f_T(T_I, X_A, e_{IA})$$

$$T_I := f_T(X_I, e_I)$$

$$B_{JK} := f_B(B_J, Z_K, e_{JK})$$

$$B_J := f_B(Z_J, e_J)$$

$$e_{IA} \perp \{e_I, e_{JK}, e_J\}$$

$$e_I \perp \{e_{JK}, e_J\}$$

$$e_{JK} \perp e_J$$

(b) SCM

Figure 8: Sample-comparison model including sample and comparison mechanisms

6. Discussion

6.1. Findings

6.2. Limitations and further research

7. Conclusion

Declarations

Financial interests: The authors declare no relevant financial interests.

Non-financial interests: The authors declare no relevant non-financial interests.

Ethics approval: The University of Antwerp Research Ethics Committee confirmed that this study does not require ethical approval.

Consent to participate: Not applicable

Consent for publication: All authors have read and approved the final version of the manuscript for publication.

Data availability: This study did not use any data.

Materials and code availability: The `CODE LINK` section at the top of the digital document located at: https://jriveraespejo.github.io/paper2_manuscript/ provides access to all materials and code.

AI-assisted technologies in the writing process: The authors used various AI-based language tools to refine phrasing, optimize wording, and enhance clarity and coherence throughout the manuscript. They take full responsibility for the final content of the publication.

CRediT authorship contribution statement: *Conceptualization:* S.G., S.DM., T.vD., and J.M.R.E; *Methodology:* S.DM., T.vD., and J.M.R.E; *Software:* J.M.R.E.; *Validation:* J.M.R.E.; *Formal Analysis:* J.M.R.E.; *Investigation:* J.M.R.E; *Resources:* S.G., S.DM., and T.vD.; *Data curation:* J.M.R.E.; *Writing - original draft:* J.M.R.E.; *Writing - review and editing:* S.G., S.DM., and T.vD.; *Visualization:* J.M.R.E.; *Supervision:* S.G. and S.DM.; *Project administration:* S.G. and S.DM.; *Funding acquisition:* S.G. and S.DM.

8. Appendix

8.1. Statistical and Causal inference

This section introduces fundamental statistical and causal inference concepts necessary for understanding the core theoretical principles described in this document. It does not, however, offer a comprehensive overview of causal inference methods. Readers seeking more in-depth understanding may wish to explore introductory papers such as Pearl (2010), Rohrer (2018), Pearl (2019), and Cinelli et al. (2020). They may also find it helpful to consult introductory books like Pearl and Mackenzie (2018), Neal (2020), and McElreath (2020). For more advanced study, readers may refer to seminal intermediate papers such as Neyman (1923), Rubin (1974), Spirtes et al. (1991), and Sekhon (2009), as well as books such as Pearl (2009), Morgan and Winship (2014), and Hernán and Robins (2020).

8.1.1. Empirical research and randomized experiments

Empirical research uses evidence from observation and experimentation to address real-world challenges. In this context, researchers typically formulate their research questions as *estimands* or *targets of inference*, i.e., the specific quantities they seek to determine (Everitt and Skrondal, 2010). For instance, researchers might be interested in answering the following question: "To what extent do different teaching methods (T) influence students' ability to produce high-quality written texts (Y)?" To investigate this, researchers could randomly assign students to two groups, each exposed to a different teaching method ($T_i = \{1, 2\}$). Then, they would perform pairwise comparisons, generating a dichotomous outcome ($Y_i = \{0, 1\}$) showing which student exhibits more of the ability. In this scenario, the research question can be rephrased as the estimand, "*On average*, is there a difference in the ability to produce high-quality written texts between the two groups of students?" and this estimand can be mathematically represented by the random associational quantity in Equation 3, where $E[\cdot]$ denotes the expected value.

$$E[Y_i \mid T_i = 1] - E[Y_i \mid T_i = 2] \tag{3}$$

Researchers then proceed to identify the estimands. *Identification* determines whether an estimator can accurately compute the estimand based solely on its assumptions, regardless of random variability (Schuessler and Selb, 2023, pp. 4). An *estimator* refers to a method or function that transforms data into an estimate (Neal, 2020). *Estimates* are numerical values that approximate

the estimand derived through the process of *estimation*, which integrates data with an estimator (Everitt and Skrondal, 2010). The Identification-Estimation flowchart (McElreath, 2020; Neal, 2020), shown in Figure 9, visually represents the transition from estimands to estimates.
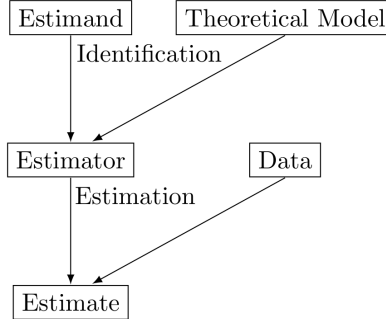

Figure 9: Identification-Estimation flowchart. Extracted and slightly modified from Neal (2020, pp. 32)

Identification is a necessary condition to ensure *consistent* estimators. An estimator achieves *consistency* when it converges to the "true" value of an estimand as the data size approaches infinity (Everitt and Skrondal, 2010). Without identification, researchers cannot achieve consistency, even with "infinite" and error-free data. As a result, deriving meaningful insights about an estimand from finite data becomes impossible (Schuessler and Selb, 2023, pp. 5). Therefore, to ensure accurate and reliable estimates, researchers prioritize estimators with desirable identification properties. For instance, the Z-test is a widely used estimator for comparing group proportions, yielding accurate estimates when its underlying assumptions are satisfied (Kanji, 2006). Furthermore, researchers can interpret estimates from the Z-test as causal, provided the data is collected through a randomized experiment.

Randomized experiments are widely recognized as the gold standard in evidence-based science (Hariton and Locascio, 2018; Hansson, 2014). This recognition stems from their ability to enable researchers interpret associational estimates as causal. They achieve this by ensuring data, and by extension an estimator, satisfies several key identification properties, such as common support, no interference, and consistency (Morgan and Winship, 2014; Neal, 2020). The most critical property, however, is the elimination of confounding. *Confounding* occurs when an external variable X simultaneously influences the outcome Y and the variable of interest T, resulting in spurious associations (Everitt and Skrondal, 2010). Randomization addresses this issue by decoupling the association between the intervention allocation T from any other variable X (Morgan and Winship, 2014; Neal, 2020).

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Nevertheless, researchers often face constraints that limit their ability to conduct randomized experiments. These constraints include ethical concerns, such as the assignment of individuals to potentially harmful interventions, and practical limitations, such as the infeasibility of, for example, assigning individuals to genetic modifications or physical impairments (Neal, 2020). In these cases, causal inference offers a valuable alternative for generating causal estimates and understanding the mechanisms underlying specific data. In addition, the framework can provide significant theoretical insights that can enhance the design of experimental and observational studies (McElreath, 2020).

8.1.2. Identification under causal inference

Unlike classical statistical modeling, which focuses primarily on summarizing data and inferring associations, the *causal inference* framework is designed to identify causes and estimate their effects using data (Shaughnessy et al., 2010; Neal, 2020). The framework uses rigorous mathematical techniques to address the *fundamental problem of causality* (Pearl, 2009; Pearl et al., 2016; Morgan and Winship, 2014). This problem revolves around the question, "What would have happened 'in the world' under different circumstances?" This question introduces the concept of counterfactuals, which are instrumental in defining and identifying causal effects.

Counterfactuals are hypothetical scenarios that are *contrary to fact*, where alternative outcomes resulting from a given cause are neither observed nor observable (Neal, 2020; Counterfactual, 2024). The structural approach to causal inference (Pearl, 2009; Pearl et al., 2016) provides a formal framework for defining counterfactuals. For instance, in the scenario described in Section 8.1.1, the approach begins by defining the *individual causal effect* (ICE) as the difference between each student's potential outcomes, as in Equation 4.

$$\tau_i = Y_i \mid do(T_i = 1) - Y_i \mid do(T_i = 2) \tag{4}$$

where $do(T_i = t)$ represents the intervention operator, $Y_i \mid do(T_i = 1)$ represents the potential outcome under intervention $T_i = 1$, and $Y_i \mid do(T_i = 1)$ represents the potential outcome under intervention $T_i = 2$. Here, an *intervention* involves assigning a constant value to the treatment variable for each student's potential outcomes. Note that if a student is assigned to intervention $T_i = 1$, the potential outcome under $T_i = 2$ becomes a counterfactual, as it is no longer observed nor observable. To address this challenge, the structural approach extends the ICE to the *average causal effect* (ACE, Equation 5), representing the average difference between the students' observed potential outcomes and their counterfactual counterparts.

$$\tau = E[\tau_i]$$
$$= E[Y_i \mid do(T_i = 1)] - E[Y_i \mid do(T_i = 2)]$$

(5)

Even though counterfactuals are unobservable, researchers can still identify the ACE from associational estimates by leveraging the structural approach. The approach identifies the ACE by statistically conditioning data on a *sufficient adjustment set* of variables X (Pearl, 2009; Pearl et al., 2016; Morgan and Winship, 2014). This *sufficient* set (potentially empty) must block all non-causal paths between T to Y without opening new ones. When such a set exists, then T and Y are *d-separated* by X ($T \perp Y \mid X$) (Pearl, 2009), and X satisfies the *backdoor criterion* (Neal, 2020, pp 37). Here, *conditioning* describes the process of restricting the focus to the subset of the population defined by the conditioning variable (Neal, 2020, pp. 32) (see Equation 6).

Conditioning on a sufficient adjustment set enables researchers to estimate the ACE, even when the data comes from an observational study. This process is feasible because such conditioning ensures that the ACE estimator satisfies several critical properties, including confounding elimination (Morgan and Winship, 2014). Naturally, the validity of claims about the causal effects of T on Y now hinges on the assumption that X serves as a sufficient adjustment set. However, as Kohler et al. (2019, pp. 150) noted, drawing conclusions about the real world from observed data inevitably requires assumptions. This requirement holds true for both observational and experimental data.

For instance, if researchers cannot conduct the randomized experiments described in Section 8.1.1 and must instead rely on observational data, they can still identify the ACE as long as an observed variable X, such as the socio-economic status of the school, satisfies the backdoor criterion. Under these circumstances, researchers first identify the *conditional average causal effect* (CACE, Equation 6)

$$CACE_t = E[Y_i \mid T_i = t, X]$$

(6)

From the CACE, researchers can identify the ACE from associational quantities as in Equation 7. This identification process is commonly known as the *backdoor adjustment*. Here, $E_X[\cdot]$ represents the marginal expected value over X (Morgan and Winship, 2014).

$$\tau = E[Y_i \mid do(T_i = 1)] - E[Y_i \mid do(T_i = 2)]$$

$$= E_X[CACE_1 - CACE_2] \tag{7}$$

$$= E_X [E[Y_i \mid T_i = 1, X] - E[Y_i \mid T_i = 2, X]]$$

Notably, the approach extends the ACE identification for a continuous variable T as in Equation 8, ensuring broad applicability across different causal scenarios (Neal, 2020, pp. 45)

$$\tau = E[Y_i \mid do(T_i = t)]$$

$$= dE_X [E[Y_i \mid T_i = t, X]] / dt \tag{8}$$

8.1.3. Diving into the specifics

The structural approach to causal inference uses SCMs and DAGs to formally and graphically represent the presumed causal structure underlying the ACE (Pearl, 2009; Pearl et al., 2016; Gross et al., 2018; Neal, 2020). Essentially, these tools serve as *conceptual (theoretical) models* on which identification analysis rests (Schuessler and Selb, 2023, pp. 4). Thus, using these tools, researchers can determine which statistical models can identify (ACE, CACE, or other), assuming the depicted causal structure is correct (McElreath, 2020), thus enabling valid causal inference. Figure 9 shows the role of theoretical models in the inference process.

SCMs and DAGs support identification analysis through two key advantages. First, regardless of complexity, they can represent various causal structures using only five fundamental building blocks (Neal, 2020; McElreath, 2020). This feature allows researchers to decompose complex structures into manageable components, facilitating their analysis (McElreath, 2020). Second, they depict causal relationships in a non-parametric and fully interactive way. This flexibility enables feasible ACE identification strategies without defining the variables' data types, the functional form between them, or their parameters (Pearl et al., 2016, pp. 35).

Thus, Section 8.1.3.1 and Section 8.1.3.2 elaborate on the first advantage, while Section 8.1.3.2 and Section 8.1.3.3 do so for the second. Finally, Section 8.1.3.4 explains how researchers use SCMs and DAGs alongside Bayesian inference methods in the estimation process.

8.1.3.1. The five fundamental block for SCMs and DAGs.

Figures 10, 11, 12, 13, and 14 display the five fundamental building blocks for SCMs and DAGs. The left panels of the figures show the formal mathematical models, represented by the SCMs,

defined in terms of a set of *endogenous* variables $V = \{X_1, X_2, X_3\}$, a set of *exogenous* variables $E = \{e_{X1}, e_{X2}, e_{X3}\}$, and a set of functions $F = \{f_{X1}, f_{X2}, f_{X3}\}$ (Pearl, 2009; Cinelli et al., 2020). Endogenous variables are those whose causal mechanisms a researcher chooses to model (Neal, 2020). In contrast, exogenous variables represent *errors* or *disturbances* arising from omitted factors that the investigator chooses not to model explicitly (Pearl, 2009, pp. 27,68). Lastly, the functions, referred to as *structural equations*, express the endogenous variables as non-parametric functions of other variables. These functions use the symbol ':=' to denote the asymmetrical causal dependence of the variables and the symbol ' $\perp$ ' to represent *d-separation*, a concept akin to (conditional) independence.

Notably, every SCM has an associated DAG (Pearl et al., 2016; Cinelli et al., 2020). The right panels of the figures display these DAGs. A DAG is a graph consisting of nodes connected by edges, where the nodes represent random variables. The term *directed* means that the edges extend from one node to another, with arrows indicating the direction of causal influence. The term *acyclic* implies that the causal influences do not form loops, ensuring the influences do not cycle back on themselves (McElreath, 2020). DAGs represent observed variables as solid black circles, while they use open circles for unobserved (latent) variables (Morgan and Winship, 2014). Although the *standard representation* of DAGs typically omits exogenous variables for simplicity, the *magnified representation* depicted in the figures offers one key advantage: including exogenous variables can help researchers highlight potential issues related to conditioning and confounding (Cinelli et al., 2020).
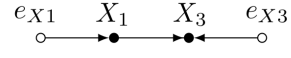
$$X_1 := f_{X1}(e_{X1})$$

$$X_3 := f_{X3}(e_{X3})$$

$$e_{X1} \perp e_{X3}$$

(a) SCM

(b) DAG

Figure 10: Two unconnected nodes

$$X_1 := f_{X1}(e_{X1})$$

$$X_3 := f_{X3}(X_1, e_{X3})$$

$$e_{X1} \perp e_{X3}$$

(b) DAG

(a) SCM

Figure 11: Two connected nodes or descendant

$$X_1 := f_{X1}(e_{X1})$$

$$X_2 := f_{X2}(X_1, e_{X2})$$
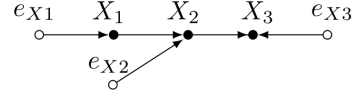
$$X_3 := f_{X3}(X_2, e_{X3})$$

$$e_{X1} \perp e_{X2}$$

$$e_{X1} \perp e_{X3}$$

$$e_{X2} \perp e_{X3}$$

(b) DAG

(a) SCM

Figure 12: Chain or mediator
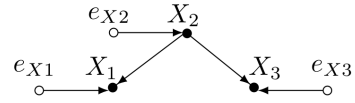
$$X_1 := f_{X1}(X_2, e_{X1})$$

$$X_2 := f_{X2}(e_{X2})$$

$$X_3 := f_{X3}(X_2, e_{X3})$$

$$e_{X1} \perp e_{X2}$$

$$e_{X1} \perp e_{X3}$$

$$e_{X2} \perp e_{X3}$$

(b) DAG

(a) SCM

Figure 13: Fork or confounder

$$X_1 := f_{X1}(e_{X1})$$

$$X_2 := f_{X2}(X_1, X_3, e_{X2})$$

$$X_3 := f_{X3}(e_{X3})$$

$$e_{X1} \perp e_{X2}$$

$$e_{X1} \perp e_{X3}$$

$$e_{X2} \perp e_{X3}$$

(a) SCM
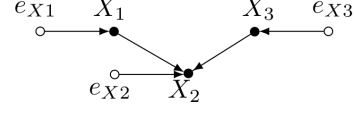
(b) DAG

Figure 14: Collider or inmorality

A careful examination of these building blocks highlights the theoretical assumptions underlying their observed variables. SCM 10a and DAG 10b depict two unconnected nodes, representing a scenario where variables X_1 and X_3 are independent or not causally related. SCM 11a and DAG 11b illustrate two connected nodes, representing a scenario where a *parent* node X_1 exerts a causal influence on a *child* node X_3. In this setup, X_3 is considered a *descendant* of X_1. Additionally, X_1 and X_3 are described as *adjacent* because there is a *direct path* connecting them. SCM 12a and DAG 12b depict a *chain*, where X_1 influences X_2, and X_2 influences X_3. In this configuration, X_1 is a parent node of X_2, which is a parent node of X_3. This structure creates a *directed path* between X_1 and X_3. Consequently, X_1 is an *ancestor* of X_3, and X_2 fully *mediates* the relationship between the two. SCM 13a and DAG 13b illustrate a *fork*, where variables X_1 and X_3 are both influenced by X_2. Here, X_2 is a parent node that *confounds* the relationship between X_1 and X_3. Finally, SCM 14a and DAG 14b show a *collider*, where variables X_1 and X_3 are concurrent causes of X_2. In this configuration, X_1 and X_3 are not causally related to each other but both influence X_2 (an *inmorality*). Notably, all building blocks assume the errors are independent of each other and from all other variables in the graph, as evidenced by the pairwise relations $e_{X1} \perp e_{X2}$, $e_{X1} \perp e_{X3}$, and $e_{X2} \perp e_{X3}$.

Researchers can then use these building blocks to represent the scenario outlined in Section 8.1.2. SCM 15a and DAG 15b depict the plausible causal structure for this example. In this context, the variable X (socio-economic status of the school) is thought to be a confounder in the relationship between the teaching method T and the outcome Y. The figures display multiple descendant relationships such as $X \rightarrow T$, $X \rightarrow Y$, and $T \rightarrow Y$. They also highlight unconnected node pairs,

evident from the relationships $e_T \perp e_X$, $e_T \perp e_Y$, and $e_X \perp e_Y$. Additional, the figures show one fork, $X \to \{T, Y\}$, and two colliders: $\{X, e_T\} \to T$ and $\{X, T, e_Y\} \to Y$.

$$X := f_X(e_X)$$

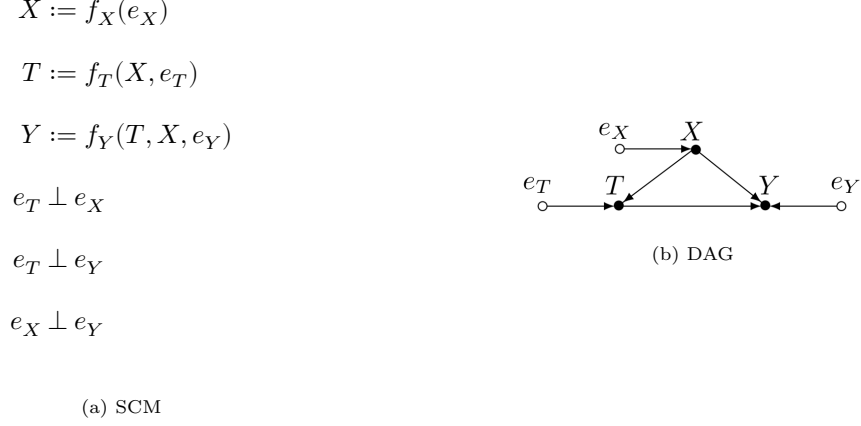$$T := f_T(X, e_T)$$

$$Y := f_Y(T, X, e_Y)$$

$$e_T \perp e_X$$

$$e_T \perp e_Y$$

$$e_X \perp e_Y$$

(b) DAG

(a) SCM

Figure 15: Plausible causal structure the scenario outlined in Section 8.1.2.

8.1.3.2. The probabilistic implications of these blocks.

Beyond their graphical capabilities, SCMs and DAGs can encode the probabilistic information embedded within a causal structure. They achieve this encoding by relying on three fundamental assumptions: the local Markov, the minimality, the causal edges assumption. The *local Markov assumption* encodes probabilistic independencies between variables by declaring that nodes in a graph are independent of all its non-descendants, given its parents (Neal, 2020, pp. 20). Meanwhile, the *minimality assumption* encodes probabilistic dependencies among variables by stating that every pair of adjacent nodes exhibits a dependency (Neal, 2020, pp. 21). Finally, the *causal edges assumption* encodes causal relationships between variables by declaring that each parent node acts as a direct cause of its children (Neal, 2020, pp. 22). Figure 16 illustrates how these assumptions influence the statistical and causal interpretations of graphs.
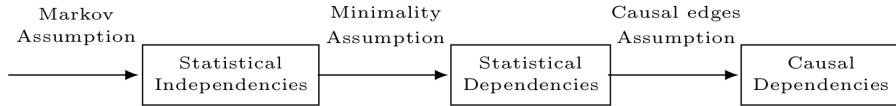

Figure 16: The flow of association and causation in graphs. Extracted and slightly modified from Neal (2020, pp. 31)

A notable implication of the assumptions underlying the probabilistic encoding is that any conceptual model described by an SCM and DAG can represent the joint distribution of variables more efficiently (Pearl et al., 2016, pp. 29). This expression takes the form of a product of conditional probability distributions (CPDs) of the type $P(child \mid parents)$. This property is formally known

as the *Bayesian Network factorization* (BNF, Equation 9) (Pearl et al., 2016, pp. 29; Neal, 2020, pp. 21). In this expression, $pa(X_i)$ denotes the set of variables that are the parents of X_i.

$$
\begin{aligned}
P(X_1, X_2, \ldots, X_P) &= X_1 \cdot \prod_{p=2}^{P} P(X_i \mid X_{i-1}, \ldots, X_1) \quad \text{(by chain rule)} \\
&= X_1 \cdot \prod_{p=2}^{P} P(X_i \mid pa(X_i)) \qquad \text{(by BNF)}
\end{aligned}
\tag{9}
$$

This encoding enables researchers with conceptual (theoretical) knowledge in the form of an SCM and DAG to predict patterns of (in)dependencies in the data. As highlighted by Pearl et al. (2016, pp. 35), these predictions depend solely on the structure of these conceptual models without requiring the quantitative details of the equations or the distributions of the errors. Moreover, once researchers observe empirical data, the patterns of (in)dependencies in the data can provide significant insights into the validity of the proposed conceptual model.

The five fundamental building blocks described in Section 8.1.3.1 clearly illustrate which (in)dependencies can SMCs and DAGs predict. For instance, applying the BNF to the causal structure shown in the SCM 10a and DAG 10b enables researchers to express the joint probability distribution of the observed variables as $P(X_1, X_3) = P(X_1)P(X_3)$, supporting the theoretical assumption that the observed variables X_1 and X_3 are unconditionally independent $(X_1 \perp X_3)$ (Neal, 2020, pp. 24). Conversely, when X_3 is unconditionally dependent on X_1 $(X_1 \not\perp X_3)$, as depicted in the SCM 11a and DAG 11b, the BNF express their joint probability distribution as $P(X_1, X_3) = P(X_3 \mid X_1)P(X_1)$. Notably, these descriptions demonstrate the clear correspondence between the structural equations illustrated in Section 8.1.3.1 and the CPDs.

Beyond the insights gained from two-node structures, researchers can uncover more nuanced patterns of(in)dependencies from chains, forks, and colliders. These (in)dependencies apply to any data set generated by a causal model with those structures, regardless of the specific functions attached to the SCM (Pearl et al., 2016, pp. 36). For instance, applying the BNF to the chain structure depicted in the SCM 12a and DAG 12b allow researchers to represent the joint distribution for the observed variables as $P(X_1, X_2, X_3) = P(X_1)P(X_2 \mid X_1)P(X_3 \mid X_2)$. This expression implies that X_1 and X_3 are unconditionally dependent $(X_1 \not\perp X_3)$, but conditionally independent when controlling for X_2 $(X_1 \perp X_3 \mid X_2)$. Moreover, in the fork structure shown in the SCM 13a and DAG 13b, researchers can express the joint distribution of the observed variables as $P(X_1, X_2, X_3) = P(X_1 \mid X_2)P(X_2)P(X_3 \mid X_2)$. Similar to the chain structure, this expression allows researchers to

further infer that X_1 and X_3 are unconditionally dependent ($X_1 \not\perp X_3$), but conditionally independent when controlling for X_2 ($X_1 \perp X_3 \mid X_2$). Finally, researchers analyzing the collider structure illustrated in the SCM 14a and DAG 14b can express the joint distribution of the observed variables as $P(X_1, X_2, X_3) = P(X_1)P(X_2 \mid X_1, X_3)P(X_3)$. This representation allows researchers to infer that X_1 and X_3 are unconditionally independent ($X_1 \perp X_3$), but conditionally dependent when controlling for X_2 ($X_1 \not\perp X_3 \mid X_2$). The authors Pearl et al. (2016, pp. 37, 40, 41) and Neal (2020, pp. 25–26) provide the mathematical proofs for these conclusions.

Using these additional probabilistic insights, researchers can revisit the scenario in Section 8.1.2. In this context, applying the BNF to the SCM 17a structure, enables the representation of the joint probability distribution of the observed variables as $P(Y, T, X) = P(Y \mid T, X)P(T \mid X)P(X)$. From this expression, researchers can infer that the outcome Y is unconditionally dependent on the teaching method T ($Y \not\perp T$). This dependency arises from two key structures: a direct causal path from the teaching method T to the outcome Y, represented by the two-connected-nodes structure $T \to Y$ (black path in DAG 17b), and a confounding non-causal path from the teaching method T to the outcome Y through the socio-economic status of the school X, represented by the fork structure $T \leftarrow X \to Y$ (gray path in DAG 17b).

$$X := x$$
$$T := f_T(x, e_T)$$
$$Y := f_Y(T, x, e_Y)$$
$$e_T \perp e_X$$
$$e_T \perp e_Y$$
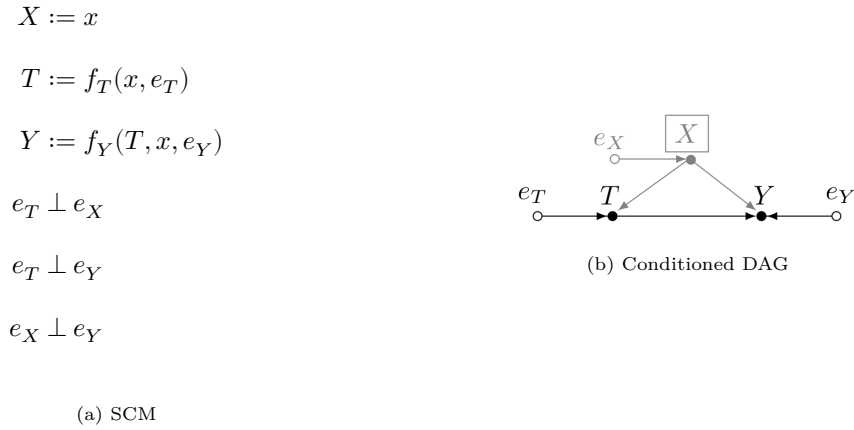$$e_X \perp e_Y$$

(a) SCM

(b) Conditioned DAG

Figure 17: Plausible causal structure the scenario outlined in Section 8.1.2.

8.1.3.3. From probability to causality.

The structural approach to causal inference translates probabilistic insights into actionable strategies seeking to identify the ACE from associational quantities. The approach achieves this by relying on the *modularity assumption*, which posits that intervening on a node alters only the causal mechanism of that node, leaving others unchanged (Neal, 2020, pp. 34).

The modularity assumption underpins the concepts of manipulated graphs and Truncated Factorization, which are essential for representing interventions $P(Y_i \mid do(T_i = t))$ within SCMs and DAGs. *Manipulated graphs* simulate physical interventions by removing specific edges from a DAG, while preserving the remaining structure unchanged (Neal, 2020, pp. 34). In parallel, *Truncated Factorization* (TF) achieves a similar simulation by removing specific functions from the conceptual model and replacing them with constants, while keeping the rest of the structure unchanged (Pearl, 2010). The probabilistic implications of this factorization are formalized in Equation 10, where S represents the subset of variables X_p directly influenced by the intervention, while an example illustrating these concepts follows below.

$$P(X_1, X_2, \ldots, X_P \mid do(S)) = \begin{cases} \prod P(X_p \mid pa(X_p)) & \text{if } p \notin S \\ 1 & \text{otherwise} \end{cases} \tag{10}$$

Using the TF, researchers can define the *backdoor adjustment* to identify the ACE. This adjustment states that if a variable $X_p \in S$ serves as a *sufficient adjustment set* for the effect of X_a on X_b, then the ACE can be identified using Equation 11. The sufficient adjustment set (potentially empty) must block all non-causal paths between X_a and X_b without introducing new paths. If such a set exists, then X_a and X_b are *d-separated* by X_p ($X_a \perp X_b \mid X_p$) (Pearl, 2009), and X_p satisfies the *backdoor criterion* (Neal, 2020, pp. 37).

$$P(X_a \mid do(X_b = x)) = \sum_{Xp} P(X_a \mid X_b = x, X_p)P(X_p) \tag{11}$$

Ultimately, the backdoor adjustment enables researchers to express the ACE as:

$$\begin{aligned} \tau &= E[X_a \mid do(X_b = 1)] - E[X_a \mid do(X_b = 2)] \\ &= E_{Xp}\left[E[X_a \mid do(X_b = 1), X_p] - E[X_a \mid do(X_b = 2), X_p]\right] \\ &= \sum_{Xp} X_a \cdot P(X_a \mid X_b = 1, X_p) \cdot P(X_p) - \sum_{Xp} X_a \cdot P(X_a \mid X_b = 2, X_p) \cdot P(X_p) \end{aligned} \tag{12}$$

With these new insights, researchers revisiting the scenario in Section 8.1.3.2 can infer that the socio-economic status of the school, X, satisfies the backdoor criterion, assuming the causal structure depicted by the SCM 17a and DAG 17b is correct. This means that X serves as a sufficient adjustment set, as it effectively blocks all confounding non-causal paths introduced by the fork

structure. Nevertheless, since Y remains dependent on T even after conditioning $(Y \not\perp T \mid X)$, this dependency can only be attributed to the direct causal effect $T \to Y$. Notably, for the purpose of identification, the conditioned DAG 17b is equivalent to the manipulated DAG 18b, because X satisfies the backdoor criterion.
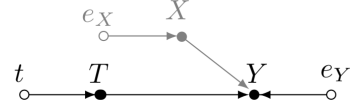
$$X := f_X(e_X)$$

$$T := t$$

$$Y := f_Y(t, X, e_Y)$$

$$e_T \perp e_X$$

$$e_T \perp e_Y$$

$$e_X \perp e_Y$$

(a) SCM

(b) Manipulated DAG

Figure 18: Plausible causal structure the scenario outlined in Section 8.1.3.2.

Researchers can then apply the *backdoor adjustment* to identify the ACE of T on Y. They achieve this by first identifying the CACE of T on Y by conditioning on X, and then marginalizing this effect over X to obtain the ACE. This process is expressed in Equation 13 (see Section 8.1.2).

$$
\begin{aligned}
\tau &= E[Y_i \mid do(T_i = 1)] - E[Y_i \mid do(T_i = 2)] \\
&= E_X \left[E[Y_i \mid T_i = 1, X] - E[Y_i \mid T_i = 2, X] \right] \\
&= \sum_X Y_i \cdot P(Y_i \mid T_i = 1, X) \cdot P(X) - \sum_X Y_i \cdot P(Y_i \mid T_i = 2, X) \cdot P(X)
\end{aligned}
\tag{13}
$$

8.1.3.4. The estimation process.

Ultimately, researchers can use Bayesian inference methods to estimate the ACE. The approach begins by defining two probability distributions: the likelihood of the data, $P(X_1, X_2, \dots, X_P \mid \theta)$, and the prior distribution, $P(\theta)$ (Everitt and Skrondal, 2010), where X_P represents a random variable, and θ represents a one-dimensional parameter space for simplicity. After observing empirical data, researchers can update the priors to posterior distributions using Bayes' rule in Equation 14:

$$
P(\theta \mid X_1, X_2, \dots, X_P) = \frac{P(X_1, X_2, \dots, X_P \mid \theta) \cdot P(\theta)}{P(X_1, X_2, \dots, X_P)}
\tag{14}
$$

Given that the denominator on the right-hand side of Equation 14 serves as a normalizing constant independent of the parameter θ, researchers can simplify the posterior updating process into three steps. First, they integrate new empirical data through the likelihood. Second, they update the parameters' priors to a posterior distribution according to Equation 15. Ultimately, they normalize these results to obtain a valid probability distribution.

$$P(\theta \mid X_1, X_2, \ldots, X_P) \propto P(X_1, X_2, \ldots, X_P \mid \theta) \cdot P(\theta) \tag{15}$$

Temporarily setting aside the definition of prior distributions $P(\theta)$, note that the posterior updating process depends heavily on the assumptions underlying the likelihood of the data. However, as the number of random variables, P, increases, this joint distribution quickly becomes intractable (Neal, 2020). This intractability is evident from Equation 16, where the likelihood distribution is expressed by multiple chained CPDs.

$$P(X_1, X_2, \ldots, X_P \mid \theta) = P(X_1 \mid \theta) \prod_{p=2}^{P} P(X_i \mid X_{i-1}, \ldots, X_1, \theta) \tag{16}$$

Nevertheless, researchers can manage the complexity of the likelihood by assuming specific local (in)dependencies among variables. SCMs and DAGs provide a formal framework to represent these assumptions, as detailed in Section 8.1.3.2. These assumptions improve model tractability and simplify the estimation process by enabling the derivation of the BNF of the likelihood (Equation 17). With this simplified structure, any probabilistic programming language can model the system and compute the parameter's posterior distribution using Equation 14.

$$P(X_1, X_2, \ldots, X_P \mid \theta) = P(X_1 \mid \theta) \prod_{p=2}^{P} P(X_i \mid pa(X_i), \theta) \tag{17}$$

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