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GLLAMM:

method, bayesian estimation, advantages, and applications to educational data.

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Leuven, September 1, 2021

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1. Preliminary consideration

IRT local independence

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Comprised of two parts [1, 10]:

- local item independence
- local individual independence.

IRT models are **not robust** to the violation of local independence [28, 3, 11].

Educational data

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often display **several** types of dependencies, violating the local item and/or individual independence, e.g.

- testlets [27];
- the measurement of multiple latent traits within individuals [24];
- cluster effects [23].

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Proposed model

The GLLAMM follow a multilevel/hierarchical multidimensional approach to account for different dependencies.

- (**good**) control for dependencies in educational data
- (**important**) reach appropriate conclusion from the parameters

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GLLAMM under the Bayesian framework, but the implementation will be:

- Complex and highly dimensional (in parameters)
- On sparse binary data

Implementation (cont.)

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However, complex parametrizations introduce pathologies that prevent MCMC methods to achieve ergodicity [5, 6, 17, 18, 2], i.e. reach stationarity, convergence, and good mixing [15].

Although there are proposed solutions, still **no simple rotation/rescaling of the parameter, or (sometimes) the amount of data, allow to visit the posterior distribution properly** [2].

Implementation (cont.)

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Multiple authors showed that changing the **posterior sampling geometries**, i.e. removing the dependence of the parameters on other sampled parameters, **improves** the performance of the MCMC methods [5, 6, 17, 18, 2]

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2. The GLAMM for dichotomous outcomes

Model definition

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Following Rabe-Hesketh et al. [20, 21], we define the GLLAMM in two parts:

1. the response model
2. the latent structure

Moreover, **the response model (1)** can be represented by a Generalized Linear Model (GLM) [16, 14] with:

1. a distributional part
2. a systematic part

1. The response model

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Conditional to all parameters $\mathbf{\Omega} = \{\beta, \mathbf{\Lambda}, \mathbf{\Theta}, \mathbf{\Psi}, \mathbf{\Gamma}\}$; and the "stacked" vector of covariates $\mathbf{X}$ and $\mathbf{W}$; **the distributional part** is defined by:

$$f\left(y_{jkd} = 1 \mid \mathbf{X}, \mathbf{W}, \mathbf{\Omega}\right) = \pi_{jkd}^{n}(1 - \pi_{jkd})^{1-n} \qquad (1)$$

Furthermore, **the systematic part** is defined in the following form:

$$P\left(y_{jkd} = 1 \mid \mathbf{X}, \mathbf{W}, \mathbf{\Omega}\right) = \pi_{jkd} = h(\tau_k + \upsilon_{jkd}) \qquad (2)$$

where τ_k is k'th item threshold, assumed to be zero for the binary case [20].

1. The response model (cont.)

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Moreover, the inverse-link function $h(\cdot)$ can be defined in three ways:

$$h(x) = \begin{cases} \exp(x)[1+\exp(x)]^{-1} \\ \Phi(x) \\ \exp(-\exp(x)) \end{cases} \tag{3}$$

corresponding to the logistic, standard normal $\Phi(x)$, and Gumbel (extreme value type I) cumulative distributions, respectively.

1. The response model (cont.)

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Finally, the linear predictor is defined by:

$$v_{jkd} = \mathbf{X}_j\,\boldsymbol{\beta} + \sum_{m=2}^{M+1} \boldsymbol{\eta}^{(m)}\boldsymbol{\alpha}^{(m)}\mathbf{A}_j^{(m)} + \sum_{l=2}^{L+1} \boldsymbol{\theta}_j^{(l)}\boldsymbol{\lambda}^{(l)}\mathbf{B}_j^{(l)} \quad (4)$$

2. The latent structure

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The structural model for the latent variables is represented in the following form:

$$\boldsymbol{\Theta} = \underset{(S\times S)}{\boldsymbol{\Psi}}\underset{(S\times 1)}{\boldsymbol{\Theta}} + \underset{(S\times Q)}{\boldsymbol{\Gamma}}\underset{(Q\times 1)}{\mathbf{W}} + \underset{(S\times 1)}{\boldsymbol{\zeta}} \tag{5}$$

where $S = K + D$, $K = \sum_m K_m$, and $D = \sum_l D_l$.

Notice equation (5) is the generalization of a single-level Structural Equation Models (SEM) to a multilevel setting.

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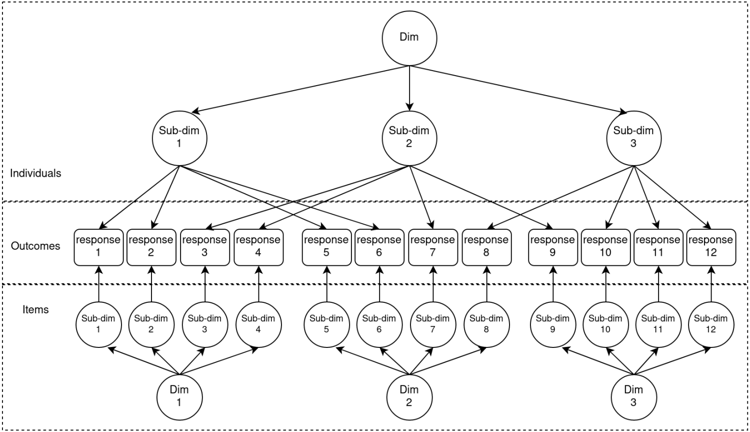

Motivating example (cont.)

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- Empty level-1 covariates matrix $\mathbf{X}_j$ $(P = 0)$
- $M = 2$ levels at the items block, with $K_2 = 12$ and $K_3 = 3$, i.e. $\boldsymbol{\eta}^{(2)} = [\eta_1^{(2)}, \ldots, \eta_{12}^{(2)}]^T$ and $\boldsymbol{\eta}^{(3)} = [\eta_1^{(3)}, \eta_2^{(3)}, \eta_3^{(3)}]^T$
- $L = 2$ levels in the individuals block, with $D_2 = 3$ and $D_3 = 1$, i.e. $\boldsymbol{\theta}^{(2)} = [\theta_1^{(2)}, \theta_2^{(2)}, \theta_3^{(2)}]^T$ and $\boldsymbol{\theta}^{(3)} = \theta_1^{(3)}$.
- Specific regression relationship among latents $\boldsymbol{\Psi}$, i.e. $\boldsymbol{\alpha}^{(3)} = [\alpha_{11}^{(3)}, \ldots, \alpha_{15}^{(3)}, \alpha_{21}^{(3)}, \ldots, \alpha_{25}^{(3)}, \alpha_{31}^{(3)}, \ldots, \alpha_{35}^{(3)}]^T$ and $\boldsymbol{\lambda}^{(3)} = [\lambda_1^{(3)}, \lambda_2^{(3)}, \lambda_3^{(3)}]^T$
- Empty structural covariates $\mathbf{W}$.

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3. Bayesian Estimation

Bayesian GLLAMM for dichotomous outcomes

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1. **Posterior distribution.** Given that $\mathbf{Y}$ is the observed data and $\mathbf{\Omega} = \{\beta, \mathbf{\Lambda}, \mathbf{\Theta}, \mathbf{\Psi}, \mathbf{\Gamma}\}$ the parameters:

$$P(\mathbf{\Omega} \mid \mathbf{Y}) = \frac{P(\mathbf{Y} \mid \mathbf{\Omega})\ P(\mathbf{\Omega})}{\int P(\mathbf{Y} \mid \mathbf{\Omega})\ P(\mathbf{\Omega})\ d\mathbf{\Omega}} \tag{6}$$

2. **Prior distributions** $P(\mathbf{\Omega})$. Similar to Patz and Junker [19], we use an independent distributional structure for the joint priors:

$$\begin{aligned} P(\mathbf{\Omega}) = & P(\boldsymbol{\beta})\ [P(\boldsymbol{\alpha})\ P(\boldsymbol{\lambda})]\ [P(\boldsymbol{\eta})\ P(\boldsymbol{\theta})] \\ & [P(\mathbf{\Psi}_\eta)\ P(\mathbf{\Psi}_\theta)]\ [P(\mathbf{\Gamma}_\eta)\ P(\mathbf{\Gamma}_\theta)] \end{aligned} \tag{7}$$

Bayesian GLLAMM for dichotomous outcomes (cont.)

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3. **Likelihood** $P(\mathbf{Y} \mid \mathbf{\Omega})$. Following Rabe-Hesketh et al. [20], the likelihood function is build in a recursive way.

$$f\left(\mathbf{y}=\mathbf{1} \mid \mathbf{X}, \mathbf{W}, \mathbf{\Omega}\right)=\prod_{j=1}^{J} \prod_{d=1}^{D} \prod_{k=1}^{K} f\left(y_{jkd}=1 \mid \mathbf{X}, \mathbf{W}, \mathbf{\Omega}\right) \tag{8}$$

$$f_{(m)}^{(l)}\left(\mathbf{y}=\mathbf{1} \mid \mathbf{X}, \mathbf{W}, \mathbf{\Omega}\right)=\int\left[\prod f_{(m-1)}^{(l-1)}\left(\mathbf{y}=\mathbf{1} \mid \mathbf{X}, \mathbf{W}, \mathbf{\Omega}\right)\right] P(\mathbf{\Theta}_{(m)}^{(l)} \tag{9}$$

$$\mathcal{L}(\mathbf{X}, \mathbf{W}, \mathbf{\Omega})=\prod_{m=2}^{M+1} \prod_{l=2}^{L+1} f_{(m)}^{(l)}\left(\mathbf{y}=\mathbf{1} \mid \mathbf{X}, \mathbf{W}, \mathbf{\Omega}\right) \tag{10}$$

$$\ell(\mathbf{X}, \mathbf{W}, \mathbf{\Omega})=\log \mathcal{L}(\mathbf{X}, \mathbf{W}, \mathbf{\Omega}) \tag{11}$$

Computational implementation

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1. Hamiltonian Monte Carlo (HMC) and `Stan` [26].
2. **No burn-in and thinning**. Warm-up phase to "tune-up" the number of steps (`leapfrogs`), and the `step size` [26].
3. We use $3,000$ **effective iterations**: 3 chains of $2,000$ iterations each, where $1,000$ are spend on warm-up.
4. **Initial starts** sampled from the priors defined in the model.
5. Prior distributions selected based on **prior predictive simulations**.
6. We test the **centered (CP)** and **non-centered parametrizations (NCP)**.

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4. Simulation studies

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Objectives

1. **Performance.** In terms of achieving ergodicity, under the CP and NCP,
2. **Recovery capacity.** Capacity to recover the parameters of interest, especially the structural regression parameters.
3. **Retrodictive accuracy.** Capacity to retrodict the data of interest, according to a set of aggregating dimensions.

Results (Performance)

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1. the non-centered parametrization (NCP) largely improved the performance of the MCMC chains, towards achieving ergodicity.

   This is true across models, simulated sample sizes, and simulated replicas.

2. No large difference in performance was observed in either the sub-dimensions' correlation or loading parameters.

   However, no evidence supported the idea the parameters suffered from a further lack of identification.

Results (Performance, cont.)

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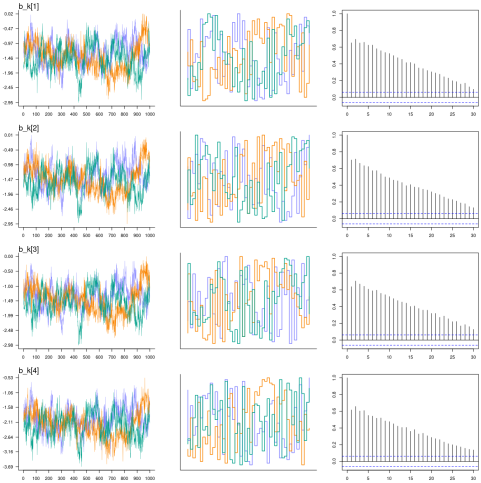

Results (Performance, cont.)

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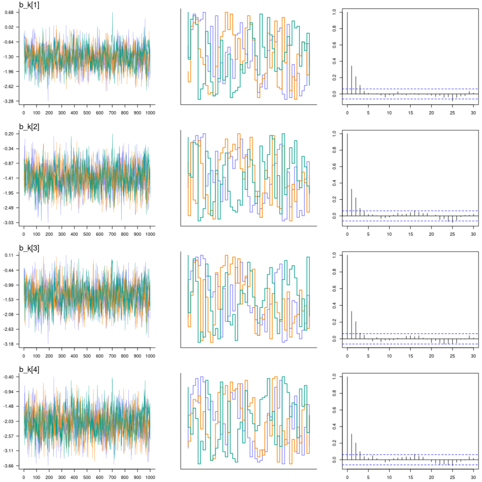

Results
(Recovery capacity)

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1. The GLLAMM was able to recover most of the simulated parameters with good precision.
2. However, the model still had issues estimating the sub-dimensions correlations and loadings.

   Under the Confirmatory Factor Analysis theory (CFA), a SOLV model is only justified, if the lower-level correlations are high enough, usually above 0.8.
3. It is surprising that CP and NCP achieved similar levels of recovery capacity.

Results
(Recovery capacity, cont.)

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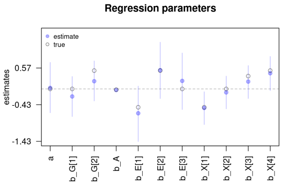

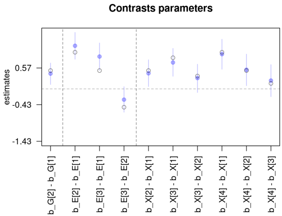

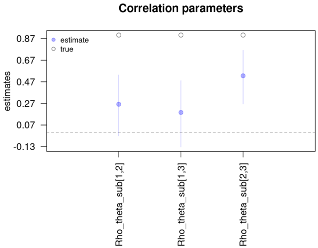

Results
(Retrodictive accuracy)

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1. the models managed to capture the traits of the data, while avoiding its exact replication.

   These result were consistent across models, simulated sample sizes and replicas.

Results (Retrodictive accuracy, cont.)

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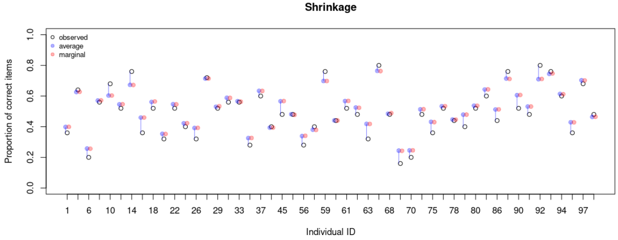

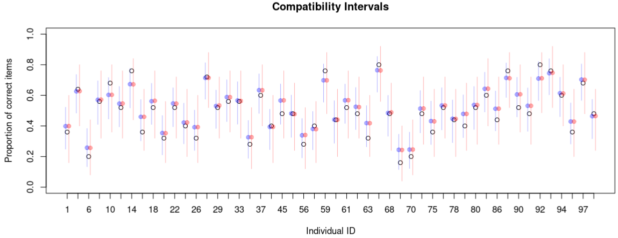

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Objectives

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1. **Evaluate the performance of the parametrizations.** Ergodicity of CP vs NCP.
2. **Evaluate the retrodictive accuracy.**
3. **Assess the psychometric properties.** Special interest in determine how difficult the items were.
4. **Test research hypothesis.** About the explanatory power a set of covariates had on the latent dimensions, and their implications for the educational authority.

Instrument and data

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1. **Instrument.** Reading comprehension sub-test composed of 25 binary scored items, from the Peruvian public teaching career national assessment.
2. **Access.** Legal requirement of open information to the Ministry of Education of Peru (MINEDU).
3. **Sample scheme.** Simple random sample of $2,000$ (from approx. $195,0000$).

Results (Performance)

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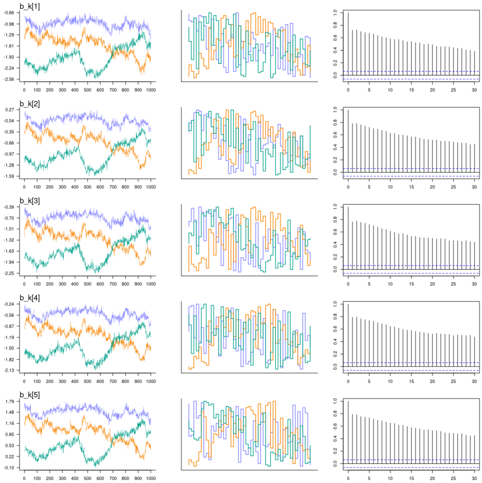
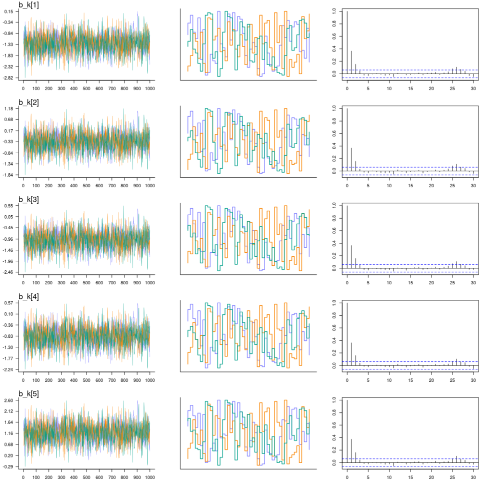
1. the non-centered parametrization (NCP) largely improved the performance of the MCMC chains, towards achieving ergodicity.
2. No large difference in performance was observed in either the sub-dimensions' correlation or loading parameters.

   results are similar to simulation study, albeit in some cases more extreme.

Results (Performance, cont.)



Results (Performance, cont.)



Results
(Retrodictive accuracy)

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1. the models managed to capture the traits of the data, while avoiding its exact replication.

   Similar to simulation study.

Results (Retrodictive accuracy, cont.)

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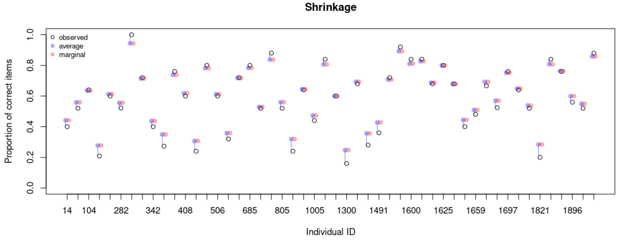

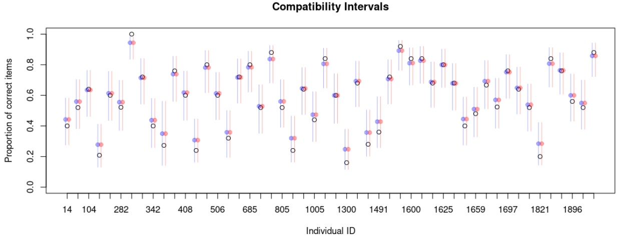

Results (Psychometric properties)

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1. the items were scattered throughout a significant portion of the abilities range.
2. Specific benefit of the implementation allowed us to assess how difficult the texts were.
3. (a word of advice) A more sound psychometric analysis requires access to the items and texts description, but it was not possible due to legal restrictions.

Results (Psychometric properties, cont.)

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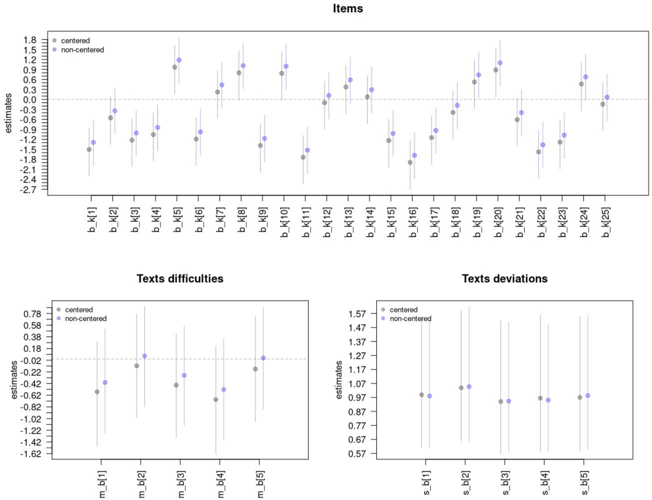

Results (Hypothesis testing)

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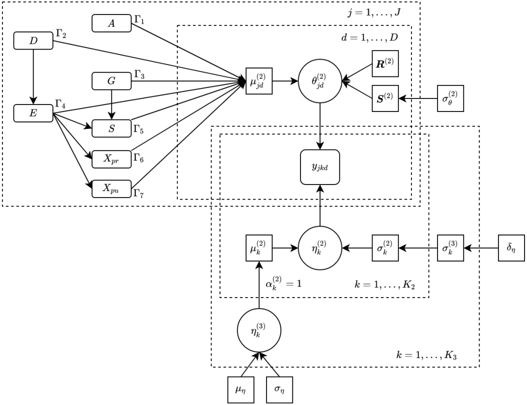

Figure: Directed Acyclig Graph (DAG). Application's first-order latent variable model (FOLV).

Results (Hypothesis testing, cont.)

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1. Age (style of teaching proxy) explains negatively reading comprehension $-0.036[-0.04, -0.03]$.
2. Disability also explained the the reading comprehension sub-dimensions, although the results were mildly unexpected.
3. the statistical evidence seem to support the lower quality of training on pedagogical institutes.
4. Experience improved the reading comprehension abilities. Private experience effects were larger than the public. We observe also diminishing returns on abilities, as the years of experience increases.
5. Instructing students from secondary education improves reading comprehension.

Results (Hypothesis testing, cont.)

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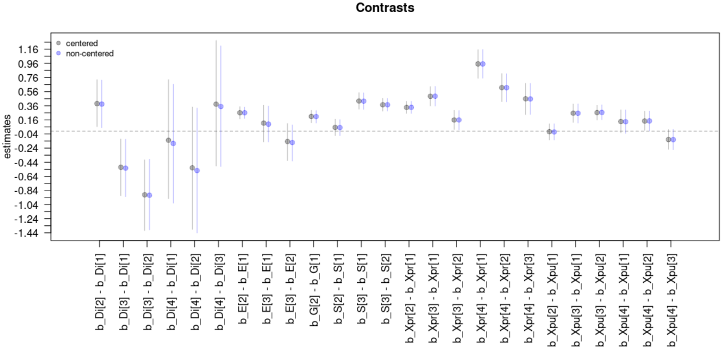

Figure: Application's first-order latent variable model. CP and NCP comparison plot.

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6. Conclusions and further development

Conclusions

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1. The NCP largely improved the performance of the MCMC chains, towards achieving ergodicity. True across models, simulated sample sizes, simulated replicas, and even under the real application, albeit with some caveats.
2. Our proposed model was able to recover most of the simulated parameters with good precision. The model still had issues estimating the sub-dimensions correlations and loadings.
3. The models managed to capture the traits of the data, while avoiding its exact replication. Consistent across models, simulated sample sizes and replicas, and even under the final educational application.

Conclusions (cont.)

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1. The NCP was slightly faster than the CP, although the magnitudes of the differences in running time were not large.
2. On the application, the model provided an extra benefit, to asses the psychometric properties of texts.
3. In relation to our hypothesis of interest, the model produced sound statistical results, supported not only by the statistical application, but also from a DAG guiding our interpretation and causal assumptions.

Further investigation

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1. Why the benefits of the NCP did not fully extend to correlations and loadings?.
2. Test other simulation scenarios to find why the CP (no ergodicity) managed recover the parameters as well as the NCP.
3. Related to the previous, test results for Variational Inference methods (VI)
4. The statistical evidence remains valid with a better sample scheme?
5. Test the model also with response explanatory variables, e.g. response time, number of alternatives, etc.
6. Test the model for measurement invariance.

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Improving MCMC chain performance

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Four solutions are offered to solve the previous pathologies:

- Changing the settings of the MCMC method
  1. increasing the number of iterations per chain, with large burn-in and thinning processes
  2. designing model-specific MCMC algorithms.
- Readjusting the Bayesian model
  3. re-write the model in an alternative parametrization (**simple changes**)
  4. encode prior information through the prior distributions

Cluster effects

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Individual clustering involves the addition of more random effects to the linear predictor defined in equation (**??**):

$$
\begin{aligned}
\upsilon_{jkdc} &= \upsilon_{jkd} + \sum_{c=1}^{C} \delta_c \\
&= \upsilon_{jkd} + \boldsymbol{\delta} \mathbf{Z}_j
\end{aligned} \tag{12}
$$

Example (restrictions for IRT)

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we could set the restriction $\boldsymbol{\alpha}^{(2)} = -\boldsymbol{\lambda}^{(2)}$ where $\boldsymbol{\lambda}^{(2)} > \mathbf{0}$. In that case we get a multidimensional generalization of the linear predictor observed in the archetypical Rasch [22], or 2PL [13] models, i.e. $\lambda_d^{(2)}(\theta_{jd}^{(2)} - \eta_k^{(2)})$

In addition,
$\boldsymbol{\alpha}^{(3)} = [\alpha_{11}^{(3)}, \ldots, \alpha_{15}^{(3)}, \alpha_{21}^{(3)}, \ldots, \alpha_{25}^{(3)}, , \alpha_{31}^{(3)}, \ldots, \alpha_{35}^{(3)}]^T$
$= [1, \ldots, 1]^T$, indicating texts difficulties explain directly the items difficulties at the lower level.

Example (restrictions for IRT)

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Moreover, notice that because in the IRT framework $\boldsymbol{\eta}$ and $\boldsymbol{\theta}$ should be orthogonal to each other by design, we can further decompose equation (5) in the following form:

$$\boldsymbol{\eta} = \underset{(K\times K)}{\boldsymbol{\Psi}_{\eta}}\underset{(K\times 1)}{\boldsymbol{\eta}} + \underset{(K\times Q)}{\boldsymbol{\Gamma}_{\eta}}\underset{(Q\times 1)}{\mathbf{W}_{\eta}} + \underset{(K\times 1)}{\boldsymbol{\zeta}_{\eta}} \tag{13}$$

$$\boldsymbol{\theta} = \underset{(D\times S)}{\boldsymbol{\Psi}_{\theta}}\underset{(D\times 1)}{\boldsymbol{\theta}} + \underset{(D\times Q)}{\boldsymbol{\Gamma}_{\theta}}\underset{(Q\times 1)}{\mathbf{W}_{\theta}} + \underset{(D\times 1)}{\boldsymbol{\zeta}_{\theta}} \tag{14}$$

Model Assumptions

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Following Skrondal and Rabe-Hesketh [25], the framework has two main assumptions:

(M1) **Complete latent space.**[9] In the GLLAMM representation, the space is complete if we consider all latent variables $\boldsymbol{\Theta}$ at levels $l > 1$ and $m > 1$.

(M2) **Local Independence.** It assumes independence conditional on all the latent dimensions and covariates, at different hierarchical levels; effectively modeling all the observed dependencies.

Model Assumptions (cont.)

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Local Independence is defined as follows:

$$f\left(\mathbf{y}=\mathbf{1} \mid \mathbf{X}, \mathbf{W}, \mathbf{\Omega}\right)=\prod_{j=1}^{J} \prod_{d=1}^{D} \prod_{k=1}^{K} f\left(y_{jkd}=1 \mid \mathbf{X}, \mathbf{W}, \mathbf{\Omega}\right) \quad (15)$$

Model Assumptions (cont.)

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but comes from:

1. **Local item independence,**

$$f\left(y_{j..}=1 \mid \mathbf{X}, \mathbf{W}, \boldsymbol{\Omega}\right)=\prod_{d=1}^{D} \prod_{k=1}^{K} f\left(y_{jkd}=1 \mid \mathbf{X}, \mathbf{W}, \boldsymbol{\Omega}\right) \tag{16}$$

2. **Local individual independence,**

$$f\left(y_{.kd}=1 \mid \mathbf{X}, \mathbf{W}, \boldsymbol{\Omega}\right)=\prod_{j=1}^{J} f\left(y_{jkd}=1 \mid \mathbf{X}, \mathbf{W}, \boldsymbol{\Omega}\right) \tag{17}$$

Why bayesian?

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- It is built on a simulation-based estimation method, therefore, it can handle all kinds of priors and data-generating processes [4].
- While the likelihood for the data and priors for the parameters are used to define the posterior sampling distributions, they can also be used in a generative way [15].
- Because the procedure integrates prior knowledge about the parameters, it can produce results even in scenarios where the Maximum Likelihood methods (ML) have issues of non-convergence or improper estimation [25, 4, 15]

There is nothing wrong?

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- It exposes the user to somewhat-arbitrary decisions about the running of the chains, in order to ensure a proper performance (**solution:** Hamiltonian Monte Carlo (HMC) [2]).
- The user can include all type of information through the priors distributions, making their elicitation convenient for manipulation (**solution:** prior predictive simulations and/or sensitivity analysis).
- Visual evaluation of performance, making it hard to assess if a proper posterior investigation have been made [8] (**solution:** help of `Rhat`, `n_eff`, and change of posterior sampling geometry).

There is nothing wrong? (cont.)

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- The procedure makes it hard to discover parameters' lack of identification [25] (**solution:** regularizing priors).
- Oftentimes the posterior sampling geometry of the model makes it hard to find proper solutions for the parameter space [2] (**solution:** change the posterior sampling geometry).
- The greater the complexity of the model, the harder it is to communicate/share and takes more time (**solution:** No solution, but it is a small "price" to pay).

Prior elicitation

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Un-informative priors are not the solution:

1. Uninformative / weakly-informative latent prior:

$$\begin{aligned} \theta &\sim N(0, 100) & \theta &\sim N(0, 1) \\ \text{logit}(p) &= \theta & \text{logit}(p) &= \theta \end{aligned}$$

2. Uninformative / weakly-informative hierarchical latent prior:

$$\begin{aligned} v &\sim \log N(0, 3) & v &\sim \log N(0, 0.5) \\ \theta &\sim N(0, v) & \theta &\sim N(0, v) \\ \text{logit}(p) &= \theta & \text{logit}(p) &= \theta \end{aligned}$$

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Prior elicitation (cont.)

Un-informative priors are not the solution:

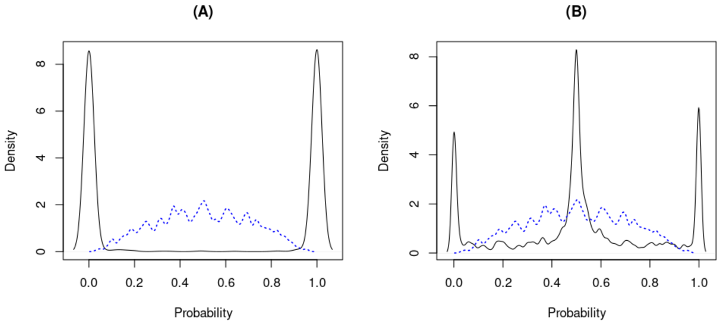

Figure: Prior predictive simulation. Examples of uninformative and mildly informative priors.

To center or not to center

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Even the most simple hierarchical models present formidable pathologies, that no simple rotation/rescaling of the parameter can be performed to visit the posterior distribution properly [2].

Example, the devil's funnel [15]:

$$\begin{aligned} v &\sim N(0, 3) \\ \theta &\sim N(0, \exp(v)) \end{aligned} \tag{18}$$

Equation (18) describes a centered parametrization (CP)

To center or not to center (cont.)

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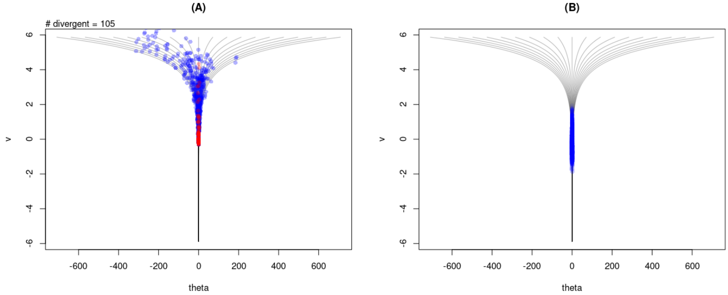

Figure: Posterior sampling geometry. Centered Parametrization.

To center or not to center (cont.)

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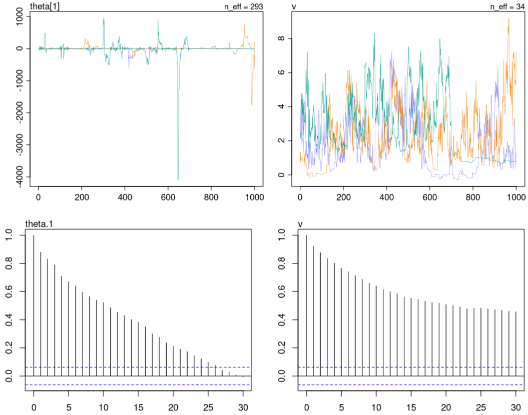

Figure: The Devil's funnel. Centered Parametrization. Trace and AFC plots.

To center or not to center (cont.)

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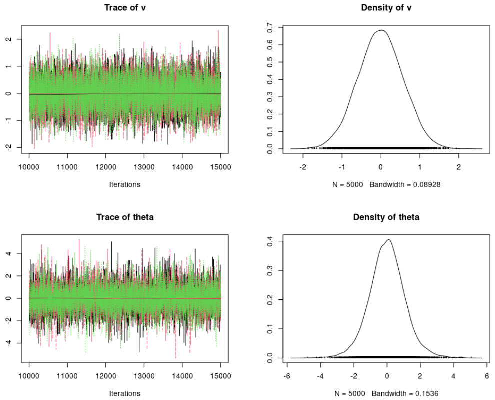

Figure: The Devil's funnel. Centered Parametrization implemented in JAGS.

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How can we solve this?:

1. Adapt HMC warm-up (`adapt_delta`$= 0.99$).
2. Use **regularizing priors**
3. Use the non-centered parametrization.

To center or not to center (regularizing priors)

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Consider the use of a regularizing prior on equation (18):

$$\begin{aligned} v &\sim N(0,1) \\ \theta &\sim N(0, \exp(v)) \end{aligned} \tag{19}$$

To center or not to center (regularizing priors, cont.)

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versus figure 4.

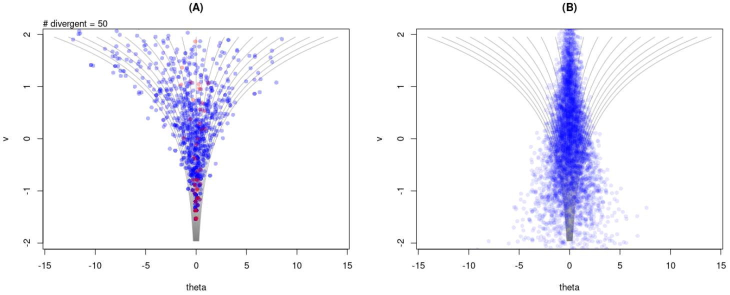

Figure: Posterior sampling geometry. Centered Parametrization with mildly informative priors.

To center or not to center (regularizing priors, cont.)

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versus figure 5.

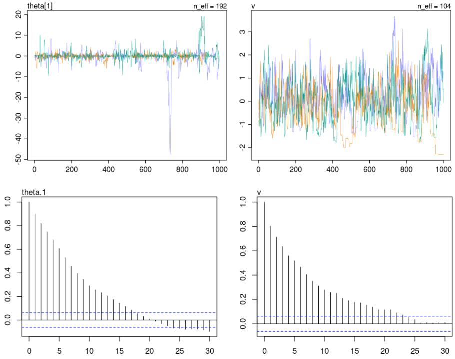

Figure: The Devil's funnel. Centered Parametrization with mildly informative priors. Trace and AFC plots.

To center or not to center (cont.)

How can we solve this?:

1. Adapt HMC warm-up (`adapt_delta`$= 0.99$).
2. Use regularizing priors
3. Use the **non-centered parametrization**.

To center or not to center (non-centered parametrization)

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Changing the posterior sampling geometry means to modify equation (18) in the following way:

$$\begin{aligned} v &\sim N(0,3) \\ z &\sim N(0,1) \\ \theta &= \exp(v)\ z \end{aligned} \tag{20}$$

To center or not to center (non-centered parametrization, cont.)

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versus figure 4 and 7.

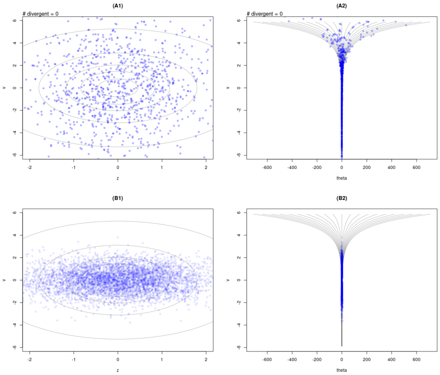

Figure: Posterior sampling geometry. Non-Centered Parametrization.

To center or not to center (non-centered parametrization, cont.)

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versus figure 5 and 8.

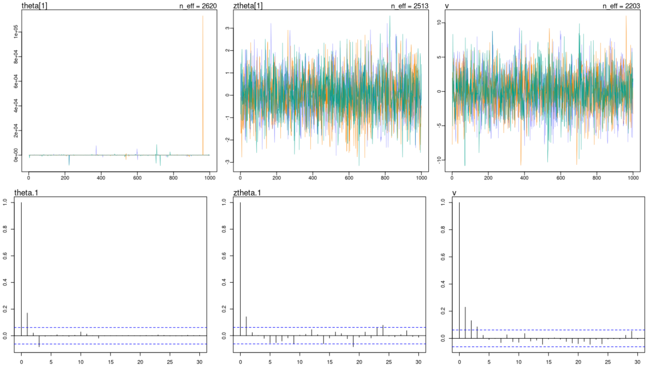

Figure: The Devil's funnel. Non-centered Parametrization. Trace and AFC plots.

Simulation study design

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A $3 \times 2 \times 2$ fractional factorial design:

- Three different samples sizes to generate the data under analysis: 500, 250, and 100.
- Two parametrization of the models: CP and NCP.
- Two models of interest: the first- and second-order latent variable model.

Ten (10) data sets were generated for each study condition. Each data set resembled responses to 25 binary scored items, conforming to the SOLV model defined in figure 12. The model was motivated by the hypothesized structure of the reading comprehension sub-test, from the Peruvian public teaching career national assessment

Simulation study design (cont.)

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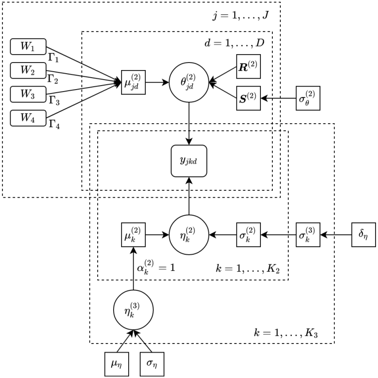

Figure: Directed Acyclig Graph (DAG). First-order latent variable model (SOLV).

Simulation study design (cont.)

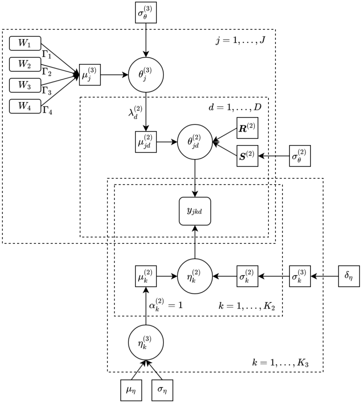

Figure: Directed Acyclig Graph (DAG). Second-order latent variable model (SOLV).

Evaluation criteria

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1. **Performance.** Trace, trank and ACF plots with support of `Rhat` and `n_eff` statistics developed by Gelman et al. [7] (pp. $284 - 287$).
2. **Recovery capacity.** we used the between replica root mean squared error (RMSE_B).
3. **Retrodictive accuracy.** we used the average within $\overline{\text{RMSE}}_W$ and between prediction root mean squared error RMSE_B, of the responses' predictive proportion $\hat{p}$, versus the observed proportion p.

Likelihood, priors and hyper-priors (centered parametrization)

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$$y_{jkd} \sim \text{Bernoulli}(\pi_{jkd}) \tag{21}$$
$$\text{logit}(\pi_{jkd}) = \upsilon_{jkd} \tag{22}$$
$$\upsilon_{jkd} = \theta^{(2)}_{jd} - \eta^{(2)}_{k} \tag{23}$$

$$\boldsymbol{\theta}^{(2)}_{j} = \left[\theta^{(2)}_{j1}, \theta^{(2)}_{j2}, \theta^{(2)}_{j3}\right] \tag{24}$$
$$\boldsymbol{\theta}^{(2)}_{j} \sim \text{MVNormal}\left(\boldsymbol{\mu}^{(2)}_{j}, \boldsymbol{\Sigma}^{(2)}\right) \tag{25}$$

$$\boldsymbol{\Sigma}^{(2)} = \boldsymbol{S}^{(2)} \cdot \boldsymbol{R}^{(2)} \cdot \boldsymbol{S}^{(2)} \tag{26}$$
$$\boldsymbol{S}^{(2)} = \boldsymbol{\sigma}^{(2)}_{\theta}\mathbf{I} \tag{27}$$

Likelihood, priors and hyper-priors (centered parametrization, cont.)

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For the FOLV model:

$$\boldsymbol{\mu}_j^{(2)} = \left[\mu_{j1}^{(2)},\ \mu_{j2}^{(2)}, \mu_{j3}^{(2)}\right] \tag{28}$$

$$\mu_{jd}^{(2)} = \Gamma_0 + \Gamma_1 W_{1j} + \Gamma_2 (W_{2j} - W_{2\mathsf{min}}) + \Gamma_3 W_{3j} + \Gamma_4 W_{4j} \tag{29}$$

Likelihood, priors and hyper-priors (centered parametrization, cont.)

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For the SOLV model:

$$\boldsymbol{\mu}_j^{(2)} = \left[\mu_{j1}^{(2)},\ \mu_{j2}^{(2)}, \mu_{j3}^{(2)}\right] \tag{30}$$

$$\boldsymbol{\lambda}^{(2)} = \left[\lambda_1^{(2)},\ \lambda_2^{(2)}, \lambda_3^{(2)}\right] \tag{31}$$

$$\mu_{jd}^{(2)} = \lambda_d^{(2)}\ \theta_j^{(3)} \tag{32}$$

$$\theta_j^{(3)} \sim \text{Normal}\left(\mu_j^{(3)}, \sigma_\theta^{(3)}\right) \tag{33}$$

$$\mu_j^{(3)} = \Gamma_0 + \Gamma_1 W_{1j} + \Gamma_2 (W_{2j} - W_{2\text{min}}) + \Gamma_3 W_{3j} + \Gamma_4 W_{4j} \tag{34}$$

Likelihood, priors and hyper-priors (centered parametrization, cont.)

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For the items:

$$\eta_k^{(2)} \sim \text{Normal}\left(\mu_k^{(2)}, \sigma_k^{(2)}\right) \tag{35}$$

$$\mu_k^{(2)} = \boldsymbol{\eta}^{(3)}\mathbf{A} \tag{36}$$

$$\sigma_k^{(2)} = \boldsymbol{\sigma}^{(3)}\mathbf{A} \tag{37}$$

$$\boldsymbol{\eta}^{(3)} = \left[\eta_1^{(3)}, \eta_2^{(3)}, \eta_3^{(3)}, \eta_4^{(3)}, \eta_5^{(3)}\right] \tag{38}$$

$$\boldsymbol{\sigma}^{(3)} = \left[\sigma_1^{(3)}, \sigma_2^{(3)}, \sigma_3^{(3)}, \sigma_4^{(3)}, \sigma_5^{(3)}\right] \tag{39}$$

$$\eta_k^{(3)} \sim \text{Normal}\left(\mu_\eta, \sigma_\eta\right) \tag{40}$$

$$\sigma_k^{(3)} \sim \text{Exponential}\left(\delta_\eta\right) \tag{41}$$

Likelihood, priors and hyper-priors (centered parametrization, cont.)

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Remaining priors and hyper-priors

$$\boldsymbol{R}^{(2)} \sim \text{LkjCorrelation}(2) \tag{42}$$

$$\Gamma_{1c} \sim \text{Normal}(0, 0.5) \tag{43}$$

$$\Gamma_{2} \sim \text{Normal}(0, 0.5) \tag{44}$$

$$\Gamma_{3c} \sim \text{Normal}(0, 1) \tag{45}$$

$$\Gamma_{4c} \sim \text{Normal}(0, 0.5) \tag{46}$$

Likelihood, priors and hyper-priors (Non-centered parametrization)

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Under the NCP, equation (35) was re-defined as follows:

$$\eta_k^{(2)} = \mu_k^{(2)} + \sigma_k^{(2)} z_k^{(2)} \tag{47}$$

$$z_k^{(2)} \sim \text{Normal}(0, 1) \tag{48}$$

Equation (25) was re-defined as follows:

$$\boldsymbol{\theta}_j^{(2)} = \boldsymbol{\mu}_j^{(2)} + \boldsymbol{S}^{(2)} \cdot \boldsymbol{L}_\Sigma^{(2)} \cdot (\mathbf{z}_j \mathbf{I}) \tag{49}$$

$$\mathbf{z}_j = [z_{j1}, \ldots, z_{jd}]^T \tag{50}$$

$$z_{jd} \sim \text{Normal}(0, 1) \tag{51}$$

$$\boldsymbol{L}_\Sigma^{(2)} \sim \text{LKJCorrelationCholesky}(2) \tag{52}$$

Likelihood, priors and hyper-priors (Non-centered parametrization, cont.)

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Finally, equation (33) was re-defined as follows:

$$\theta_j^{(3)} = \mu_j^{(3)} + \sigma_\theta^{(3)} z_j \tag{53}$$

$$z_j \sim \text{Normal}(0, 1) \tag{54}$$

Identification

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We used the unit variance identification scheme (UVI), that is, to set the scale of the higher-order dimension and sub-dimensions to one:

- $\sigma_{\theta}^{(3)} = 1$
- $\mathbf{S}^{(2)} = \boldsymbol{\sigma}_{\theta}^{(2)}\mathbf{I}$ with $\boldsymbol{\sigma}_{\theta}^{(2)} = [1, 1, 1]^T$
- $\mu_{\eta} = 0,\ \sigma_{\eta} = 1,\ \delta_{\eta} = 2$

The second turned the covariance matrix into a correlation, i.e. $\boldsymbol{\Sigma}^{(2)} = \boldsymbol{R}^{(2)}$.

Prior predictive investigation

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From two perspectives:

- the IRT perspective
- the outcome perspective

Prior predictive investigation (cont.)

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From the IRT perspective:

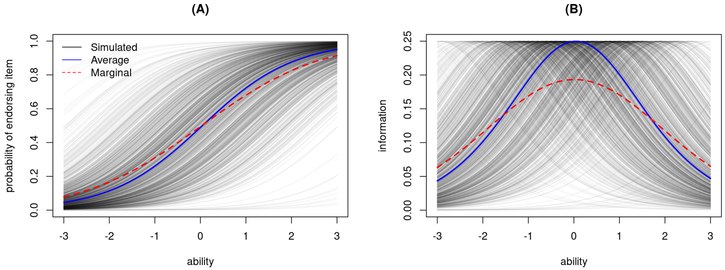

Figure: First-order latent variable model (FOLV). (A) Item Characteristics Curve, ICC. (B) Item Information Function, IIF.

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Prior predictive investigation (cont.)

From the outcome perspective:

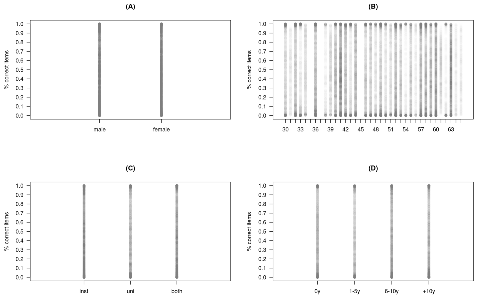

Figure: First-order latent variable model (FOLV). Aggregated endorsement rate per simulated covariate: (A) gender, (B) age, (C) education, and (D) experience.

Running time

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			Time (min.)		
	Parametrization	Sample	mean	min	max
1	CP	100	4.80	1.89	7.56
2	CP	250	7.85	5.60	16.57
3	CP	500	19.15	17.20	22.53
4	NCP	100	1.94	1.74	2.23
5	NCP	250	7.18	6.78	8.40
6	NCP	500	22.38	20.12	28.62

Table: First-order latent variable model (FOLV). Running time statistics.

Running time (cont.)

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			Time (min.)		
	Parametrization	Sample	mean	min	max
1	CP	100	4.04	1.39	8.55
2	CP	250	7.13	5.22	11.49
3	CP	500	16.14	14.45	19.73
4	NCP	100	1.87	1.62	2.28
5	NCP	250	5.92	5.20	6.79
6	NCP	500	16.58	13.37	18.26

Table: Second-order latent variable model (SOLV). Running time statistics.

Application model fit

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They try to approximate the out-of-sample KL-divergence [12]:

	Model	Param.	WAIC	lppd	penalty
1	FOLV	CP	54,632.4	-25,429.7	1,886.5
2	FOLV	NCP	54,631.7	-25,427.6	1,888.3
3	SOLV	CP	54,610.3	-25,348.4	1,956.7
4	SOLV	NCP	54,614.7	-25,337.9	1,969.4

Table: Model fit. Widely Applicable Information Criterion (WAIC).

Application model fit (cont.)

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	Model	Param.	PSIS	lppd	penalty
1	FOLV	CP	54,657.4	-27,328.7	1,904.4
2	FOLV	NCP	54,656.9	-27,328.5	1,898.1
3	SOLV	CP	54,627.3	-27,313.7	1,940.1
4	SOLV	NCP	54,642.5	-27,321.2	1,990.2

Table: Model fit. Pareto-smoothed importance sampling cross-validation (PSIS).

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