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Generalized Linear Latent and Mixed Model (GLLAMM):

Method, bayesian estimation, advantages, and applications to educational data.

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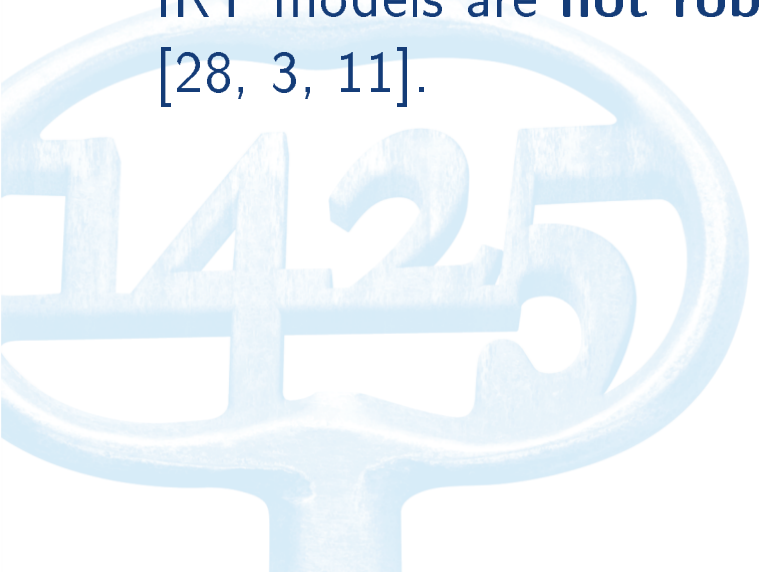
1. Preliminary considerations

Local independence

Item Response Theory (IRT) model's assumption comprised of two parts [1, 10]:

- local item independence
- local individual independence.

IRT models are **not robust** to the violation of local independence [28, 3, 11].



Educational data

often display **several** types of dependencies, violating the local item and/or individual independence, e.g.

- testlets [27];
- the measurement of multiple latent traits within individuals [24];
- cluster effects [23].

Proposed model

The GLLAMM follow a multilevel/hierarchical multidimensional approach to account for different dependencies.

- (**good**) control for dependencies in educational data
- (**important**) reach appropriate conclusion from the parameters

Implementation

GLLAMM under the Bayesian framework will be:

- Complex and highly dimensional (in parameters)
- On sparse binary data

However, **complex parametrizations** (and even simple) introduce pathologies that prevent MCMC methods to achieve **ergodicity** [5, 6, 17, 18, 2], i.e. reach stationarity, convergence, and good mixing [15].

There are (simple) proposed solutions, but **still we cannot properly visit** the posterior distribution [2]

Implementation (cont.)

Luckily changing the **posterior sampling geometries**, i.e. removing the dependence of the parameters on other sampled parameters, seem to **improve** the performance of the MCMC methods [5, 6, 17, 18, 2]

2. The GLAMM for dichotomous outcomes

Model definition

Following Rabe-Hesketh et al. [20, 21], we define the GLLAMM in two parts:

1. the response model
2. the latent structure

Moreover, **the response model (1)** can be represented by a Generalized Linear Model (GLM) [16, 14] with:

1. a distributional part
2. a systematic part

1. The response model

Conditional to all parameters $\mathbf{\Omega} = \{\beta, \mathbf{\Lambda}, \mathbf{\Theta}, \mathbf{\Psi}, \mathbf{\Gamma}\}$; and the "stacked" vector of covariates X and W; **the distributional part** is defined by:

$$f\left(y_{jkd} = 1 \mid \mathrm{X}, \mathrm{W}, \mathbf{\Omega}\right) = \pi_{jkd}^{n}(1 - \pi_{jkd})^{1-n} \tag{1}$$

Furthermore, **the systematic part** is defined in the following form:

$$P\left(y_{jkd} = 1 \mid \mathrm{X}, \mathrm{W}, \mathbf{\Omega}\right) = \pi_{jkd} = h(\tau_k + v_{jkd}) \tag{2}$$

where τ_k is k'th item threshold, assumed to be zero for the binary case [20].

1. The response model (cont.)

Moreover, the inverse-link function $h(\cdot)$ can be defined in three ways:

$$h(x) = \begin{cases} \exp(x)[1+\exp(x)]^{-1} \\ \Phi(x) \\ \exp(-\exp(x)) \end{cases} \tag{3}$$

corresponding to the logistic, standard normal $\Phi(x)$, and Gumbel (extreme value type I) cumulative distributions, respectively.

Finally, the linear predictor is defined by:

$$v_{jkd} = \mathrm{X}_j \, \boldsymbol{\beta} + \sum_{m=2}^{M+1} \boldsymbol{\eta}^{(m)} \boldsymbol{\alpha}^{(m)} \mathrm{A}_j^{(m)} + \sum_{l=2}^{L+1} \boldsymbol{\theta}_j^{(l)} \boldsymbol{\lambda}^{(l)} \mathrm{B}_j^{(l)} \tag{4}$$

2. The latent structure

The structural model for the latent variables is represented in the following form:

$$\underset{(S\times S)(S\times 1)}{\boldsymbol{\Psi}}\ \underset{}{\boldsymbol{\Theta}} + \underset{(S\times Q)(Q\times 1)}{\boldsymbol{\Gamma}}\ \underset{}{\mathbf{W}} + \underset{(S\times 1)}{\boldsymbol{\zeta}} \tag{5}$$

where $S = K + D$, $K = \sum_m K_m$, and $D = \sum_l D_l$.

Notice equation (5) is the generalization of a single-level Structural Equation Models (SEM) to a multilevel setting.

Motivating example

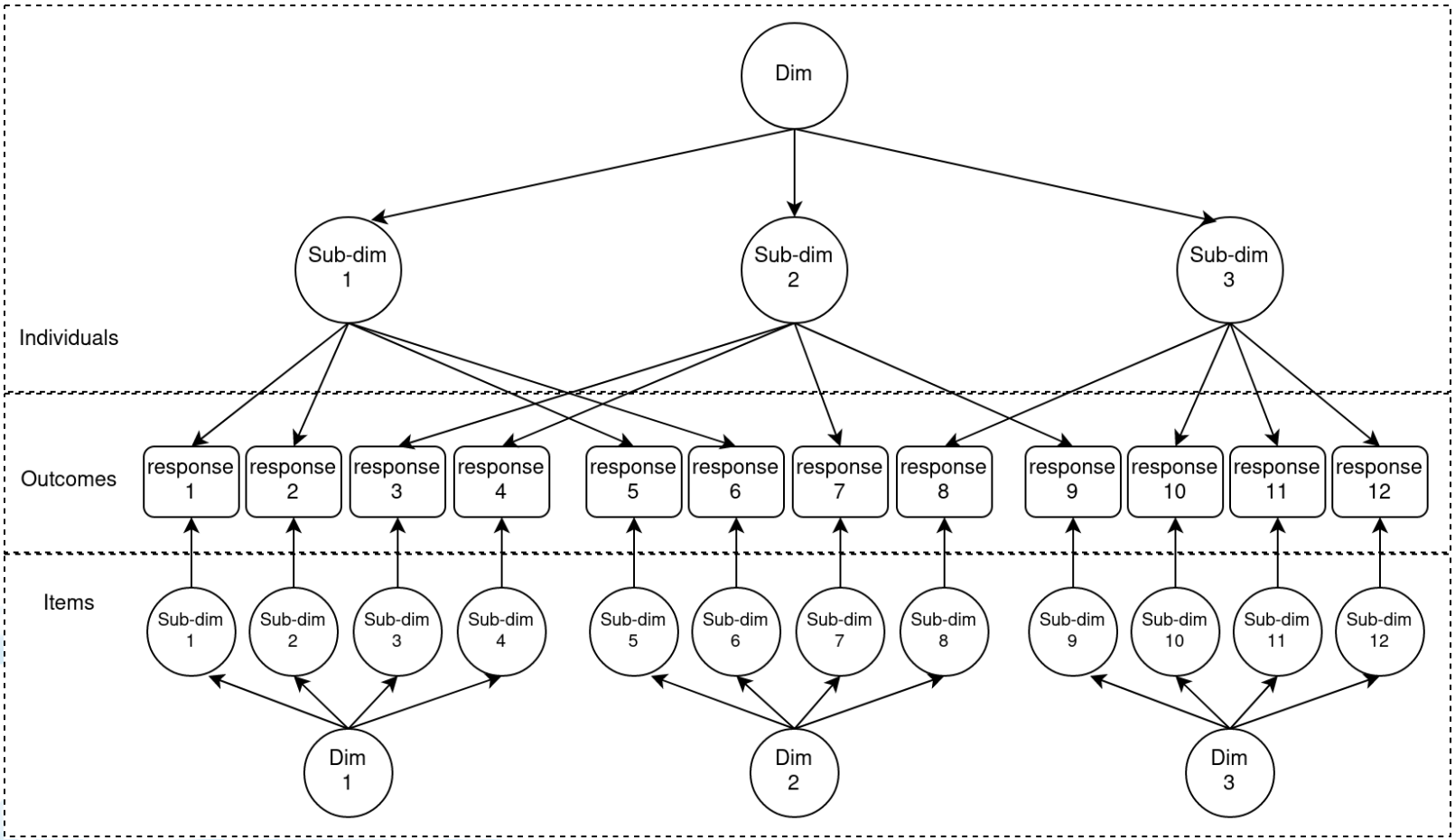


Figure: Path diagram of the dimensional structure for a hierarchical cross-classified IRT model.

Motivating example (cont.)

- Empty level-1 covariates matrix X_j ($P = 0$)
- $M = 2$ levels at the items block, with $K_2 = 12$ and $K_3 = 3$, i.e. $\boldsymbol{\eta}^{(2)} = [\eta_1^{(2)}, \ldots, \eta_{12}^{(2)}]^T$ and $\boldsymbol{\eta}^{(3)} = [\eta_1^{(3)}, \eta_2^{(3)}, \eta_3^{(3)}]^T$
- $L = 2$ levels in the individuals block, with $D_2 = 3$ and $D_3 = 1$, i.e. $\boldsymbol{\theta}^{(2)} = [\theta_1^{(2)}, \theta_2^{(2)}, \theta_3^{(2)}]^T$ and $\boldsymbol{\theta}^{(3)} = \theta_1^{(3)}$.
- Specific regression relationship among latents $\boldsymbol{\Psi}$, i.e. $\boldsymbol{\alpha}^{(3)} = [\alpha_{11}^{(3)}, \ldots, \alpha_{15}^{(3)}, \alpha_{21}^{(3)}, \ldots, \alpha_{25}^{(3)}, \alpha_{31}^{(3)}, \ldots, \alpha_{35}^{(3)}]^T$ and $\boldsymbol{\lambda}^{(3)} = [\lambda_1^{(3)}, \lambda_2^{(3)}, \lambda_3^{(3)}]^T$
- Empty structural covariates W.

3. Bayesian Estimation

Bayesian GLLAMM for dichotomous outcomes

1. **Posterior distribution.** Given that Y is the observed data and $\Omega = \{\beta, \Lambda, \Theta, \Psi, \Gamma\}$ the parameters:

$$P(\Omega \mid \mathrm{Y}) = \frac{P(\mathrm{Y} \mid \Omega)\, P(\Omega)}{\int P(\mathrm{Y} \mid \Omega)\, P(\Omega)\, d\Omega} \tag{6}$$

2. **Prior distributions** $P(\Omega)$. Similar to Patz and Junker [19], we use an independent distributional structure for the joint priors:

$$\begin{aligned} P(\Omega) =& P(\beta)\, P(\Lambda)\, P(\Theta)\, P(\Psi)\, P(\Gamma) \\ =& P(\beta)\, [P(\alpha)\, P(\lambda)]\, [P(\eta)\, P(\theta)] \\ & [P(\Psi_\eta)\, P(\Psi_\theta)]\, [P(\Gamma_\eta)\, P(\Gamma_\theta)] \end{aligned} \tag{7}$$

Bayesian GLLAMM for dichotomous outcomes (cont.)

❸ **Likelihood** $P(\mathrm{Y} \mid \boldsymbol{\Omega})$. Following Rabe-Hesketh et al. [20], the likelihood function is build in a recursive way.

$$f\left(\mathrm{y}=1 \mid \mathrm{X}, \mathrm{W}, \boldsymbol{\Omega}\right)=\prod_{j=1}^{J} \prod_{d=1}^{D} \prod_{k=1}^{K} f\left(y_{jkd}=1 \mid \mathrm{X}, \mathrm{W}, \boldsymbol{\Omega}\right) \tag{8}$$

$$f_{(m)}^{(l)}\left(\mathrm{y}=1 \mid \mathrm{X}, \mathrm{W}, \boldsymbol{\Omega}\right)=\int\left[\prod f_{(m-1)}^{(l-1)}\left(\mathrm{y}=1 \mid \mathrm{X}, \mathrm{W}, \boldsymbol{\Omega}\right)\right] P(\boldsymbol{\Theta}_{(m)}^{(l)})\, d\boldsymbol{\Theta}_{(m)}^{(l)} \tag{9}$$

$$\mathcal{L}(\mathrm{X}, \mathrm{W}, \boldsymbol{\Omega})=\prod_{m=2}^{M+1} \prod_{l=2}^{L+1} f_{(m)}^{(l)}\left(\mathrm{y}=1 \mid \mathrm{X}, \mathrm{W}, \boldsymbol{\Omega}\right) \tag{10}$$

$$\ell(\mathrm{X}, \mathrm{W}, \boldsymbol{\Omega})=\log \mathcal{L}(\mathrm{X}, \mathrm{W}, \boldsymbol{\Omega}) \tag{11}$$

Computational implementation

1. Hamiltonian Monte Carlo (HMC) and `Stan` [26].
2. **No burn-in and thinning**. Warm-up phase to "tune-up" the number of steps (`leapfrogs`), and the `step size` [26].
3. We use $3,000$ **effective iterations**: 3 chains of $2,000$ iterations each, where $1,000$ are spend on warm-up.
4. **Initial starts** sampled from the priors defined in the model.
5. Prior distributions selected based on **prior predictive simulations**.
6. We test the **centered (CP)** and **non-centered parametrizations (NCP)**.

4. Simulation studies

Objectives

1. **Performance.** In terms of achieving ergodicity, under the CP and NCP,
2. **Recovery capacity.** Capacity to recover the parameters of interest, especially the structural regression parameters.
3. **Retrodictive accuracy.** Capacity to retrodict the data of interest, according to a set of aggregating dimensions.

Results (Performance)

1. the non-centered parametrization (NCP) largely improved the performance of the MCMC chains, towards achieving ergodicity.

   This is true across models, simulated sample sizes, and simulated replicas.

2. No large difference in performance was observed in either the sub-dimensions' correlation or loading parameters.

   However, no evidence supported the idea the parameters suffered from a further lack of identification.

Results
(Performance, cont.)

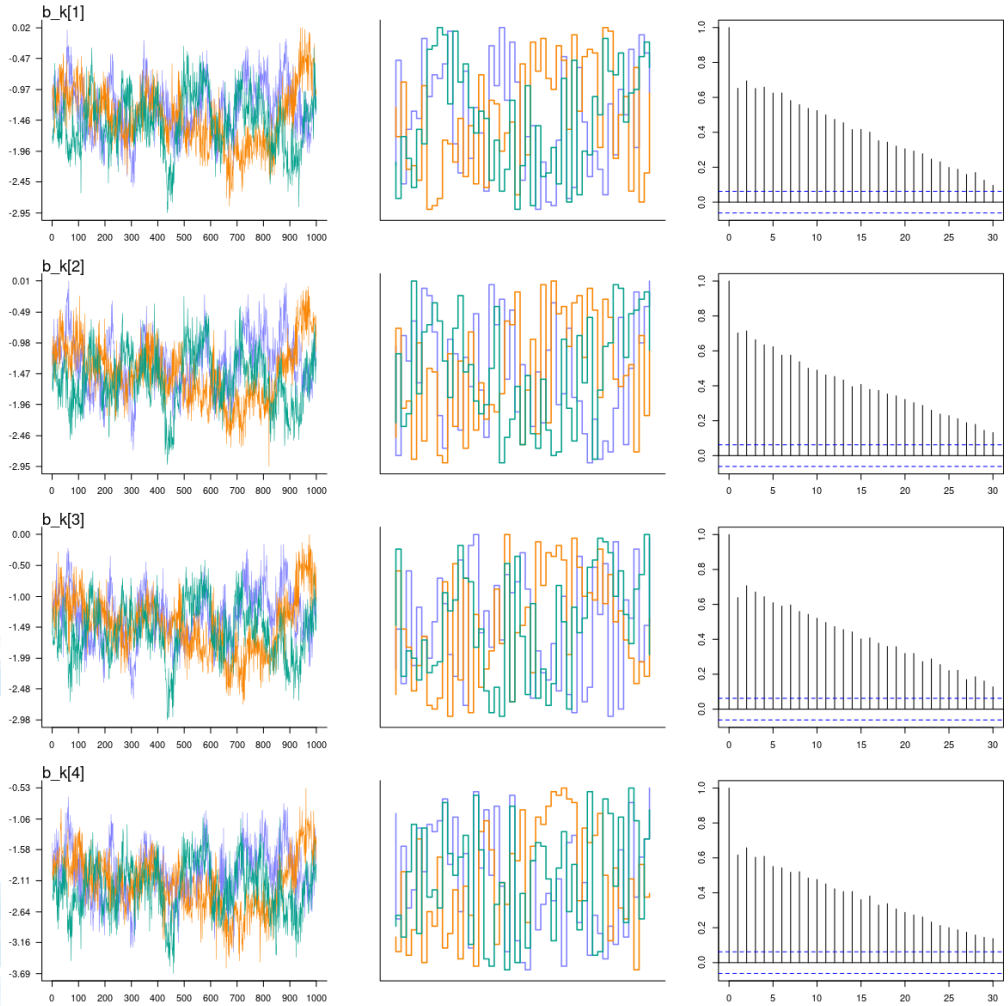


Figure: First-order latent variable model (FOLV). Sample size 100, replica number 1. Centered parametrization. Item Difficulties.

Results
(Performance, cont.)

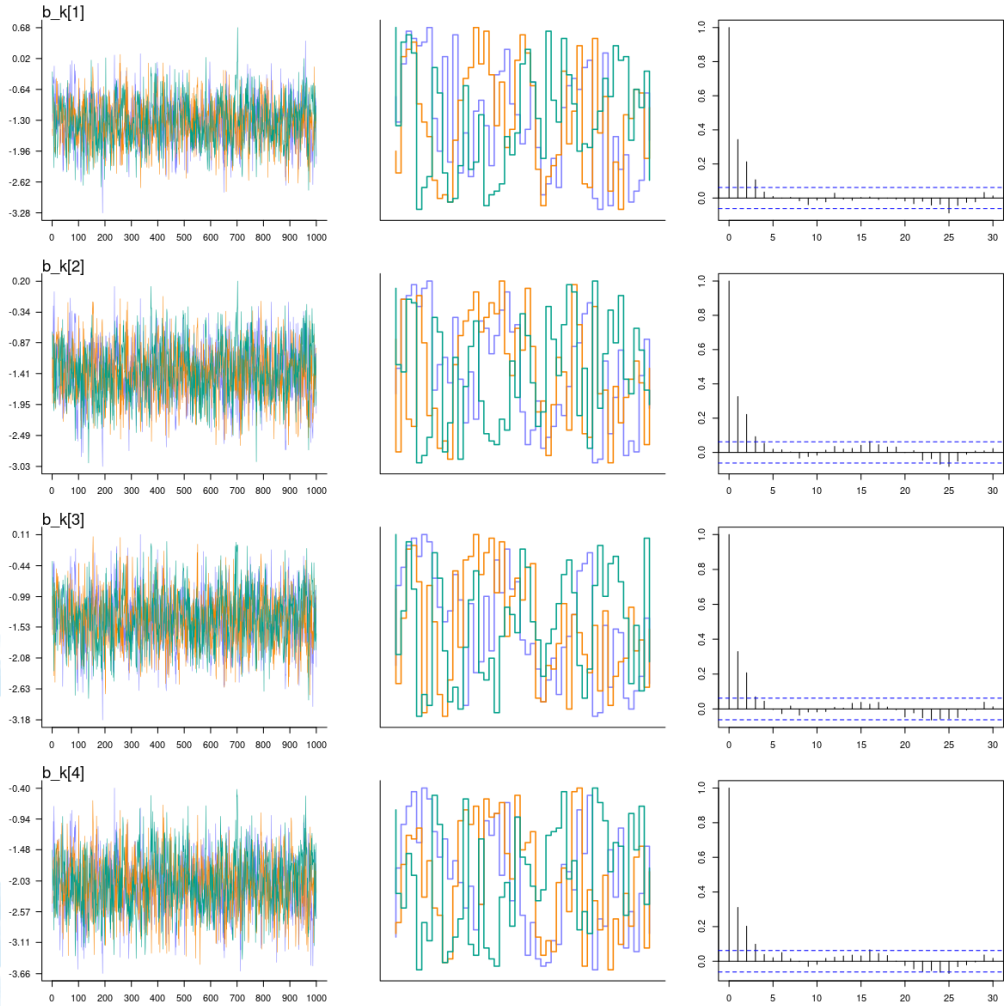


Figure: First-order latent variable model (FOLV). Sample size 100, replica number 1. Non-centered parametrization. Item Difficulties.

Results (Recovery capacity)

1. The GLLAMM was able to recover most of the simulated parameters with good precision.
2. However, the model still had issues estimating the sub-dimensions correlations and loadings.

   Under the Confirmatory Factor Analysis theory (CFA), a SOLV model is only justified, if the lower-level correlations are high enough, usually above 0.8.
3. It is surprising that CP and NCP achieved similar levels of recovery capacity.

Results
(Recovery capacity, cont.)

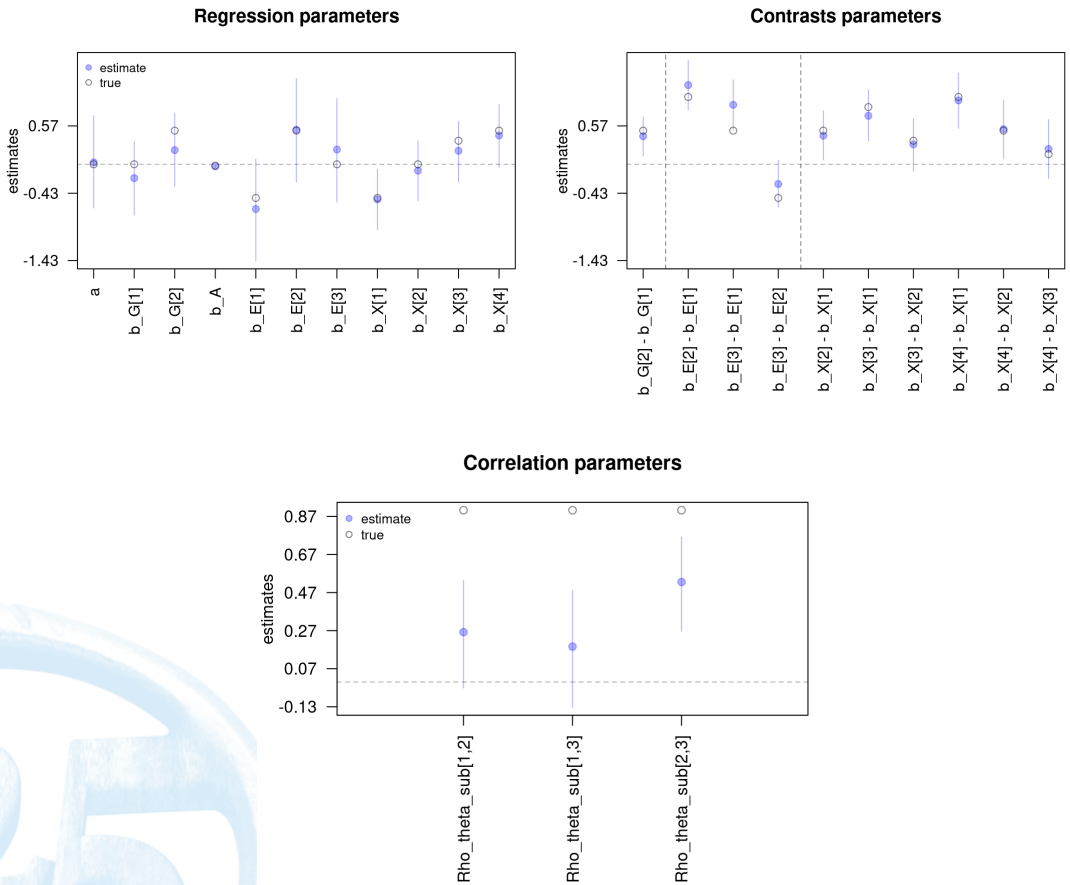
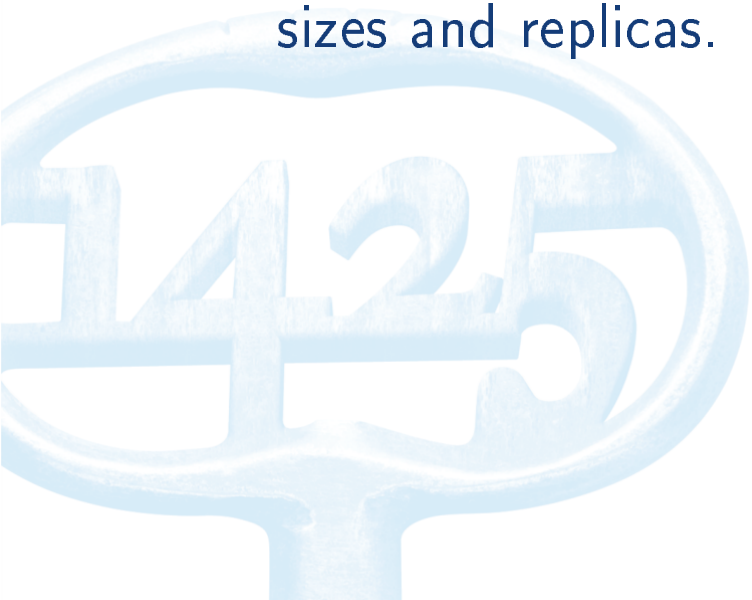


Figure: First-order latent variable model (FOLV). Non-centered parametrization. Sample size 100, replica 1. Regression, contrast, and correlation parameters.

Results
(Retrodictive accuracy)

1. the models managed to capture the traits of the data, while avoiding its exact replication.

   These result were consistent across models, simulated sample sizes and replicas.



Results
(Retrodictive accuracy, cont.)

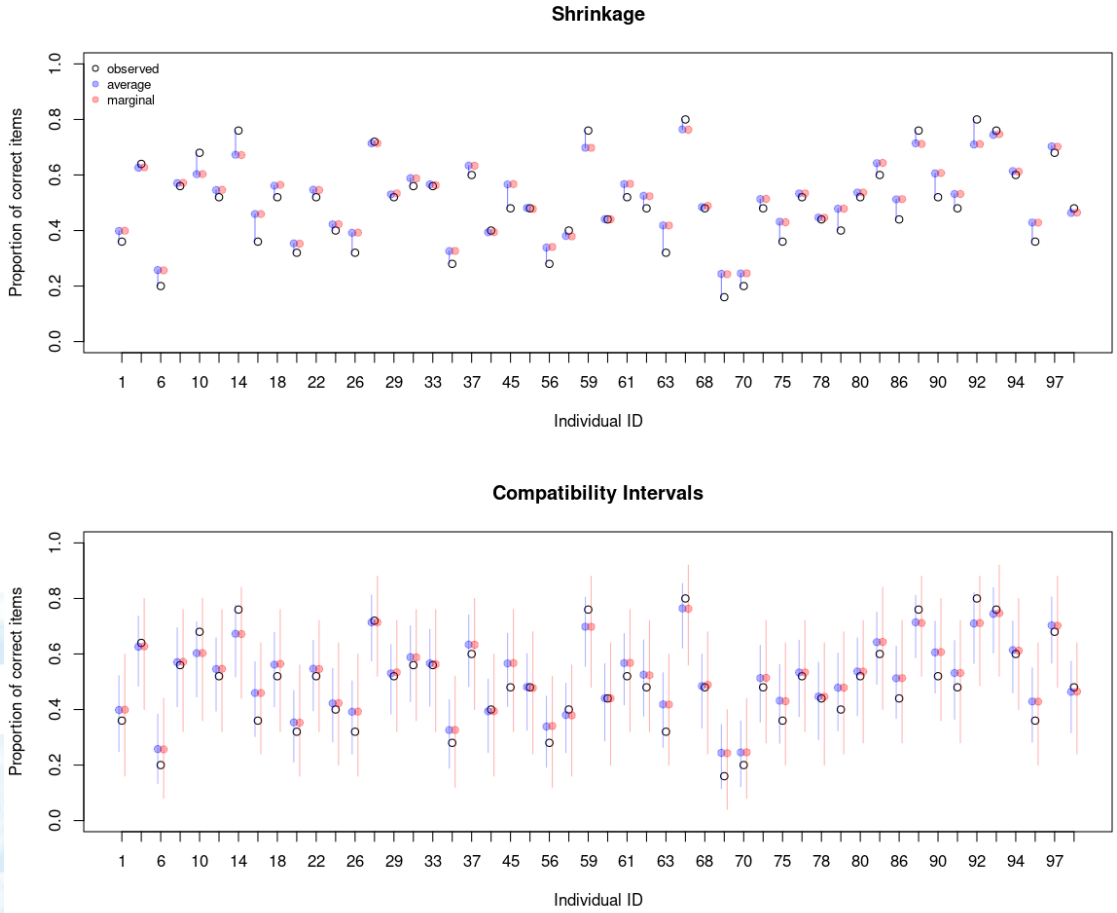


Figure: First-order latent variable model (FOLV). Sample size 100, replica number 4. Non-centered parametrization. Individual predictive plot.

5. Application

Objectives

1. **Evaluate the performance of the parametrizations.** Ergodicity of CP vs NCP.
2. **Evaluate the retrodictive accuracy.**
3. **Assess the psychometric properties.** Special interest in determine how difficult the items were.
4. **Test research hypothesis.** About the explanatory power a set of covariates had on the latent dimensions, and their implications for the educational authority.

Instrument and data

1. **Instrument.** Reading comprehension sub-test composed of 25 binary scored items, from the Peruvian public teaching career national assessment.
2. **Access.** Legal requirement of open information to the Ministry of Education of Peru (MINEDU).
3. **Sample scheme.** Simple random sample of $2,000$ (from approx. $195,0000$).

Results (Performance)

1. the non-centered parametrization (NCP) largely improved the performance of the MCMC chains, towards achieving ergodicity.
2. No large difference in performance was observed in either the sub-dimensions' correlation or loading parameters.

results are similar to simulation study, albeit in some cases more extreme.

Results
(Performance, cont.)

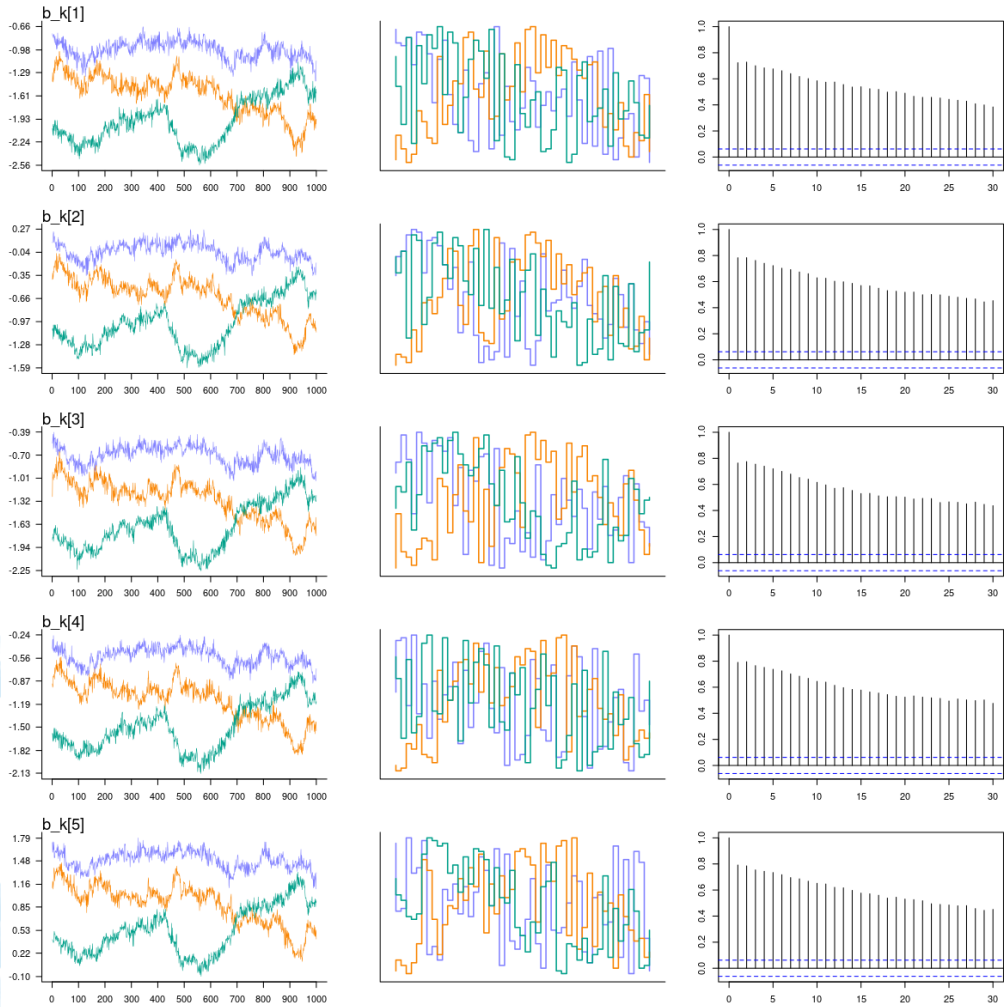


Figure: Application's first-order latent variable model (FOLV). Centered parametrization. Item difficulties.

Results
(Performance, cont.)

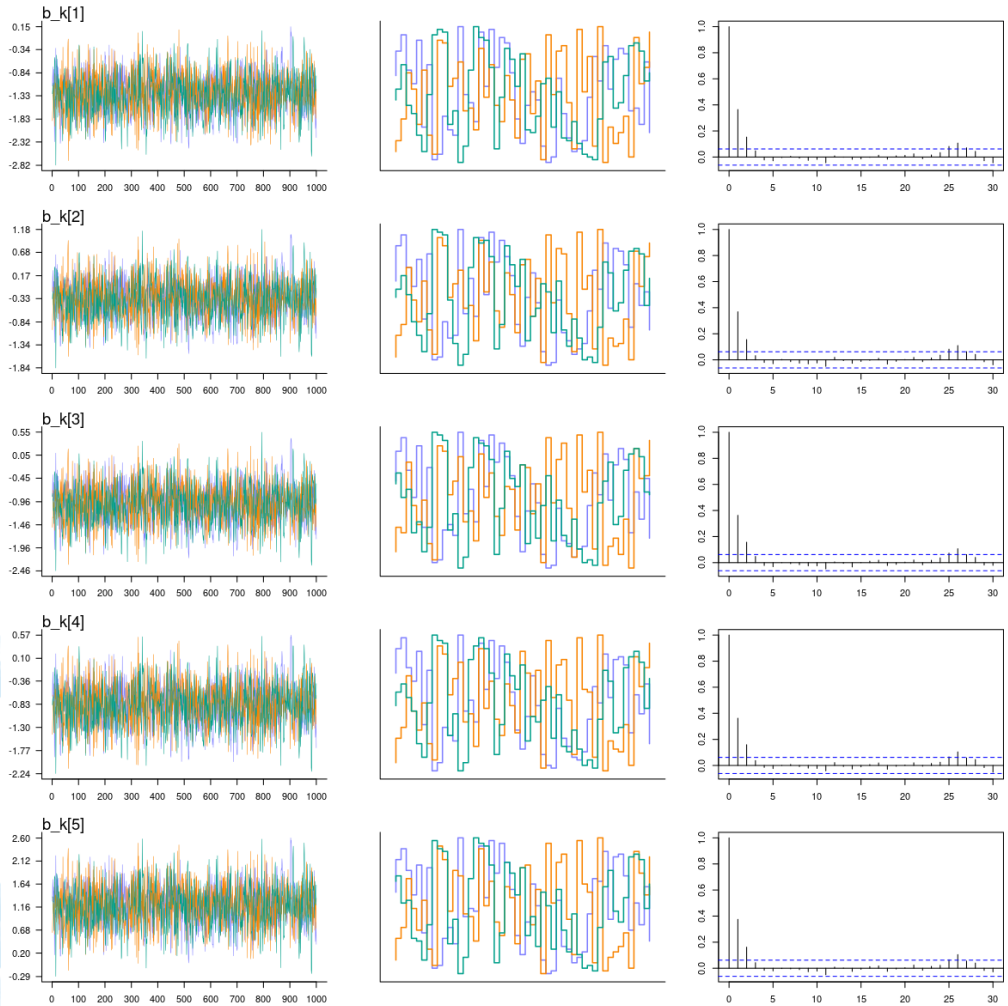


Figure: Application's first-order latent variable model (FOLV). Non-centered parametrization. Item difficulties.

Results
(Retrodictive accuracy)

1. the models managed to capture the traits of the data, while avoiding its exact replication.

   Similar to simulation study.

Results
(Retrodictive accuracy, cont.)

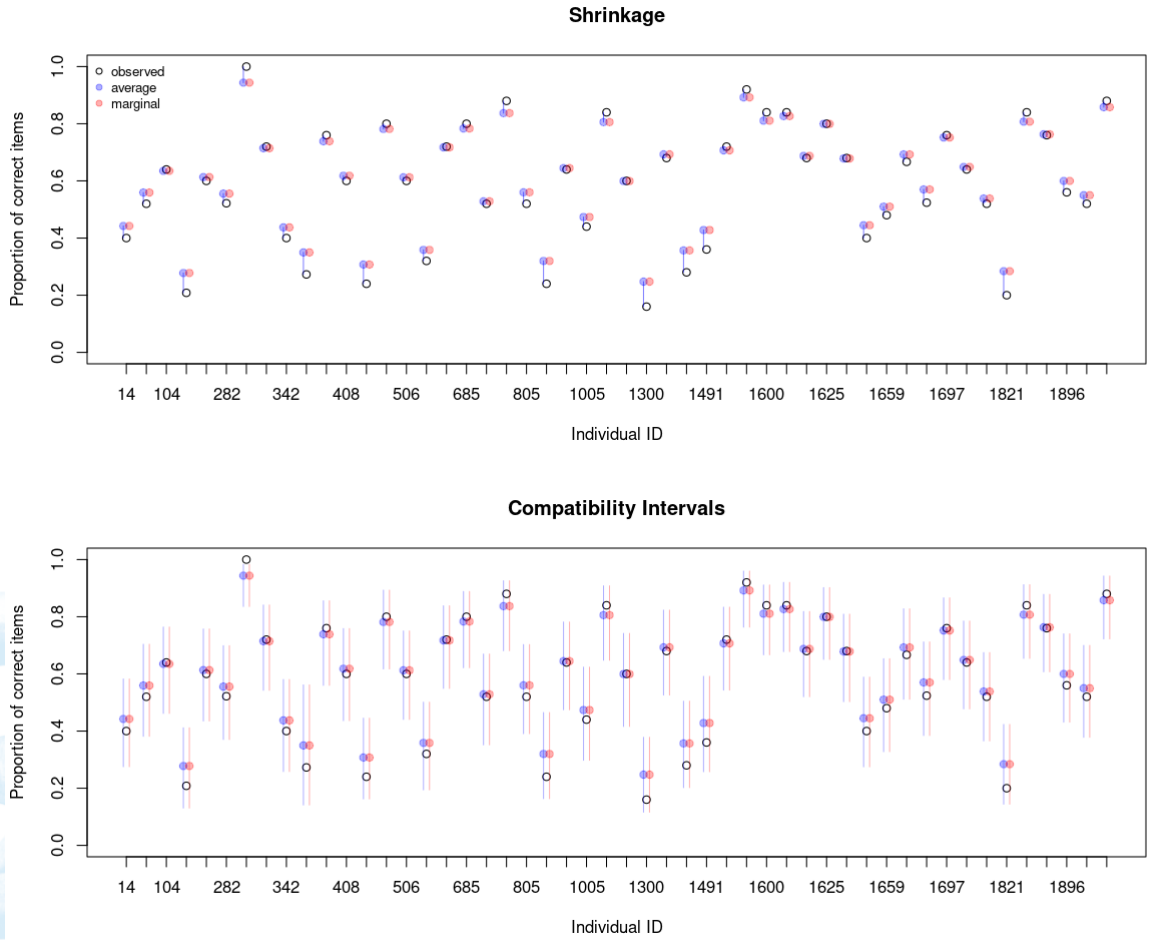


Figure: First-order latent variable model (FOLV). Non-centered parametrization. Individual predictive plot.

Results
(Psychometric properties)

1. the items were scattered throughout a significant portion of the abilities range.
2. Specific benefit of the implementation allowed us to assess how difficult the texts were.
3. (a word of advice) A more sound psychometric analysis requires access to the items and texts description, but it was not possible due to legal restrictions.

Results
(Psychometric properties, cont.)

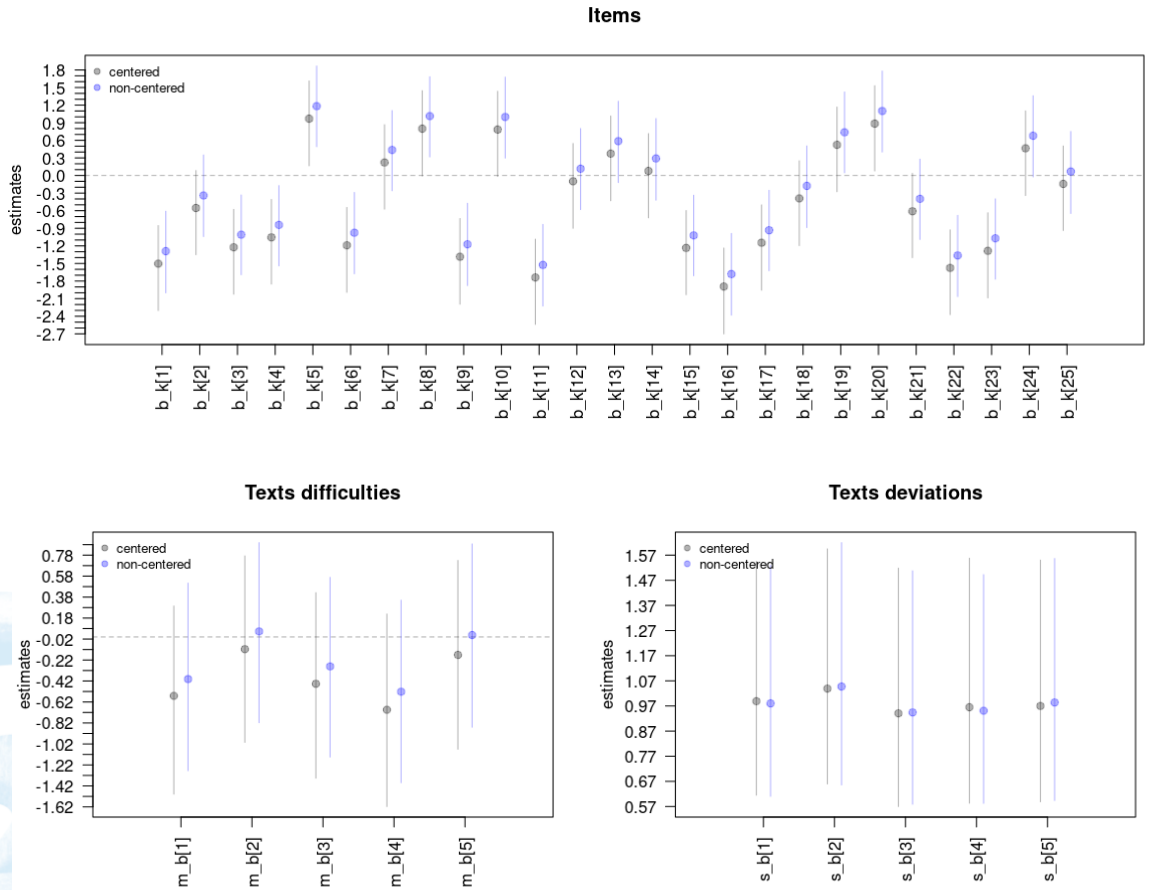


Figure: First-order latent variable model (FOLV). Centered and non-centered parametrization. Items, texts difficulties, and texts deviations.

Results
(Hypothesis testing)

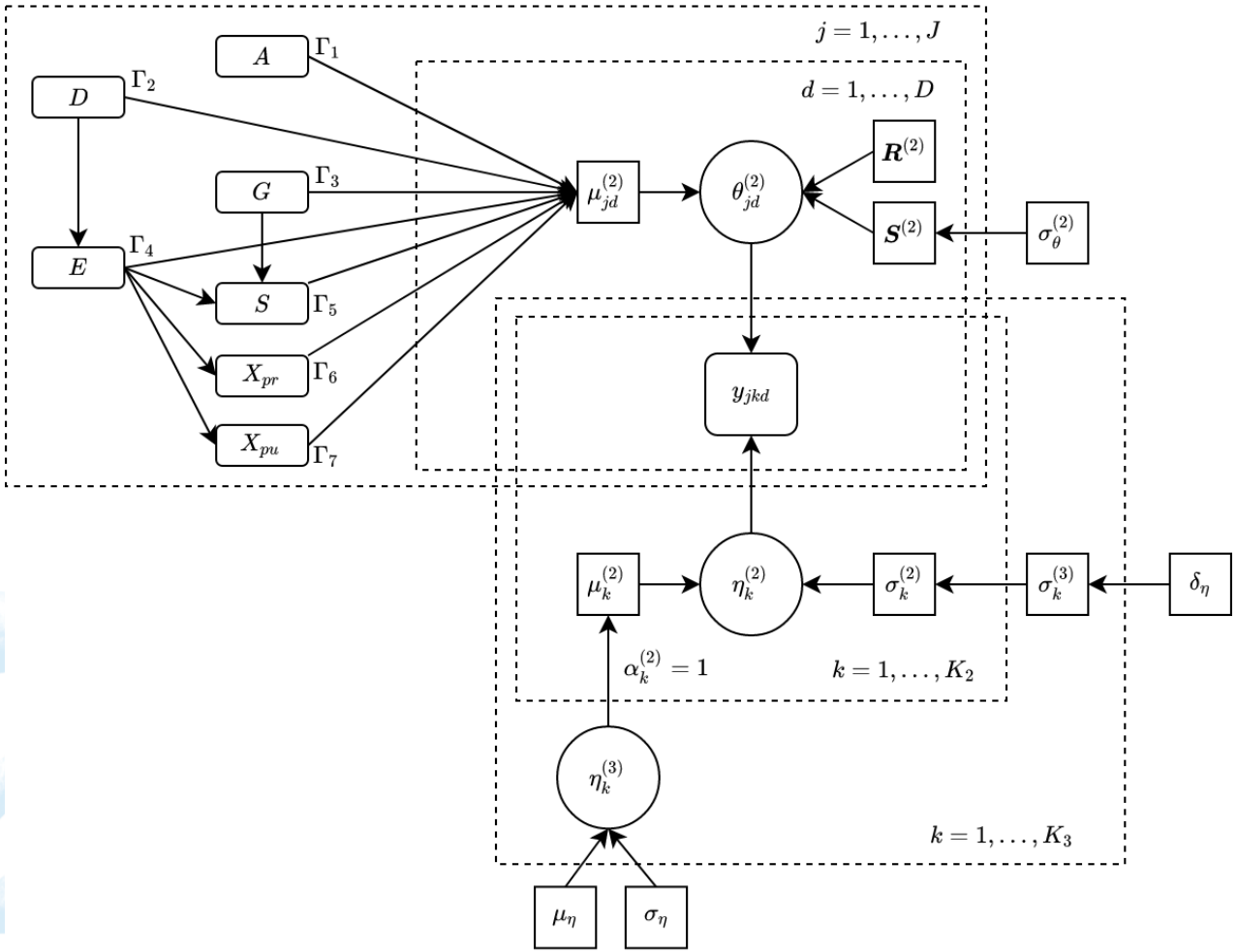


Figure: Directed Acyclig Graph (DAG). Application's first-order latent variable model (FOLV).

Results
(Hypothesis testing, cont.)

1. Age (style of teaching proxy) explains negatively reading comprehension $-0.036[-0.04, -0.03]$.
2. Disability also explained the the reading comprehension sub-dimensions, although the results were mildly unexpected.
3. the statistical evidence seem to support the lower quality of training on pedagogical institutes.
4. Experience improved the reading comprehension abilities. Private experience effects were larger than the public. We observe also diminishing returns on abilities, as the years of experience increases.
5. Instructing students from secondary education improves reading comprehension.

Results
(Hypothesis testing, cont.)

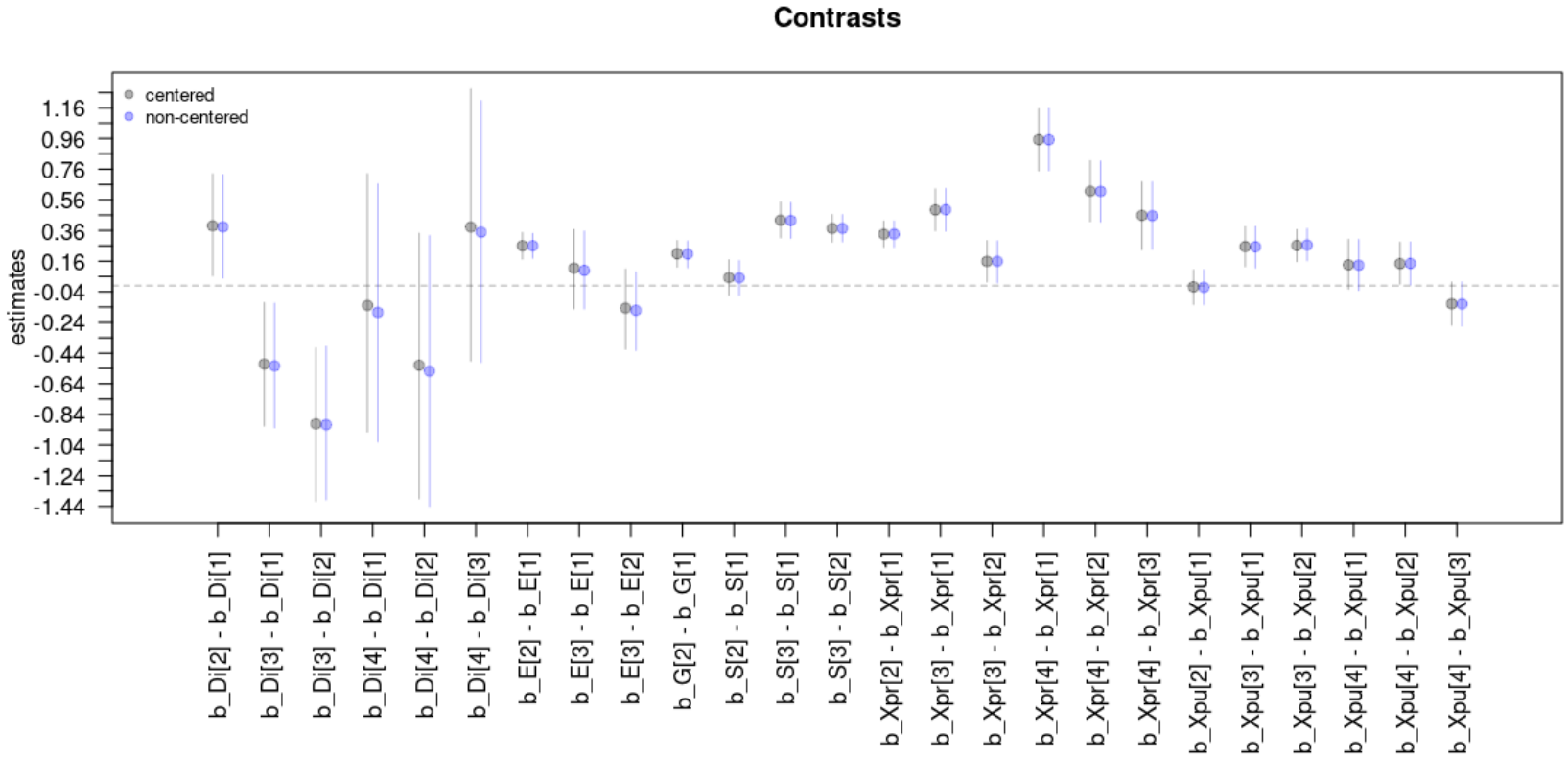


Figure: Application's first-order latent variable model (FOLV). CP and NCP comparison plot. Contrasts.

6. Conclusions and further development

Conclusions

1. The NCP **largely improved the performance** of the MCMC chains. True in simulations, and under the real application, albeit with some caveats.
2. The proposed model **recovered most of the simulated parameters** with good precision. The model still had issues with the sub-dimensions correlations and loadings.
3. The models managed to **capture the traits of the data**, while **avoiding its exact replication**. Consistent in simulations, and under the real application.
4. The NCP was **slightly faster** than the CP, although the magnitudes of the differences in running time were not large.
5. On the real application, the model **provided the (extra) benefit** to asses the psychometric properties of texts.
6. Finally, the model produced **interesting statistical results**, supported by the statistical application and the DAG guiding our interpretation and causal assumptions.

Further investigation

1. Why the benefits of the NCP did not fully extend to correlations and loadings?.
2. Why the CP (no ergodicity) managed recover the parameters as good as the NCP.
3. Similar to HMC, test results for Variational Inference methods (VI)
4. The statistical evidence remains valid with a better sample scheme?
5. Test the model also with response explanatory variables, e.g. response time, number of alternatives, etc.
6. Test the model for measurement invariance.

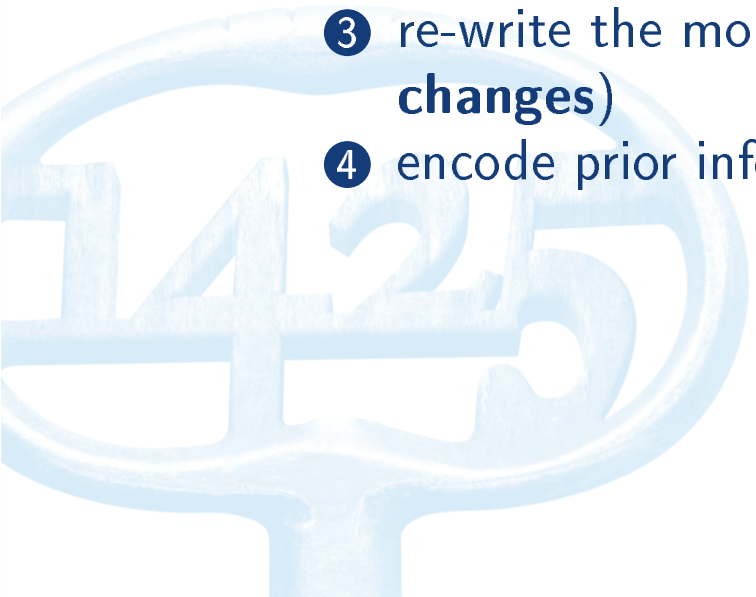
Thank you!

Appendix

Improving MCMC chain performance

Four solutions are offered to solve the previous pathologies:

- Changing the settings of the MCMC method
  1. increasing the number of iterations per chain, with large burn-in and thinning processes
  2. designing model-specific MCMC algorithms.
- Readjusting the Bayesian model
  3. re-write the model in an alternative parametrization (**simple changes**)
  4. encode prior information through the prior distributions



Cluster effects

Individual clustering involves the addition of more random effects to the linear predictor defined in equation (4):

$$\begin{aligned} \nu_{jkdc} &= \nu_{jkd} + \sum_{c=1}^{C} \delta_c \\ &= \nu_{jkd} + \boldsymbol{\delta} \mathsf{Z}_j \end{aligned} \tag{12}$$

Example (restrictions for IRT)

we could set the restriction $\boldsymbol{\alpha}^{(2)} = -\boldsymbol{\lambda}^{(2)}$ where $\boldsymbol{\lambda}^{(2)} > 0$. In that case we get a multidimensional generalization of the linear predictor observed in the archetypical Rasch [22], or 2PL [13] models, i.e. $\lambda_d^{(2)}(\theta_{jd}^{(2)} - \eta_k^{(2)})$

In addition, $\boldsymbol{\alpha}^{(3)} = [\alpha_{11}^{(3)}, \ldots, \alpha_{15}^{(3)}, \alpha_{21}^{(3)}, \ldots, \alpha_{25}^{(3)}, , \alpha_{31}^{(3)}, \ldots, \alpha_{35}^{(3)}]^T = [1, \ldots, 1]^T$, indicating texts difficulties explain directly the items difficulties at the lower level.

Example (restrictions for IRT)

Moreover, notice that because in the IRT framework $\boldsymbol{\eta}$ and $\boldsymbol{\theta}$ should be orthogonal to each other by design, we can further decompose equation (5) in the following form:

$$\underset{(K\times 1)}{\boldsymbol{\eta}} = \underset{(K\times K)}{\boldsymbol{\Psi}_{\eta}}\underset{(K\times 1)}{\boldsymbol{\eta}} + \underset{(K\times Q)}{\boldsymbol{\Gamma}_{\eta}}\underset{(Q\times 1)}{\mathrm{W}_{\eta}} + \underset{(K\times 1)}{\boldsymbol{\zeta}_{\eta}} \tag{13}$$

$$\underset{(D\times 1)}{\boldsymbol{\theta}} = \underset{(D\times S)}{\boldsymbol{\Psi}_{\theta}}\underset{(D\times 1)}{\boldsymbol{\theta}} + \underset{(D\times Q)}{\boldsymbol{\Gamma}_{\theta}}\underset{(Q\times 1)}{\mathrm{W}_{\theta}} + \underset{(D\times 1)}{\boldsymbol{\zeta}_{\theta}} \tag{14}$$

Model Assumptions

Following Skrondal and Rabe-Hesketh [25], the framework has two main assumptions:

(M1) Complete latent space.[9] In the GLLAMM representation, the space is complete if we consider all latent variables $\boldsymbol{\Theta}$ at levels $l > 1$ and $m > 1$.

(M2) Local Independence. It assumes independence conditional on all the latent dimensions and covariates, at different hierarchical levels; effectively modeling all the observed dependencies.

Model Assumptions (cont.)

Local Independence is defined as follows:

$$f\left(\mathrm{y}=1 \mid \mathrm{X}, \mathrm{W}, \boldsymbol{\Omega}\right)=\prod_{j=1}^{J} \prod_{d=1}^{D} \prod_{k=1}^{K} f\left(y_{jkd}=1 \mid \mathrm{X}, \mathrm{W}, \boldsymbol{\Omega}\right) \tag{15}$$

and comes from:

1. **Local item independence,**

$$f\left(y_{j..}=1 \mid \mathrm{X}, \mathrm{W}, \boldsymbol{\Omega}\right)=\prod_{d=1}^{D} \prod_{k=1}^{K} f\left(y_{jkd}=1 \mid \mathrm{X}, \mathrm{W}, \boldsymbol{\Omega}\right) \tag{16}$$

2. **Local individual independence,**

$$f\left(y_{.kd}=1 \mid \mathrm{X}, \mathrm{W}, \boldsymbol{\Omega}\right)=\prod_{j=1}^{J} f\left(y_{jkd}=1 \mid \mathrm{X}, \mathrm{W}, \boldsymbol{\Omega}\right) \tag{17}$$

Why bayesian?

- It is built on a simulation-based estimation method, therefore, it can handle all kinds of priors and data-generating processes [4].
- While the likelihood for the data and priors for the parameters are used to define the posterior sampling distributions, they can also be used in a generative way [15].
- Because the procedure integrates prior knowledge about the parameters, it can produce results even in scenarios where the Maximum Likelihood methods (ML) have issues of non-convergence or improper estimation [25, 4, 15]

There is nothing wrong?

- Somewhat-arbitrary decisions about the running of the chains (**solution:** Hamiltonian Monte Carlo (HMC) [2]).
- Convenient elicitation of priors(**solution:** prior predictive simulations and/or sensitivity analysis).
- It is hard to assess if a proper posterior investigation have been made [8] (**solution:** help of `Rhat`, `n_eff`, and change of posterior sampling geometry).
- It makes it hard to discover parameters' lack of identification [25] (**solution:** regularizing priors).
- The posterior sampling geometry of the model makes it hard to find proper solutions for the parameter space [2] (**solution:** change the posterior sampling geometry).
- The greater the complexity of the model, the harder it is to communicate/share and takes more time (**solution:** No solution, but it is a small "price" to pay).

Prior elicitation

Un-informative priors are not the solution:

1. Uninformative / weakly-informative latent prior:

$$\theta \sim N(0, 100) \qquad \theta \sim N(0, 1)$$
$$\text{logit}(p) = \theta \qquad \text{logit}(p) = \theta$$

2. Uninformative / weakly-informative hierarchical latent prior:

$$v \sim \log N(0, 3) \qquad v \sim \log N(0, 0.5)$$
$$\theta \sim N(0, v) \qquad \theta \sim N(0, v)$$
$$\text{logit}(p) = \theta \qquad \text{logit}(p) = \theta$$

Prior elicitation (cont.)

Un-informative priors are not the solution:

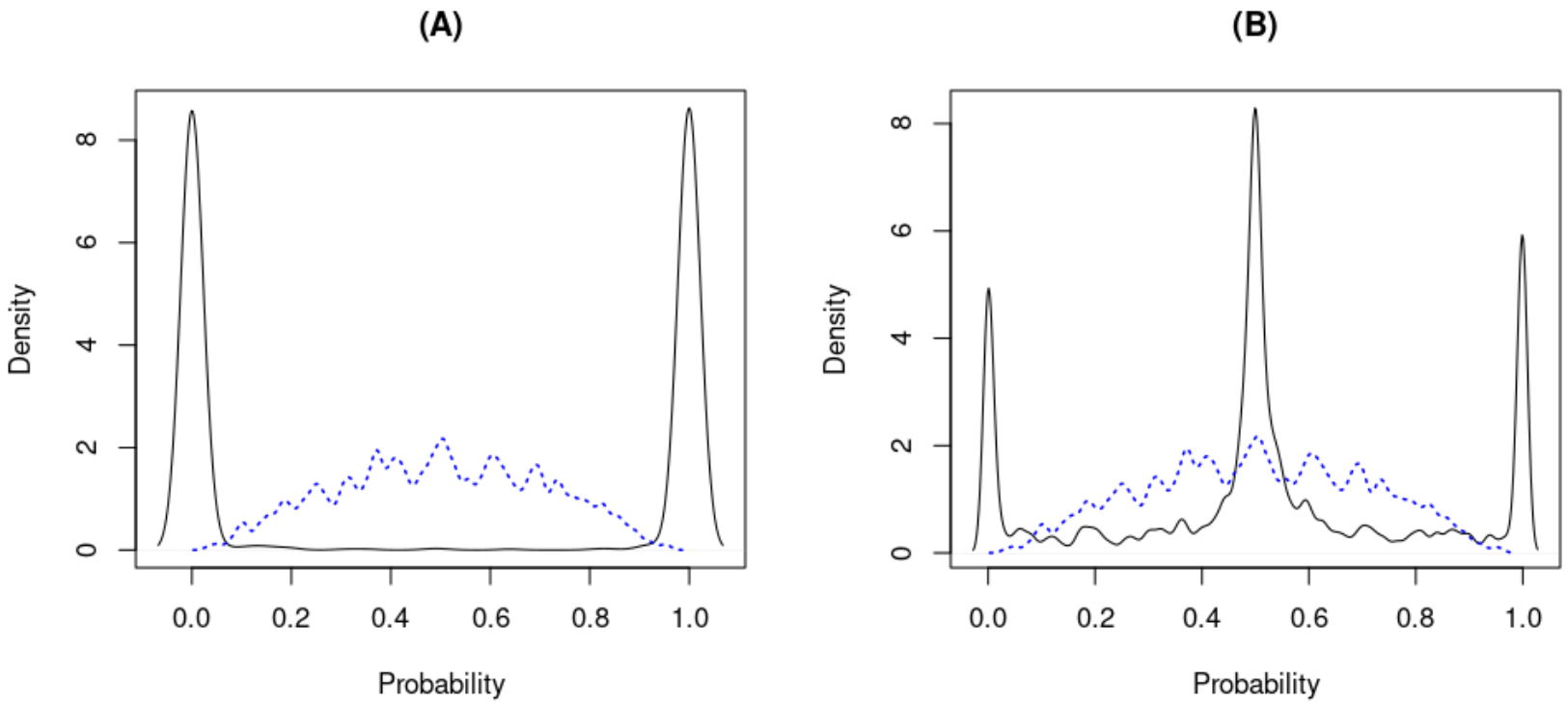


Figure: Prior predictive simulation. Examples of uninformative and mildly informative priors.

To center or not to center

Even the most simple hierarchical models present formidable pathologies, that no simple rotation/rescaling of the parameter can be performed to visit the posterior distribution properly [2].

Example, the devil's funnel [15]:

$$\begin{aligned} v &\sim N(0, 3) \\ \theta &\sim N(0, \exp(v)) \end{aligned} \tag{18}$$

Equation (18) describes a centered parametrization (CP)

To center or not to center (cont.)

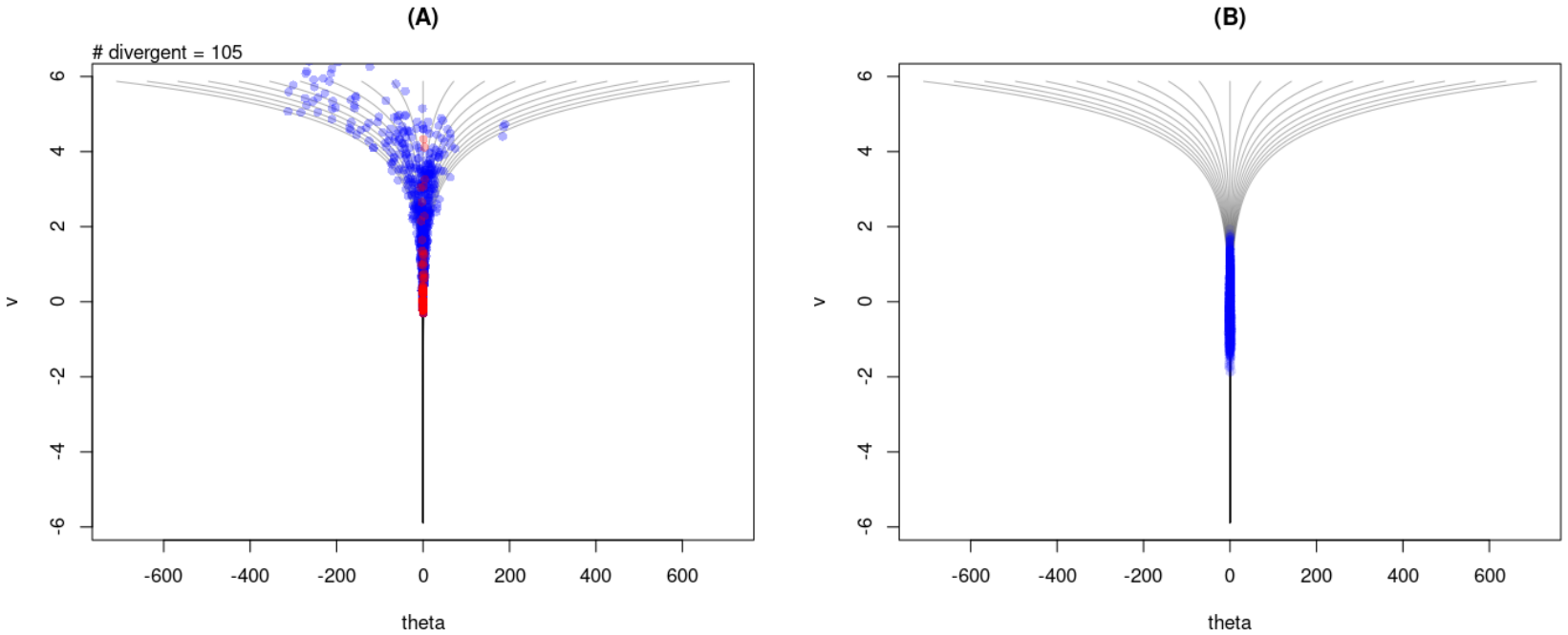


Figure: Posterior sampling geometry. Centered Parametrization.

To center or not to center (cont.)

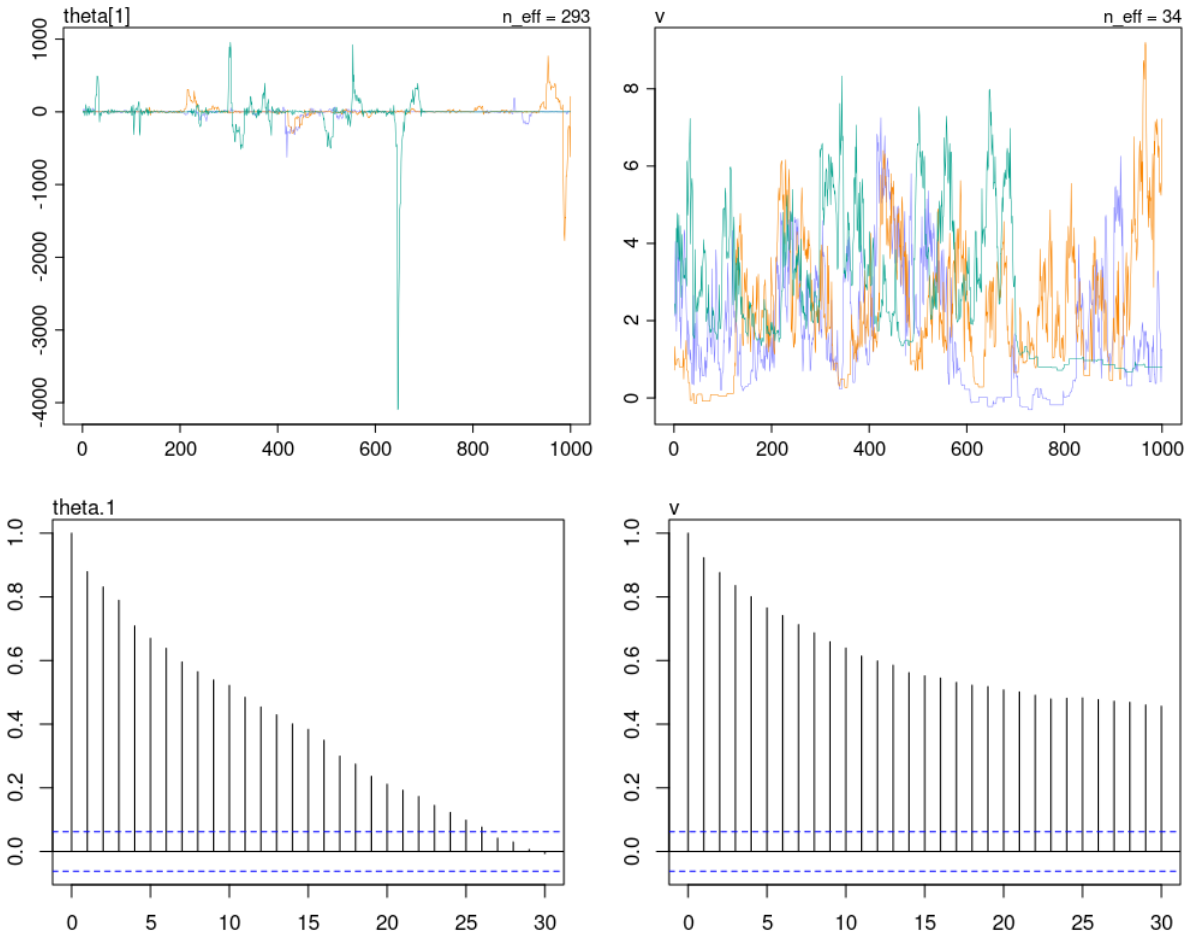


Figure: The Devil's funnel. Centered Parametrization. Trace and AFC plots.

To center or not to center (cont.)

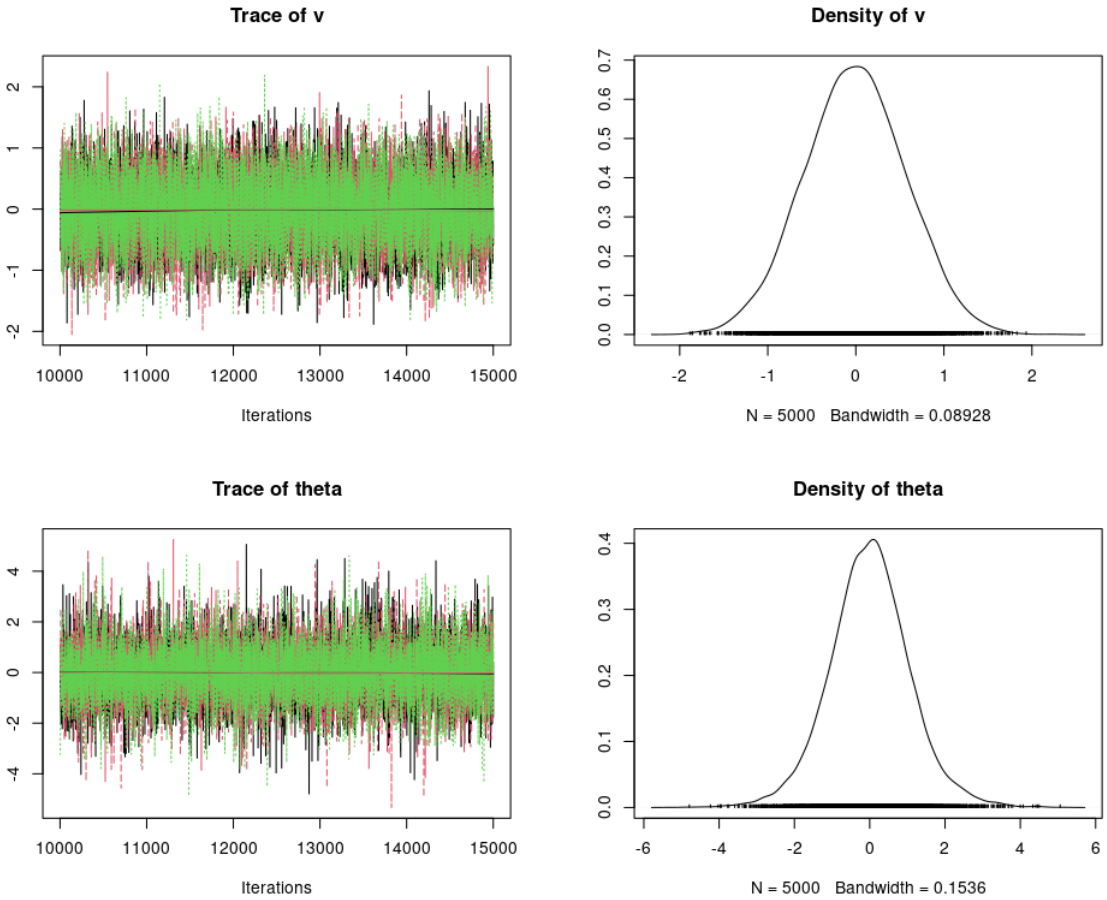


Figure: The Devil's funnel. Centered Parametrization implemented in JAGS.

To center or not to center (cont.)

How can we solve this?:

1. Adapt HMC warm-up (`adapt_delta`= 0.99).
2. Use **regularizing priors**
3. Use the non-centered parametrization.

To center or not to center (regularizing priors)

Consider the use of a regularizing prior on equation (18):

$$\begin{aligned} v &\sim N(0,1) \\ \theta &\sim N(0,\exp(v)) \end{aligned} \tag{19}$$

To center or not to center (regularizing priors, cont.)

versus figure 13.

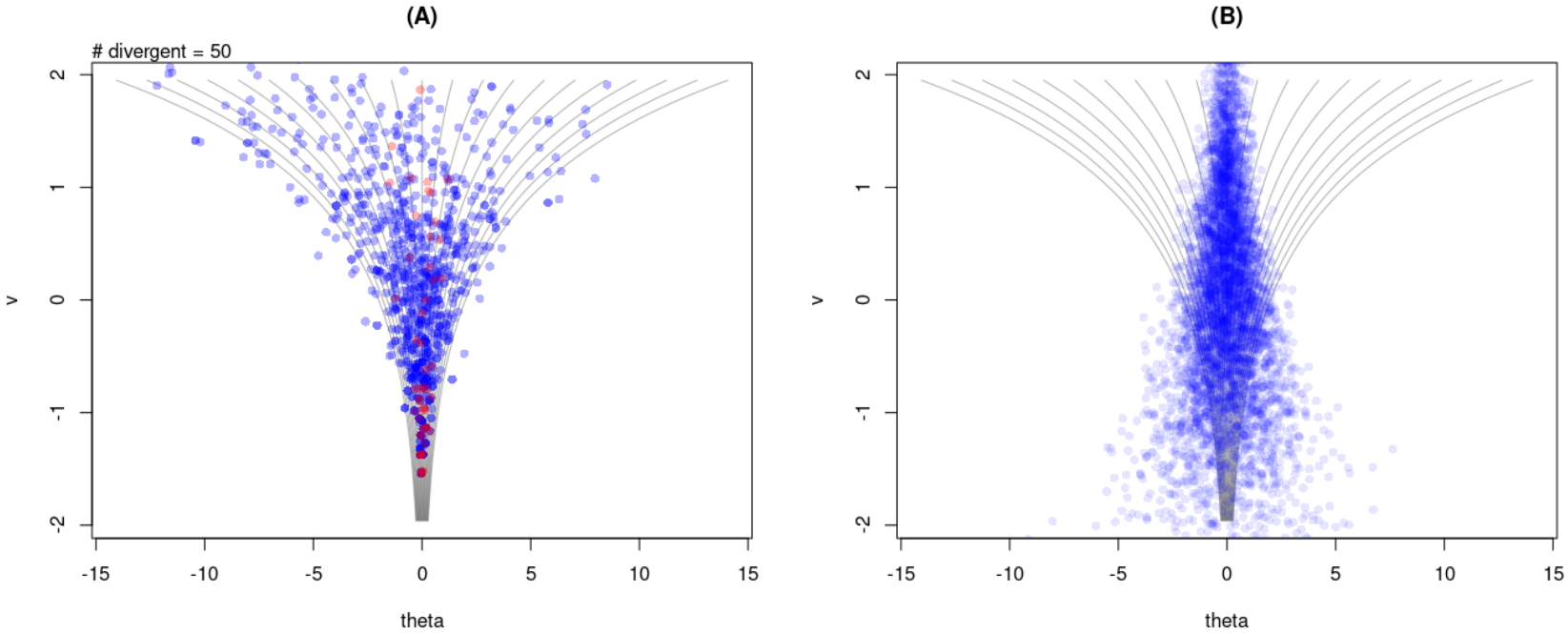


Figure: Posterior sampling geometry. Centered Parametrization with mildly informative priors.

To center or not to center (regularizing priors, cont.)

versus figure 14.

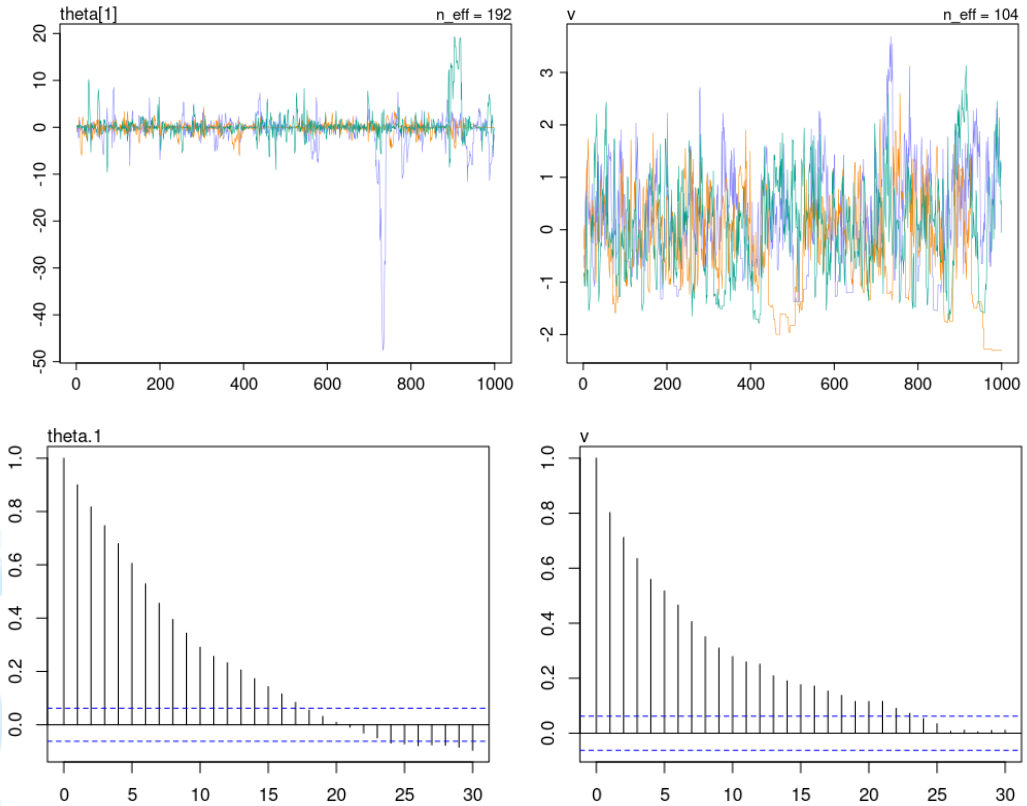


Figure: The Devil's funnel. Centered Parametrization with mildly informative priors. Trace and AFC plots.

To center or not to center (cont.)

How can we solve this?:

1. Adapt HMC warm-up (`adapt_delta`= 0.99).
2. Use regularizing priors
3. Use the **non-centered parametrization**.

To center or not to center (non-centered parametrization)

Changing the posterior sampling geometry means to modify equation (18) in the following way:

$$\begin{aligned} v &\sim N(0, 3) \\ z &\sim N(0, 1) \\ \theta &= \exp(v)\, z \end{aligned} \tag{20}$$

To center or not to center (non-centered parametrization, cont.)

versus figure 13 and 16.

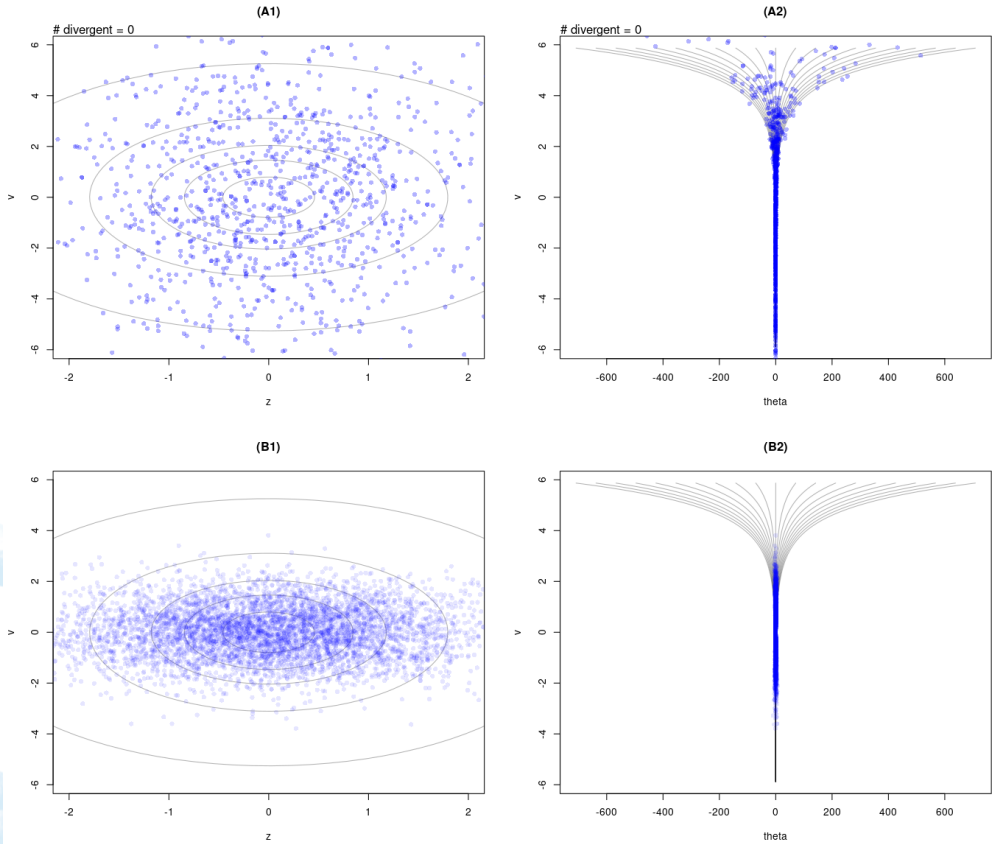


Figure: Posterior sampling geometry. Non-Centered Parametrization.

To center or not to center (non-centered parametrization, cont.)

versus figure 14 and 17.

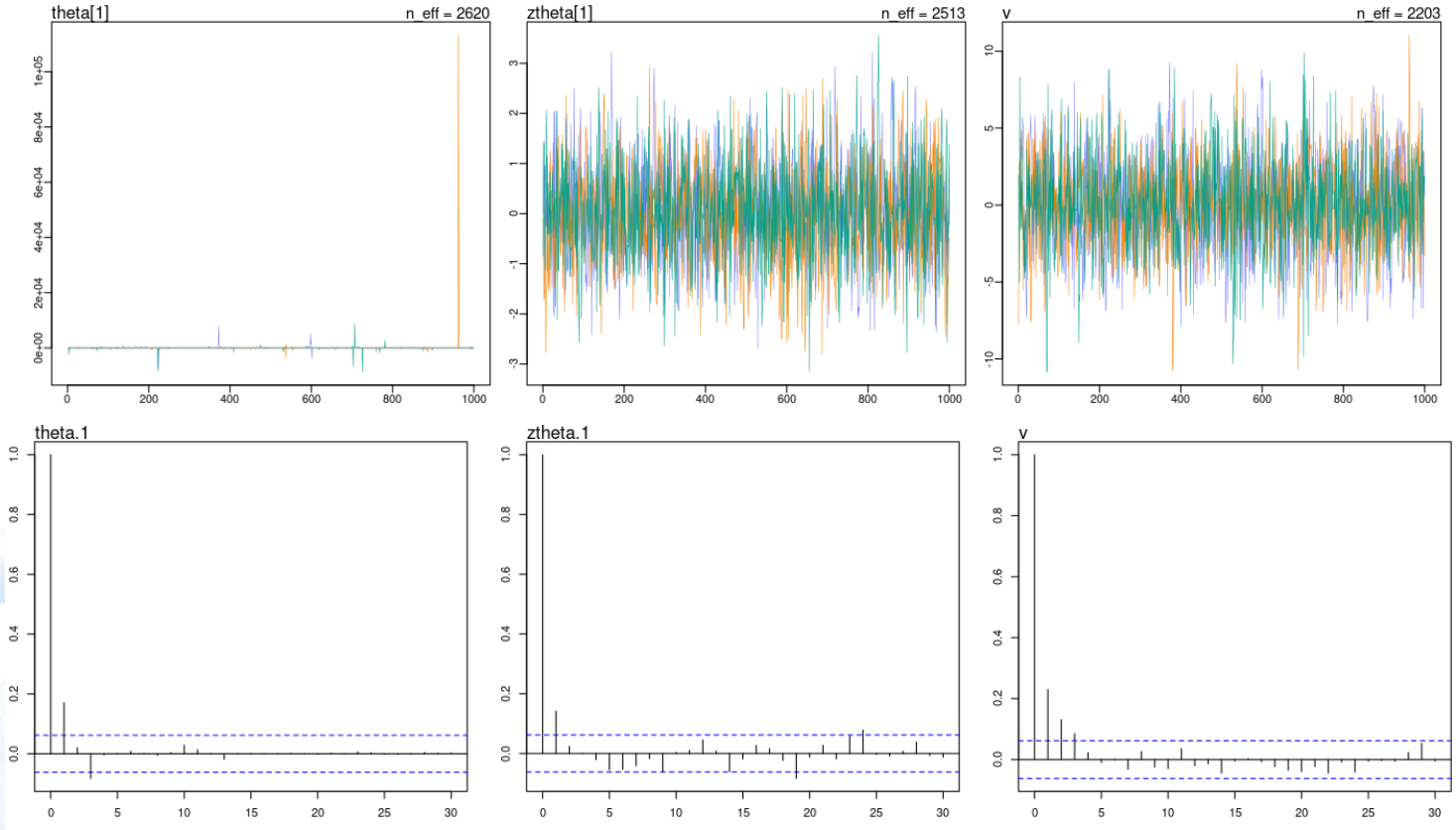


Figure: The Devil's funnel. Non-centered Parametrization. Trace and AFC plots.

Simulation (study design)

A $3 \times 2 \times 2$ fractional factorial design:

- Three different samples sizes to generate the data under analysis: 500, 250, and 100.
- Two parametrization of the models: CP and NCP.
- Two models of interest: the first- and second-order latent variable model.

Ten (10) data sets were generated for each study condition. Each data set resembled responses to 25 binary scored items, conforming to the SOLV model defined in figure 21. The model was motivated by the hypothesized structure of the reading comprehension sub-test, from the Peruvian public teaching career national assessment

Simulation
(study design, cont.)

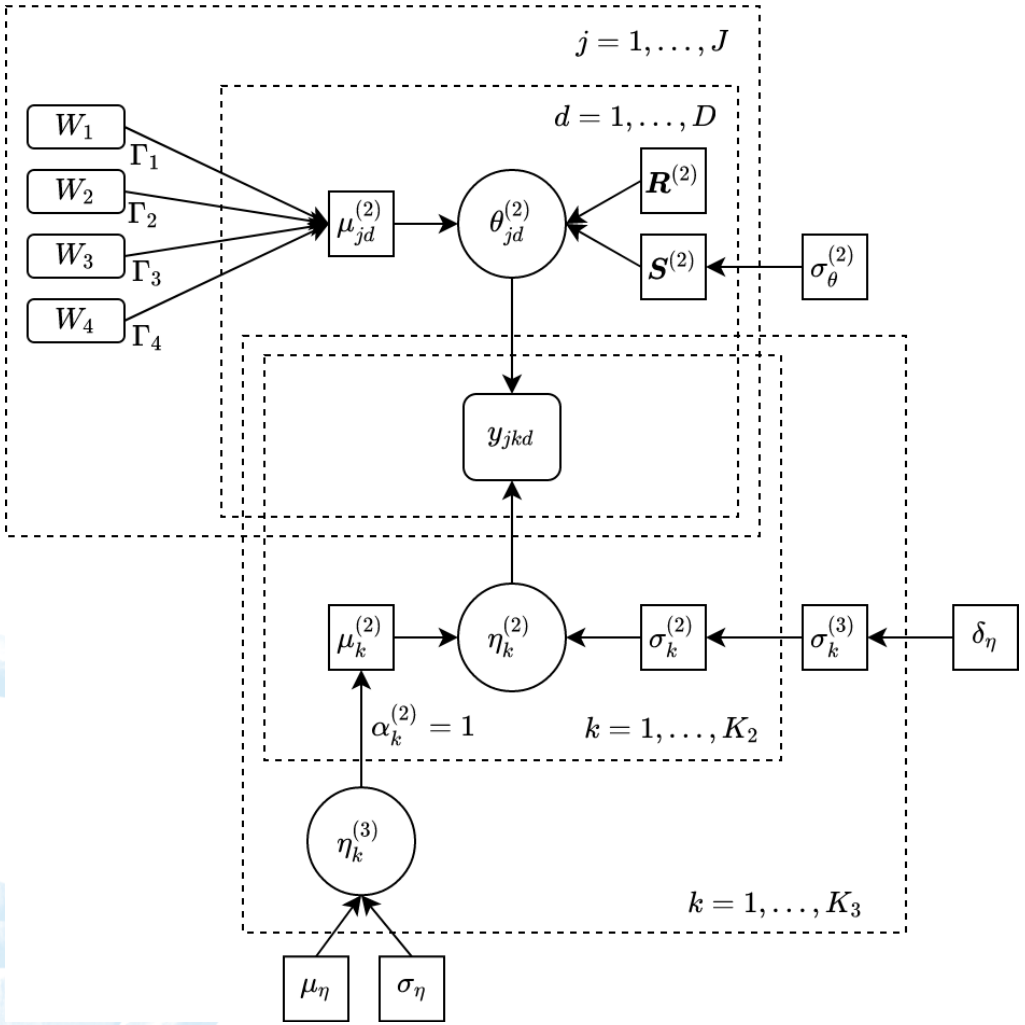


Figure: Directed Acyclig Graph (DAG). First-order latent variable model (SOLV).

Simulation
(study design, cont.)

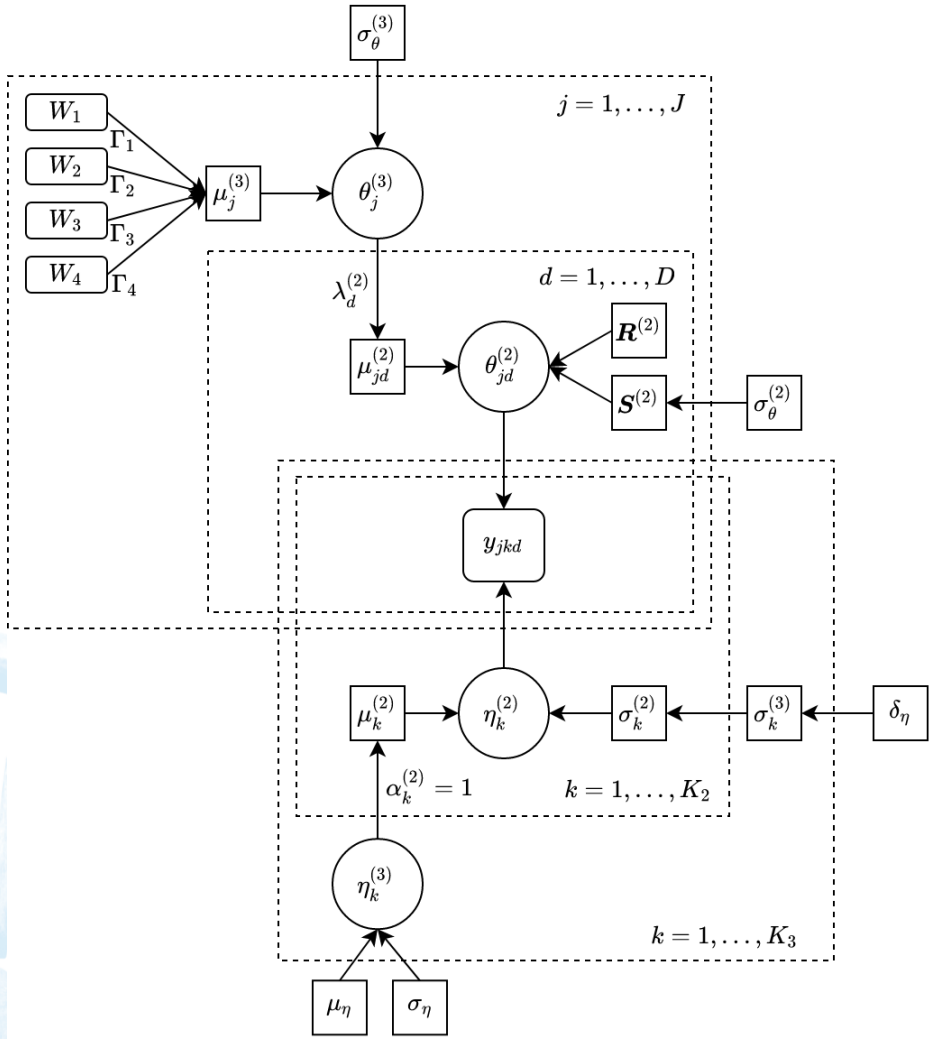


Figure: Directed Acyclig Graph (DAG). Second-order latent variable model (SOLV)

Simulation
(evaluation criteria)

1. **Performance.** Trace, trank and ACF plots with support of `Rhat` and `n_eff` statistics developed by Gelman et al. [7] (pp. 284 – 287).
2. **Recovery capacity.** we used the between replica root mean squared error (RMSE_B).
3. **Retrodictive accuracy.** we used the average within $\overline{\text{RMSE}}_W$ and between prediction root mean squared error RMSE_B, of the responses' predictive proportion $\hat{p}$, versus the observed proportion p.

Likelihood, priors and hyper-priors (centered parametrization)

$$y_{jkd} \sim \text{Bernoulli}(\pi_{jkd}) \tag{21}$$

$$\text{logit}(\pi_{jkd}) = v_{jkd} \tag{22}$$

$$v_{jkd} = \theta^{(2)}_{jd} - \eta^{(2)}_{k} \tag{23}$$

$$\boldsymbol{\theta}^{(2)}_{j} = \left[\theta^{(2)}_{j1}, \theta^{(2)}_{j2}, \theta^{(2)}_{j3}\right] \tag{24}$$

$$\boldsymbol{\theta}^{(2)}_{j} \sim \text{MVNormal}\left(\boldsymbol{\mu}^{(2)}_{j}, \Sigma^{(2)}\right) \tag{25}$$

$$\Sigma^{(2)} = \text{S}^{(2)} \cdot \text{R}^{(2)} \cdot \text{S}^{(2)} \tag{26}$$

$$\text{S}^{(2)} = \boldsymbol{\sigma}^{(2)}_{\theta} \text{I} \tag{27}$$

Likelihood, priors and hyper-priors (centered parametrization, cont.)

For the FOLV model:

$$\boldsymbol{\mu}_j^{(2)} = \left[\mu_{j1}^{(2)},\ \mu_{j2}^{(2)}, \mu_{j3}^{(2)}\right] \tag{28}$$

$$\mu_{jd}^{(2)} = \Gamma_0 + \Gamma_1 W_{1j} + \Gamma_2 (W_{2j} - W_{2\text{min}}) + \Gamma_3 W_{3j} + \Gamma_4 W_{4j} \tag{29}$$

Likelihood, priors and hyper-priors (centered parametrization, cont.)

For the SOLV model:

$$\boldsymbol{\mu}_j^{(2)} = \left[\mu_{j1}^{(2)},\ \mu_{j2}^{(2)}, \mu_{j3}^{(2)}\right] \tag{30}$$

$$\boldsymbol{\lambda}^{(2)} = \left[\lambda_1^{(2)},\ \lambda_2^{(2)}, \lambda_3^{(2)}\right] \tag{31}$$

$$\mu_{jd}^{(2)} = \lambda_d^{(2)}\ \theta_j^{(3)} \tag{32}$$

$$\theta_j^{(3)} \sim \text{Normal}\left(\mu_j^{(3)}, \sigma_\theta^{(3)}\right) \tag{33}$$

$$\mu_j^{(3)} = \Gamma_0 + \Gamma_1 W_{1j} + \Gamma_2 (W_{2j} - W_{2\text{min}}) + \Gamma_3 W_{3j} + \Gamma_4 W_{4j} \tag{34}$$

Likelihood, priors and hyper-priors (centered parametrization, cont.)

For the items:

$$\eta_k^{(2)} \sim \text{Normal}\left(\mu_k^{(2)}, \sigma_k^{(2)}\right) \tag{35}$$

$$\mu_k^{(2)} = \boldsymbol{\eta}^{(3)}\mathrm{A} \tag{36}$$

$$\sigma_k^{(2)} = \boldsymbol{\sigma}^{(3)}\mathrm{A} \tag{37}$$

$$\boldsymbol{\eta}^{(3)} = [\eta_1^{(3)}, \eta_2^{(3)}, \eta_3^{(3)}, \eta_4^{(3)}, \eta_5^{(3)}] \tag{38}$$

$$\boldsymbol{\sigma}^{(3)} = [\sigma_1^{(3)}, \sigma_2^{(3)}, \sigma_3^{(3)}, \sigma_4^{(3)}, \sigma_5^{(3)}] \tag{39}$$

$$\eta_k^{(3)} \sim \text{Normal}\left(\mu_\eta, \sigma_\eta\right) \tag{40}$$

$$\sigma_k^{(3)} \sim \text{Exponential}\left(\delta_\eta\right) \tag{41}$$

Likelihood, priors and hyper-priors (centered parametrization, cont.)

Remaining priors and hyper-priors

$$R^{(2)} \sim \text{LkjCorrelation}(2) \tag{42}$$
$$\Gamma_{1c} \sim \text{Normal}(0, 0.5) \tag{43}$$
$$\Gamma_{2} \sim \text{Normal}(0, 0.5) \tag{44}$$
$$\Gamma_{3c} \sim \text{Normal}(0, 1) \tag{45}$$
$$\Gamma_{4c} \sim \text{Normal}(0, 0.5) \tag{46}$$

Likelihood, priors and hyper-priors (Non-centered parametrization)

Under the NCP, equation (35) was re-defined as follows:

$$\eta_k^{(2)} = \mu_k^{(2)} + \sigma_k^{(2)} z_k^{(2)} \tag{47}$$

$$z_k^{(2)} \sim \text{Normal}(0, 1) \tag{48}$$

Equation (25) was re-defined as follows:

$$\boldsymbol{\theta}_j^{(2)} = \boldsymbol{\mu}_j^{(2)} + \mathrm{S}^{(2)} \cdot \mathrm{L}_{\Sigma}^{(2)} \cdot (\mathrm{z}_j \mathrm{I}) \tag{49}$$

$$\mathrm{z}_j = [z_{j1}, \ldots, z_{jd}]^T \tag{50}$$

$$z_{jd} \sim \text{Normal}(0, 1) \tag{51}$$

$$\mathrm{L}_{\Sigma}^{(2)} \sim \text{LKJCorrelationCholesky}(2) \tag{52}$$

Likelihood, priors and hyper-priors (Non-centered parametrization, cont.)

Finally, equation (33) was re-defined as follows:

$$\theta_j^{(3)} = \mu_j^{(3)} + \sigma_\theta^{(3)} z_j \tag{53}$$

$$z_j \sim \text{Normal}(0, 1) \tag{54}$$

Identification

We used the unit variance identification scheme (UVI), that is, to set the scale of the higher-order dimension and sub-dimensions to one:

- $\sigma_\theta^{(3)} = 1$
- $\mathsf{S}^{(2)} = \boldsymbol{\sigma}_\theta^{(2)}\mathsf{I}$ with $\boldsymbol{\sigma}_\theta^{(2)} = [1, 1, 1]^T$
- $\mu_\eta = 0$, $\sigma_\eta = 1$, $\delta_\eta = 2$

The second turned the covariance matrix into a correlation, i.e. $\Sigma^{(2)} = \mathsf{R}^{(2)}$.

Prior predictive investigation

From two perspectives:

- the IRT perspective
- the outcome perspective

Prior predictive investigation (cont.)

From the IRT perspective:

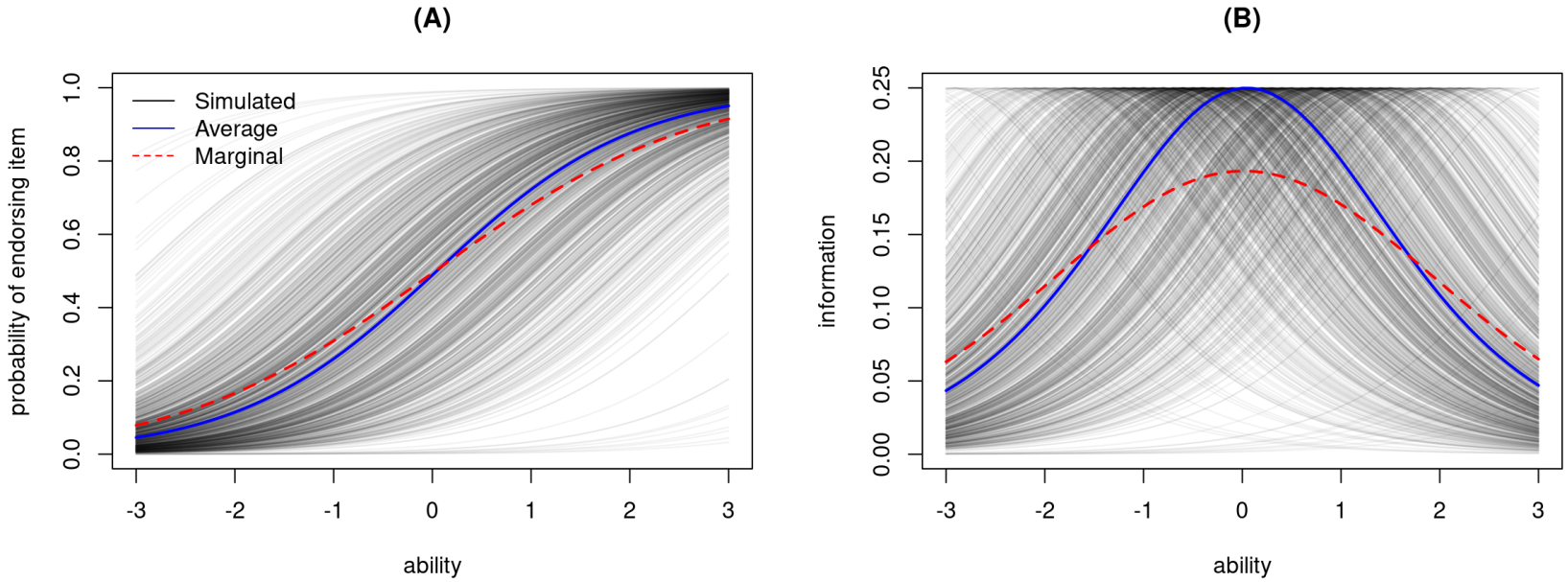


Figure: First-order latent variable model (FOLV). (A) Item Characteristics Curve, ICC. (B) Item Information Function, IIF.

Prior predictive investigation (cont.)

From the outcome perspective:

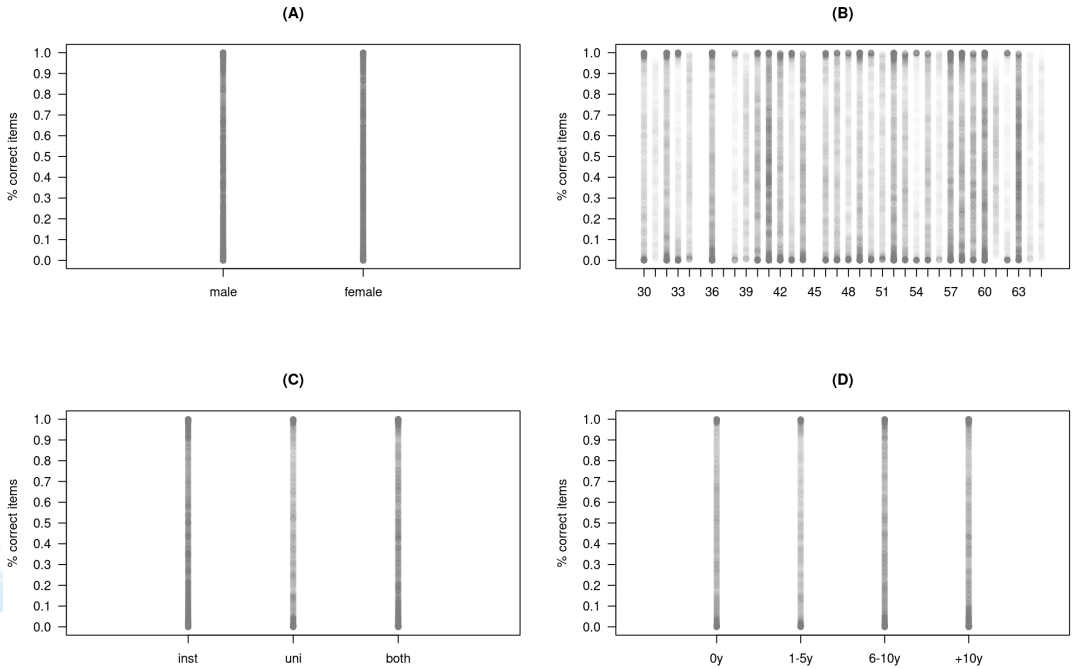


Figure: First-order latent variable model (FOLV). Aggregated endorsement rate per simulated covariate: (A) gender, (B) age, (C) education, and (D) experience.

Running time

	Parametrization	Sample	Time (min.) mean	min	max
1	CP	100	4.80	1.89	7.56
2	CP	250	7.85	5.60	16.57
3	CP	500	19.15	17.20	22.53
4	NCP	100	1.94	1.74	2.23
5	NCP	250	7.18	6.78	8.40
6	NCP	500	22.38	20.12	28.62

Table: First-order latent variable model (FOLV). Running time statistics.

Running time (cont.)

	Parametrization	Sample	Time (min.) mean	min	max
1	CP	100	4.04	1.39	8.55
2	CP	250	7.13	5.22	11.49
3	CP	500	16.14	14.45	19.73
4	NCP	100	1.87	1.62	2.28
5	NCP	250	5.92	5.20	6.79
6	NCP	500	16.58	13.37	18.26

Table: Second-order latent variable model (SOLV). Running time statistics.

Application (model fit)

They try to approximate the out-of-sample KL-divergence [12]:

	Model	Param.	WAIC	lppd	penalty
1	FOLV	CP	54,632.4	-25,429.7	1,886.5
2	FOLV	NCP	54,631.7	-25,427.6	1,888.3
3	SOLV	CP	54,610.3	-25,348.4	1,956.7
4	SOLV	NCP	54,614.7	-25,337.9	1,969.4

Table: Model fit. Widely Applicable Information Criterion (WAIC).

Application
(model fit, cont.)

	Model	Param.	PSIS	lppd	penalty
1	FOLV	CP	54,657.4	-27,328.7	1,904.4
2	FOLV	NCP	54,656.9	-27,328.5	1,898.1
3	SOLV	CP	54,627.3	-27,313.7	1,940.1
4	SOLV	NCP	54,642.5	-27,321.2	1,990.2

Table: Model fit. Pareto-smoothed importance sampling cross-validation (PSIS).

References I

[1] Baker, F. [2001]. The basic of item response theory, *Technical report*, ERIC Clearinghouse on Assessment and Evaluation.

[2] Betancourt, M. and Girolami, M. [2012]. Hamiltonian monte carlo for hierarchical models.
url: arxiv.org/abs/1312.0906v1.

[3] Chen, W. and Thissen, D. [1997]. Local dependence indexes for item pairs using item response theory, *Journal of Educational and Behavioral Statistics* **22**(3): 265–289.
doi: https://doi.org/10.3102/10769986022003265.

[4] Fox, J. [2010]. *Bayesian Item Response Modeling, Theory and Applications*, Statistics for Social and Behavioral Sciences, fienberg, s. and van der linden, w. edn, Springer Science+Business Media, LLC.

References II

[5] Gelfand, A., Sahu, S. and Carlin, B. [1995]. Efficient parametrisations for normal linear mixed models, *Biometrika* **82**(3): 479–488.
doi: https://doi.org/10.1093/biomet/82.3.479.

[6] Gelfand, A., Sahu, S. and Carlin, B. [1996]. Efficient parameterizations for generalised linear models (with discussion), *in* J. Bernardo, J. Berger, A. Dawid and a. Smith (eds), *Bayesian Statistics*, Vol. 5, pp. 165–180.

[7] Gelman, A., Carlin, J., Stern, H., Dunson, D., Vehtari, A. and Rubin, D. [2014]. *Bayesian Data Analysis*, Texts in Statistical Science, third edn, Chapman and Hall/CRC.

[8] Gelman, A. and Rubin, D. [1996]. Markov chain monte carlo methods in biostatistics, *Statistical Methods in Medical Research* **5**(4): 339–355.
doi: https://doi.org/10.1177/096228029600500402.

References III

[9] Hambleton, R. and Swaminathan, H. [1991]. *Item Response Theory*, Evaluation in Education and Human Services series, Springer Science+Business Media, LLC.

[10] Hambleton, R., Swaminathan, H. and Rogers, H. [1991]. *Fundamentals of Item Response Theory*, SAGE Publications Inc.

[11] Jiao, H., Kamata, A., Wang, S. and Jin, Y. [2012]. A multilevel testlet model for dual local dependence, *Journal of Educational Measurement* **49**(1): 82–100. **doi:** https://doi.org/10.1111/j.1745-3984.2011.00161.x.

[12] Kullback, S., . L. R. [1951]. On information and sufficiency, *The Annals of Mathematical Statistics* **22**(1): 79–86. **url:** http://www.jstor.org/stable/2236703.

[13] Lord, F. and Novik, M. [2008]. *Statistical Theories of Mental Test Scores*, Information Age Publishing.

References IV

[14] McCullagh, P. and Nelder, J. [1989]. *Generalized Linear Models*, Monographs on Statistics Applied Probability, Chapman Hall/CRC Press.

[15] McElreath, R. [2020]. *Statistical Rethinking: A Bayesian Course with Examples in R and Stan*, Texts in Statistical Science, 2 edn, Chapman and Hall/CRC.
doi: https://doi.org/10.1201/9780429029608.

[16] Nelder, J. and Wedderburn, W. [1972]. Generalized linear models, *Royal Statistical Society* **135**(3): 370–384.
doi: https://doi.org/10.2307/2344614.
url: https://www.jstor.org/stable/2344614.

[17] Papaspiliopoulos, O., Roberts, G. and Skold, M. [2003]. Non-centered parameterisations for hierarchical models and data augmentation, *Bayesian Statistics* **7**: 307–326.
url: http://econ.upf.edu/ omiros/papers/val7.pdf.

References V

[18] Papaspiliopoulos, O., Roberts, G. and Skold, M. [2007]. A general framework for the parametrization of hierarchical models, *Statistical Science* **22**(1): 59–73.
doi: https://www.doi.org/10.1214/088342307000000014.

[19] Patz, R. J. and Junker, B. W. [1999]. A straightforward approach to markov chain monte carlo methods for item response models, *Journal of Educational and Behavioral Statistics* **24**(2): 146–178.
doi: 10.3102/10769986024002146.

[20] Rabe-Hesketh, S., Skrondal, A. and Pickles, A. [2004a]. Generalized multilevel structural equation modeling, *Psychometrika* **69**(2): 167–190.
doi: https://www.doi.org/10.1007/BF02295939.

References VI

[21] Rabe-Hesketh, S., Skrondal, A. and Pickles, A. [2004b]. Maximum likelihood estimation of limited and discrete dependent variable models with nested random effects, *Journal of Econometrics* **128**(2): 301–323.
doi: https://www.doi.org/10.1016/j.jeconom.2004.08.017.
url:
http://www.sciencedirect.com/science/article/pii/S030440760400159

[22] Rasch, G. [1980]. *Probabilistic Models for Some Intelligence and Attainment Tests*, University of Chicago Press.

[23] Raudenbush, S. and Bryk, A. [2002]. *Hierarchical linear models: Applications and data analysis methods (Vol. 1)*, Advanced Quantitative Techniques in the Social Sciences, SAGE Publications Inc.

References VII

[24] Reckase, M. [2009]. *Multidimensional Item Response Theory*, Statistics for Social and Behavioral Sciences, Springer Science+Business Media, LLC.

[25] Skrondal, A. and Rabe-Hesketh, S. [2004]. *Generalized Latent Variable Modeling: Multilevel, Longitudinal, and Structural Equation Models*, Interdisciplinary Statistics, Chapman Hall/CRC Press.

[26] Stan Development Team. [2021]. *Stan Modeling Language Users Guide and Reference Manual, version 2.26*, Vienna, Austria.
url: https://mc-stan.org.

[27] Wainer, H., Bradlow, E. and Wang, X. [2007]. *Testlet response theory and its applications*, Cambridge University Press.

References VIII

[28] Yen, W. [1984]. Effects of local item dependence on the fit and equating performance of the three-parameter logistic model, *Applied Psychological Measurement* **8**(2): 125–145. **doi:** https://doi.org/10.1177/014662168400800201.

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