Transform Methods - So Far

Recall, for LTI system:

Total Response = Zero-Input Response + Zero-State Response

Phasor Analysis: Gives zero-state response only when input is a sinusoid, AND after all transients have died down (steady state)

Bilateral Laplace Transform: Gives zero-state response for any* input

[*Any input that has a Laplace Transform]

- **Unilateral Laplace Transform**: Gives total response for any* input
  - Requires initial conditions (@ $t=0$)
  - Useful for time $t \geq 0$

Definition of Unilateral Laplace Transform

$$\mathcal{UL}\{x(t)\} = X(s) = \int_{t=0^-}^{\infty} x(t)\, e^{-st}\, dt$$

- Differs from Bilateral Laplace only by lower limit of integral
  - If $x(t<0)=0$
    $\Longrightarrow \mathcal{L}\{x(t)\} = \mathcal{UL}\{x(t)\}$

- Signals differing for $t<0$, but same for $t \geq 0$ have same $\mathcal{UL}\{\cdot\}$.

- $\mathcal{UL}$ used with right-sided (usually causal) signals
  $\Longrightarrow$ ROC is $Re(s) > \sigma_0$
  $\Longrightarrow$ Seldom bother to specify ROC

- <u>Most</u> (not all) transforms, properties same as bilateral transform

  - Key difference: Derivative property

$\mathcal{UL}$ - Transform of Exponential

- Find $\mathcal{UL}\{e^{-\lambda t}u(t)\}$

- Sol'n

$$X(s) = \int_{t=0^-}^{\infty} e^{-\lambda t} e^{-st}\, dt$$

$$= \int_{t=0^-}^{\infty} e^{-t(s+\lambda)}\, dt = \left. \frac{e^{-t(s+\lambda)}}{-(s+\lambda)} \right|_{t=0}^{\infty}$$

$$= \frac{e^{-\infty(s+\lambda)} - e^{0}}{-(s+\lambda)} = \frac{e^{-\infty(\sigma+\lambda)} \cdot e^{-\infty \cdot j\omega} - 1}{-(s+\lambda)}$$

Note 1: $\left| e^{-j\omega\infty} \right| \leq 1$

Note 2: $e^{-\infty(\sigma+\lambda)} \rightarrow 0$ for $\sigma > -\lambda$, else diverges

So,

$$X(s) = \frac{0-1}{-(s+\lambda)}, \quad Re(s) > -\lambda$$

$$\Rightarrow \boxed{X(s) = \frac{1}{s+\lambda}}$$

Don't write ROC

Unilateral Laplace Transform

UL-Transform of Impulse

- Find $\mathcal{UL}\{\delta(t)\}$

- Sol'n

$$X(s) = \int_{t=0^-}^{\infty} \delta(t)\, e^{-st}\, dt$$

Impulse included in integral

$$= e^{-s \cdot 0}$$

$$X(s) = 1, \quad \forall s$$

For UL, don't write ROC

$$\Downarrow$$

$$\boxed{X(s) = 1}$$

Unilateral Laplace Transform

Common Unilateral Laplace Transforms

A Short Table of (Unilateral) Laplace Transforms

$x(t)$	$X(s)$
$\delta(t)$	1
$u(t)$	$\dfrac{1}{s}$
$tu(t)$	$\dfrac{1}{s^2}$
$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$
$e^{\lambda t} u(t)$	$\dfrac{1}{s-\lambda}$
$te^{\lambda t} u(t)$	$\dfrac{1}{(s-\lambda)^2}$
$t^n e^{\lambda t} u(t)$	$\dfrac{n!}{(s-\lambda)^{n+1}}$
$\cos bt\, u(t)$	$\dfrac{s}{s^2+b^2}$
$\sin bt\, u(t)$	$\dfrac{b}{s^2+b^2}$
$e^{-at} \cos bt\, u(t)$	$\dfrac{s+a}{(s+a)^2+b^2}$
$e^{-at} \sin bt\, u(t)$	$\dfrac{b}{(s+a)^2+b^2}$
$re^{-at} \cos(bt+\theta)\, u(t)$	$\dfrac{(r\cos\theta)s + (ar\cos\theta - br\sin\theta)}{s^2+2as+(a^2+b^2)}$
$re^{-at} \cos(bt+\theta)\, u(t)$	$\dfrac{0.5re^{j\theta}}{s+a-jb} + \dfrac{0.5re^{-j\theta}}{s+a+jb}$
$re^{-at} \cos(bt+\theta)\, u(t)$ $r = \sqrt{\dfrac{A^2c+B^2-2ABa}{c-a^2}}$ $\theta = \tan^{-1}\left(\dfrac{Aa-B}{A\sqrt{c-a^2}}\right)$ $b = \sqrt{c-a^2}$	$\dfrac{As+B}{s^2+2as+c}$
$e^{-at}\left[A\cos bt + \dfrac{B-Aa}{b}\sin bt\right] u(t)$ $b=\sqrt{c-a^2}$	$\dfrac{As+B}{s^2+2as+c}$

[Lathi, Table 4.1]

Unilateral Laplace Transform

Table of UL-Transform Properties

TABLE 4.2 The Laplace Transform Properties (Unilateral)

Operation	$x(t)$	$X(s)$
Addition	$x_1(t) + x_2(t)$	$X_1(s) + X_2(s)$
Scalar multiplication	$kx(t)$	$kX(s)$
Time differentiation	$\frac{dx}{dt}$	$sX(s) - x(0^-)$
	$\frac{d^2x}{dt^2}$	$s^2X(s) - sx(0^-) - \dot{x}(0^-)$
	$\frac{d^3x}{dt^3}$	$s^3X(s) - s^2x(0^-) - s\dot{x}(0^-) - \ddot{x}(0^-)$
	$\frac{d^nx}{dt^n}$	$s^nX(s) - \sum_{k=1}^{n} s^{n-k}x^{(k-1)}(0^-)$
Time integration	$\int_{0^-}^{t} x(\tau)\,d\tau$	$\frac{1}{s}X(s)$
	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(t)\,dt$
Time shifting	$x(t - t_0)u(t - t_0)$	$X(s)e^{-st_0} \quad t_0 \geq 0$
Frequency shifting	$x(t)e^{s_0t}$	$X(s - s_0)$
Frequency differentiation	$-tx(t)$	$\frac{dX(s)}{ds}$
Frequency integration	$\frac{x(t)}{t}$	$\int_{s}^{\infty} X(z)\,dz$
Scaling	$x(at), a \geq 0$	$\frac{1}{a}X\left(\frac{s}{a}\right)$
Time convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$
Frequency convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi j}X_1(s) * X_2(s)$
Initial value	$x(0^+)$	$\lim_{s\to\infty} sX(s) \quad (n > m)$
Final value	$x(\infty)$	$\lim_{s\to 0} sX(s) \quad$ [poles of $sX(s)$ in LHP]

[Lathi]

Note difference from bilateral:

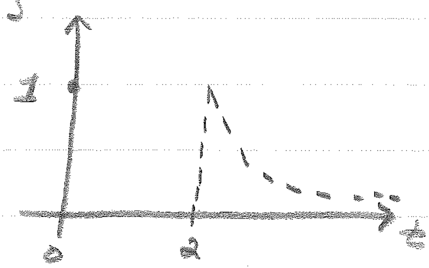
$$\boxed{\frac{dx(t)}{dt} \longleftrightarrow sX(s) - x(0^-)}$$

Initial condition!!

Unilateral Laplace Transform

Example: Time Shifting Property

∘ Find $\mathcal{UL}\{ e^{-3(t-2)} u(t-2) \}$



A) From Definition:

$$X(s) = \int_{t=0^-}^{\infty} e^{6} e^{-3t} u(t-2)\, e^{-st}\, dt$$

$$= e^{6} \int_{t=2}^{\infty} e^{-t(s+3)}\, dt = \left. \frac{e^{6} e^{-t(s+3)}}{-(s+3)} \right|_{t=2}^{\infty}$$

$$= \frac{e^{6} e^{-\infty(s+3)} - e^{6} e^{-2(s+3)}}{-(s+3)} = \frac{e^{6} e^{-6} e^{-2s}}{s+3}$$

Goes to zero for $Re(s) > -3$

$$= \boxed{\frac{e^{-2s}}{s+3} = X(s)}$$

B) i) $e^{-\lambda t} u(t) \longleftrightarrow \dfrac{1}{s+\lambda}$

ii) $x(t-T)\, u(t-T) \longleftrightarrow e^{-sT} X(s)$

$(T \geq 0)$

$$\boxed{X(s) = \frac{e^{-2s}}{s+3}}$$

Frequency Differentiation Property

° **Know**: $t \cdot x(t) \longleftrightarrow -\dfrac{d}{ds} X(s)$

° **If** $e^{-\lambda t} u(t) \longleftrightarrow \dfrac{1}{s+\lambda}$

Find $\mathcal{UL}\{t e^{-\lambda t} u(t),\ t^2 e^{-\lambda t} u(t)\}$

Sol'n

a) $\mathcal{UL}\{t e^{-\lambda t} u(t)\} = -\dfrac{d}{ds}\left\{\dfrac{1}{s+\lambda}\right\} = -\dfrac{d}{ds}\{(s+\lambda)^{-1}\}$

$$= +1(s+\lambda)^{-2} \cdot 1 = \boxed{\frac{1}{(s+\lambda)^2}}$$

b) $\mathcal{UL}\{t^2 e^{-\lambda t} u(t)\} = -\dfrac{d}{ds}\left\{\dfrac{1}{(s+\lambda)^2}\right\} = -\dfrac{d}{ds}\{(s+\lambda)^{-2}\}$

$$= +2(s+\lambda)^{-3} \cdot 1 = \boxed{\frac{2}{(s+\lambda)^3}}$$

$$\Rightarrow t^N e^{-\lambda t} u(t) \longleftrightarrow \frac{N!}{(s+\lambda)^{N+1}}$$

MATLAB "laplace()" Command

- Symbolic Toolbox
- Unilateral Transform (NO ROC provided)

Example

```
>>syms t a
>> laplace( exp(-a*t) )
ans =
1/(s+a)
```

MATLAB DEMO

laplace_demo.txt

```
>> syms t a w
>>
>> laplace( exp(-a*t) )

ans =

1/(s+a)


>> laplace( exp(-4*t) )

ans =

1/(s+4)
```

$e^{-\lambda t} u(t) \longleftrightarrow \frac{1}{s+\lambda}$

NOT MATTER

```
>> laplace( sym(1) )

ans =

1/s
```

$u(t) \longleftrightarrow \frac{1}{s}$

```
>> laplace( t )

ans =

1/s^2
```

$t\,u(t) \longleftrightarrow \frac{1}{s^2}$

```
>> laplace( t^4 * exp(-a*t) )

ans =

24/(s+a)^5
```

$t^N e^{-\lambda t} u(t) \longleftrightarrow \frac{N!}{(s+\lambda)^{N+1}}$

```
>> laplace( exp(-a*t) * cos(w*t) )

ans =

(s+a)/((s+a)^2+w^2)
```

$e^{-\lambda t} \cdot \cos(w t)\, u(t) \longleftrightarrow \frac{s+\lambda}{(s+\lambda)^2 + w^2}$

```
>> laplace( cos(5*t) )

ans =

s/(s^2+25)
```

$\cos(w t)\, u(t)$

```
>> laplace( cos(5*t) * sin(5*t) )

ans =

5/(s^2+100)


>> laplace( abs( cos(5*t) ) )

ans =

1/(s^2+25)*(s+5*csch(1/10*pi*s))
```

Please Excuse My Dear Aunt Sally ???

Unilateral Laplace Transform

Example DE with Initial Conditions - I (1)

- **Solve** $\dfrac{d^2y(t)}{dt^2} + 6\dfrac{dy(t)}{dt} + 8\,y(t) = \dfrac{dx(t)}{dt} + 2\,x(t)$

with initial conditions $y(0^-) = -1$, $\dfrac{dy(0^-)}{dt} = 1$

and input $x(t) = 2e^{-3t}u(t)$.

Sol'n

1) Transform to $U\mathcal{L}$-domain:

a) $x(t) = 2e^{-3t}u(t) \xleftrightarrow{U\mathcal{L}} \dfrac{2}{s+3}$

b) $$\left[s^2Y(s) - s\cdot y(0^-) - \frac{dy(0^-)}{dt}\right] + 6\left[s\,Y(s) - y(0^-)\right]$$

$$+8\,Y(s) = s\,X(s) - x(0^-) + 2\,X(s)$$

where $x(0^-) = 0$.

2) Solve in $U\mathcal{L}$-domain:

$$\underbrace{Y(s)\left[s^2+6s+8\right]}_{\text{System characteristics}} + \underbrace{\left[s+5\right]}_{\text{Due to initial conditions}} = \underbrace{\frac{2s+4}{s+3}}_{\text{Due to input}}$$

(Continued)

Example DE with Inital Conditions - I (2)

2 continued)

$$Y(s) =$$

$$= \left[\frac{2s+4}{s+3} - (s+5)\right]\left[\frac{1}{s^2+6s+8}\right] \qquad = \left[\frac{2s+4}{s+3} - (s+5)\right]\left[\frac{1}{s^2+6s+8}\right]$$

$$= \underbrace{\frac{-s^2-6s-11}{(s+2)(s+4)(s+3)}}_{\text{Total response}} \qquad = \underbrace{\frac{2s+4}{(s+3)(s+2)(s+4)}}_{\text{zero-state response}} - \underbrace{\frac{s+5}{(s+2)(s+4)}}_{\text{zero-input response}}$$

3) Transform result back to time domain:

$$Y(s) =$$

$$= \frac{-1.5}{s+2} + \frac{-1.5}{s+4} + \frac{2}{s+3}$$

$$y(t) = [\underbrace{-1.5e^{-2t} - 1.5e^{-4t}}_{\text{Transient response}} \underbrace{+2e^{-3t}}_{\text{Forced response}}]u(t)$$

cancellation

$$= \left[\frac{2}{s+3} + \frac{0}{s+2} + \frac{-2}{s+4}\right] + \left[\frac{-1.5}{s+2} + \frac{1/2}{s+4}\right]$$

$$y(t) = \underbrace{[2e^{-3t} - 2e^{-4t}]u(t)}_{\text{zero-state response}} + \underbrace{\left[-1.5e^{-2t} + \frac{1}{2}e^{-4t}\right]u(t)}_{\text{zero-input response}}$$

Solving LTI Circuits

Time Domain

1) Write KVL, KCL using time-domain terminal laws.

E.g., $i_c(t) = C\,\dfrac{dv_c(t)}{dt}$, $v_c(t) = \dfrac{1}{C}\displaystyle\int_{\tau=t_0}^{t} i_c(\tau)\,d\tau + v(t_0)$

2) Solve in time domain
   - Homogeneous solution, particular solution
   - → Form of particular solution depends on input

Frequency Domain

1) a) Write time-domain KVL, KCL or b) Convert components to frequency domain (Impedance)

   - Then, convert DE to frequency domain
   - Write KVL, KCL

2) Solve in frequency domain

3) Convert result back to time domain

Unilateral Laplace Transform

$\mathcal{UL}$ - Transform and Impedance

R:

$$v_R(t) = i_R(t) \cdot R$$

$$\Big\updownarrow \mathcal{UL}$$

$$V_R(s) = I_R(s) \cdot R \quad \text{or} \quad \frac{V_R(s)}{I_R(s)} = R$$

L: $$v_L(t) = L \frac{d\, i_L(t)}{dt}$$

$$\Big\updownarrow \mathcal{UL}$$

$$V_L(s) = L\left[s I_L(s) - i_L(0^-)\right]$$

or
$$\begin{cases} V_L(s) = sL\, I_L(s) - L \cdot i_L(0^-) \\ I_L(s) = \dfrac{V_L(s) + L \cdot i_L(0^-)}{sL} \end{cases}$$

C:

$$i_C(t) = C \frac{d\, v_C(t)}{dt}$$

$$\Big\updownarrow \mathcal{UL}$$

$$I_C(s) = C\left[s V_C(s) - v_C(0^-)\right]$$

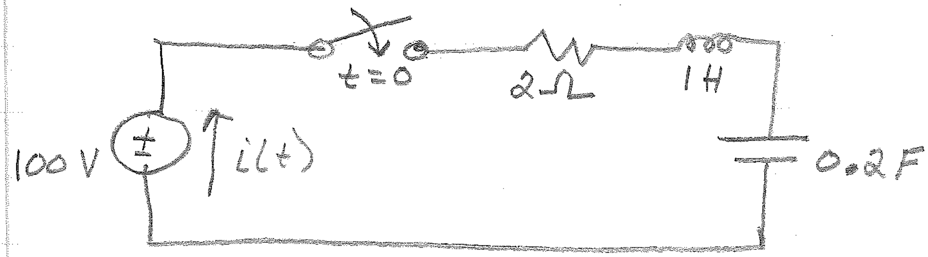
- No simple impedance
- USE TERMINAL LAWS (in frequency domain

or
$$\begin{cases} I_C(s) = sC\, V_C(s) - C \cdot v_C(0^-) \\ V_C(s) = \dfrac{I_C(s) + C \cdot v_C(0^-)}{sC} \end{cases}$$

Unilateral Laplace Transform

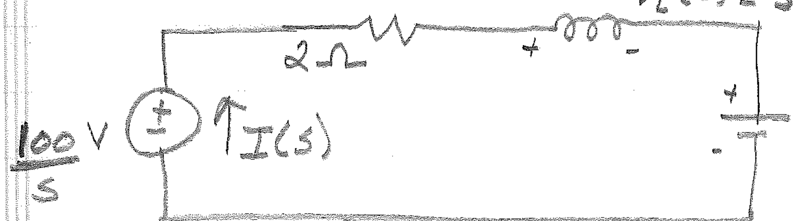
Electrical Circuit Example 1

Find $i(t)$ for $t>0$ if $v_c(0)=50$ V.



1) Convert to frequency domain ($t>0$):

$V_L(s) = sI(s)$



$$V_c(s) = \frac{I(s)+0.2\cdot 50}{0.2\cdot s} = \frac{5\cdot I(s)+50}{s}$$

KVL:

$$\frac{100}{s} = 2I(s) + sI(s) + \frac{5I(s)+50}{s}$$

2) Solve in frequency domain:

$$I(s) = \frac{50}{s^2+2s+5}$$

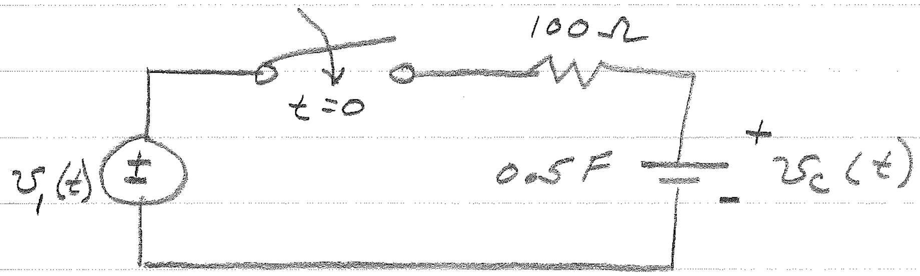
3) Convert result back to time domain:

$$I(s) = \frac{50}{(s+1)^2+4}$$

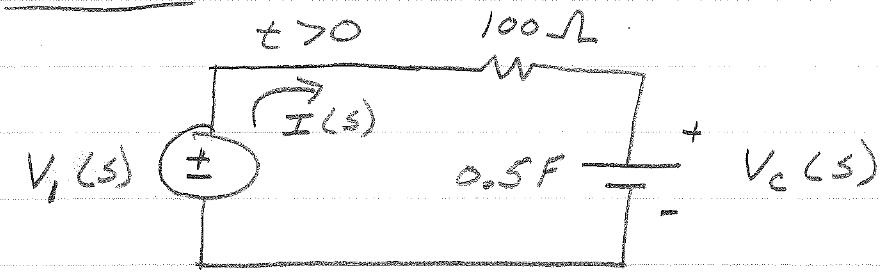
$$\updownarrow \quad \boxed{i(t) = 25e^{-t}\sin(2t)\,u(t)}$$

Electrical Circuit Example Ia (I)

- **Find** the (a) impulse response, (b) step response and (c) ramp response
  if $v_1(t) \Rightarrow$ input, $v_c(t) \rightarrow$ output, $v_c(0^-) = 4$ V.



Sol'n



By KVL:

$$V_1(s) = 100 \cdot I(s) + \frac{I(s) + \frac{1}{2} \cdot v_c(0^-)}{s \cdot \frac{1}{2}} \qquad (1)$$

and

$$I(s) = s \cdot \frac{1}{2} V_c(s) - \frac{1}{2} \cdot v_c(0^-) \qquad (2)$$

or

$$\left.\begin{aligned} &(1)\ s \cdot V_1(s) = I(s)\left[100s + 2\right] + 4 \\ &(2)\ I(s) = \frac{s}{2} V_c(s) - 2 \end{aligned}\right\} \longrightarrow$$

(continued)

Electrical Circuit Example 1a (2)

(Continued)

- Solving for $V_c(s)$ as a function of $V_1(s)$:

$$V_c(s) = \frac{V_1(s) + 200}{50s + 1}$$

(a) Impulse $\Rightarrow$ $V_1(s) = 1$

$$V_c(s) = \frac{201}{50s+1} = \frac{201/50}{s + 1/50} \Rightarrow \boxed{v_c(t) = \frac{201}{50} e^{-t/50} u(t)}$$

(b) Step $\Rightarrow$ $V_1(s) = 1/s$

$$V_c(s) = \frac{1/s + 200}{50s+1} = \frac{200s+1}{50s^2+s} = \frac{3}{s+1/50} + \frac{1}{s}$$

$$\Rightarrow \boxed{v_c(t) = \left[3e^{-t/50} + 1\right] u(t)}$$

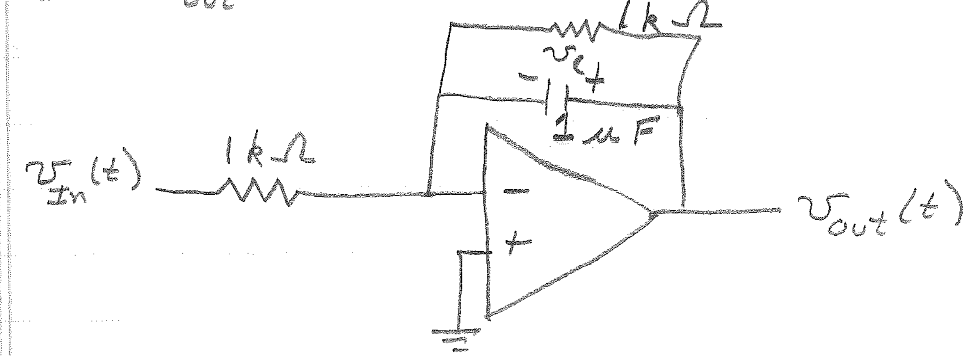
(c) Ramp $\Rightarrow$ $V_1(s) = 1/s^2$

$$V_c(s) = \frac{1/s^2 + 200}{50s+1} = \frac{200s^2+1}{50s^3+s^2} = \frac{54}{s+1/50} - \frac{50}{s} + \frac{1}{s^2}$$

$$\Rightarrow \boxed{v_c(t) = \left[54e^{-t/50} - 50 + t\right] u(t)}$$

Electrical Circuit Example 2

Find $v_{out}(t)$ for $t>0$ if $v_c(0^-)=10V$ and $v_{In}(t)=\sin(t)\mu(t)\,V$



1) Convent to frequency domain:

$$V_{In}(s)=\frac{1}{s^2+1}\quad,\quad v_c(0^-)=10$$

By KCL at inverting input:

$$\frac{V_{In}(s)}{1k}+\frac{V_{out}(s)}{1k}+1\mu\left[s\cdot V_{out}(s)-v_c(0^-)\right]=0$$

2) Solve in frequency domain:

$$V_{out}(s)=\frac{10s^2-990}{s^3+1000s^2+s+1000}$$

3) Convert result back to time domain:

$$V_{out}(s)=\frac{9.999}{s+1000}+\frac{0.001s-1}{(s^2+1)}$$ ← Two terms

$$\boxed{v_{out}(t)=\left[9.999e^{-1000t}+0.001\cos(t)-\sin(t)\right]\mu(t)\ V}$$

Example DE (1)

• Find $y(t)$ for $(D^2+5D+6)\,y(t) = D\,x(t)$

with $y(0)=-1$, $\dot{y}(0)=1$, $x(t)=u(t)$,

(Same ZIR as "Zero Input Response: ZIR - Real Roots")

Sol'n

1) Transform to $\mathcal{UL}$-domain:

$$\left[s^2Y(s) - s\cdot y(0^-) - \dot{y}(0^-)\right] + \left[5sY(s) - 5y(0^-)\right] + 6Y(s)$$

$$= s\cdot X(s) - x(0^-)$$

$$Y(s)\left[s^2+5s+6\right] = s\cdot X(s) - x(0^-) + s\,y(0^-) + \dot{y}(0^-) + 5\cdot y(0^-)$$

2) Solve in $\mathcal{UL}$ domain:

$$x(t) = u(t) \Longrightarrow X(s) = \frac{1}{s}, \quad x(0^-) = 0$$

$$Y(s) = \frac{s\cdot\frac{1}{s} - 0 + s\cdot(-1) + 1 + 5\cdot(-1)}{s^2+5s+6}$$

$$Y(s) = \frac{-s-3}{s^2+5s+6}$$

continued

Example DE (2)

Continued

3) Transform results back to time domain:

$$Y(s) = \frac{-s-3}{s^2+5s+6} = \underbrace{\frac{0}{s+3} + \frac{-1}{s+2}}$$

$$\Downarrow$$

$$= \frac{-(s+3)}{(s+3)(s+2)}$$

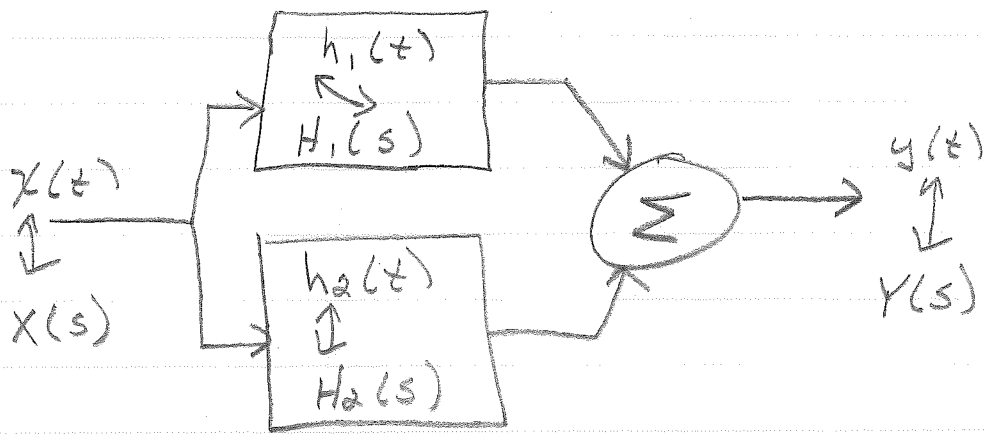
so,

$$Y(s) = \frac{-1}{s+2}$$

$$\Downarrow$$

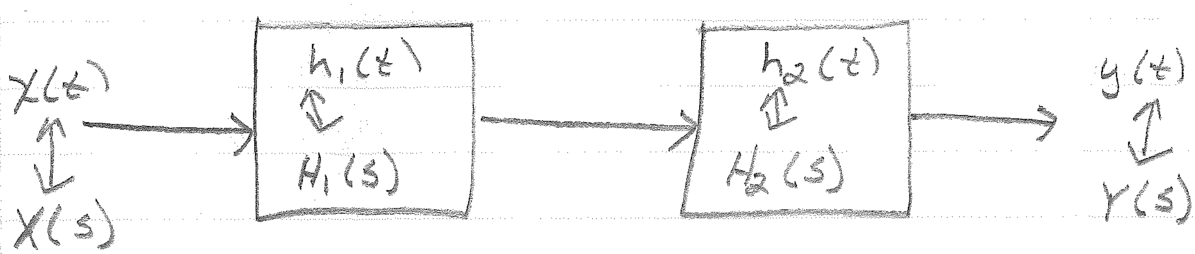
$$\boxed{y(t) = -e^{-2t}\, u(t)}$$

Interconnected LTI Systems



$$y(t) = x(t) * h_1(t) + x(t) * h_2(t) = x(t) * [h_1(t) + h_2(t)]$$

$$Y(s) = X(s) \cdot H_1(s) + X(s) \cdot H_2(s) = X(s) \cdot [H_1(s) + H_2(s)]$$



$$y(t) = x(t) * h_1(t) * h_2(t)$$

$$Y(s) = X(s) \cdot H_1(s) \cdot H_2(s)$$

Unilateral Laplace Transforms