Imaginary Unit *j*

- **Define:** $j = \sqrt{-1}$

- **Then,**

$$j^2 = \left(\sqrt{-1}\right)^2 = \quad -1$$

$$j^3 = j^2 \bullet j = (-1) \bullet j = \quad -j$$

$$j^4 = j^2 \bullet j^2 = (-1)(-1) = \quad 1$$

$$j^5 = j^4 \bullet j = 1 \bullet j = \quad j$$

$$\ddots$$

- **So,**

$$j = i \longleftarrow \sqrt{-1}$$

$$j^2 = -1$$

$$j^3 = -j$$

$$j^4 = 1$$

Rectangular Form of Complex Number

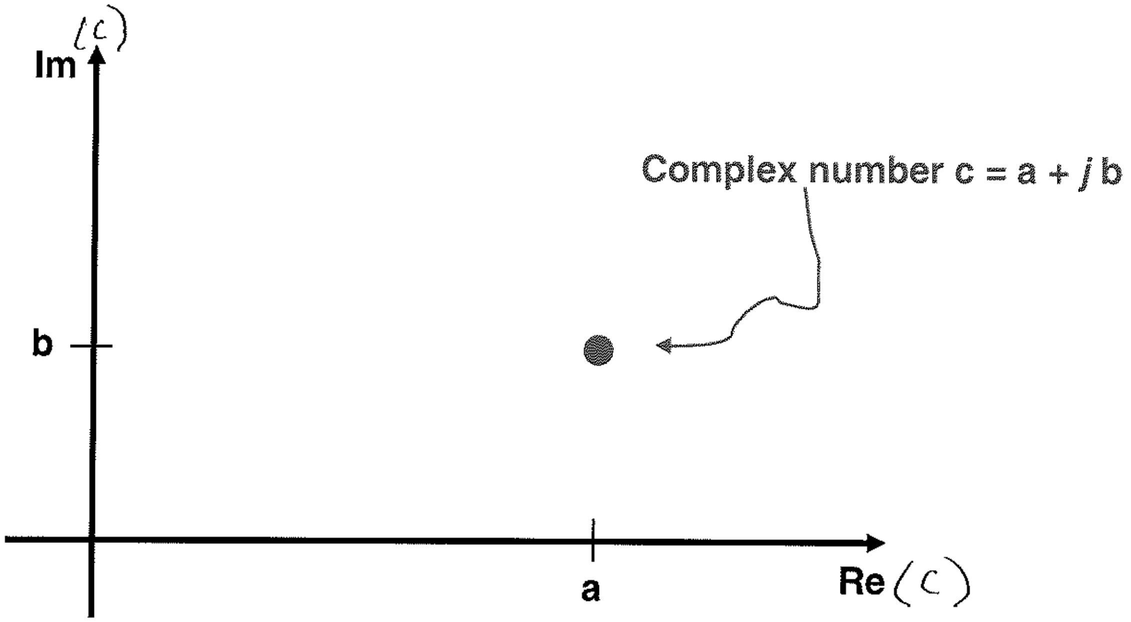
- **Complex number:** $c = a + j\,b$

Real Part

Re {c} = a

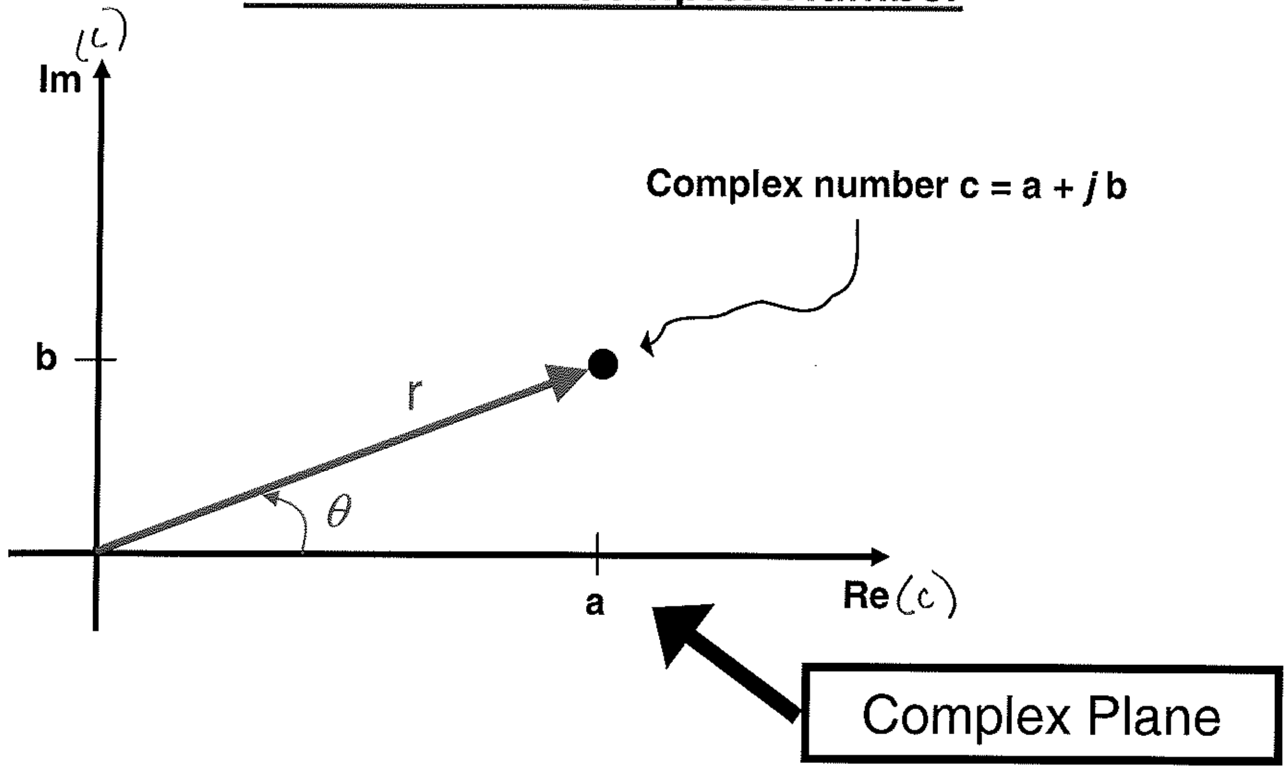
Imaginary Part

Im {c} = b



Complex Plane

Polar Form of Complex Number



Complex Plane

- **Polar form written as:** $r\angle\theta$

  Magnitude

  Angle

- **<u>Rectangular-to-polar</u> conversion:**

$$
\boxed{\begin{aligned} r &= \sqrt{a^2+b^2} \\ \theta &= \begin{cases} \tan^{-1}\left(\dfrac{b}{a}\right), & a \ge 0 \\ \tan^{-1}\left(\dfrac{b}{a}\right)+\pi, & a<0 \end{cases} \end{aligned}}
$$

Note: $\frac{-\pi}{2} \le \tan^{-1}(\cdot) \le \frac{\pi}{2}$

Exception: Newer calculators

- **<u>Polar-to-rectangular</u> conversion:**

$a = r \bullet \cos(\theta)$ $\qquad$ $b = r \bullet \sin(\theta)$

Rectangular-Polar Conversions — Examples

- **Ex. 1 Rectangular to polar:**

$c = 20 + j30$

Therefore:

$$r = \sqrt{20^2 + 30^2} = \sqrt{400 + 900} = \sqrt{1300} = \underline{36.06}$$

$$\theta = \tan^{-1}\left(\frac{30}{20}\right) = \tan^{-1}(1.5) = \underline{0.983 \text{ rads}}$$

$$c = 36.06 \, \underline{|\, 0.983 \text{ rads}}$$

MATLAB handles complex numbers implicitly !!

- **Ex. 2 Polar to rectangular:**

$$c = 36.06 \, \underline{|\, 0.983 \text{ rads}}$$

Therefore:

a = 36.06 cos(0.983 rad) = 20.00

b = 36.06 sin(0.983 rad) = 30.01

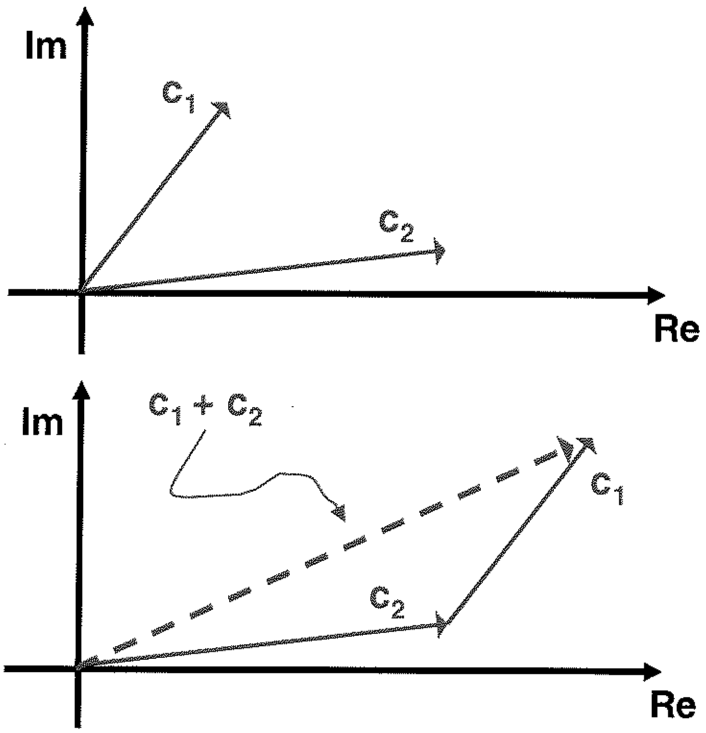
$c = 20.00 + j30.01$

Complex Number Addition

- **Rectangular form** ➔ (Sum Real Parts) + j (Sum Imag Parts)
  - So: $c_1 = 18 + j\,11$
    $c_2 = -4 - j\,5$

$$c_1 + c_2 = (18 - 4) + j\,(11 - 5) = 14 + j\,6$$

- Graphical Addition:



- **Polar Form:**
  - Convert to rectangular form
  - Add
  - Convert result back to polar form

Complex Number Multiplication

- **Rectangular Form:**
  - Let: $c_1 = a_1 + j\,b_1, \quad c_2 = a_2 + j\,b_2$

$$c_1 \bullet c_2 = (a_1 + j\,b_1)\,(a_2 + j\,b_2)$$

$$= a_1\,a_2 + j\,b_1\,a_2 + j\,a_1\,b_2 + j^2\,b_1\,b_2$$

$$c_1 \bullet c_2 = (a_1\,a_2 - b_1\,b_2) + j\,(b_1\,a_2 + a_1\,b_2)$$

Example: $c_1 = 3 - j\,4, \quad c_2 = 2 + j\,5$

$$c_1 \bullet c_2 = (3 \bullet 2 - j \bullet 4 \bullet 2 + j \bullet 5 \bullet 3 - j^2 \bullet 5 \bullet 4)$$

$$= 6 - j\,8 + j\,15 + 20) = \underline{26 + j\,7}$$

- **Polar Form:**
  - Let: $c_1 = r_1 \angle \theta_1 \;, \quad c_2 = r_2 \angle \theta_2$
  - Can show: $c_1 \bullet c_2 = r_1 \bullet r_2 \angle \theta_1 + \theta_2$

Example: $c_1 = 5 \angle 30^\circ \;, \quad c_2 = 6 \angle 15^\circ$

$$c_1 \bullet c_2 = 5 \bullet 6 \angle 30^\circ + 15^\circ = 30 \angle 45^\circ$$

Complex Conjugate

If $c = a + jb = r\angle\theta$

then $c^* = a - jb = r\angle -\theta$

"Complex Conjugate"

Note 1:

$$c + c^* = (a+jb) + (a-jb) = 2a = 2 \cdot \mathrm{Re}\{c\}$$

Note 2:

$$c \cdot c^* = (a+jb)\cdot(a-jb)$$

$$= a^2 + jab - jab - j^2 b^2$$

$$= a^2 + b^2 = |c|^2$$

Complex Number Division – Rectangular Form

- Let $c_1 = a_1 + jb_1$ $\qquad c_2 = a_2 + jb_2$

- Then:

$$\frac{c_1}{c_2} = \frac{a_1 + jb_1}{a_2 + jb_2} \longleftarrow \text{"Irrational" denominator}$$

- Rationalize denominator → Multiply by:

$$1 = \frac{a_2 - jb_2}{a_2 - jb_2} = \frac{c_2^*}{c_2^*}$$

Ex.

$c_1 = 1 + j2$ $\qquad c_2 = 3 - j4$

$$\frac{c_1}{c_2} = \frac{1+j2}{3-j4} \cdot \frac{3+j4}{3+j4}$$

$$= \frac{3 + j6 + j4 - 8}{9 - j12 + j12 + 16} = \frac{-5 + j10}{\underset{(a_2^2 + b_2^2)}{25}} = \frac{-1+j2}{5}$$

Complex Number Division – Polar Form

- Let $c_1 = r_1 \angle \theta_1$ $\qquad c_2 = r_2 \angle \theta_2$

  - Can show:

$$\frac{c_1}{c_2} = \frac{r_1}{r_2} \angle \theta_1 - \theta_2$$

Ex.

$c_1 = \sqrt{5} \angle 63.4°$ $\qquad c_2 = 5 \angle -53.1°$

$$\frac{c_1}{c_2} = \frac{\sqrt{5}}{5} \angle 63.4° - -53.1°$$

$$= \frac{1}{\sqrt{5}} \angle 116.5° \cong \boxed{0.447 \angle 116.5°}$$

General Sinusoid

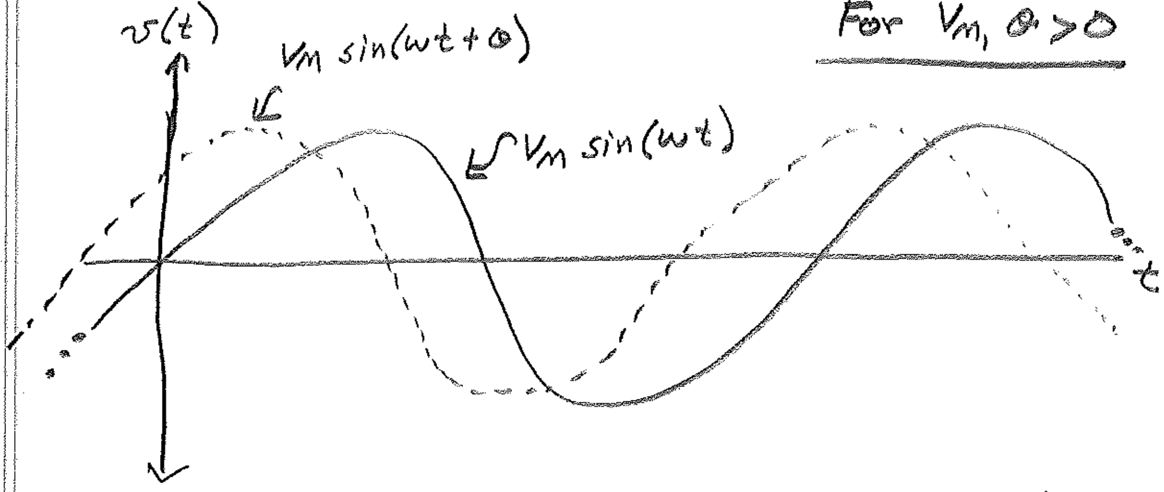
- $$v(t) = \underbrace{V_m}_{\text{Amplitude}} \ \sin(\underbrace{\omega t}_{\text{Frequency}} + \underbrace{\theta}_{\text{Phase}})$$

Frequency: $\omega \Rightarrow$ radians/s, $f \Rightarrow$ Hertz

Phase: $\theta^\circ \Rightarrow$ degrees, else $\Rightarrow$ radians

∘ Recall: $\omega = 2\pi f$, $360^\circ = 2\pi$ radians

$$\text{period} = T = \frac{1}{f} = \frac{2\pi}{\omega}$$

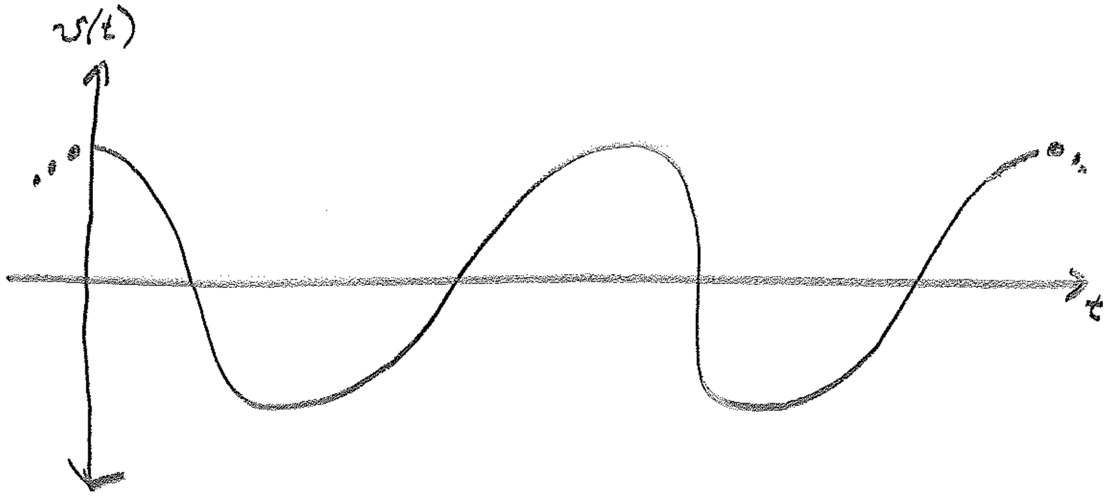


For $V_m, \theta > 0$

$V_m \sin(\omega t + \theta)$ **leads** $V_m \sin(\omega t)$ by θ

Cosines

- $\cos(\omega t) = \sin(\omega t + \frac{\pi}{2})$



Cosine __leads__ sine by $\frac{\pi}{2}$ radians = 90 degrees

Useful Trigonometric Identities

$$\cos(wt) = \sin(wt + \pi/2)$$
$$\sin(wt) = \cos(wt - \pi/2)$$

$$\left.\begin{aligned}\sin(wt \pm n\cdot 2\pi) &= \sin(wt)\\ \cos(wt \pm n\cdot 2\pi) &= \cos(wt)\end{aligned}\right\}\text{ For n an integer}$$

$$\left.\begin{aligned}\sin(wt \pm n\pi) &= -\sin(wt)\\ \cos(wt \pm n\pi) &= -\cos(wt)\end{aligned}\right\}\text{ For n an }\underline{\underline{\text{odd}}}\text{ integer}$$

Exponential Form of Complex Numbers

- Euler's Identity:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

- Therefore:

$$r \cdot e^{j\theta} = \underbrace{r\cos(\theta)}_{\text{Real part}} + j\,\underbrace{r\sin(\theta)}_{\text{Imag part}}$$

- Same equations as polar to rectangular conversion

$$\Rightarrow \quad r\angle\theta = r\,e^{j\theta}$$

↗

Exponential form of complex number

(a.k.a. exponential polar form)

Complex Exponential Function

- Let: $\theta = \omega t$,

$$r\;e^{j\omega t} = r\cos(\omega t) + j\,r\sin(\omega t) = r\angle\omega t$$

- Takes on complex value at each instant in time
  - Real part varies in time as cosine
  - Imag part varies in time as sine

$$Re[r \cdot e^{j\omega t}] = r\cos(\omega t), \quad Im[r \cdot e^{j\omega t}] = r\sin(\omega t)$$

MATLAB

- If lecture time permits, quick demonstration of MATLAB
- Launching
- Windows
    - MATLAB
        - Command Window
        - Command History
        - Workspace
    - Help
    - Edit
- Scalars, vectors, matrices
    - Simple math (addition, matrix multiply, element-by-element multiply
- Commands "residue()", "roots()"
- Simple function M-File, debugger
- Plotting
- Complex numbers and exp().

MATLAB