Continuous-Time Signals and Systems

Introduction to Digital Signal Processing

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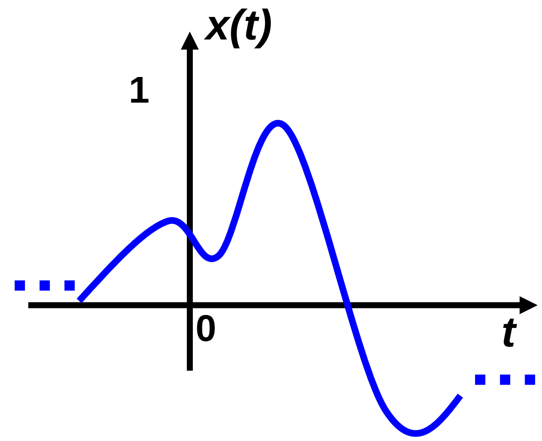
What is a Signal?

- **Continuous time:**



"t" ➔ time variable

Energy:

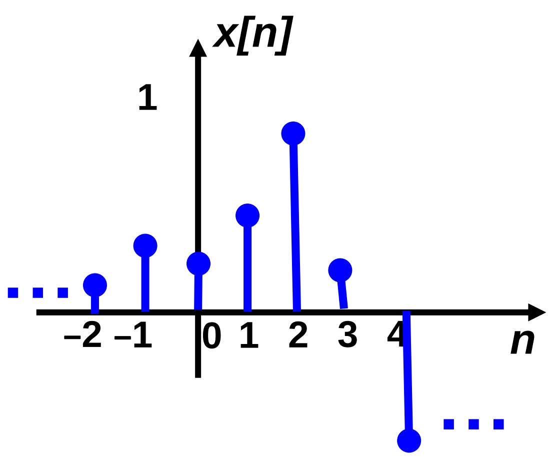
$$E_x = \int_{t=-\infty}^{\infty} |x(t)|^2 \, dt$$

Ave. Power in *x(t)* Periodic:

$$P_x = \frac{1}{T} \int_{T} |x(t)|^2 \, dt,$$

"*T*" is the period

- **Discrete time:**



Energy:

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Ave. Power in *x[n]* Periodic:

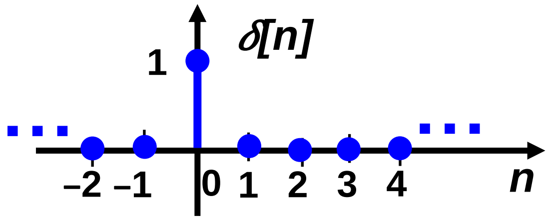
$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2 ,$$

"*N*" is the period (integer)

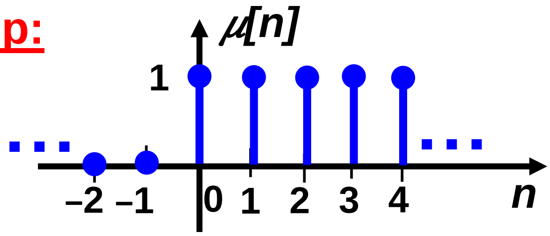
- "*n*" ➔ Discrete-time sample index (integer)
- *x[n]* often formed from periodic samples of a continuous-time signal

Basic Discrete-Time Signals

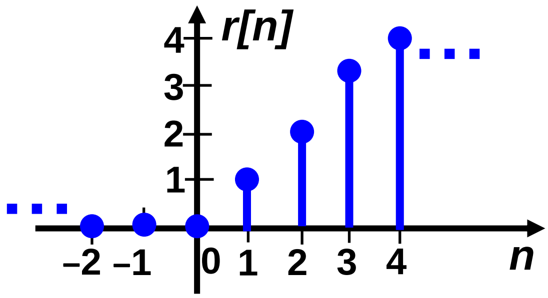
- **Unit impulse:**
  - Regular function!!



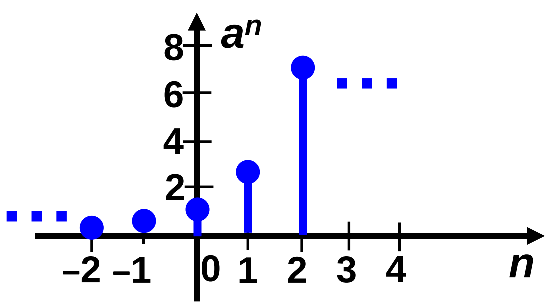
- **Unit step:**



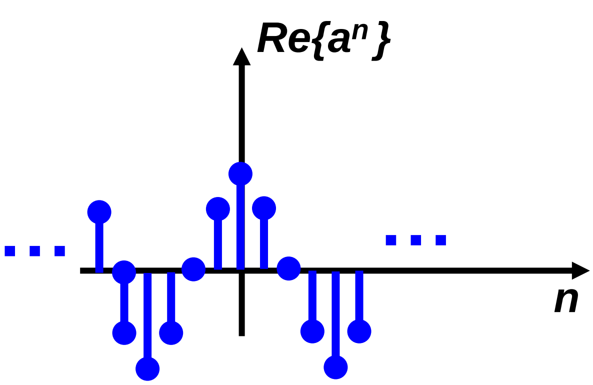
- **Unit ramp:**

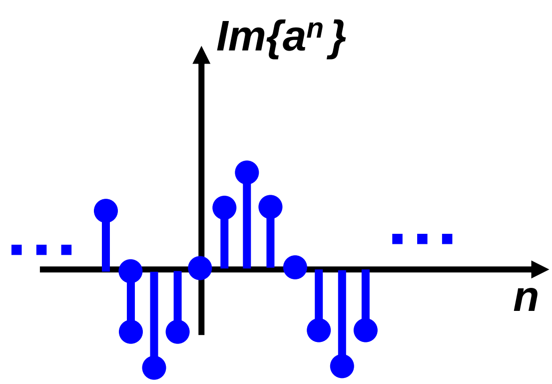


- **Real exponential:**
  - E.g., for a > 0, real



- **Complex exponential:** E.g., for $a = r\, e^{j\,\omega}$, $\omega \neq 0$, $r = 1$





Signal Operations

- **Scalar multiplication:** $y[n] = a \cdot x[n]$

- **Time shifting:** $y[n] = x[n - m]$

  - **If *m* ≠ integer $\longrightarrow$ Interpolation**

- **Time scaling:** $y[n] = x[a \cdot n]$

  - **If *a* ≠ integer $\longrightarrow$ Interpolation**

- **Time reversal:** $y[n] = x[-n]$

Continuous-Time: Derivatives, integrals

$\updownarrow$

Discrete-Time: Differences, sums

Signal Classification

Topic	Continuous	Discrete
Periodic	$x(t) = x(t+T)$, $\forall\, t$	$x[n] = x[n+N], \forall\, n$ n, N integer Note: Some sinusoids NOT periodic; e.g., ω irrational
Causal	$x(t) = 0,\ t < 0$	$x[n] = 0,\ n < 0$
Even Symmetric	$x_e(t) = x_e(-t)$ $x_e(t) = \frac{x(t)+x(-t)}{2}$	$x_e[n] = x_e[-n]$ $x_e[n] = \frac{x[n]+x[-n]}{2}$
Odd Symmetric	$x_o(t) = -x_o(-t)$ $x_o(t) = \frac{x(t)-x(-t)}{2}$	$x_o[n] = -x_o[-n]$ $x_o[n] = \frac{x[n]-x[-n]}{2}$

Systems

Property	Continuous	Discrete
Time Invariance	$x(t) \xrightarrow{\mathcal{T}} y(t)$ *implies* $x(t-T) \xrightarrow{\mathcal{T}} y(t-T)$	$x[n] \xrightarrow{\mathcal{T}} y[n]$ *implies* $x[n-m] \xrightarrow{\mathcal{T}} y[n-m]$ "Shift Invariant"
Linear	$\mathcal{T}[a_1\, x_1(t) + a_2\, x_2(t)] = a_1\, y_1(t) + a_2\, y_2(t)$	$\mathcal{T}\{a_1\, x_1[n] + a_2\, x_2[n]\} = a_1\, y_1[n] + a_2\, y_2[n]$
Impulse Response	$\delta(t-T) \rightarrow \boxed{\mathcal{T}} \rightarrow h(t,T)$	$\delta[n-m] \rightarrow \boxed{\mathcal{T}} \rightarrow h[n,m]$
	IF time invariant	
	$h(t,T) \rightarrow h(t-T)$	$h[n,m] \rightarrow h[n-m]$

Convolution—For LTI Systems

- **Continuous-time:**

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = x(t) * h(t)$$

- **Discrete-time:**

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]\, h[n-m] = x[n] * h[n]$$

- **Similar properties:**

  – **Commutative:**

$$x[n] * h[n] = h[n] * x[n]$$

  – **Associative:**

$$\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$$

  – **Distributive:**

$$x[n] * \{h_1[n] + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$$

Differential and Difference Equations

• Continuous-time:

$$a_R \frac{d^R\, y(t)}{dt^R} + a_{R-1} \frac{d^{R-1}\, y(t)}{dt^{R-1}} + \cdots + a_1 \frac{d\, y(t)}{dt} + a_0\, y(t)$$
$$= b_P \frac{d^P\, x(t)}{dt^P} + \cdots + b_1 \frac{d\, x(t)}{dt} + b_0\, x(t)$$

• Discrete-time:

$$a_R\, y[n-R] + a_{R-1}\, y[n-R-1] + \cdots + a_1\, y[n-1] + a_0\, y[n]$$
$$= b_P\, x[n-P] + \cdots + b_1\, x[n-1] + b_0\, x[n]$$

Or, for $a_0 = 1$ (typical):

$$y[n] = \sum_{m=0}^{P} b_m \cdot x[n-m] - \sum_{m=1}^{R} a_m \cdot y[n-m]$$

Note: For non-causal system, allow terms with future inputs, e.g.,

$$x[n+1], x[n+2], \quad \ldots$$

Solving Differential/Difference Equations with NULL Initial Conditions

- Continuous-time $\longrightarrow$ Bilateral Laplace Transform

$$\mathcal{L}\{x(t)\} = X(s) = \int_{t=-\infty}^{\infty} x(t)\, e^{-s\,t}\, dt$$

$$\underbrace{s = \sigma + j\,\omega}_{\text{complex}}$$

- Discrete-time $\longrightarrow$ z-Transform

$$z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$\underbrace{z = r \cdot e^{j\,\omega}}_{\text{complex}}$$

Note: X(z) is continuous-valued !!!

Common (Bilateral) Laplace- and z-Transforms

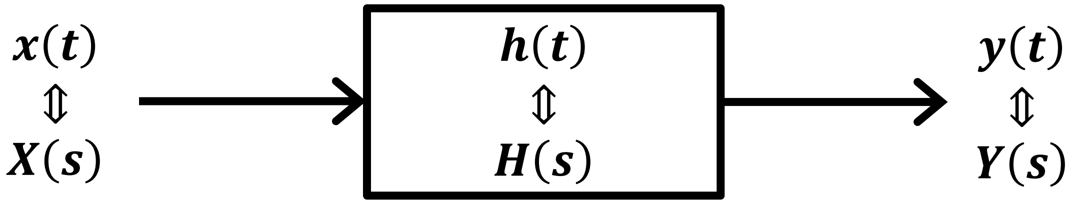
Function	Continuous-Time Bilateral Laplace	Discrete-time z-Transform
Impulse	$\mathbf{1}, \ \forall s$	$\mathbf{1}, \ \forall z$
Step	$\frac{1}{s}, \ Re(s) > 0$	$\frac{1}{1-z^{-1}}, \ \lvert z \rvert > 1$
Ramp	$t \cdot \mu(t) \Updownarrow \frac{1}{s^2}, \ Re(s) > 0$	$n \cdot \mu[n] \Updownarrow \frac{z^{-1}}{(1-z^{-1})^2}, \ \lvert z \rvert > 1$

Properties

Convolution	$x(t) * h(t) = X(s) \cdot H(s)$	$x[n] * h[n] = X(z) \cdot H(z)$

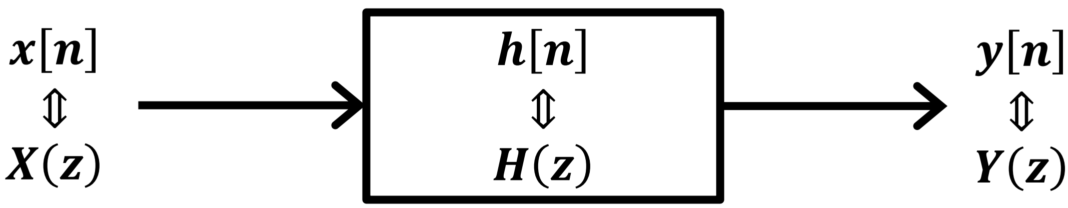
System Function

- **Continuous-time:**



$$H(s) = \frac{Y(s)}{X(s)}$$

- **Discrete-time:**



$$H(z) = \frac{Y(z)}{X(z)}$$

Rational Transfer Functions

- **Continuous-time:**

$$H(s) = \frac{Y(s)}{X(s)} = G \cdot \frac{\prod_{m=1}^{P}(s - z_m)}{\prod_{m=1}^{R}(s - p_m)}$$

- **Causal system stable iff** $Re\{p_m\} < 0$

- **Discrete-time:**

$$H(z) = \frac{Y(z)}{X(z)} = b_0 \cdot z^{R-P} \cdot \frac{\prod_{m=1}^{P}(z - z_m)}{\prod_{m=1}^{R}(z - p_m)}$$

$$= b_0 \cdot \frac{\prod_{m=1}^{P}\left(1 - z_m\, z^{-1}\right)}{\prod_{m=1}^{R}\left(1 - p_m\, z^{-1}\right)}$$

- **Causal system stable iff** $|p_m| < 1$

Solving Differential/Difference Equations WITH Initial Conditions

- **Continuous-time** $\longrightarrow$ Unilateral Laplace Transform

$$\mathcal{UL}\{x(t)\} = X(s) = \int_{t=0}^{\infty} x(t)\, e^{-s\,t}\, dt$$

» **Property:** $\dfrac{d\,x(t)}{dt} \leftrightarrow s \cdot X(s) - x(0^-)$

Initial Condition

- **Discrete-time** $\longrightarrow$ One-Sided z-Transform

$$z^+\{x[n]\} = X^+(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

» **Properties:** *(m > 0)*

$$x[n-m] \leftrightarrow z^{-m}\left[X^+(z) + \sum_{n=1}^{m} x[-n]\, z^{n}\right]$$

$$x[n+m] \leftrightarrow z^{m}\left[X^+(z) - \sum_{n=0}^{m-1} x[n]\, z^{-n}\right]$$

Initial Conditions

Steady State Response to Sinusoids

• Continuous-time $\longrightarrow$ Fourier Transform

$$X(\omega) = \int_{t=-\infty}^{\infty} x(t)\, e^{-j\,\omega\,t}\, dt$$

$$j\,\omega = s\Big|_{s=j\,\omega}$$

• Discrete-time $\longrightarrow$ Discrete Time Fourier Transform (DTFT)

$$X\left(e^{j\,\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\,\omega\,n}$$

Continuous-valued in ω !!!

$$e^{-j\,\omega\,n} = z^{-n}\Big|_{z=e^{j\,\omega}}$$

– **ISSUE:** Digital computers cannot store continuous-valued variable

➔ Must sample $X\left(e^{j\,\omega}\right) \Longrightarrow X[k]$

Discrete Fourier Transform (DFT)

Fourier Series Representations

• **Continuous-time** $\longrightarrow$ **Fourier Series**

$$x(t) = \sum_{m=-\infty}^{\infty} d_m \, e^{jm2\pi f_o t}$$

where

$$d_m = \frac{1}{T_o} \int_{T_o} x(t) \, e^{-j\,m2\pi f_o t} \, dt$$

• **Discrete-time** $\longrightarrow$

– **Sampling of DTFT "treats" *x[n]* as periodic !!**

– **So, Discrete Fourier Transform (DFT):**

$$X[k] = \sum_{n=0}^{N-1} x[n] \, e^{-j\frac{2\pi k n}{N}}$$

• **where:**

– **"*N*" is the sequence length**

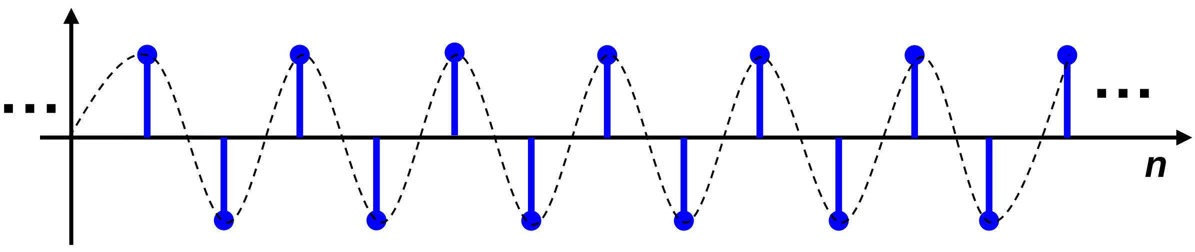
– **Integer "*k*" indexes from *0* to *(N–1)***

– **Finite sum appropriate for numerical computation on computer**

Signal Sampling

- Continuous-time signal frequency range $\longrightarrow$ Infinite

- Consider creating discrete-time signal by sampling a continuous-time signal.
  - Maximum possible discrete-time frequency represented by alternating high-low samples:



  - $\Longrightarrow f_{Max} = \dfrac{f_{Sample}}{2}$

  - Continuous-time frequencies above $\frac{f_s}{2}$ "misrepresented"

    $\longrightarrow$ ALIASING

  - $X(e^{j\omega})$ and $X[k]$ axis only unique over ranges corresponding to:

$$\frac{-f_s}{2} \le f \le \frac{f_s}{2}$$

More and more (and more!) signal processing migrating to digital domain