Solution

Worcester Polytechnic Institute — Department of Electrical and Computer Engineering
ECE2311 — Continuous-Time Signal and System Analysis — Term B'17

Homework 8: Due Tuesday, 28 November 2017 (3:00 P.M.)

Write your name and ECE box at the top of each page.

General Reminders on Homework Assignments:

- Always complete the reading assignments *before* attempting the homework problems.
- Show all of your work. Use written English, where applicable, to provide a log or your steps in solving a problem. (For numerical homework problems, the writing can be brief.)
- A solution that requires physical units is *incorrect* unless the units are listed as part of the result.
- Get in the habit of underlining, circling or boxing your result.
- Always write neatly. Communication skills are essential in engineering and science. "If you didn't write it, you didn't do it!"

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1) Unilateral Laplace Transform:

a) Use direct integration to find the Unilateral Laplace Transform of: $x(t)=7+2e^{-3t}$. (For Unilateral Laplace Transforms, you need *not* list the region of convergence.)

$$X(s)=\int_{t=0^-}^{\infty}\left(7+2e^{-3t}\right)e^{-st}\,dt = 7\int_{t=0^-}^{\infty} e^{-st}\,dt + 2\int_{t=0^-}^{\infty} e^{-t(3+s)}\,dt$$

$$=\left.\frac{7e^{-st}}{-s}\right|_{t=0^-}^{\infty} + \left.\frac{2e^{-t(3+s)}}{-(3+s)}\right|_{t=0^-}^{\infty} = \underbrace{\left(0-\frac{7}{-s}\right)}_{\mathrm{Re}(s)>0}+\underbrace{\left(0-\frac{2}{-(3+s)}\right)}_{\mathrm{Re}(s)>-3}$$

$$\boxed{X(s)=\frac{7}{s}+\frac{2}{s+3}}$$

Note: Function values for t<0 not relevant.

b) Use the transform tables to find the Unilateral Laplace Transform of: $h(t)=t^2-3\sin(4t)$.

From the transform tables:

$$\boxed{H(s)=\frac{2}{s^3}-\frac{12}{s^2+16}}$$

Note: Function values for t<0 not relevant.

2) Unilateral Laplace Transform to Solve LTI Differential Equations with Initial Conditions:

a) Use the Unilateral Laplace Transform to solve: $\frac{d^2y(t)}{dt^2}+3\frac{d\,y(t)}{dt}+9\,y(t)=3\,x(t)$ with initial conditions $y(0^-)=-2$ and $\frac{d\,y(0^-)}{dt}=1$, and input $x(t)=\cos(5t)u(t)$. Use MATLAB's numerical routines [residue(), roots(), etc.] as needed to aid in the steps of the computation. Hint: Manipulate complex conjugate roots together so that they can be inverse transformed using the transform table. (This problem has a fair amount of computation. Take your time, follow the general steps, use MATLAB to aid in the numerical computation.)

Convert to Unilateral Laplace domain:

$$x(t)=\cos(5t)u(t)\leftrightarrow X(s)=\frac{s}{s^2+25}$$

$$\left[s^2\,Y(s)-s\,y(0^-)-\dot{y}(0^-)\right]+3\left[s\,Y(s)-y(0^-)\right]+9\,Y(s)=3\,X(s)$$

$$s^2\,Y(s)+2\,s-1+3\,s\,Y(s)+6+9\,Y(s)=\frac{3\,s}{s^2+25}$$

$$Y(s)\left[s^2+3\,s+9\right]=\frac{3\,s}{s^2+25}-2\,s-5$$

$$Y(s)=\frac{-2\,s^3-5\,s^2-47\,s-125}{(s^2+25)(s^2+3\,s+9)}$$

Using conv() and residue():

$$Y(s)=\frac{-0.95+j\,0.45}{s+1.5-j\,2.60}+\frac{-0.95-j\,0.45}{s+1.5+j\,2.60}+\frac{-0.05-j\,0.05}{s-j\,5}+\frac{-0.05+j\,0.05}{s+j\,5}$$

Writing the numerators in polar form:

$$Y(s)=\frac{1.05\,e^{j154.8^o}}{s+1.5-j\,2.60}+\frac{1.05\,e^{-j154.8^o}}{s+1.5+j\,2.60}+\frac{0.068\,e^{-j136.8^o}}{s-j\,5}+\frac{0.068\,e^{j136.8^o}}{s+j\,5}$$

The first two terms match the Unilateral Laplace Transform of $e^{-\lambda t}\cos(\omega_o\,t+\theta)u(t)$, with $\lambda=1.5$, $\omega_o=2.60$, and $\theta=154.8^o$.

The second two terms also match the Unilateral Laplace Transform of $e^{-\lambda t}\cos(\omega_o\,t+\theta)u(t)$, with $\lambda=0$, $\omega_o=5$, and $\theta=-136.8^o$.

Transforming back to the time domain:

$$\boxed{y(t)=2.1e^{-1.5t}\cos(2.60t+154.8^o)u(t)+0.136\cos(5t-136.8^o)u(t)}$$

b) Use the Unilateral Laplace Transform to solve: $4\frac{d\,y(t)}{dt}+6\,y(t)=x(t)$ with initial conditions $y(0^-)=2$ and input $x(t)=4\,e^{-2t}\,u(t)$. Use MATLAB routines [residue(), roots(), etc.] as needed to aid in the steps of the computation.

Convert to Unilateral Laplace domain:

$$x(t) = 4e^{-2t}u(t) \leftrightarrow X(s) = \frac{4}{s+2}$$

$$4\left[sY(s) - y(0^-)\right] + 6Y(s) = \frac{4}{s+2}$$

Substitute the initial conditions and solve in the frequency domain:

$$4sY(s) - 8 + 6Y(s) = \frac{4}{s+2}$$

$$Y(s)[4s+6] = \frac{4}{s+2} + 8 = \frac{8s+20}{s+2}$$

$$Y(s) = \frac{8s+20}{(s+2)(4s+6)}$$

Using residue():

$$Y(s) = \frac{-2}{s+2} + \frac{4}{s+1.5}$$

Converting to the time domain:

$$\boxed{y(t) = \left(-2e^{-2t} + 4e^{-1.5t}\right)u(t)}$$

3) Unilateral Laplace Transform to Solve LTI Electrical Circuits with Initial Conditions:

a) Consider the electronic circuit shown below with input $v_1(t)$ and output $v_2(t)$. Let $\mathrm{R}_1 = 2\,\Omega$, $\mathrm{R}_\mathrm{F} = 5\,\Omega$ and $\mathrm{C} = 2\ \mathrm{F}$. Let the initial output voltage be $v_2(0^-) = -2\ \mathrm{V}$. Find:

By KCL at the inverting terminal

$$\frac{-V_1(s)}{R_1} - \frac{V_2(s)}{R_F} - \left[V_2(s)\cdot sC - C\,v_2(0^-)\right] = 0, \text{ where } v_2(t) = v_C(t)$$

$$V_2(s)\left[\frac{-1}{R_F} - sC\right] = \frac{V_1(s)}{R_1} - C\,v_2(0^-)$$

$$V_2(s) = \frac{\dfrac{V_1(s)}{R_1} - C\cdot v_2(0^-)}{-sC - \dfrac{1}{R_F}} = \frac{V_1(s)\cdot R_F - C\cdot R_1\cdot R_F\cdot v_2(0^-)}{-sC\,R_1\,R_F - R_1}$$

Substituting known values:

$$\boxed{V_2(s) = \frac{5V_1^+(s) + 40}{-20s - 2}}$$

i) The response of this system when the input is an impulse function.

Impulse response:

$$v_1(t)=\delta(t)\leftrightarrow V_1(s)=1$$

$$V_2(s)=\frac{45}{-20\,s-2}=\frac{-9/4}{s+\frac{1}{10}}$$

Converting back to the time domain:

$$\boxed{v_2(t)=\frac{-9}{4}e^{\frac{-t}{10}}\,u(t)\quad V}$$

ii) The response of this system when the input is a step function.

Step response:

$$v_1(t)=u(t)\leftrightarrow V_1(s)=\frac{1}{s}$$

Therefore:

$$V_2(s)=\frac{\frac{5}{s}+40}{-20\,s-2}=\frac{40\,s+5}{-20\,s^2-2\,s}=\underset{\substack{\uparrow\\ \text{residue()}}}{\frac{0.5}{s+\frac{1}{10}}-\frac{2.5}{s}}$$

Converting back to the time domain:

$$\boxed{v_2(t)=\left[0.5\,e^{\frac{-t}{10}}-2.5\right]u(t)\quad V}$$

iii) The response of this system when the input is $v_1(t)=t\,e^{-2t}\,u(t)$ V .

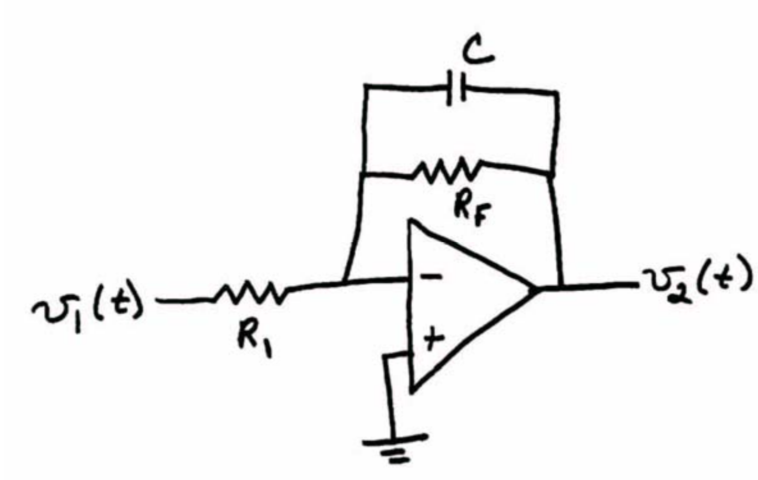
$$v_1(t)=t\,e^{-2t}\,u(t)\leftrightarrow V_1(s)=\frac{1}{(s+2)^2}$$

$$V_2(s)=\frac{\frac{5}{(s+2)^2}+40}{-20\,s-2}=\frac{40\,s^2+160\,s+165}{-20\,s^3-82\,s^2-88\,s-8}$$

Using residue():

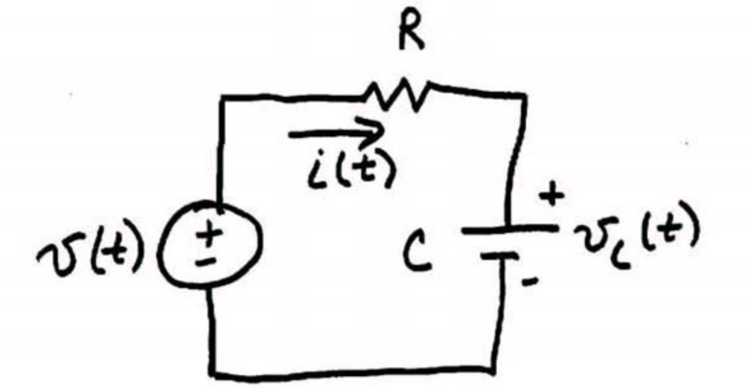
$$V_2(s)=\frac{0.0693}{s+2}+\underset{\substack{\uparrow\\ \text{repeated}}}{\frac{0.1316}{(s+2)^2}}-\frac{2.0693}{s+\frac{1}{10}}$$

$$\boxed{v_2(t)=\left[0.0693\,e^{-2t}+0.1316\,t\,e^{-2t}-2.0693\,e^{\frac{-t}{10}}\right]u(t)\quad A}$$



4) A MATLAB Function to Write:

a) Consider the RC circuit shown below, with *v(t)* as the input and *i(t)* as the output.



If we write KVL in the frequency domain using the Unilateral Laplace Transform, we find the relation that: $V(s) = I(s)\cdot R + \dfrac{I(s) + C\cdot v_C(0^-)}{s\cdot C}$. Solving this equation for the circuit current in the frequency domain gives: $I(s) = \dfrac{1}{R}\cdot\dfrac{V(s)\cdot s - v_C(0^-)}{s + \dfrac{1}{R\cdot C}}$. Consider the step input:

$v(t) = u(t) \quad \Leftrightarrow \quad V(s) = \dfrac{1}{s}$. If we substitute this input into the circuit response equation, we get the frequency-domain representation of the step response as: $I(s) = \dfrac{1 - v_C(0^-)}{R}\cdot\dfrac{1}{s + \dfrac{1}{R\cdot C}}$. By means of the inverse transform, we find that the time-domain step response current is:

$$i(t) = \frac{1 - v_C(0^-)}{R}\cdot e^{\frac{-t}{R\cdot C}}\cdot u(t).$$

As expected, the time response of the current is a function of the component (R and C) values as well as the initial capacitor voltage.

For your homework problem, write a numerical MATLAB function ("m-file") that accepts the resistor value, capacitor value and initial capacitor voltage as inputs. The function then produces one plot axis with two curves. The first curve is the current step response with null initial conditions [i.e., $v_C(0^-)$ set to zero]. The second curve is the current step response utilizing the initial capacitor voltage supplied to the function. Create each plot over the time range from 0 seconds to five time constants (for this circuit, one time constant equals $R \cdot C$ seconds). Plot the two curves in such a way that they are easily distinguished from each other. Label each axis (including physical units). These two simultaneous plots allow a convenient visual evaluation of the influence of the initial capacitor voltage on the circuit current.

Hand in a copy of your MATLAB m-file *and* your plot output for the following case: R = 10 kΩ, C = 0.1 μF and $v_C(0^-)$ = 2. V.

Each student will develop their own MATLAB code.
Here is my version of the plot: