Solution

Worcester Polytechnic Institute — Department of Electrical and Computer Engineering
ECE2311 — Continuous-Time Signal and System Analysis — Term B'17

Homework 9: Due Tuesday, 5 December 2017 (3:00 P.M.)

Write your name and ECE box at the top of each page.

General Reminders on Homework Assignments:

- Always complete the reading assignments *before* attempting the homework problems.
- Show all of your work. Use written English, where applicable, to provide a log or your steps in solving a problem. (For numerical homework problems, the writing can be brief.)
- A solution that requires physical units is *incorrect* unless the units are listed as part of the result.
- Get in the habit of underlining, circling or boxing your result.
- Always write neatly. Communication skills are essential in engineering and science. "If you didn't write it, you didn't do it!"

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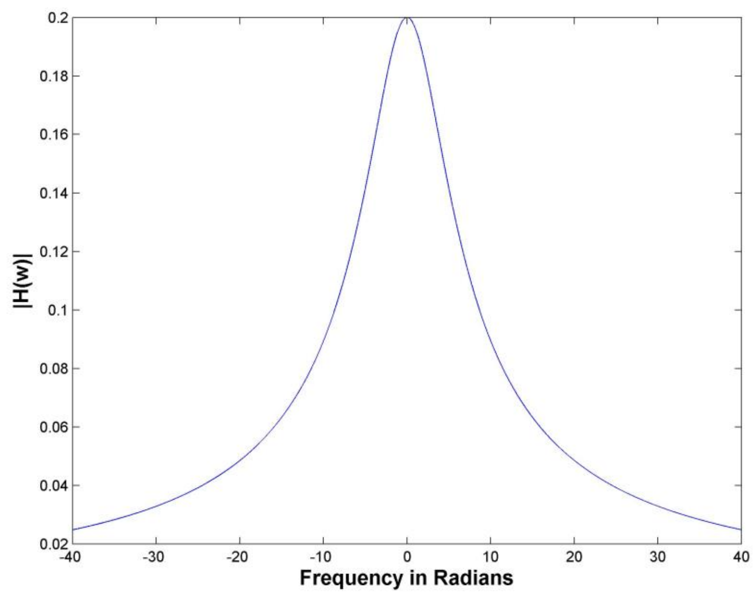
1) Computing the Fourier Transform:

a) Use direct integration to find the Fourier Transform of: $x(t) = \begin{cases} e^{-5t}, & 0 \le t \le 5 \\ 0, & \text{otherwise} \end{cases}$. Plot the magnitude of this transform. (Use MATLAB for the plotting, if you find it helpful.)

$$X(\omega) = \int_{t=-\infty}^{\infty} x(t) e^{-j\omega t}\, dt = \int_{t=0}^{5} e^{-5t} e^{-j\omega t}\, dt = \int_{t=0}^{5} e^{-t(j\omega+5)}\, dt$$

$$= \left. \frac{e^{-t(j\omega+5)}}{-(j\omega+5)} \right|_{t=0}^{5} = \frac{e^{-5(j\omega+5)} - e^{0}}{-(j\omega+5)} = \frac{1 - e^{-5(j\omega+5)}}{j\omega+5}$$

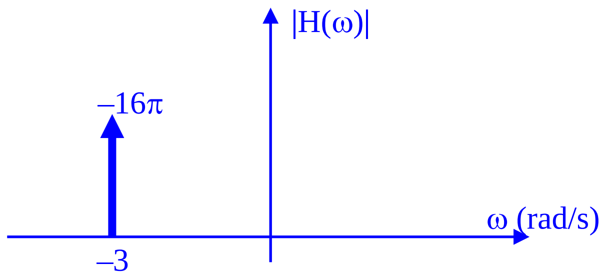


b) Use the transform tables to find the Fourier Transform of: $h(t) = -8e^{-j3t}$. Plot the magnitude of this transform. (Use MATLAB for the plotting, if you find it helpful.) Note: Since the time-domain function is complex-valued, the frequency-domain function magnitude is *not* symmetric about 0 Hertz.

From class notes: $e^{-j\omega_o t} \Leftrightarrow 2\pi\delta(\omega+\omega_o)$. Thus:

$$h(t) = -8e^{-j3t} \Leftrightarrow H(\omega) = -8[2\pi\delta(\omega+3)] = -16\,\pi\,\delta(\omega+3).$$

Hence, the Fourier Transform consists of an impulse of area –8π, located at a frequency of $\omega = -3$ radians/sec.



2) Computing the Inverse Fourier Transform:

a) Use direct integration to find the time-domain function whose Fourier Transform is:

$$G_\tau(\omega) = \begin{cases} 1, & |\omega| \le \tau/2 \\ 0, & |\omega| > \tau/2 \end{cases}$$. If this function is real-valued, write it without using the imaginary unit "*j*".

$$g(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} G(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{\omega=-\tau/2}^{\tau/2} e^{j\omega t}\,d\omega = \left.\frac{e^{j\omega t}}{2\pi\cdot jt}\right|_{\omega=-\tau/2}^{\tau/2}$$

$$= \frac{e^{j\frac{\tau}{2}t} - e^{-j\frac{\tau}{2}t}}{t\pi\cdot 2j} = \frac{1}{\pi}\cdot\frac{\sin\left(\frac{\tau}{2}t\right)}{t} = \frac{\tau}{2\pi}\cdot \mathrm{sinc}\left(\frac{\tau}{2}t\right)$$

Note: The sinc() function, by definition, has a defined value of 1 at $t = 0$.

b) Use direct integration to find the time-domain function whose Fourier Transform is:

$$H(\omega) = \begin{cases} 5, & 0 < \omega < 5 \\ 0, & \text{otherwise} \end{cases}$$. If this function is real-valued, write it without using the imaginary unit "*j*".

$$h(t)=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}H(\omega)e^{j\omega t}\,d\omega=\frac{1}{2\pi}\int_{\omega=0}^{5}5e^{j\omega t}\,d\omega=\left.\frac{5e^{j\omega t}}{jt\cdot 2\pi}\right|_{\omega=0}^{5}$$

$$=\frac{5e^{j5t}-5}{j\cdot 2\pi t}=j5\cdot\frac{1-e^{j5t}}{2\pi t},\quad t\neq 0$$

This function cannot be evaluated at $t = 0$ (formula gives 0/0). Hence, at $t = 0$:

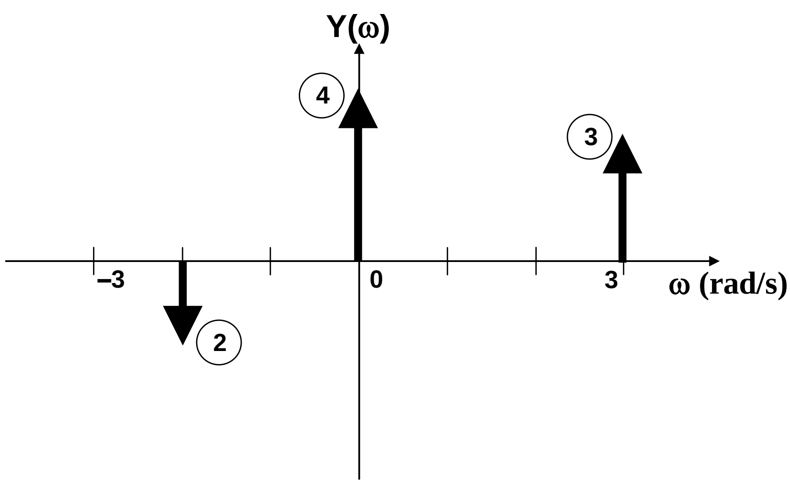
$$h(0)=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}H(\omega)e^{j\omega 0}\,d\omega=\frac{1}{2\pi}\int_{\omega=0}^{5}5\,d\omega=\left.\frac{5\omega}{2\pi}\right|_{\omega=0}^{5}$$

$$=\frac{25-0}{2\pi}=\frac{25}{2\pi},\quad t=0$$

Since $H(\omega)$ magnitude is not symmetric about 0 Hz, $h(t)$ is complex-valued. Gathering the full result gives:

$$\boxed{h(t)=\begin{cases} j5\cdot\dfrac{1-e^{j5t}}{2\pi t}, & t\neq 0\\[2mm] \dfrac{25}{2\pi}, & t=0\end{cases}}$$

c) Use the transform tables to find the inverse Fourier Transform of the function shown in the following figure:



Since $|Y(\omega)|$ is NOT symmetric about 0 Hz, we know that $y(t)$ is complex-valued.

From the transform table:

$$Y(\omega)=4\delta(\omega)+3\delta(\omega-3)-2\delta(\omega+2)$$

$$\Updownarrow$$

$$y(t)=\frac{2}{\pi}+\frac{3}{2\pi}e^{j3t}-\frac{1}{\pi}e^{-j2t}$$

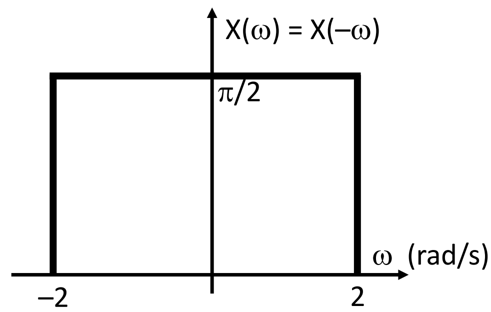
3) Properties of the Fourier Transform: Let *X(ω)* be the Fourier Transform of *x(t)* and *Y(ω)* be the Fourier Transform of *y(t)*.

a) Find *Y(ω)* as a function of *X(ω)* if: $y(t) = x(3-t) - x(-7-t)$.

$$\boxed{\begin{gathered} y(t) = x(3-t) - x(-7-t) = x\{-(t-3)\} - x\{-(t+7)\} \\ \Updownarrow \\ Y(\omega) = e^{-j\omega 3} X(-\omega) - e^{j\omega 7} X(-\omega) \end{gathered}}$$

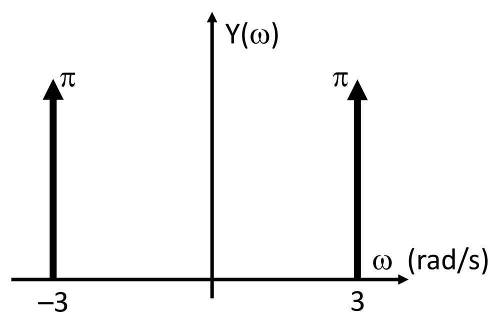
b) Use the property $x(t)\cdot y(t) \Leftrightarrow \frac{1}{2\pi} X(\omega) * Y(\omega)$ and the transform table to find the Fourier Transform of: $h(t) = \text{sinc}(2t)\cdot\cos(3t)$.

Let



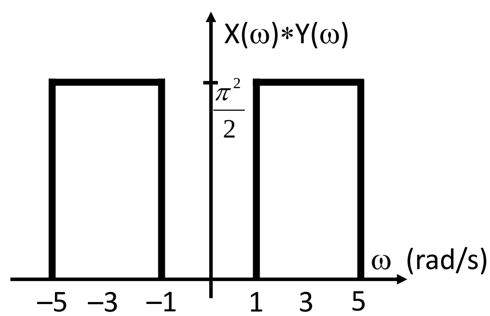
$x(t)=\text{sinc}(2t) \quad \Leftrightarrow \quad X(\omega)=\begin{cases}\dfrac{\pi}{2}, & |\omega|<2 \\ 0, & \text{otherwise}\end{cases}$ ➔

and



$y(t)=\cos(3t) \quad \Leftrightarrow \quad Y(\omega)=\pi[\delta(\omega-3)+\delta(\omega+3)]$ ➔

Note that $X(\omega)*\delta(\omega-3)=X(\omega-3)$ and $X(\omega)*\delta(\omega+3)=X(\omega+3)$.
Thus,



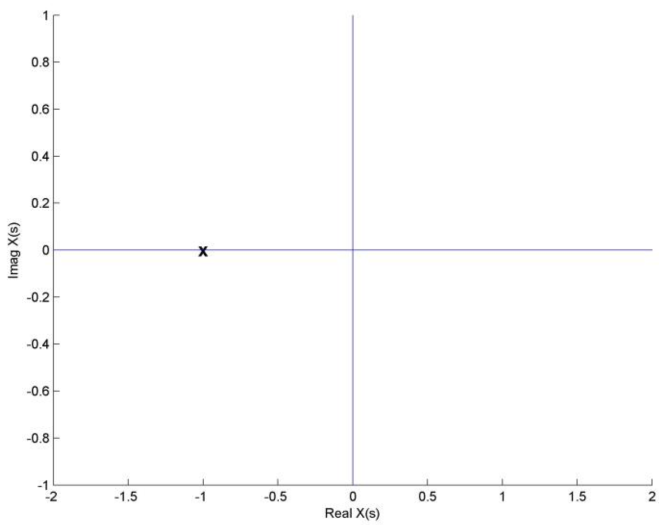
So, scale by $\dfrac{1}{2\pi}$ to get $H(\omega)$:

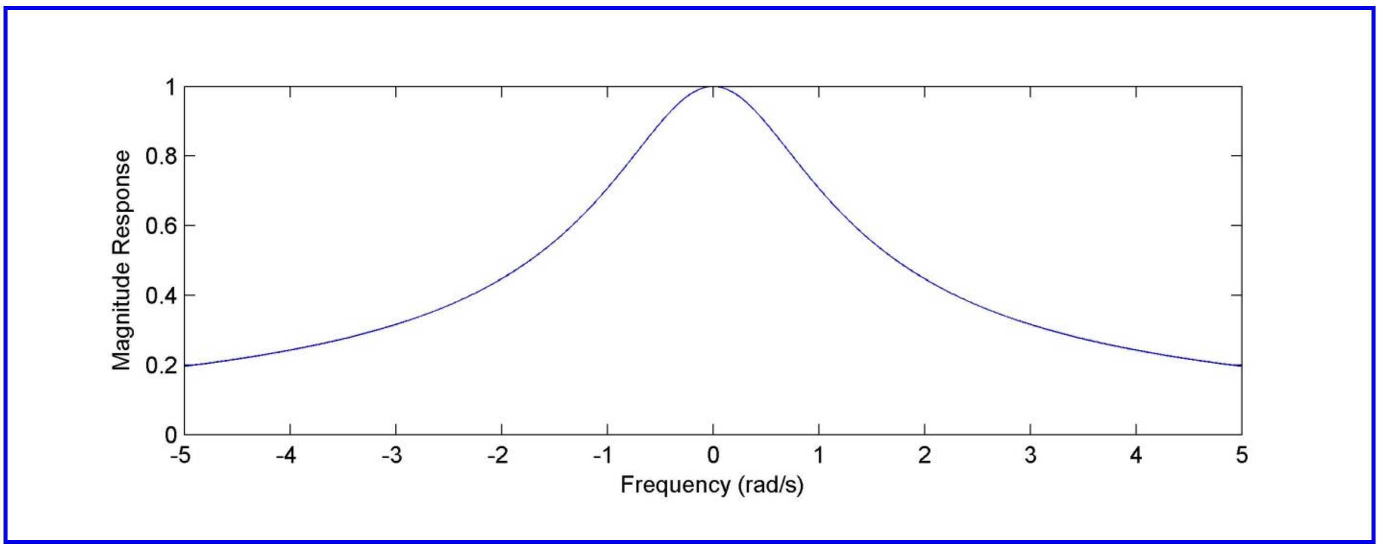
$$\boxed{H(\omega)=\begin{cases}\dfrac{\pi}{4}, & 1<|\omega|<5 \\ 0, & \text{otherwise}\end{cases}}$$

4) Magnitude Response Based on Pole and Zero Locations: For each of the following problems, you will be supplied a pole-zero diagram of a system. Sketch the **<u>magnitude response</u>** of each system, as a function of ω. **<u>Assume an unknown gain of one.</u>** Do so <u>by hand and without computational aids</u>.

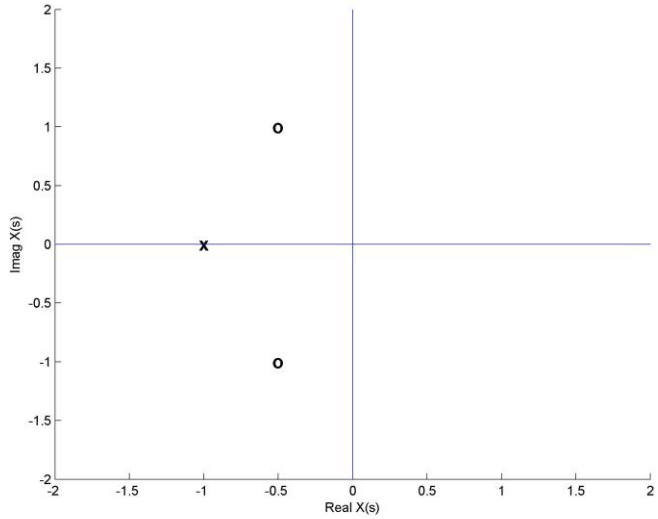
Yes, you could dump the system function into MATLAB to generate your plots. But, you *should* be able to create a reasonable estimate by hand and you *will* be expected to generate hand-drawn plots on the exam. Having a rough estimate of the frequency response based on the poles and zeros is an important skill for this area of study. (Sure, go ahead and *double-check* the shape using MATLAB's numerical routines. But, try it by hand **first!!!**)

a)

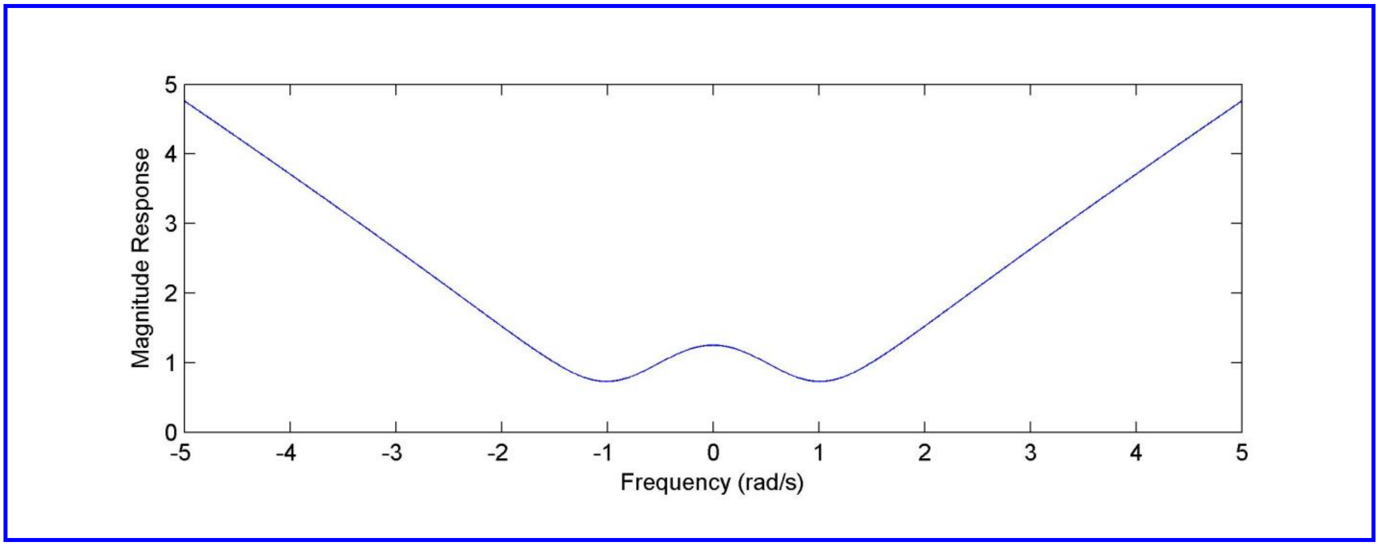




b)

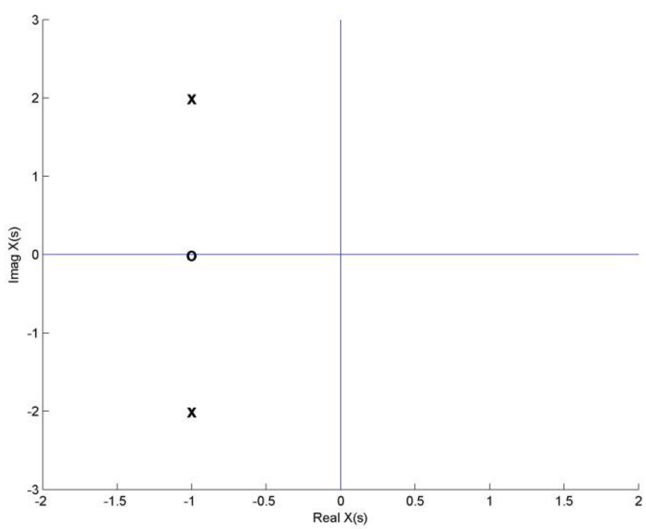


Imag X(s)
Real X(s)

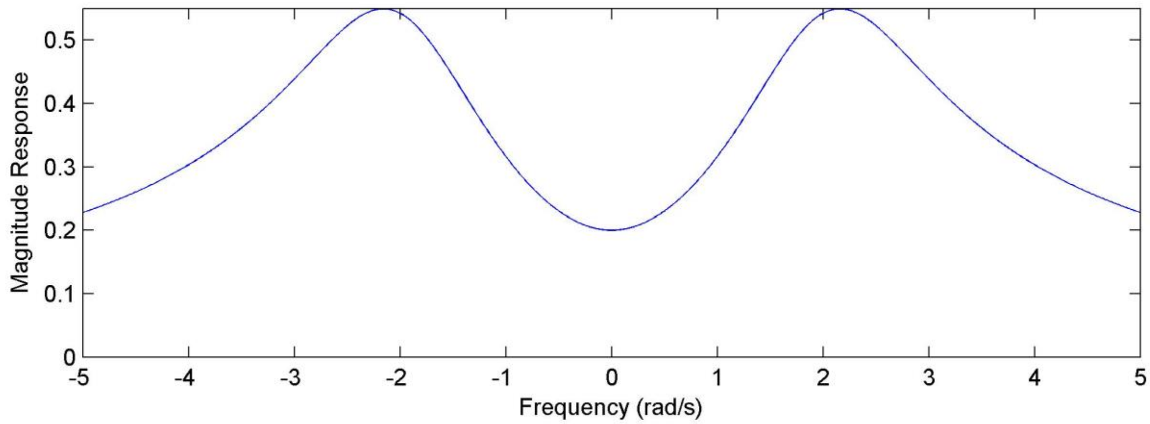


Magnitude Response
Frequency (rad/s)

c)

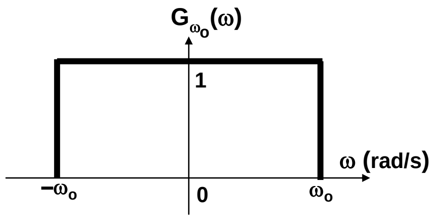


Imag X(s)
Real X(s)

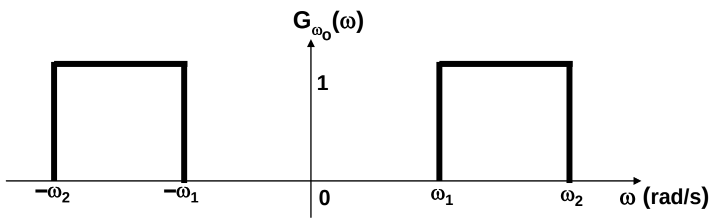


5) **Standard Band-Pass/Band-Stop Filters:**

a) Consider the Fourier Transform Pair: $\frac{\omega_o}{\pi}\cdot \text{sinc}(\omega_o t) \Leftrightarrow G_{\omega_o}(\omega) = \begin{cases} 1, & |\omega| < \omega_o \\ 0, & |\omega| > \omega_o \end{cases}$. You should be able to see that this frequency response, if viewed as a filter, is the frequency response of an ideal low pass filter, as shown here:



Now, consider the frequency response: $H_{\omega_1,\omega_2}(\omega) = \begin{cases} 1, & \omega_1 < |\omega| < \omega_2 \\ 0, & \text{otherwise} \end{cases}$. This filter is an ideal bandpass filter, as shown here:



Use direct integration of the inverse Fourier Transform to show that the impulse response of this system is: $h(t) = \begin{cases} \frac{\sin(\omega_2 t) - \sin(\omega_1 t)}{\pi t}, & t \neq 0 \\ \frac{\omega_2 - \omega_1}{\pi}, & t = 0 \end{cases}$.

$$h(t)=\frac{1}{2\pi}\int_{\omega=-\omega_2}^{-\omega_1}e^{j\omega t}\,d\omega+\frac{1}{2\pi}\int_{\omega=\omega_1}^{\omega_2}e^{j\omega t}\,d\omega=\left.\frac{e^{j\omega t}}{2\pi\,jt}\right|_{\omega=-\omega_2}^{-\omega_1}+\left.\frac{e^{j\omega t}}{2\pi\,jt}\right|_{\omega=\omega_1}^{\omega_2}$$

$$=\frac{e^{-j\omega_1 t}-e^{-j\omega_2 t}+e^{j\omega_2 t}-e^{j\omega_1 t}}{2\pi\,jt}=\frac{1}{\pi t}\left[\frac{e^{j\omega_2 t}-e^{-j\omega_2 t}}{2j}-\frac{\left(e^{j\omega_1 t}-e^{-j\omega_1 t}\right)}{2j}\right]$$

$$=\frac{\sin(\omega_2 t)-\sin(\omega_1 t)}{\pi t},\quad t\neq 0$$

Above cannot be evaluated at t=0. At this time:

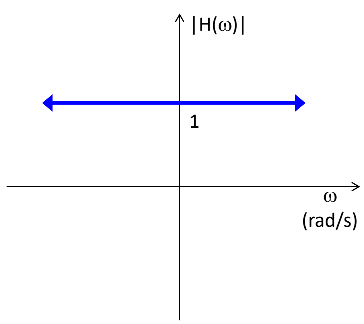
$$h(0)=\frac{1}{2\pi}\int_{\omega=-\omega_2}^{-\omega_1}e^{j\omega 0}\,d\omega+\frac{1}{2\pi}\int_{\omega=\omega_1}^{\omega_2}e^{j\omega 0}\,d\omega=\frac{1}{2\pi}\int_{\omega=-\omega_2}^{-\omega_1}d\omega+\frac{1}{2\pi}\int_{\omega=\omega_1}^{\omega_2}d\omega$$

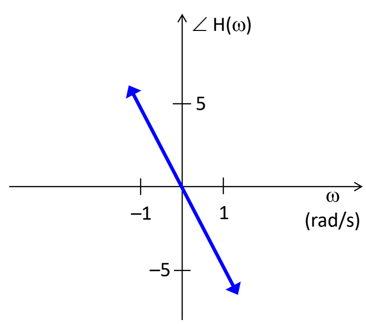
$$=\left.\frac{\omega}{2\pi}\right|_{\omega=-\omega_2}^{-\omega_1}+\left.\frac{\omega}{2\pi}\right|_{\omega=\omega_1}^{\omega_2}=\frac{-\omega_1-(-\omega_2)+\omega_2-\omega_1}{2\pi}=\frac{\omega_2-\omega_1}{\pi}$$

Combining these two results gives the full solution shown in the problem statement.

b) The "Group Delay" of a linear system is defined as the rate of change of the phase of the system's frequency response: $\tau(\omega)=-\frac{d\{\angle H(\omega)\}}{d\omega}$. Sketch the magnitude and phase response, and find the group delay for the linear systems defined by: $H(\omega)=e^{-5j\omega}$.

This function is already written in polar form. Thus, $|H(\omega)|=1$ and $\angle H(\omega)=-5\omega$ and:





By definition: $\tau(\omega)=\frac{-d(-5\omega)}{d\omega}=5\,\text{sec}$.