Solution

Worcester Polytechnic Institute — Department of Electrical and Computer Engineering
ECE2311 — Continuous-Time Signal and System Analysis — Term B'17

Homework 10: Due Friday, 8 December 2017 (3:00 P.M.)

Write your name and ECE box at the top of each page.

General Reminders on Homework Assignments:

- Always complete the reading assignments *before* attempting the homework problems.
- Show all of your work. Use written English, where applicable, to provide a log or your steps in solving a problem. (For numerical homework problems, the writing can be brief.)
- A solution that requires physical units is *incorrect* unless the units are listed as part of the result.
- Get in the habit of underlining, circling or boxing your result.
- Always write neatly. Communication skills are essential in engineering and science. "If you didn't write it, you didn't do it!"

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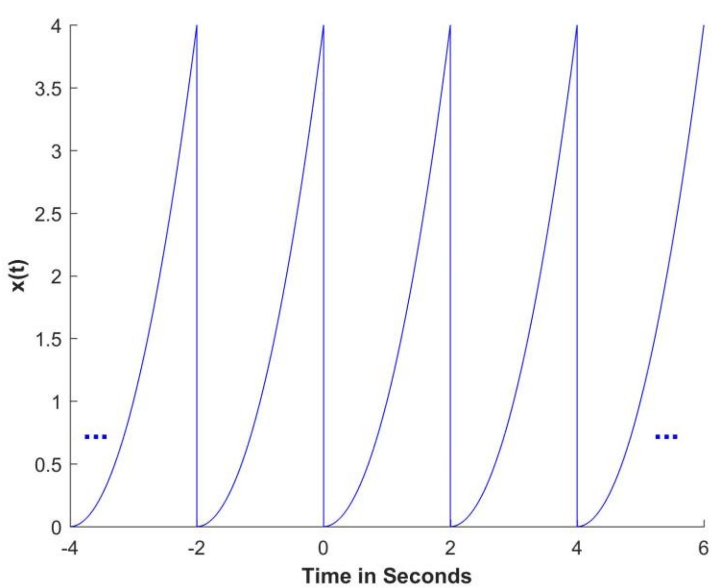
1) Fourier Series Synthesis and Analysis Equations:

a) Let *x(t)* be a periodic signal with period $T_o = 2$ seconds. Over the interval from [0, 2 seconds), define *x(t)* as: $x(t) = t^2$.

i) Sketch *x(t)*.



ii) Find the exponential form Fourier Series representation of *x(t)*.

$d_k = \frac{1}{T_o}\int_{T_o} x(t)e^{-jk\omega_o t}\,dt$, where $T_o = 2 \quad \Rightarrow \quad \omega_o = \frac{2\pi}{T_o} = \pi$.

Thus,

$$d_k = \frac{1}{2}\int_0^2 t^2\, e^{-jk\pi t}\,dt = \frac{-e^{-jk\pi t}}{2}\cdot\left(\frac{-k^2\pi^2 t^2 + 2jk\pi t + 2}{-jk^3\pi^3}\right)\Bigg|_{t=0}^{2}$$

$$= \frac{-e^{-j2\pi k}}{2}\cdot\left(\frac{-4\pi^2 k^2 + j4\pi k + 2}{-j\pi^3 k^3}\right) - -\frac{e^0}{2}\left(\frac{0+0+2}{-j\pi^3 k^3}\right).$$

$$= \frac{1}{2}\cdot\left(\frac{4\pi^2 k^2 - j4\pi k - 2 + 2}{-j\pi^3 k^3}\right) = \frac{2\pi k - 2j}{-j\pi^2 k^2} = \frac{2 + j2\pi k}{\pi^2 k^2}, \quad k \neq 0$$

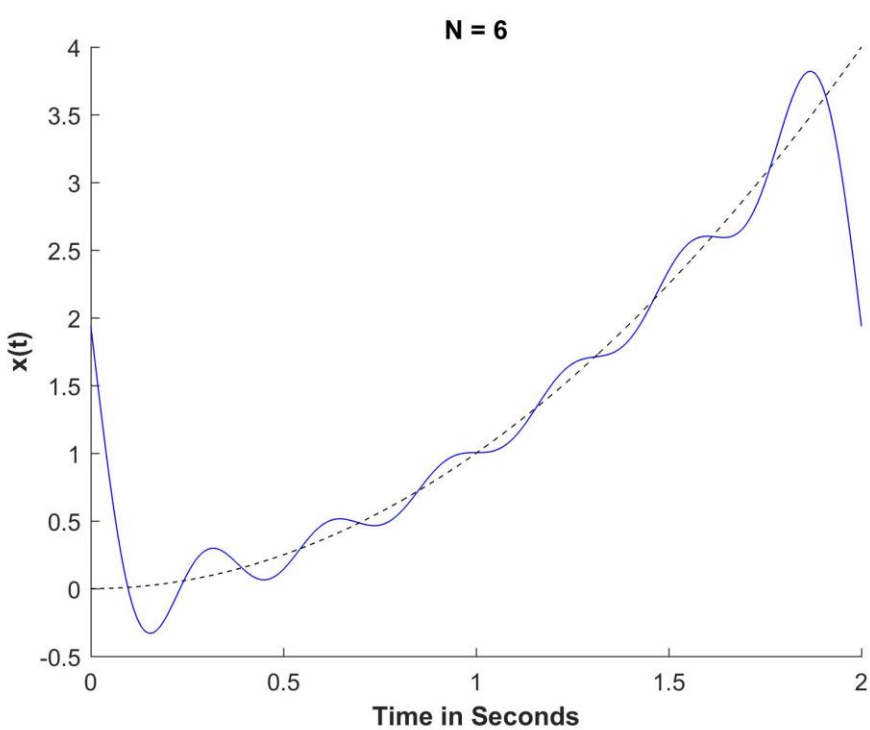
At $k=0$,

$$d_0 = \frac{1}{2}\int_0^2 t^2\, e^{-j0\cdot\pi t}\,dt = \frac{1}{2}\int_0^2 t^2\,dt = \frac{1}{2}\cdot\frac{t^3}{3}\Bigg|_{t=0}^{2} = \frac{8}{6} - \frac{0}{6} = \frac{4}{3}, \quad k = 0.$$

Combining:

$$\boxed{d_k = \begin{cases} \dfrac{2 + j2\pi k}{\pi^2 k^2}, & k \neq 0 \\ \dfrac{4}{3} & k = 0 \end{cases}}.$$

iii) Use MATLAB to plot the ideal waveform (over the period from 0 to 2 seconds) along with the Fourier Series representation utilizing up to and including the first six harmonics (i.e., k = –6, –5, –4, … +6).



b) The exponential form Fourier Series coefficients for a function are given as:
$d_k = \dfrac{(e-1)(1-j2\pi k)}{e(1+4\pi^2k^2)}$.

i) Find the trigonometric form coefficients a_0, a_k and b_k.

$$a_0 = d_0 = \frac{(e-1)(1-j2\pi\cdot 0)}{e(1+4\pi^2\cdot 0^2)} = \frac{(e-1)}{e}$$

$$a_k = d_k + d_{-k} = \frac{(e-1)(1-j2\pi k)}{e(1+4\pi^2k^2)} + \frac{(e-1)(1+j2\pi k)}{e(1+4\pi^2k^2)} = \frac{2(e-1)}{e(1+4\pi^2k^2)}$$

$$b_k = j(d_k - d_{-k}) = j\left[\frac{(e-1)(1-j2\pi k)}{e(1+4\pi^2k^2)} - \frac{(e-1)(1+j2\pi k)}{e(1+4\pi^2k^2)}\right]$$
$$= j\left[\frac{(e-1)(-2j2\pi k)}{e(1+4\pi^2k^2)}\right] = \frac{(e-1)(4\pi k)}{e(1+4\pi^2k^2)}$$

ii) Find the compact trigonometric form coefficients c_0, c_k and θ_k.

$$c_0 = a_0 = d_0 = \frac{(e-1)}{e}$$

$$c_k = \sqrt{a_k^2 + b_k^2} = \frac{2(e-1)}{e(1+4\pi^2k^2)}\cdot\sqrt{1+4\pi^2k^2}$$

$$\theta_k = \tan^{-1}\left(\frac{-b_k}{a_k}\right) = \tan^{-1}\left[\frac{-2(e-1)2\pi k}{2(e-1)}\right] = \tan^{-1}(-2\pi k)$$

2) Properties of the Fourier Series:

a) If signal $x(t)$ has (exponential form) Fourier Series coefficients d_k, show that signal $x(t-t_o)$ has Fourier Series coefficients $d_k' = d_k\, e^{-jk\omega_o t_o}$.

By definition: $x(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_o t}$.

Replace $t \to t - t_o$, giving: $x(t-t_o) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_o(t-t_o)} = \sum_{k=-\infty}^{\infty}\left\{d_k e^{-jk\omega_o t_o}\right\}e^{jk\omega_o t} = \sum_{k=-\infty}^{\infty} d_k' e^{jk\omega_o t}$,

where $d_k' = d_k\, e^{-jk\omega_o t_o}$. Q.E.D.

b) For each of the following, determine if the signal is periodic or not, and state why. For periodic signals, state the period.

i) $h(t) = \sin(t) + \sin(3t)$

$$\sin(t) \Rightarrow \omega_1 = 1\,\text{rad/s} \Rightarrow T_1 = \frac{2\pi}{\omega_1} = 2\pi\,\text{seconds}$$

and

$$\sin(3t) \Rightarrow \omega_2 = 3\,\text{rad/s} \Rightarrow T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{3}\,\text{seconds}.$$

The least common multiple of these two periods is 2π seconds.

Hence, ***h(t)* is periodic and its period = 2π seconds**.

ii) $x(t) = \sin(t) + \sin(\pi t)$

$$\sin(t) \Rightarrow \omega_1 = 1\,\text{rad/s} \Rightarrow T_1 = \frac{2\pi}{\omega_1} = 2\pi\,\text{seconds}$$

and

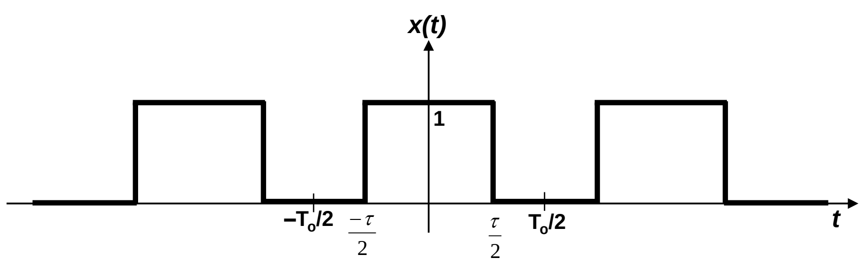
$$\sin(\pi t) \Rightarrow \omega_2 = \pi\,\text{rad/s} \Rightarrow T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{\pi} = 2\,\text{seconds}.$$

These two periods have no common multiple.

Hence, *x(t)* is **NOT PERIODIC**.

3) MATLAB:

a) In class, we showed that the Fourier Series coefficients of the square wave shown below are, using the exponential form, equal to $\begin{cases} d_0 = \dfrac{\tau}{T_o} \\ d_{k\neq 0} = \dfrac{\sin(k\,\omega_o\,\tau/2)}{k\,\pi} \end{cases}$, where τ is the duration of the "on" portion of the waveform, and $T_o = \dfrac{2\,\pi}{\omega_o}$ is the square wave period.



Write a numerical function in MATLAB (***not*** a symbolic function) that is given as input an index value *N* representing the Fourier Series harmonic. (*N* = 0 would denote the mean value term represented by d_0. *N* = 1 would represent the first harmonic terms represented by d_1 and d_{-1}. Etc.) Use the Fourier Series Synthesis Equation to represent the square wave using all terms up to and including the N^{th} harmonic. Thus, for *N* = 3, the synthesis equation would include the summing index terms: *k* = {–3, –2, –1, 0, 1, 2, 3}. Use only one "for" loop in the synthesis portion of your code. Plot the resulting sum over the interval $(-T_0/2, T_0/2)$. On the same graph, also plot the ideal square wave (over the same interval).

As your solution, hand in your MATLAB program and your plot result for {$N = 9$, $\tau = 1/4$ sec and $T_o = 1$ sec}.

Each student will produce their own MATLAB program. The plot for all students should look the same (although the choice of *y*-axis scaling and *x*-axis duration may vary student-to-student.