Fourier Represents $x(t)$ as Sum of Sines, Cosines

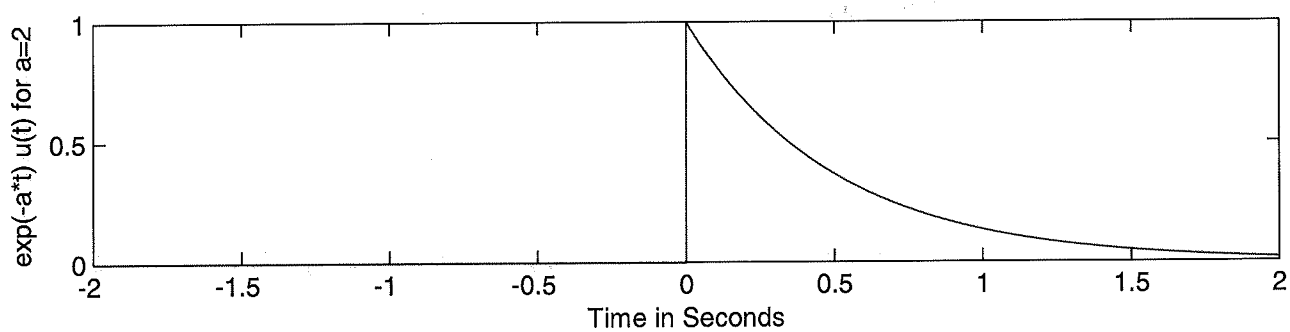
$$x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)\, e^{+j\omega t}\, d\omega = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)\left[\cos(+\omega t) + j\,\sin(+\omega t)\right] d\omega$$

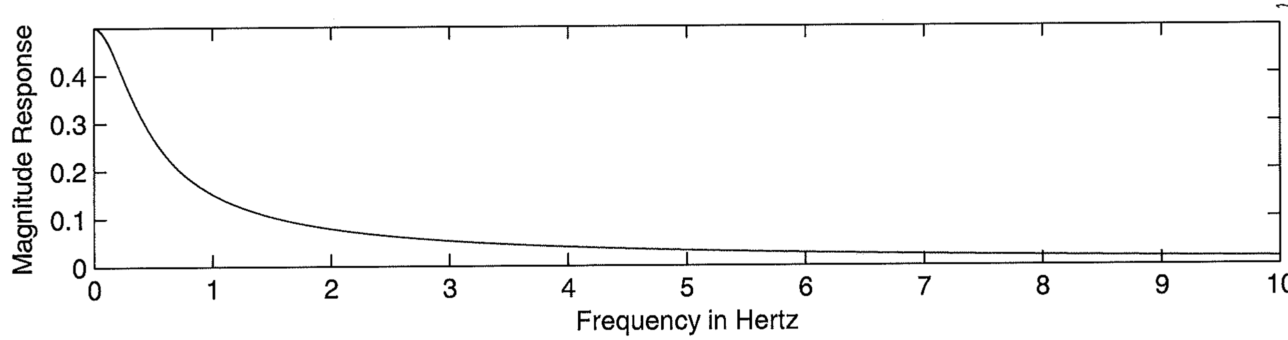
- MATLAB cannot numerically perform infinite sum
  - Approximate, understand w/ finite sum

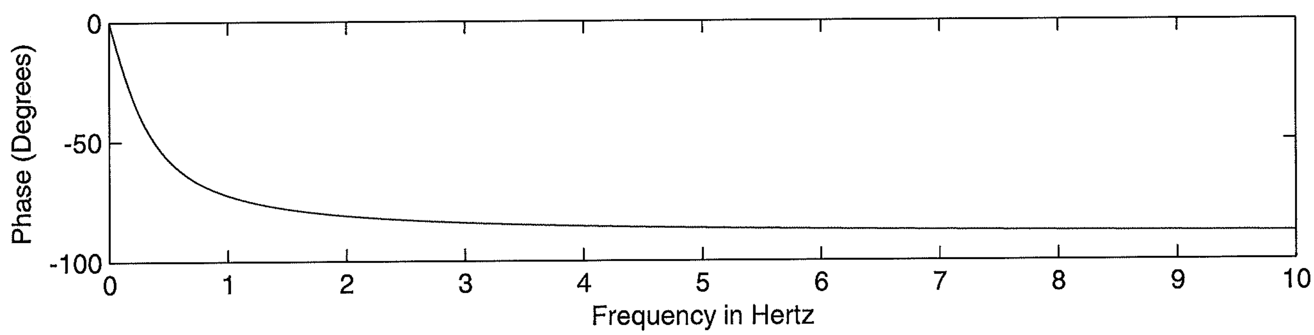
Decaying Exponential

$$e^{-at}u(t) \longleftrightarrow \frac{1}{j\omega + a}, \quad a>0$$

$e^{-2t}u(t)$ seconds

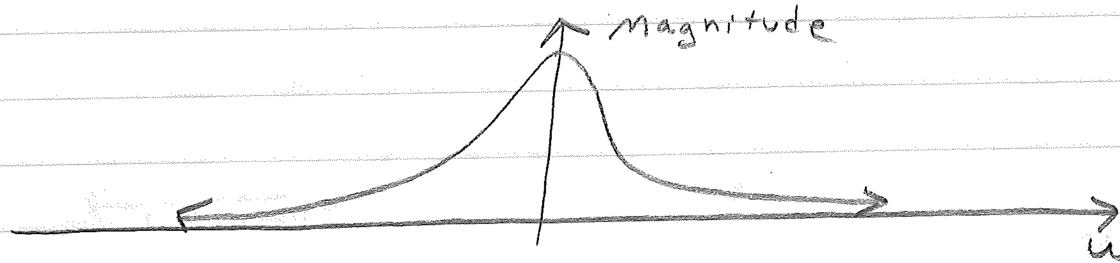


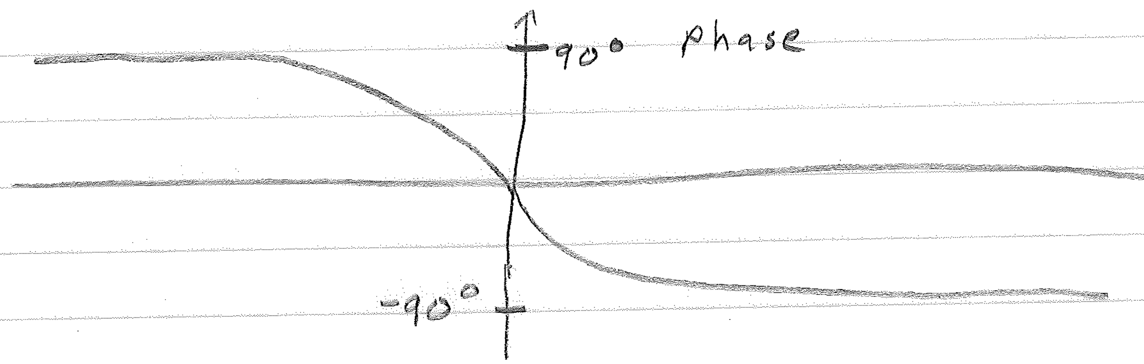




Positive-valued frequencies of Fourier Transform

Summing Concept





1) Real-valued $x(t) \Rightarrow X(\omega) = X^*(-\omega)$

∘ At any $|\omega|$, sum $X(\omega) + X(-\omega)$ contributions

$\Rightarrow$ of form: $re^{j(\omega t+\theta)} + re^{-j(\omega t+\theta)} \Longleftrightarrow \sim r\cos(\omega t + \theta)$

2) Time, frequency scale as:

$$\int_{t=-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{\omega=\infty}^{\infty} |X(\omega)|^2 d\omega$$

- Normalize magnitude squared areas

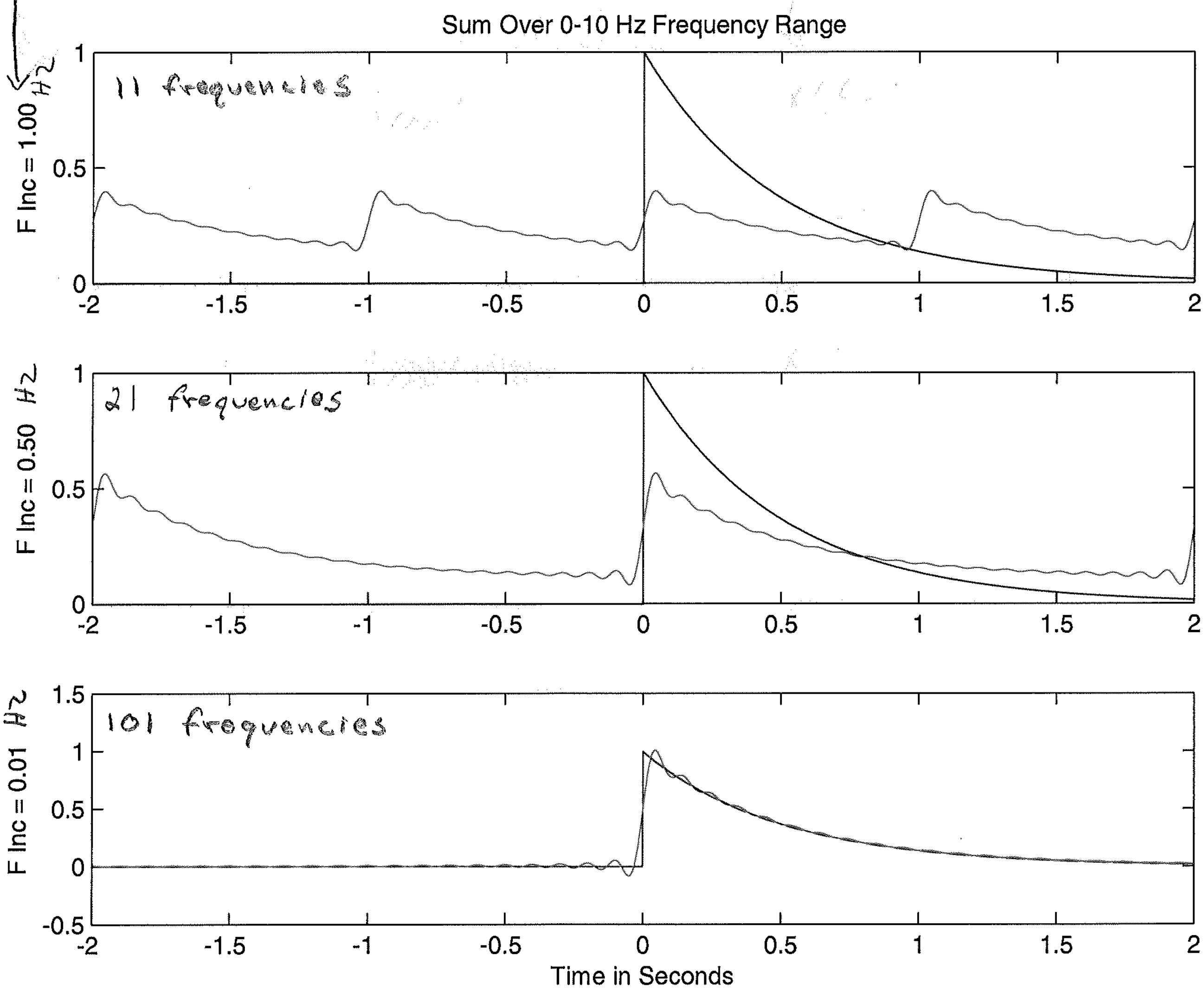
Example Fourier Sum

MATLAB Results

And, energy scaled

I.e., $|X(f=0)|\cos(0+\angle X(f=0)) + |X(f=1)|\cos(2\pi\cdot 1+\angle X(f=1))$

$+ \ldots + |X(f=10)|\cos(2\pi\cdot 10+\angle X(f=10))$



Example Fourier Sum

Recall: Rational Transforms

$$a_R \frac{d^R y(t)}{dt^R} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_P \frac{d^P x(t)}{dt^P} + \cdots + b_0 x(t)$$

$$\Updownarrow$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_P s^P + \cdots + b_1 s + b_0}{a_R s^R + \cdots + a_1 s + a_0}$$

$$= \frac{b_P}{a_R} \cdot \frac{(s-z_1)(s-z_2)\cdots(s-z_P)}{(s-p_1)(s-p_2)\cdots(s-p_R)} = G \frac{\prod_{m=1}^{P}(s-z_m)}{\prod_{m=1}^{R}(s-p_m)}$$

- So, if ROC includes $j\omega$ axis:

$$\boxed{H(j\omega) = G \frac{\prod_{m=1}^{P}(j\omega - z_m)}{\prod_{m=1}^{R}(j\omega - p_m)}}$$

In general: z_m, p_m complex.

Rational Fourier Transforms

Graphical Evaluation in the S-Plane

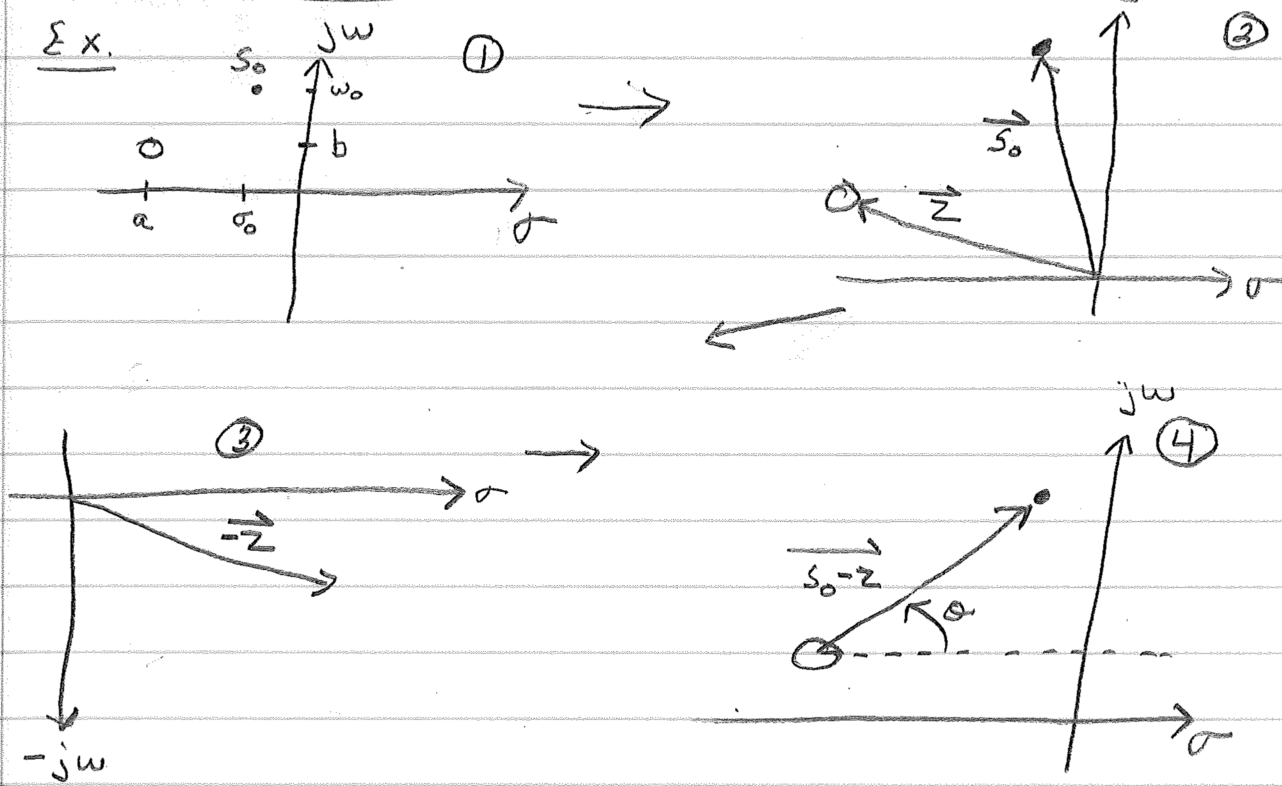
- **Consider:** System with one zero (2^{nd} quadrant)

$$X(s) = \underbrace{s}_{\sigma + j\omega} - \underbrace{z}_{a + jb}$$

> **Note:**
> $a < 0,\ b > 0$
> for 2^{nd} Quadrant

- **Graphically evaluate** at $s = s_0 = \sigma_0 + j\omega_0$

Ex.



1) $|X(s)| = |\overrightarrow{s_0 - z}|$ = Distance from z to s_0

2) $\angle X(s) = \angle \overrightarrow{s_0 - z}$ = Angle between positive real axis and $\overrightarrow{s_0 - z}$

Frequency Response from the s-Plane

- If $j\omega$ axis in Bilateral Laplace ROC:

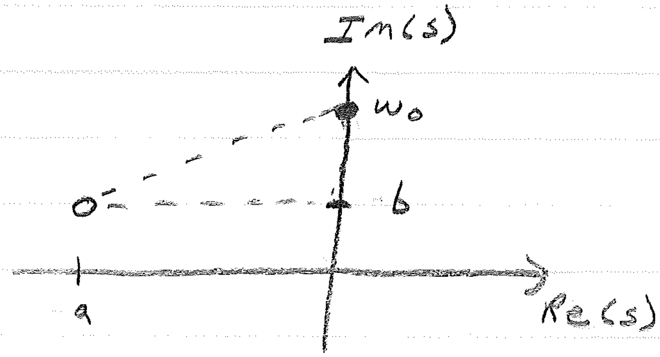
$$H(\omega) = H(s)\Big|_{s=j\omega}$$

  - So, set $\sigma = 0$, evaluate along $j\omega = j\omega_0$ axis

- Again, consider 2^{nd}-quadrant zero

$$H(\omega) = j\omega - z \;, \text{ for } z = a + jb$$

$$(a<0,\; b>0)$$



Distance minimum at $\omega_0 = b$

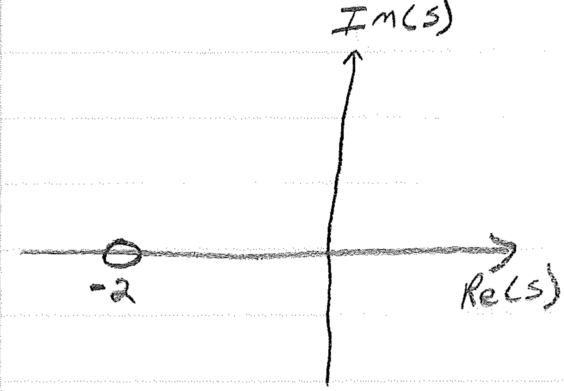
- $$d_{z\to\omega_0} = |H(\omega)| = \sqrt{(\omega_0 - b)^2 + a^2} = \sqrt{Im^2\{H(\omega)\} + Re^2\{H(\omega)\}}$$

- $$\measuredangle H(\omega) = \tan^{-1}\left(\frac{\omega_0 - b}{a}\right) = \tan^{-1}\left(\frac{Im\{H(\omega)\}}{Re\{H(\omega)\}}\right)$$

Rational Fourier Transforms

Frequency Response of LHP zero

- **Let:** $H(s) = s+2$. **Draw:** $H(\omega)$



- Magnitude minimal @ $\omega_0 = 0$ (closest point)
  - Grows as $|\omega_0|$ increases
  - Symmetric about $\omega_0 = 0$

- $H(0) = 2 \Rightarrow \measuredangle H(0) = 0°$
- Grows to $\begin{cases} +90° \text{ at } \omega_0 = \infty \\ -90° \text{ at } \omega_0 = -\infty \end{cases}$

Sol'n

$$H(\omega) = j\omega + 2$$

Helpful Points:

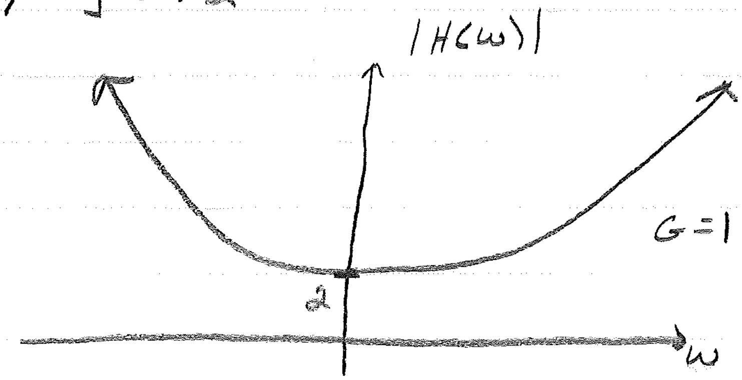
$H(0) = 2 = 2\angle 0°$

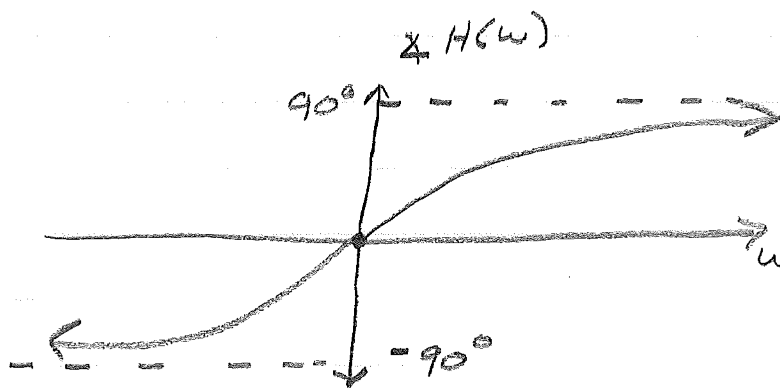
$H(2) = 2 + j2$
$= 2\sqrt{2}\angle 45°$

$H(\infty) = 2 + j\infty$
$= \infty\angle 90°$

$H(-2) = 2 - j2$
$= 2\sqrt{2}\angle -45°$

$H(-\infty) = 2 - j\infty$
$= \infty\angle -90°$

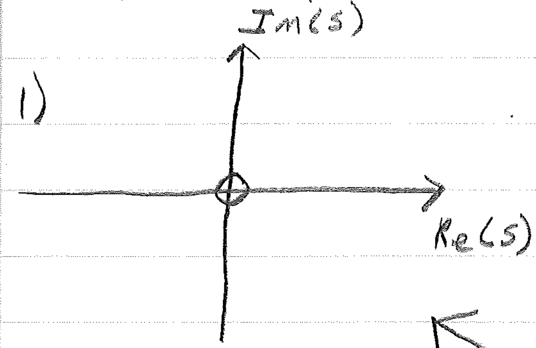




Rational Fourier Transforms

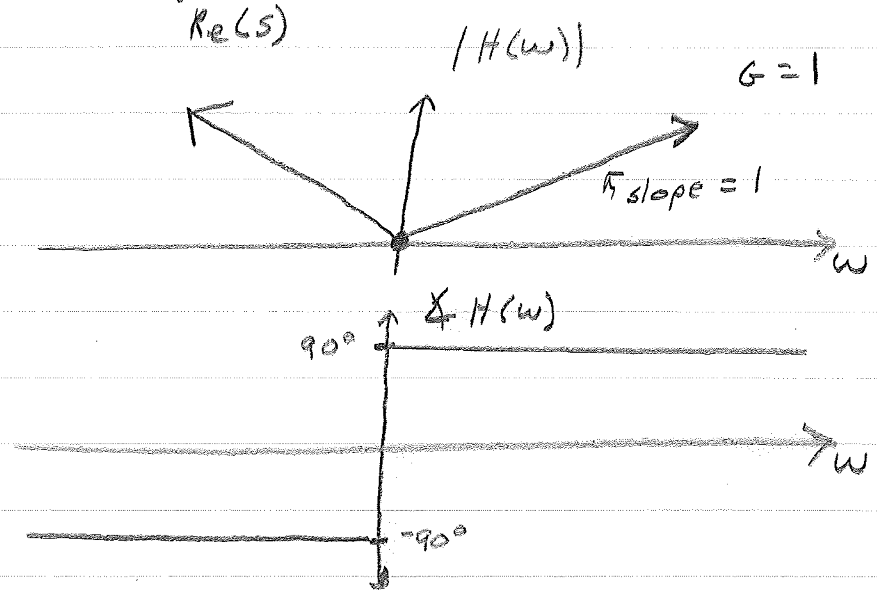
Examples: Single Zero

1)

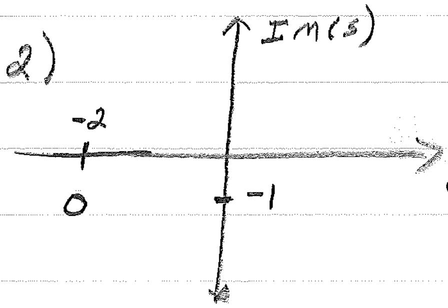


$\Rightarrow \quad H(s) = s$

$H(\omega) = j\omega$



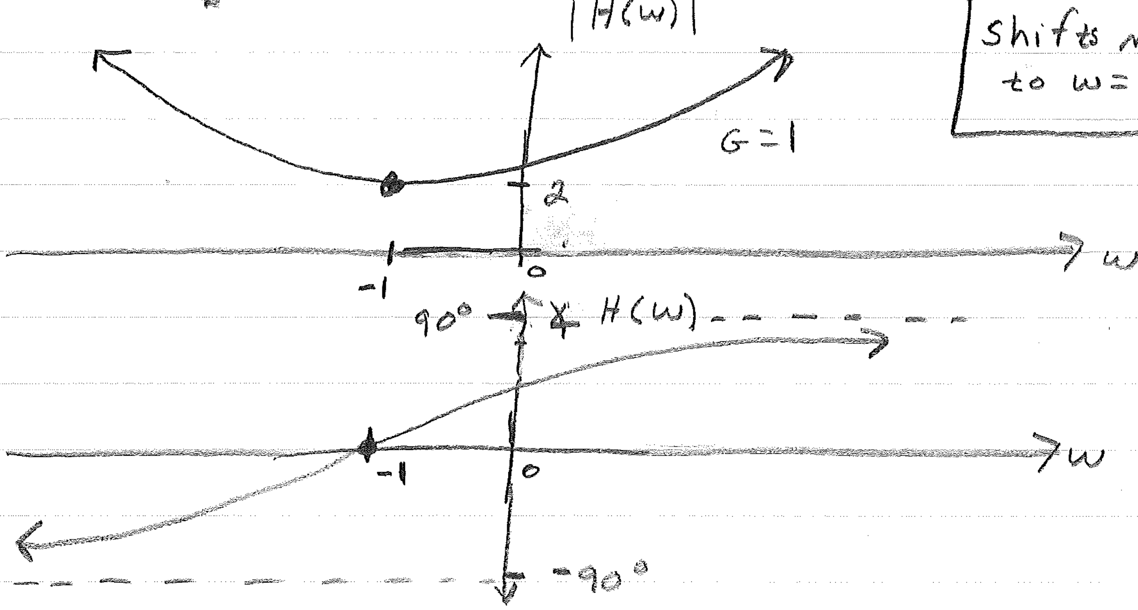
2)



$\Rightarrow \quad H(s) = s + \underbrace{2 + j}_{\text{zero}}$

$H(\omega) = j\omega + 2 + j$

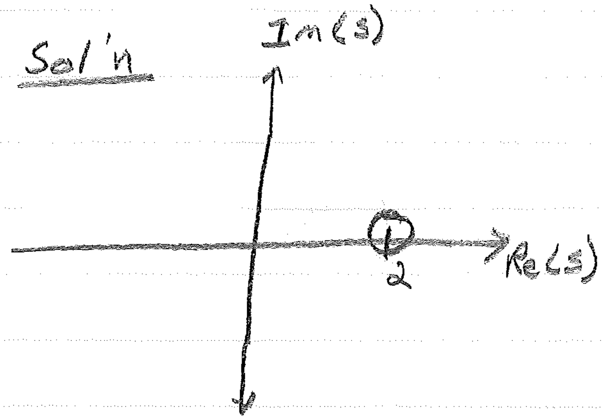
$\boxed{\text{Shifts minimum to } \omega = -1}$



Frequency Response of RHP Zero

- **Let:** $H(s) = s - 2$ **Draw:** $H(\omega)$

Sol'n



Helpful Points:

$H(\omega) = j\omega - 2$

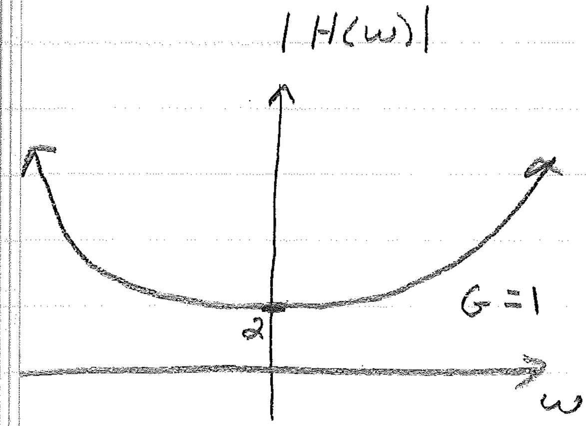
$H(0) = -2 = 2\angle 180^\circ$
$= 2\angle -180^\circ$

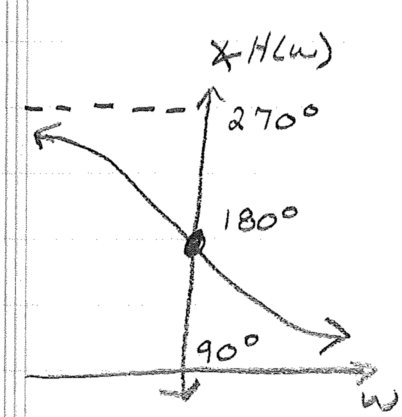
$H(2) = -2 + j2 = 2\sqrt{2}\angle 135^\circ$
$= 2\sqrt{2}\angle -225^\circ$

$H(\infty) = -2 + j\infty = \infty\angle 90^\circ$ or $\infty\angle -270^\circ$
(Approach clockwise)

$H(-2) = -2 - j2 = 2\sqrt{2}\angle 225^\circ$ or $\angle -135^\circ$

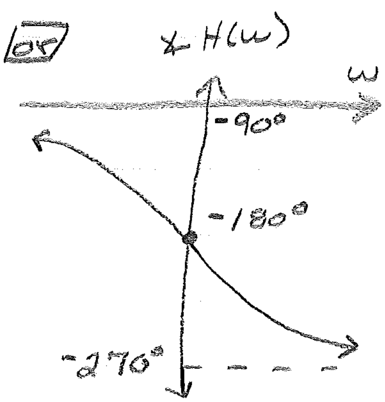
$H(-\infty) = -2 - j\infty = \infty\angle 270^\circ$ or $\angle -90^\circ$
(Approach counter-clockwise)





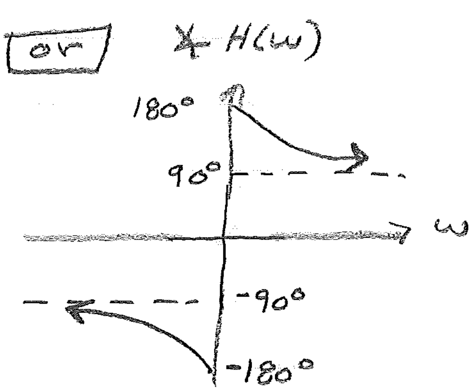
(Phase lead)

or



(Phase lag)

or



(MATLAB atan2( ))

Rational Fourier Transforms

System with one Pole

Consider: System with one pole

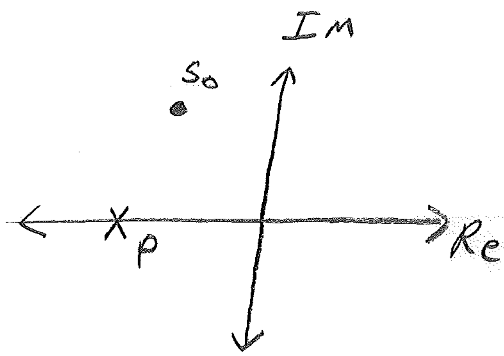
$$V(s) = \frac{1}{s-p} = |V(s)| \underline{\angle V(s)}$$

Form:

$$X(s) = \frac{1}{V(s)} = s-p = \frac{1}{|V(s)|} \underline{-\angle V(s)}$$

Gives system with one zero. Already know graphical interpretation.

$$|V(s)|\Big|_{s=s_0} = \text{Inverse distance from pole to } s_0$$

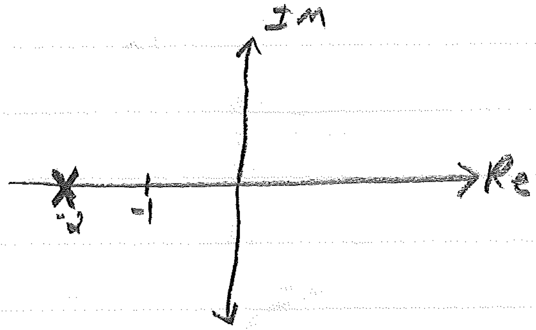


$\angle V(s)$ = Negative of angle between $\overrightarrow{s_0-p}$ and real axis

- If $j\omega$ axis in ROC
  - $H(j\omega) = H(s)\Big|_{\sigma=0}$

Graphical Evaluation of Single Pole

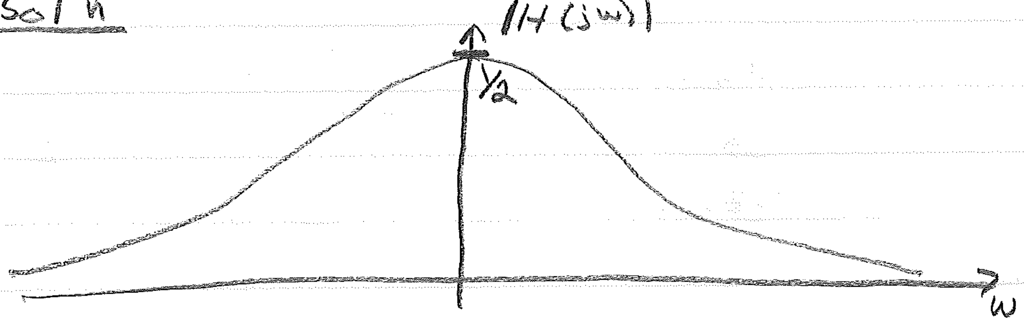
- Let $H(s) = \frac{1}{s+2}$. Draw $H(j\omega)$.

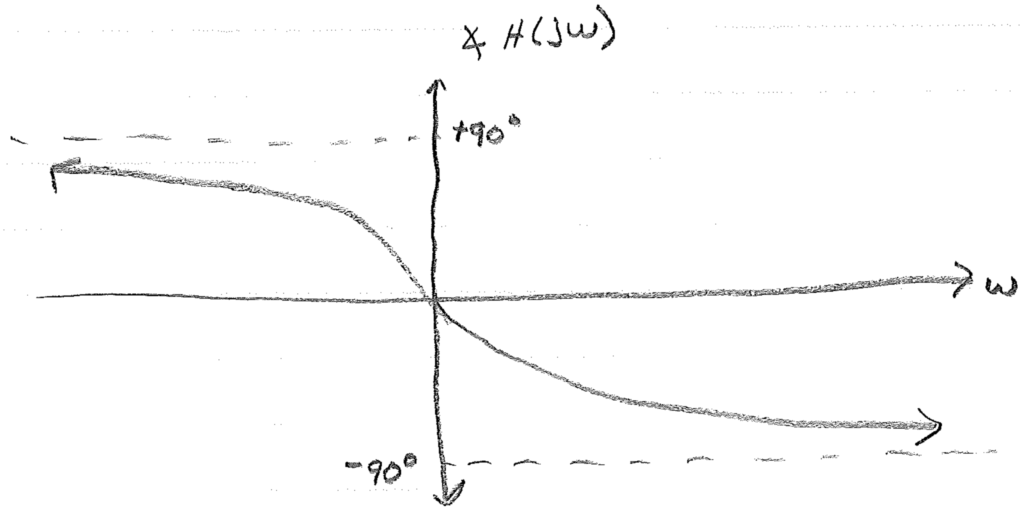


- Magnitude is maximal for $\omega_0 = 0$ (closest point)
  - Shrinks to ϕ as $\omega \to \infty$
  - Symmetric about $\omega_0 = 0$.

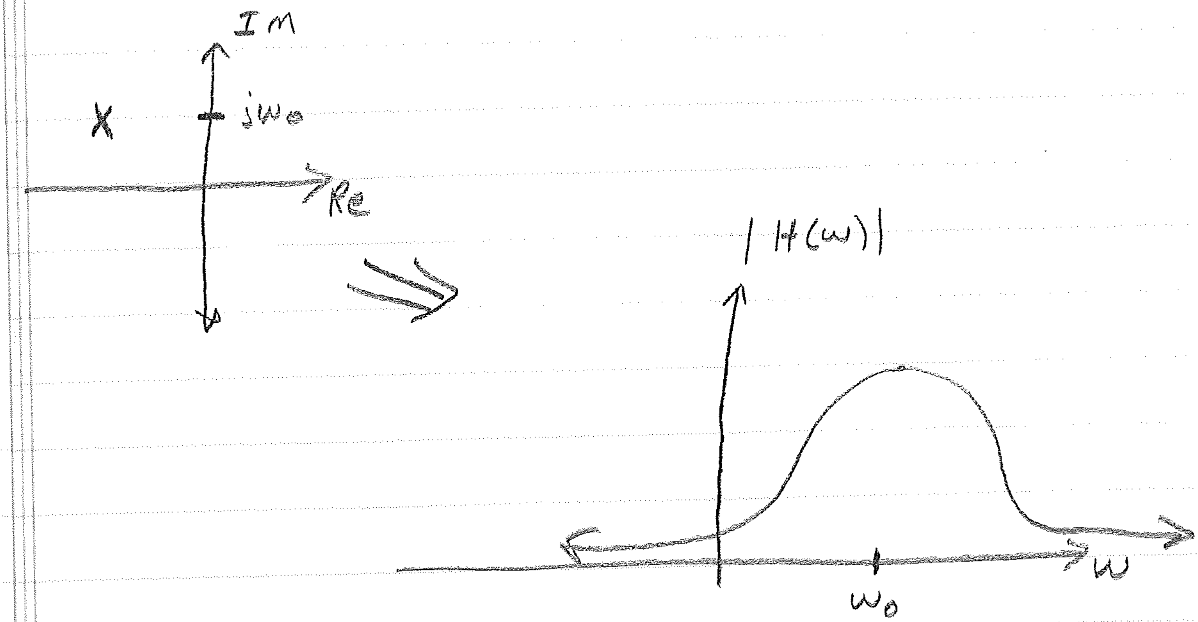
- Phase $= 0^\circ$ at $\omega_0 = 0$
- Grows to $\begin{cases} -90^\circ & \text{at } \omega_0 = \infty \\ +90^\circ & \text{at } \omega_0 = -\infty \end{cases}$

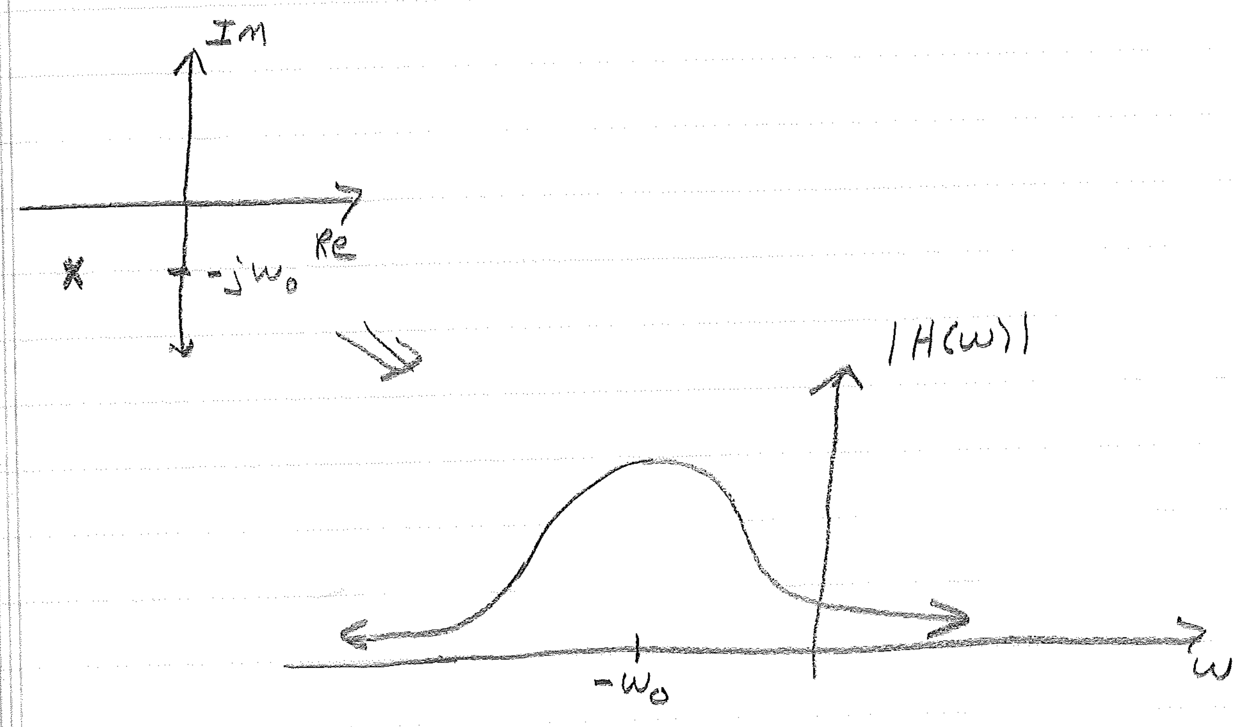
Sol'n





Different Locations of Single Pole





$\boxed{\text{Phase shifts similarly.}}$

Rational Fourier Transforms

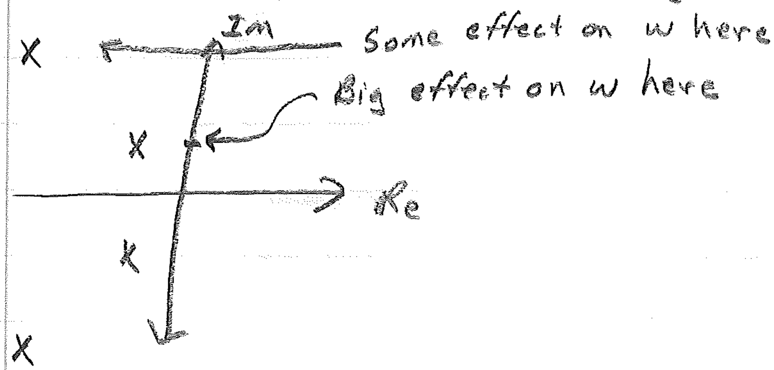
Systems with Multiple Poles & Zeros

- $$H(s) = G \frac{\prod_{m=1}^{P} (s - z_m)}{\prod_{m=1}^{R} (s - p_m)}$$

$$= G \frac{\prod_{m=1}^{P} |s - z_m|}{\prod_{m=1}^{R} |s - p_m|} \quad \Bigg| \; \sum_{m=1}^{P} \measuredangle (s - z_m) - \sum_{m=1}^{R} \measuredangle (s - p_m)$$

- Gain effect is multiplicative
- Phase effect is additive

- Frequencies nearest a pole → higher gain
- Poles near $j\omega$ axis → largest influence on $H(j\omega)$



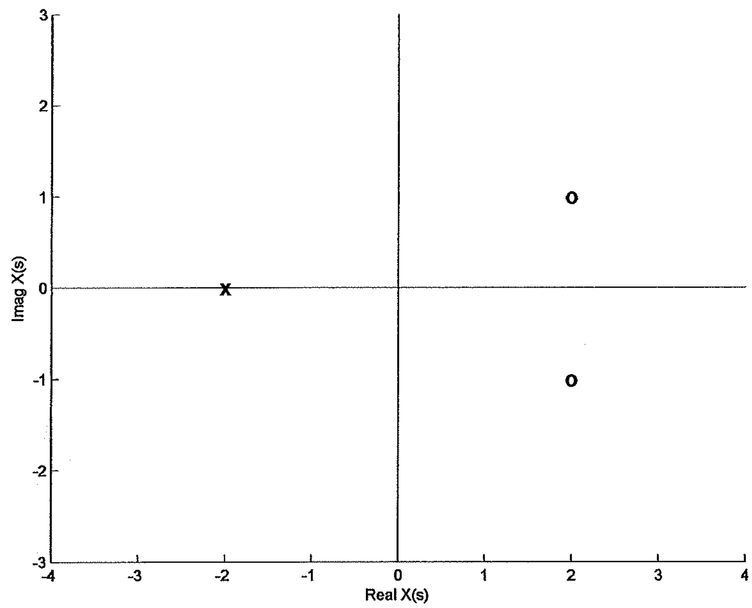
- Frequencies near a zero pull $H(j\omega)$ toward zero.

> For differential equations with real-valued coefficients, complex roots/poles come in complex-conjugate pairs.

Rational Fourier Transforms

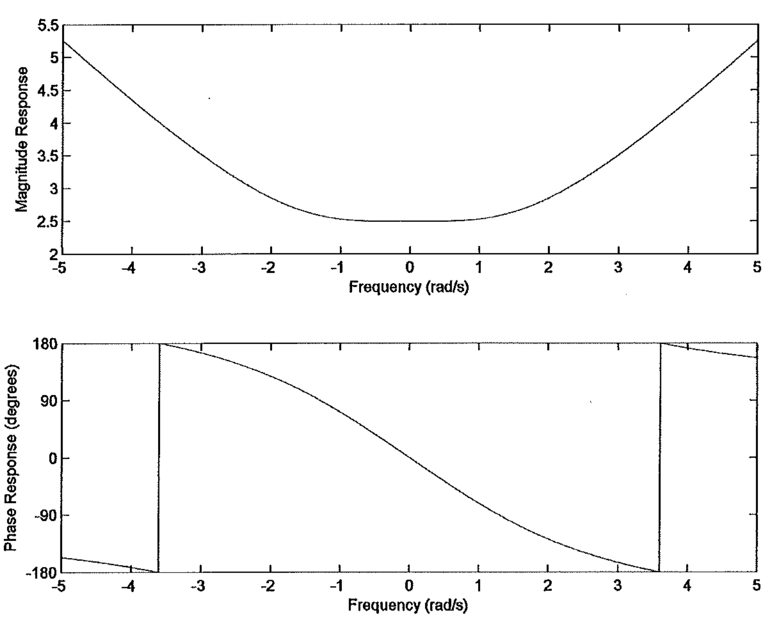
Example Pole-Zero Diagrams

- **Given** the pole-zero diagram below of an LTI system: (a) **write** the frequency response of the system and (b) **sketch** the magnitude response of the system.



- **Sol'n**

$$H(w) = G\frac{(j\omega + j - 2)(j\omega - j - 2)}{j\omega + 2}$$

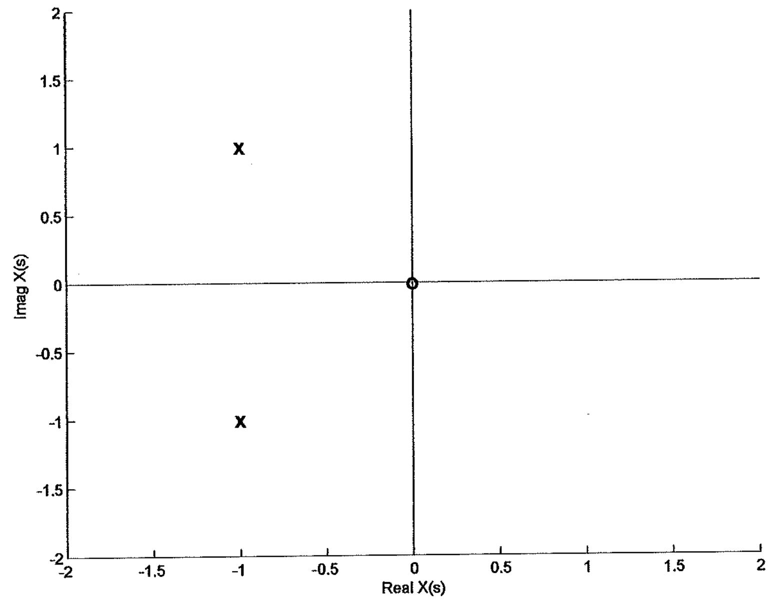


G = 1

Rational Fourier Transforms

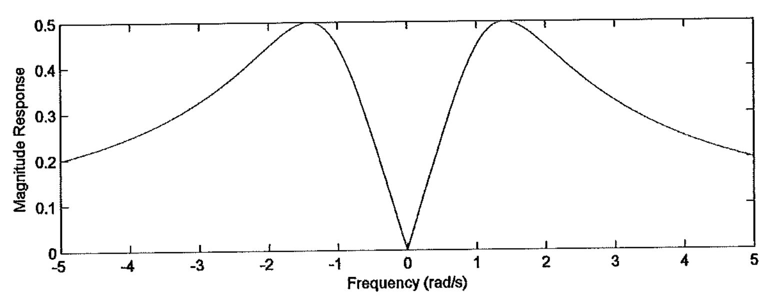
Example Pole-Zero Diagrams 2

- **Given** the pole-zero diagram below of an LTI system: (a) **write** the frequency response of the system and (b) **sketch** the magnitude response of the system.

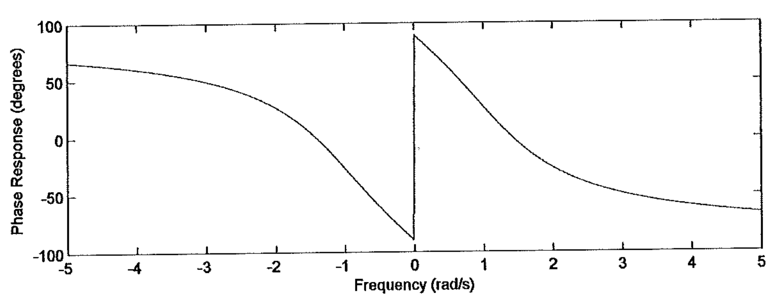


- **Sol'n**

$$H(w) = G\frac{j\omega}{(j\omega + j + 1)(j\omega - j + 1)}$$



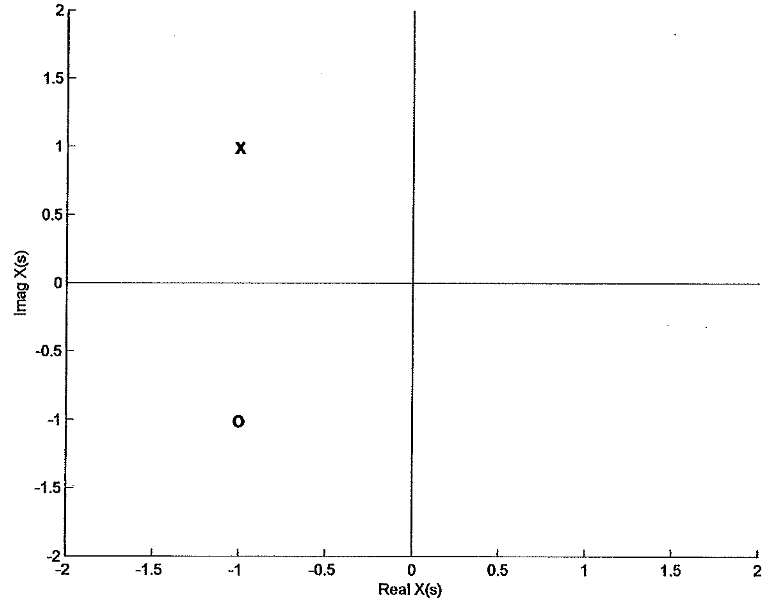
G = 1



Rational Fourier Transforms

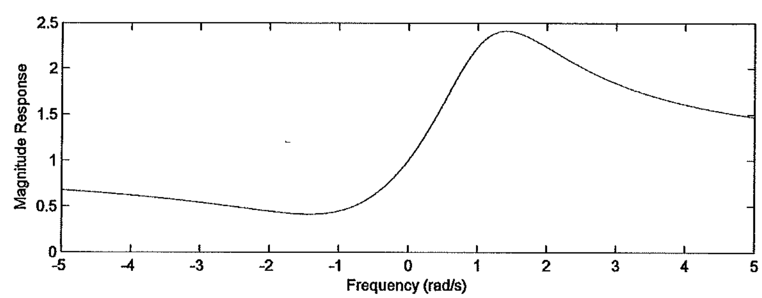
Example Pole-Zero Diagrams 3

- **Given** the pole-zero diagram below of an LTI system: (a) **write** the frequency response of the system and (b) **sketch** the magnitude response of the system.

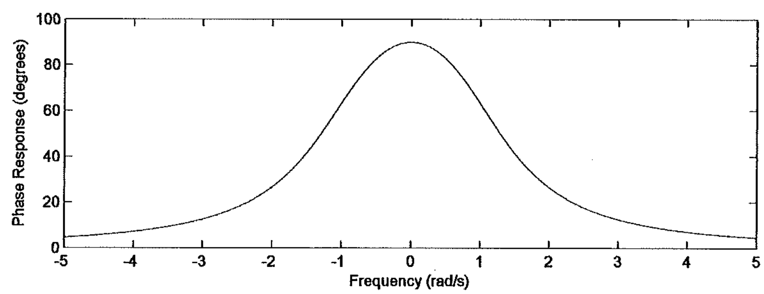


- **Sol'n**

$$H(w) = G\frac{j\omega + j + 1}{j\omega - j + 1}$$



$G = 1$



Rational Fourier Transforms

Designing Filter Shape with Poles and Zeros

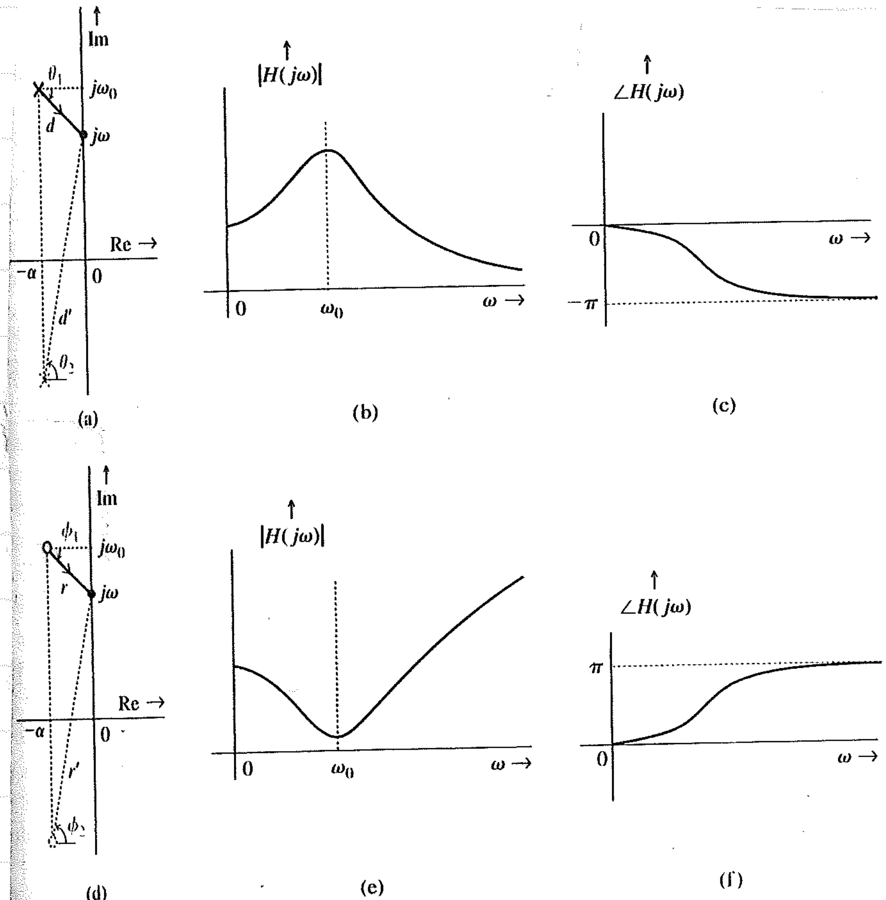


Figure 4.46 The role of poles and zeros in determining the frequency response of an LTIC system.

[Lathi]

Note: Should be able to draw $H(j\omega)$ (except for gain G) based solely on s-plane pole-zero plot.

Rational Fourier Transforms

Low-Pass Filters

- Need gain (i.e., poles) at low frequencies
- Electrical circuit: Each energy storage device (C, L) → one differential equation derivative
  → system order

- E.g.:

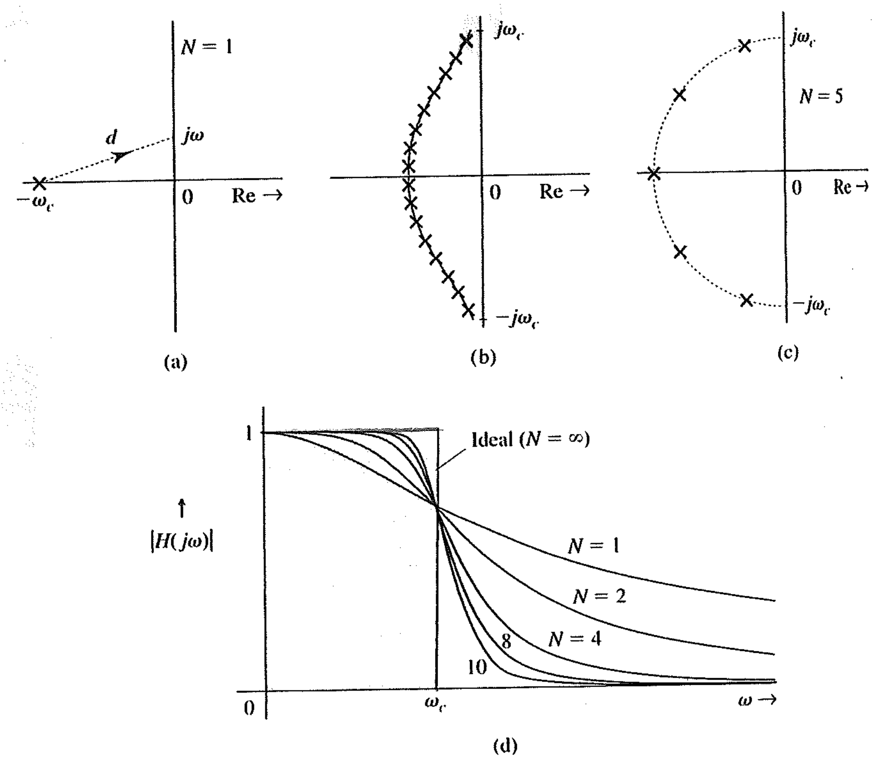


Figure 4.47 Pole–zero configuration and the amplitude response of a lowpass (Butterworth) filter.

[Lathi]

- Many LPF styles exist. Various tradeoffs include: steepest cut-off, monotonicity, deepest stop band, phase response.
  - Ideal filter not possible

Rational Fourier Transforms

Band-Pass Filters

- Need gain (i.e., poles) over limited frequency range

poles

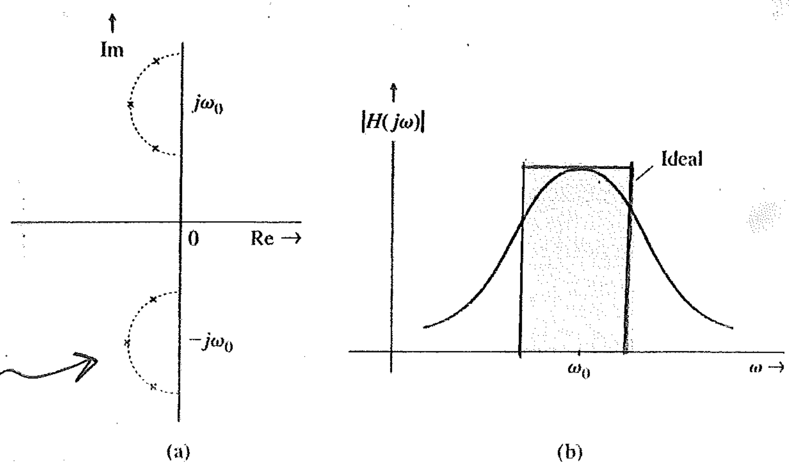


Figure 4.48 (a) Pole–zero configuration and (b) the amplitude response of a bandpass filter.

[Lathi]

Band - Stop Filter

- For $|H(jw)|\Big|_{w=\infty}=1 \rightarrow$ Need same number of poles as zeros (and $G=1$)

$$\lim_{w\to\infty} \frac{\prod_{m=1}^{Q} |jw-z_m|}{\prod_{m=1}^{Q} |jw-p_m|} = \frac{Qjw}{Qjw} = 1$$

$\boxed{\text{Same for } w=-\infty.}$

- For unity gain at $w=0$; put zeros, poles equidistant from $w=0$ (on a semi-circle)

- Put zeros at w_0 to give null

- As always, poles in LHP for stability.

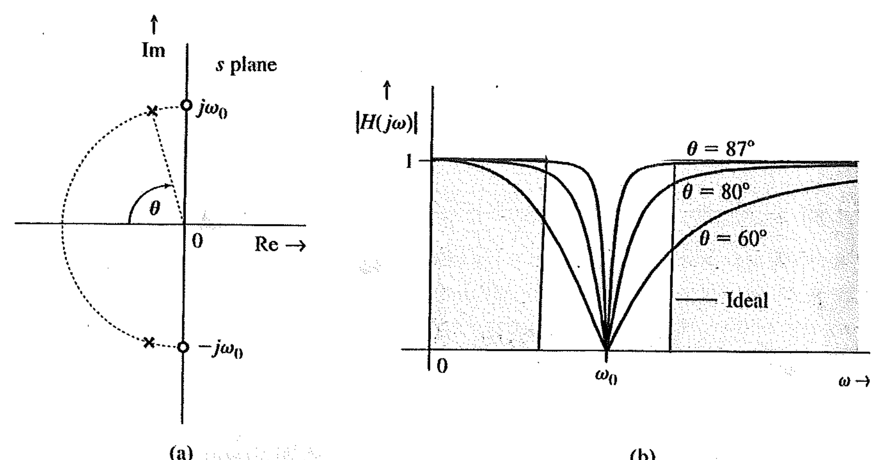


Figure 4.49 **(a)** Pole–zero configuration and **(b)** the amplitude response of a bandstop (notch) filter.

[Lathi]

Rational Fourier Transforms

Descriptors of Realizable Filters

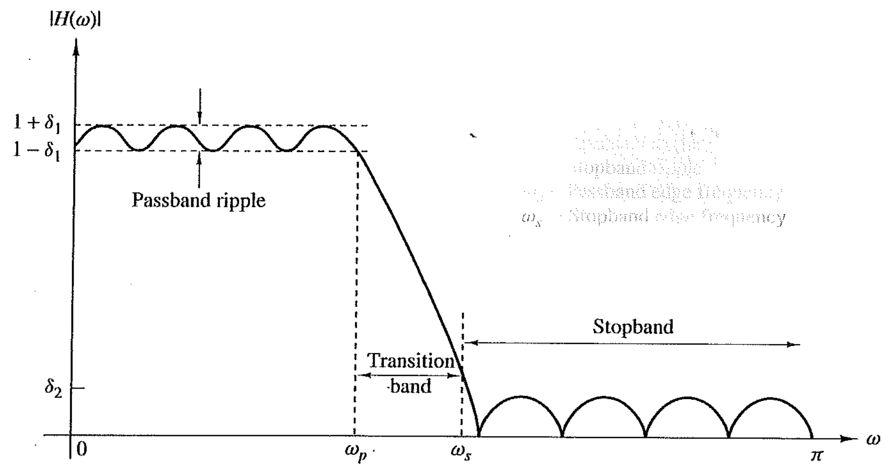


Figure 10.1.2 Magnitude characteristics of physically realizable filters.

[Proakis & Manolakis]

δ_1: ± Passband ripple

δ_2: Stopband ripple

ω_p: Passband edge frequency

ω_s: Stopband edge frequency

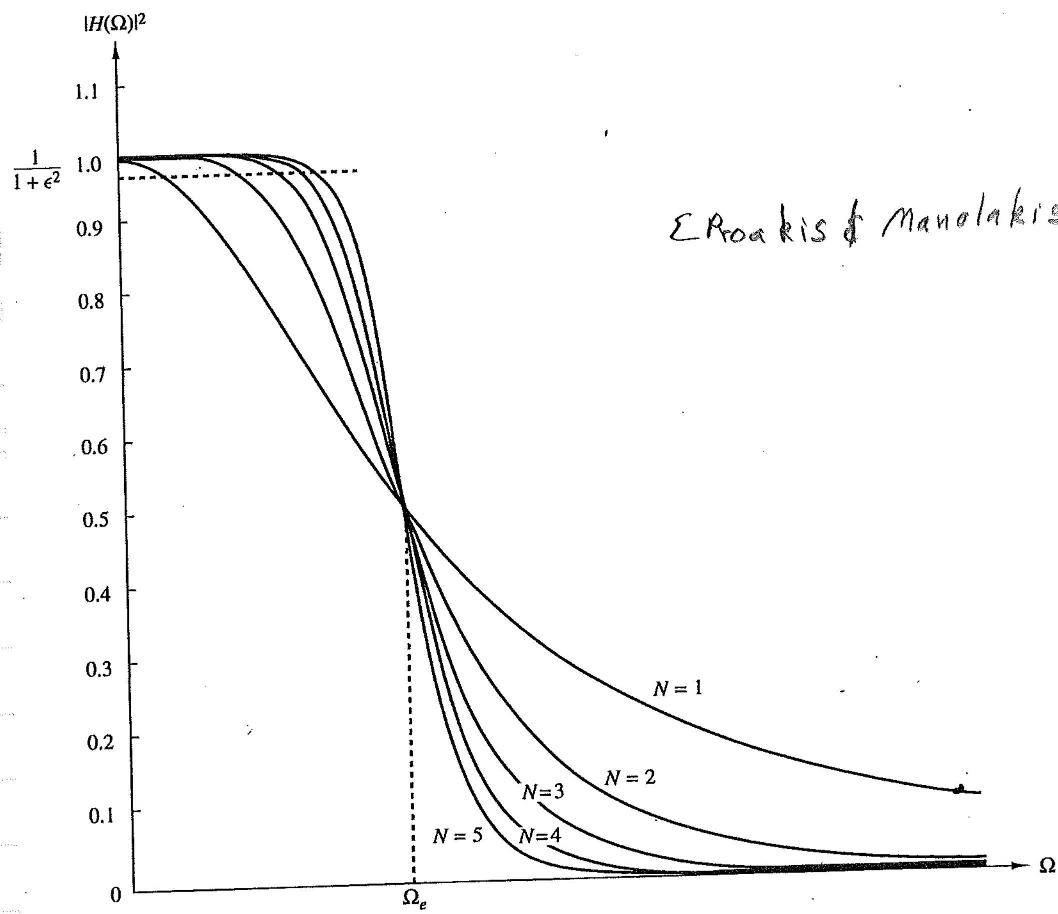
- Passband
- Stop band
- Transition band

Rational Fourier Transforms

Butterworth Filters

- Low pass: $|H(\omega)|^2 = \dfrac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2N}}$

  $N \equiv$ Filter order



[Proakis & Manolakis]

Figure 10.3.10 Frequency response of Butterworth filters.

- Monotonic in passband, stopband
- Analytic formula to determine N for desired δ_2

Rational Fourier Transforms

Chebyshev Type I Filter

- Low pass: $|H(\omega)|^2 = \dfrac{1}{1 + \epsilon^2 T_N^2\left(\frac{\omega}{\omega_c}\right)}$

Related to δ_1 → { Controls passband ripple

Chebyshev polynomial

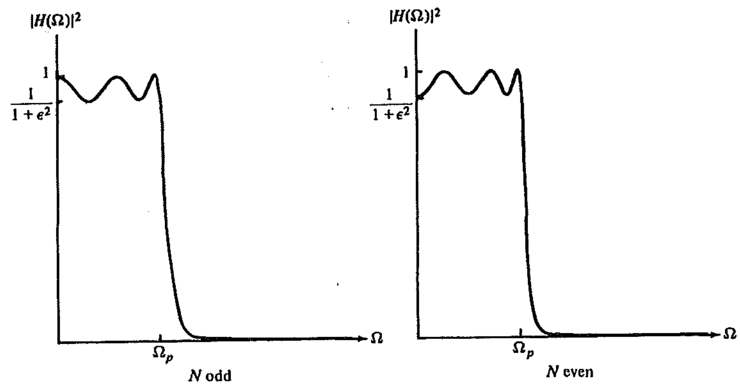


Figure 10.3.11 Type I Chebyshev filter characteristic.

[Proakis & Manolakis]

- Equiripple in passband; Monotonic in stopband

Rational Fourier Transforms

Chebyshev Type II Filter

- Low pass: $|H(\omega)|^2 = \dfrac{1}{1 + \epsilon^2 \left[T_N^2(\omega_s/\omega_p) / T_N^2(\omega_s/\omega) \right]}$

ω_s: Stopband frequency
ω_p: Passband frequency

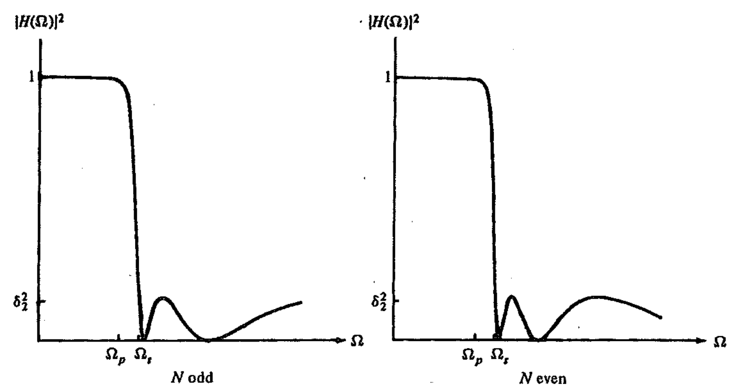


Figure 10.3.13 Type II Chebyshev filters.

[Proakis & Manolakis]

- Monotonic in passband; Equiripple in stopband
- Analytic formula to determine N from desired specifications

Rational Fourier Transforms

Elliptic Filter

- Low pass: $|H(\omega)|^2 = \dfrac{1}{1 + \epsilon^2 \, U_N(\omega/\omega_c)}$

Controls passband ripple

Jacobian elliptic function

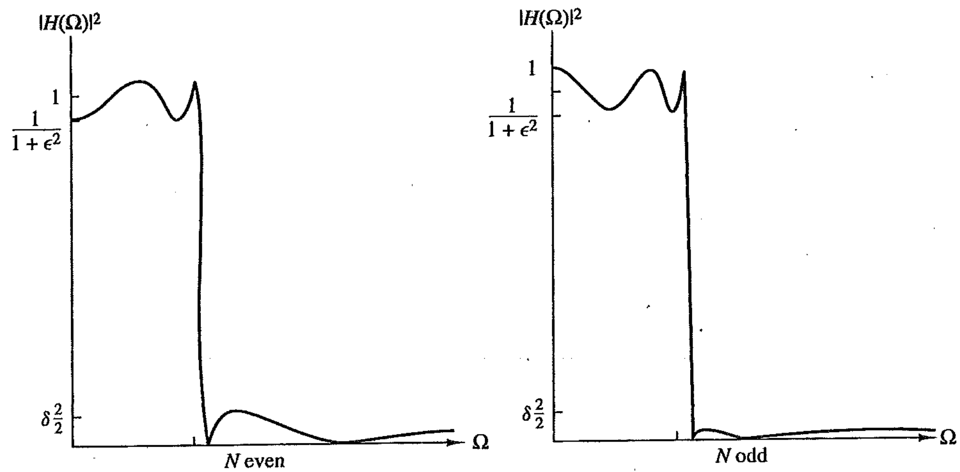


Figure 10.3.14 Magnitude-squared frequency characteristics of elliptic filters.

[Proakis & Manolakis]

- Equal ripple in passband & stopband
  - Has smallest transition bandwidth

- Analytic formula to determine N from desired specifications

Rational Fourier Transforms

Bessel Filter

- Low pass: $H(\omega) = \dfrac{1}{B_N(\omega)}$

  Bessel polynomial

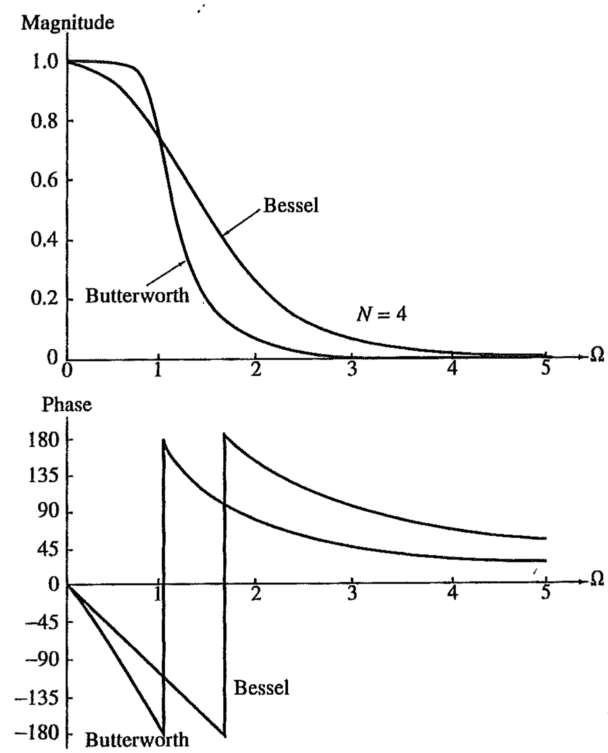


Figure 10.3.15
Magnitude and phase responses of Bessel and Butterworth filters of order $N = 4$.

[Proakis & Manolakis]

- Linear phase in passband

Rational Fourier Transforms