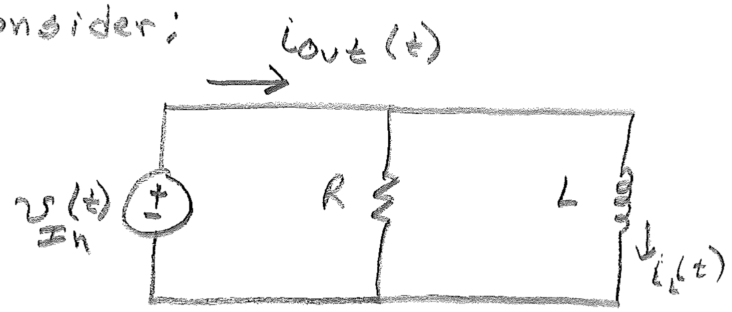
Linear System Response - Circuit Example

Consider:



Initial condition

By KCL:

$$i_{out}(t) = \frac{1}{R} v_{In}(t) + i_L(t_0) + \frac{1}{L}\int_{t_0}^{t} v_{In}(\tau)\, d\tau$$

$$i_{out}(t) = i_L(t_0) + \left[\frac{1}{R} v_{In}(t) + \frac{1}{L}\int_{t_0}^{t} v_{In}(\tau)\, d\tau\right]$$

$$\boxed{\text{Total Response} = \text{Zero-Input Response} + \text{Zero-State Response}}$$

Zero-Input Response: output due only to initial conditions → Transient Response

Zero-State Response: output if assume initial conditions $= 0$ → Steady-state Response

Linear system superposition $\Longrightarrow$ Compute separate responses; Add to get total response

Zero-Input Response

Linear Differential System Equations

General:

$$a_R \frac{d^R y(t)}{dt^R} + a_{R-1} \frac{d^{R-1} y(t)}{dt^{R-1}} + \dots + a_1 \frac{dy(t)}{dt} + a_0\, y(t)$$

$$= b_P \frac{d^P x(t)}{dt^P} + b_{P-1} \frac{d^{P-1} x(t)}{dt^{P-1}} + \dots + b_1 \frac{dx(t)}{dt} + b_0\, x(t)$$

or

$$\sum_{m=0}^{R} a_m \frac{d^m y(t)}{dt^m} = \sum_{m=0}^{P} b_m \frac{d^m x(t)}{dt^m}$$

- Most authors set $a_R = 1$ or $a_0 = 1$
- $a_m, b_m \Rightarrow$ constants
- Using operator notation: $D \rightarrow d/dt$

$$(a_R D^R + \dots + a_1 D + a_0)\, y(t) = (b_P D^P + \dots + b_1 D + b_0)\, x(t)$$

- If

$$Q(D) \equiv (a_R D^R + \dots + a_1 D + a_0)$$

$$O(D) \equiv (b_P D^P + \dots + b_1 D + b_0)$$

then $$Q(D)\, y(t) = O(D)\, x(t)$$

Zero-Input Response

Zero - Input Response: Formulation (1)

- Set $x(t) = 0$ in general formula:

$$Q(D) \cdot y(t) = 0$$

- Example: $\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 6\, y(t) = 0$

- Solution requires linear combination of $y(t)$ plus its derivatives sum to zero

$\Rightarrow$ Functional form of $y, \dot{y}, \ddot{y}, \ldots$ identical

$\Rightarrow$ Only function: exponential: $e^{\lambda t}$

[λ can be complex]

- So, guess

$$y_0(t) = c\, e^{\lambda t}, \quad c, \lambda \text{ constants}$$

then
$$\frac{d\, y_0(t)}{dt} = D y_0(t) = c \lambda e^{\lambda t}$$

$$\frac{d^2 y_0(t)}{dt^2} = D^2 y_0(t) = c \lambda^2 e^{\lambda t}$$

$$\vdots$$

$$\frac{d^R y_0(t)}{dt^R} = D^R y_0(t) = c \lambda^R e^{\lambda t}$$

(Continued)

Zero-Input Response: Formulation (2)

- Substitute guess into general formula:

$$c\, e^{\lambda t}\left(a_R \lambda^R + a_{R-1}\lambda^{R-1} + \cdots + a_1 \lambda + a_0\right) = 0$$

Note 1: $e^{\lambda t} \neq 0$

Note 2: $c = 0$ is trivial

$\Rightarrow$ Non-trivial solution requires

$$a_R \lambda^R + a_{R-1}\lambda^{R-1} + \cdots + a_1 \lambda + a_0 = 0$$

- Same polynomial coefficients as $Q(D)$
- Factor to find λ
- λ has R solutions (assuming distinct λ_i)
- General solution is sum of R solutions:

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_R e^{\lambda_R t}$$

- Roots can be
  - Real, distinct
  - Repeated
  - Complex

slightly modified solution form.

$\boxed{\text{CHARACTERISTIC EQUATION}}$

- Find c_m using initial conditions

Zero-Input Response

ZIR - Real Roots

- **Find** $y_0(t)$ for $(D^2+5D+6)\,y(t) = D\,x(t)$

  with $y_0(0) = -1$ and $\dot{y}_0(0) = 1$

- **Sol'n**

  For zero-input, solve $(D^2+5D+6)\,y(t) = 0$

  Characteristic polynomial: $\lambda^2+5\lambda+6=0$

  or $(\lambda+3)(\lambda+2)=0$

  $\lambda_i = -3,\ -2$

  So,

  $$y_0(t) = c_1 e^{-3t} + c_2 e^{-2t}$$

  Resolve c_1, c_2 with initial conditions:

  $$\dot{y}_0(t) = -3c_1 e^{-3t} - 2c_2 e^{-2t}$$

  So,

  $$\left.\begin{aligned} y_0(0) &= c_1 + c_2 &&= -1 \\ \dot{y}_0(0) &= -3c_1 - 2c_2 &&= 1 \end{aligned}\right\} \Longrightarrow \begin{aligned} c_1 &= 1 \\ c_2 &= -2 \end{aligned}$$

  Giving:

  $$y_0(t) = e^{-3t} - 2\cdot e^{-2t}$$

Example: ZIR, Real Roots

° **Find** $y_0(t)$ for $(D^2 + 7D + 12)y(t) = x(t)$

with $y_0(0) = 2$, $\dot{y}_0(0) = -4$.

° **Sol'n**

For zero-input, solve $(D^2 + 7D + 12)y(t) = 0$

Characteristic polynomial: $\lambda^2 + 7\lambda + 12 = 0$

$$(\lambda + 4)(\lambda + 3) = 0$$

$$\Rightarrow \quad \lambda_i = -4, -3$$

So,

$$y_0(t) = c_1 e^{-4t} + c_2 e^{-3t}$$

Resolve c_1, c_2 with initial conditions:

$$\dot{y}_0(t) = -4c_1 e^{-4t} - 3c_2 e^{-3t}$$

So,

$$\left.\begin{aligned} y_0(0) &= c_1 + c_2 = 2 \\ \dot{y}_0(0) &= -4c_1 - 3c_2 = -4 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= -2 \\ c_2 &= 4 \end{aligned}$$

Giving:

$$\boxed{y_0(t) = -2e^{-4t} + 4e^{-3t}}$$

ZIR - Repeated Roots

- If same root appears twice: $\cdots(\lambda+4)(\lambda+4)\cdots$

  $\Rightarrow$ Replace two sol'n portions with:

$$\left(c_i + c_{i+1} \cdot t\right) e^{-4t}$$

- In general:

  If

$$Q(\lambda) = (\lambda - \lambda_1)^r (\lambda - \lambda_{r+1}) \cdots (\lambda - \lambda_R)$$

  then

$$y_0(t) = (c_1 + c_2 t + c_3 t^2 + \cdots + c_r t^{r-1}) e^{\lambda_1 t} + c_{r+1} e^{\lambda_{r+1} t} + \cdots + c_R e^{\lambda_R t}$$

- Example: Find $y_0(t)$ as function of c_i for

$$(D^3 + 5D^2 + 8D + 4)\, y(t) = D \cdot x(t)$$

Sol'n

Characteristic polynomial: $\lambda^3 + 5\lambda^2 + 8\lambda + 4 = 0$

Facters as $(\lambda+2)(\lambda+2)(\lambda+1)$

So, $\lambda_i = -2, -2, -1$

or

$$\boxed{y_0(t) = (c_1 + c_2 t) e^{-2t} + c_3 e^{-t}}$$

(Would find c_i from initial conditions)

Zero-Input Response

Example: ZIR, Repeated Roots

- **Find** $y_0(t)$ for $(D^2+6D+9)\,y(t)=0$

  with $y(0)=0$ and $\dot{y}(0)=4$.

- Sol'n

Characteristic polynomial: $\lambda^2+6\lambda+9=0$

$$(\lambda+3)(\lambda+3)=0$$

$$\lambda_i = -3,\,-3 \leftarrow \text{repeated}$$

Repeated, so:

$$y_0(t) = c_1 e^{-3t} + c_2\, t\, e^{-3t}$$

Resolve c_1, c_2 via initial conditions:

$$\dot{y}_0(t) = -3c_1 e^{-3t} - 3c_2 t e^{-3t} + c_2 e^{-3t}$$

So,

$$\left.\begin{aligned} y_0(0) &= c_1 + c_2 \cdot 0 = 0 \\ \dot{y}_0(0) &= -3c_1 + c_2 = 4 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 0 \\ c_2 &= 4 \end{aligned}$$

Giving:

$$\boxed{y_0(t) = 4te^{-3t}}$$

ZIR - Complex Roots

- Treat as usual, just λ_i and c_i complex
- If know solution is real-valued
  $\Rightarrow$ Complex roots must come in complex conjugate pairs: $\lambda_i = \alpha + j\beta$, $\lambda_{i+1} = \alpha - j\beta$

So, for 2nd-degree characteristic polynomial:

$$y_0(t) = c_1 e^{(\alpha + j\beta)t} + c_2 e^{(\alpha - j\beta)t}$$

where $c_1 = c_2^*$

<u>or</u> can show:

$$y_0(t) = c\, e^{\alpha t} \cos(\beta t + \theta)$$

where c, θ real

ZIR – Complex Roots Example

• Find $y_0(t)$ as function of c_i for:

$$(D^3 + 3D^2 + 7D + 5)\,y(t) = D \cdot x(t)$$

Sol'n

Characteristic polynomial:

$$\lambda^3 + 3\lambda^2 + 7\lambda + 5 = 0$$

Factors as: $(\lambda + 1 - j2)(\lambda + 1 + j2)\cdot(\lambda + 1)$

So, $\lambda_i = \{-1 + j2,\ -1 - j2,\ -1\}$

Thus, $$\boxed{y_0(t) = c_1 e^{(-1+j2)t} + c_2 e^{(-1-j2)t} + c_3 e^{-t}}$$

If know $y(t)$ real, $\Rightarrow$

1) $c_1 = c_2^*$

and

2) $$\boxed{y_0(t) = c\, e^{-t} \cos(2t + \theta) + c_3 e^{-t}}$$

• Find c, θ, c_3 via initial conditions

High Order Differential Equations

E.g.

$$(D^6 + 4D^5 - 8D^2 + 11)\, y(t) = (10D^3 + 4D)\, x(t)$$

- Same general approach
- Non-repeated roots
  - Real-, complex-valued
    - For real-valued $y(t)$, $x(t)$
      $\Rightarrow$ complex-conjugate pairs
- Repeated roots

- But
  - Tedious math
  - Difficult to understand, visualize system

$\Longrightarrow$ Better approach may be transform methods

Finding Roots Using MATLAB

- Make vector with descending coefficients of characteristic polynomial.
  - Input to MATLAB

Example: $\lambda^3 + 3\lambda^2 + 7\lambda + 5 = 0$

```
>> roots([1  3  7  5])
ans =
   -1.000 + 2.000i
   -1.000 - 2.000i
   -1.000
>>
```

Cautions:

- Solution is numeric
- Roundoff error
  - Obscures natural numbers (π, e, ...)
  - Obscures true integers
  - Can make real roots look complex
    - Tiny imaginary part

Zero-Input Response

MATLAB- Symbolic Root Finding

- Use symbolic math (Maple) to find roots analytically.

Example: $\lambda^3 + 3\lambda^2 + 7\lambda + 5 = 0$

```
>> syms x;   % Define 'x' as symbolic variable

>> solve(x^3 + 3*x^2 + 7*x + 5)   % Solves: Expression = 0

ans =

    -1
-1 + 2*i
-1 - 2*i

>>
```

Zero-Input Response

MATLAB Demo Printout (1) 11

matlab_root_finding.txt

```
>> roots([1 3 7 5])

ans =

  -1.0000 + 2.0000i
  -1.0000 - 2.0000i
  -1.0000
```

Example from class notes

```
>> roots([1 3 7 5+eps])

ans =

  -1.0000 + 2.0000i
  -1.0000 - 2.0000i
  -1.0000

>> roots([1 3 7 5.00004])

ans =

  -1.0000 + 2.0000i
  -1.0000 - 2.0000i
  -1.0000
```

with low printing resolution, can not see that the roots have changed

In "File" → "Preferences", set "Numeric Format" to "long"

```
>> roots([1 3 7 5.00004])

ans =

 -0.99999500000000 + 2.00000000001875i
 -0.99999500000000 - 2.00000000001875i
 -1.00001000000000

>> roots([1 3 7 5])

ans =

 -1.00000000000000 + 2.00000000000000i
 -1.00000000000000 - 2.00000000000000i
 -1.00000000000000
```

Now, can see numeric differences

Set numeric format back to "short"

```
>> syms x
>> clear ans
>> solve(x^3 + 3*x^2 + 7*x + 5)

ans =

     -1
 -1+2*i
 -1-2*i
```

Symbolic math

```
>> solve(x^3 + 3*x^2 + 7*x + 5.0001)

ans =
```

root 1

```
-1/300*(1350+150*76800000081^(1/2))^(1/3)+400/(1350+150*76800000081^(1/2))^(1/3)-1

1/600*(1350+150*76800000081^(1/2))^(1/3)-200/(1350+150*76800000081^(1/2))^(1/3)-1+1/
20*i*3^(1/2)*(-1/30*(1350+150*76800000081^(1/2))^(1/3)-4000/(1350+150*76800000081^(1
/2))^(1/3))
```

root 2

Zero-Input Response

MATLAB Demo Printout (2)


```
1/600*(1350+150*76800000081^(1/2))^(1/3)-200/(1350+150*76800000081^(1/2))^(1/3)-1-1/
20*i*3^(1/2)*(-1/30*(1350+150*76800000081^(1/2))^(1/3)-4000/(1350+150*76800000081^(1
/2))^(1/3))
```

root 3

```
>> simplify(ans)

ans =
```

simplification NOT help.

```
-1/300*((1350+150*76800000081^(1/2))^(2/3)-120000+300*(1350+150*76800000081^(1/2))^(
1/3))/(1350+150*76800000081^(1/2))^(1/3)

-1/600*(-(1350+150*3^(1/2)*25600000027^(1/2))^(2/3)+120000+600*(1350+150*3^(1/2)*256
00000027^(1/2))^(1/3)+i*3^(1/2)*(1350+150*3^(1/2)*25600000027^(1/2))^(2/3)+120000*i*
3^(1/2))/(1350+150*3^(1/2)*25600000027^(1/2))^(1/3)

1/600*((1350+150*3^(1/2)*25600000027^(1/2))^(2/3)-120000-600*(1350+150*3^(1/2)*25600
000027^(1/2))^(1/3)+i*3^(1/2)*(1350+150*3^(1/2)*25600000027^(1/2))^(2/3)+120000*i*3^
(1/2))/(1350+150*3^(1/2)*25600000027^(1/2))^(1/3)


>>
```


Zero - Input Response