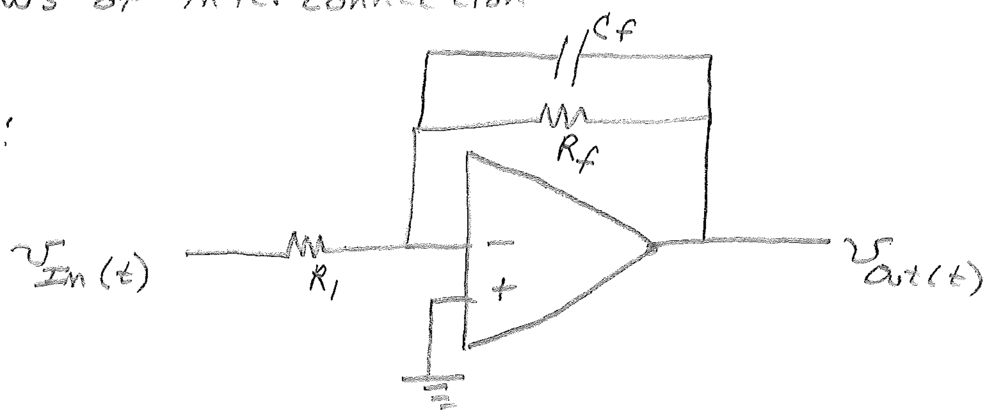
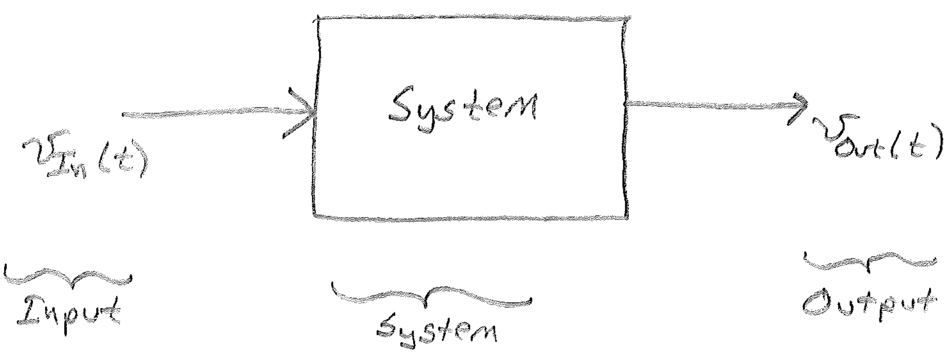
General Concept of a System

- Physical system: Input-output relationship derived from interconnected components and their laws of interconnection

- E.g.:



- "Black box" representation



Single Input Single Output (SISO) system

- Can also form Multiple Input Multiple Output (MIMO) systems

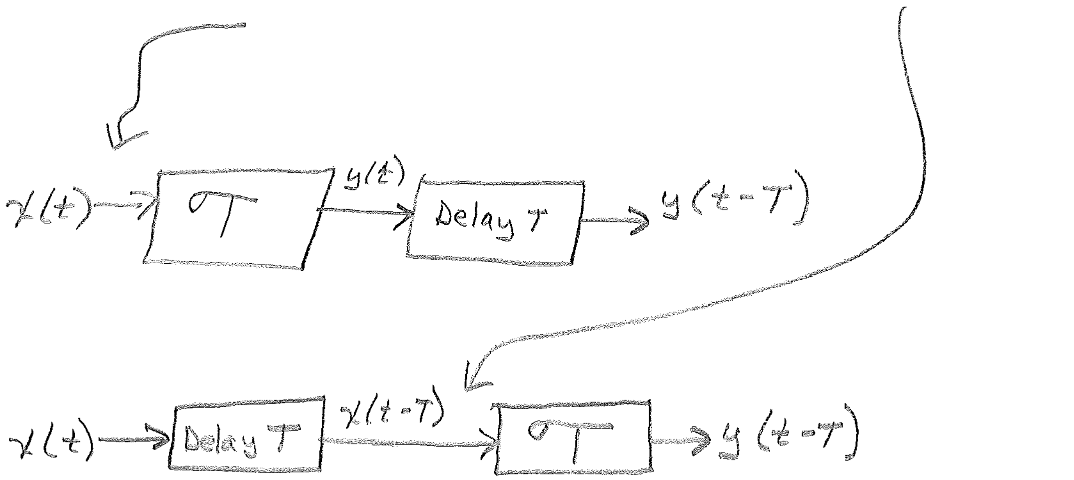
- Functional representation:

$$v_{Out}(t) = \mathcal{T}\left[v_{In}(t)\right]$$

Time Invariance

- Relaxed system <u>time invariant</u> iff

$$x(t) \xrightarrow{\mathcal{T}} y(t) \quad \text{implies} \quad x(t-T) \xrightarrow{\mathcal{T}} y(t-T)$$



- Test via definition

  ① Delay input by T, apply to system

  - Replace $x(t) \rightarrow x(t-T)$; Apply to $\mathcal{T}$

  ② Apply $x(\cdot)$ to system, then delay

  - Replace $t \rightarrow t-T$ at system output

$$\boxed{\text{Time invariant if } ① = ②}$$

Systems

Test Example 1: Time Invariace

- Determine if time invariant [$x(t)$ is input]

$$y(t) = 3 \cdot x(t-4)$$

① Delay input; T

a) $$x(t-4)\Big|_{t \to t-T} = x(t-T-4)$$

b) Apply T:

$$y(t,T) = 3 \cdot (\text{Input}) = 3 \cdot x(t-T-4)$$

② Delay output

$$y(t)\Big|_{t \to t-T} = y(t-T) = 3 \cdot x(t-T-4)$$

Same → Time invariant system

Time Invariance: Test Example 2

- Determine if time invariant [$x(t)$ is input]

$$y(t) = t \cdot x(t)$$

① Delay input; $\mathcal{T}$

a) $$x(t)\Big|_{t \to t-T} = x(t-T)$$

b) Apply $\mathcal{T}$:

$$y(t,T) = t \cdot (\text{Input}) = t \cdot x(t-T)$$

② Delay output

$$y(t)\Big|_{t \to t-T} = y(t-T) = (t-T) \cdot x(t-T)$$

$$= t \cdot x(t-T) - T \cdot x(t-T)$$

Differ → Time varying system

Time Invariance: Test Example 3

- Determine if time invariant [$x(t)$ is input]

$$y(t) = \sin[x(t)]$$

Sol'n

① Delay input; $\mathcal{T}$

a) $x(t)\Big|_{t \to t-T} = x(t-T)$

b) Apply $\mathcal{T}$:

$$y(t,T) = \sin[\text{Input}\quad] = \sin[x(t-T)]$$

② Delay output

$$y(t)\Big|_{t \to t-T} = y(t-T) = \sin[x(t-T)]$$

Same → Time invariant System.

Time Invariance: Test Example 4

- Determine if time invariant [$x(t)$ is input]

$$y(t) = 2 \cdot x(t+1) + t$$

Sol'n

① Delay input; $\mathcal{T}$

a) $$x(t+1)\Big|_{t \to t-T} = x(t-T+1)$$

b) Apply $\mathcal{T}$:

$$y(t,T) = 2 \cdot (\text{Input}) + t$$
$$= 2 \cdot x(t-T+1) + t$$

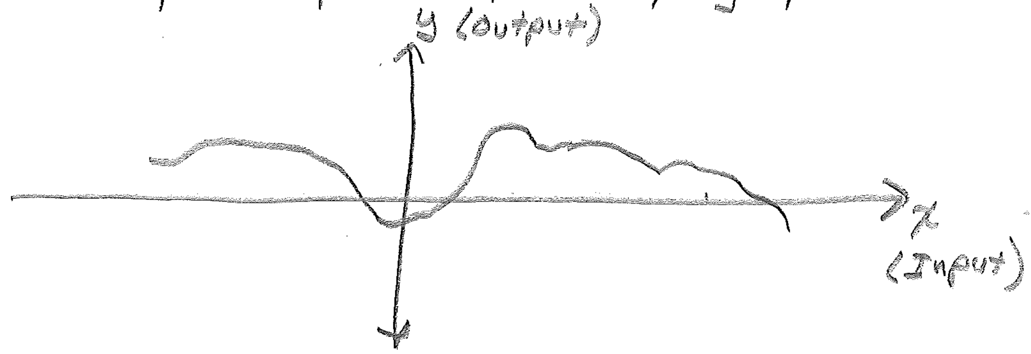
② Delay output

$$y(t)\Big|_{t \to t-T} = y(t-T) = 2 \cdot x(t-T+1) + t - T$$

Differ → Time varying system

Static vs. Dynamic Systems

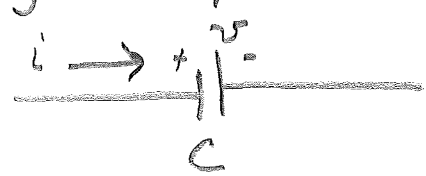
- <u>Static systems</u>: Output depends only on present value of input
  - Memory-less (not depend on time)
  - Simple input-output map/graph



E.g.: $y(t) = -4 \cdot x(t)$

- <u>Dynamic systems</u>: Output depends on past history of input

  E.g.: Capacitor terminal law



$$v(t) = \frac{1}{C} \int_{t_0}^{t} i(\tau)\,d\tau + v(t_0)$$

Examples: Static vs. Dynamic Systems

- For input $x(t)$ and output $y(t)$, state if these systems are static or dynamic:

System	Static or Dynamic?
$y(t) = \sin[x(t)]$	Static
$y(t) = \frac{d\,x(t)}{dt}$	Dynamic
$y(t) = 2\frac{d^2 x(t)}{dt^2} - 3\frac{d\,x(t)}{dt} + 2$	Dynamic
$y(t) = 4 \cdot x^2(t)$	Static

Causality

- **Causal system**: Output at time t_0 only depends on inputs from times $t \le t_0$
  - Present output depends on present, past inputs
  - Output cannot start before input is applied

ELSE: non-causal

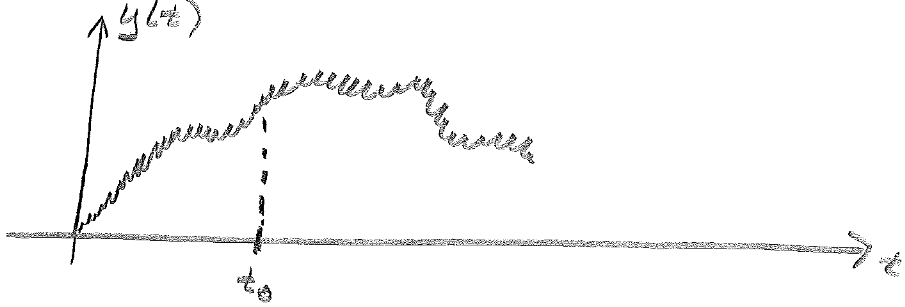
- Non-causal systems use future information
  $y(t) = x(t+2)$

  - In practice, store $x(t)$. Then, produce $y(t)$ using future values of $x(t)$

    Well, systems where time is the independent variable

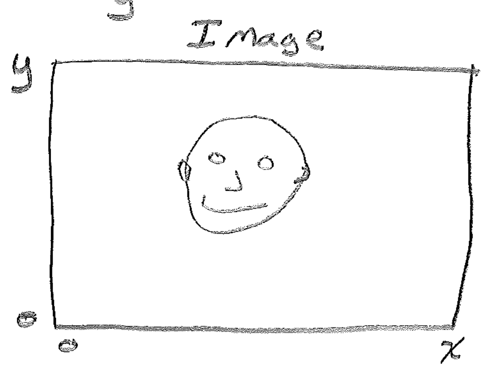
Non-Causal Examples

- Tracking a noisy target



- Let $y(t)$ be noisy position measurement
- Better estimate of position at time t_0
  $\Rightarrow$ Average position estimates over range $(t_0 - \tau, t_0 + \tau)$

- Image processing

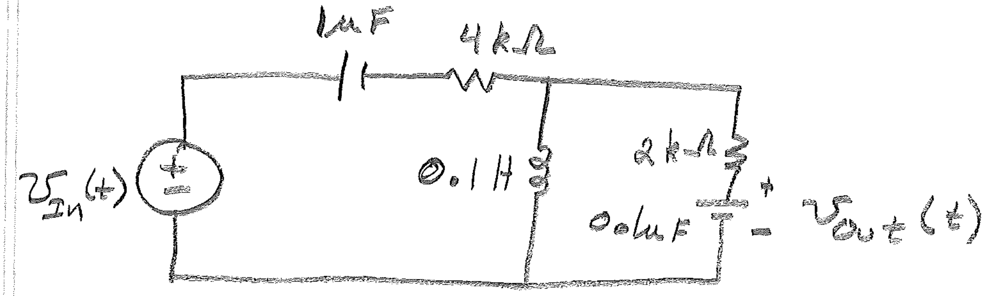


- x, y are independent variable (like t above)
  - Color at location $x, y \rightarrow C(x,y)$
  - Shift image left $C_1(x,y) = C(x+1, y+1)$

"Non-causal"

Systems

Example: Causality



- If $v_{In}(t) = 4 \sin(1000t)\, u(t)$ V

is the input and $v_{out}(t)$ is the output,

is the system causal?

Sol'n

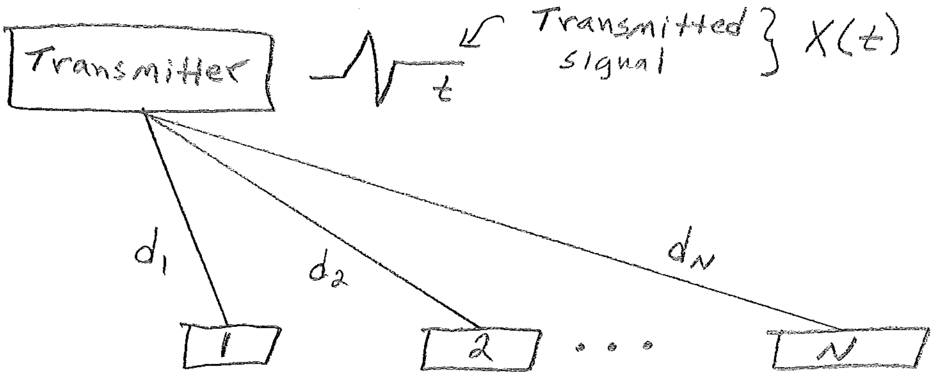
$$v_{out}(t) = a_3 \frac{d^3 v_{In}(t)}{dt^3} + a_2 \frac{d^2 v_{In}(t)}{dt^2} + a_1 \frac{d\, v_{In}(t)}{dt} + a_0$$

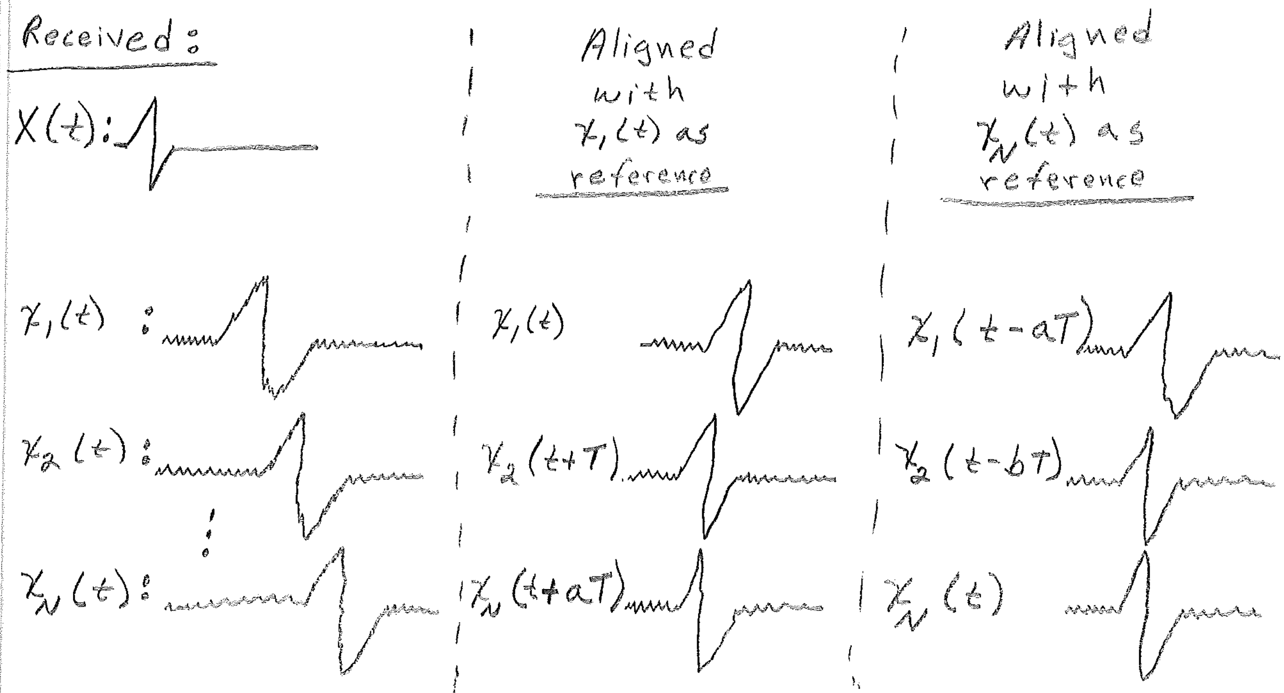
$$\{a_3, a_2, a_1, a_0\} \rightarrow f(C_1, C_2, R_1, R_2, L_1)$$

No future times $\Rightarrow$ $\boxed{\text{Causal}}$

Example: Causality

- Recall: Beam forming via phased array





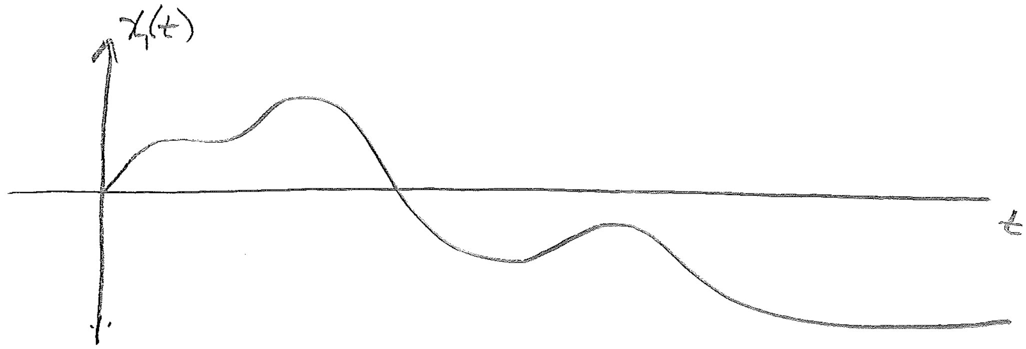
$a = N-1$, $b = N-2$

Causality:	Non-Causal	Causal

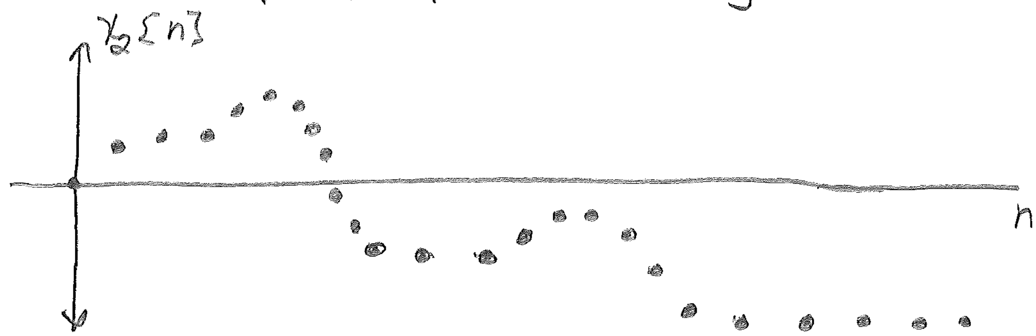
Systems

Continuous- vs. Discrete-Time Systems

- **Continuous**: Inputs, outputs defined for continuous values of time



- **Discrete**: Inputs, outputs defined only at discrete instances



- Time instances often equal-interval
- Digital computers → discrete-time signals
- Can sample continuous-time signal (e.g., from electrical analog circuit) to create discrete-time signal

Invertible Systems

- System T invertible if system S exits for all inputs $x(t)$ such that:

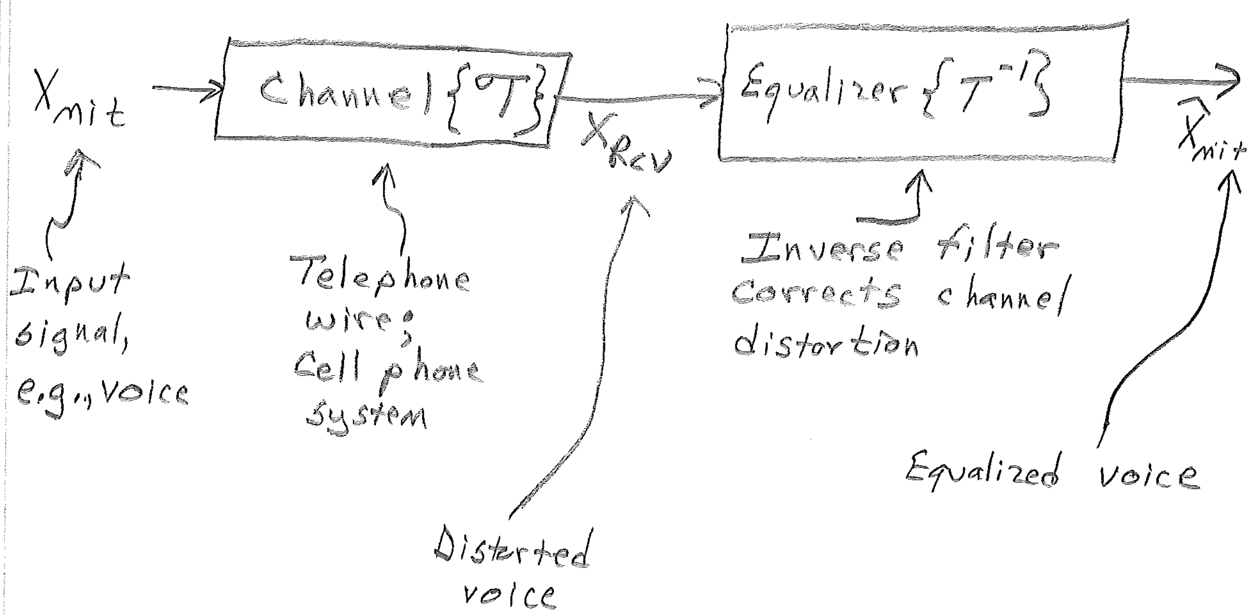
$$x(t) = S\{T[x(t)]\}$$

or

$$x(t) \longrightarrow \boxed{T} \longrightarrow \boxed{S = T^{-1}} \longrightarrow x(t)$$

- Non-invertible system examples
  - Rectifier:
    For Output = 2, Input = 2 or −2
  - Any filter that has gain = 0 at any frequency

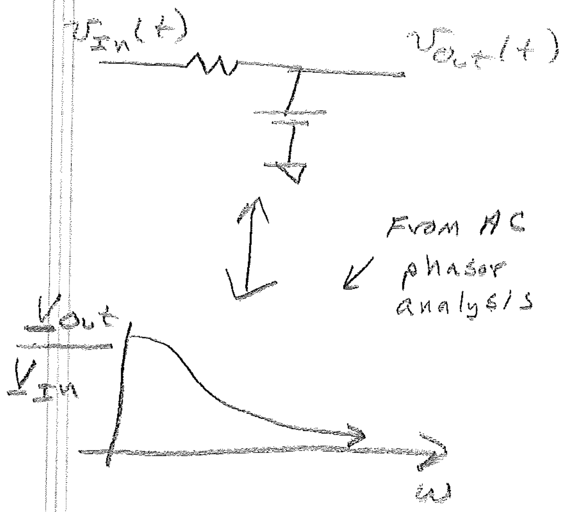
- Application example: Equalization filter



Examples: Invertible Systems

- For input $x(t)$ and output $y(t)$, state if these systems are invertible or not invertible:

System	Invertible ?
1) Low-pass filter formed by an RC circuit:	Theory: Yes, for $\omega \leq \infty$ Practice: Only for frequencies above noise floor
2) $y(t) = x^2(t)$	NOT Invertible
3) $y(t) = x^3(t)$	Invertible



System Stability

- Multiple measures of stability can be defined
- BIBO stability – Bounded Input – Bounded Output
    - Relaxed system BIBO stable iff every bounded input produces a bounded output

E.g.

Let $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

If $x(t) = u(t)$ (Unit step)

Then

$$y(t) = \int_{-\infty}^{t} u(\tau)\, d\tau = \int_{0}^{t} d\tau$$

$$= \tau \Big|_{\tau=0}^{t} = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

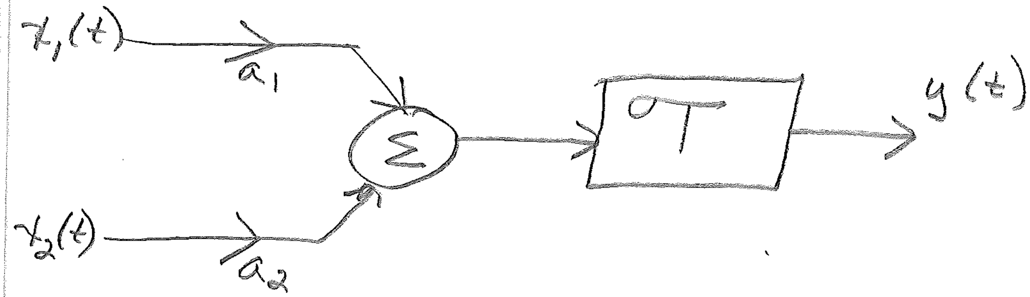
$$\boxed{\begin{array}{c} y(t) \rightarrow \infty \text{ as } t \rightarrow \infty \\ \text{Not BIBO stable} \end{array}}$$

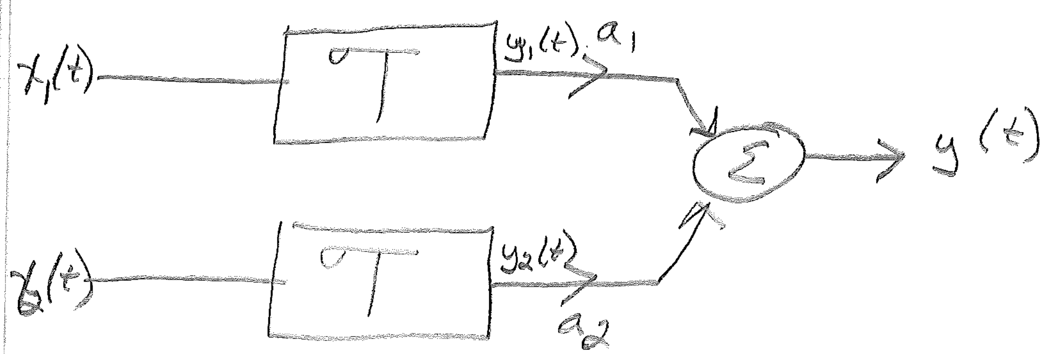
Linear Systems (I)

- For inputs $x_1(t)$, $x_2(t)$ and constants a_1, a_2; system $\mathcal{T}$ linear iff

$$\mathcal{T}\left[a_1 \cdot x_1(t) + a_2 \cdot x_2(t)\right] = \underbrace{a_1 \cdot \mathcal{T}[x_1(t)] + a_2 \cdot \mathcal{T}[x_2(t)]}_{a_1 \cdot y_1(t) + a_2 \cdot y_2(t)}$$

- Graphically:





Linear system $\Longleftrightarrow$ superposition

Linear Systems (2)

- Linear system consequences
  - Scale input by A $\longrightarrow$ Scale output by A
  - If apply constant ϕ input $\rightarrow$ Output $= \phi$

- If system <u>not</u> linear $\Longrightarrow$ Non-linear

- Determining <u>if</u> system is linear
  - Define:

$$x_1(t) \longrightarrow \boxed{T} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{T} \longrightarrow y_2(t)$$

1) In system equation, <u>replace</u>:

$$x(t) \rightarrow a_1 x_1(t) + a_2 x_2(t)$$

$$y(t) \rightarrow a_1 y_1(t) + a_2 y_2(t)$$

2) Equation still "valid"?

Valid $\rightarrow$ <u>Linear</u>

Not Valid $\rightarrow$ <u>Non-Linear</u>

Linear Systems Example 1

- Determine if linear system: $\frac{dy(t)}{dt} + y(t) = 6\,x(t)$

- **Solution**
  - Substitute:

$$\frac{d\left[a_1 y_1(t) + a_2 y_2(t)\right]}{dt} + \left[a_1 y_1(t) + a_2 y_2(t)\right] \overset{?}{=} 6\left[a_1 x_1(t) + a_2 x_2(t)\right]$$

or

$$a_1 \frac{dy_1(t)}{dt} + a_2 \frac{dy_2(t)}{dt} + a_1 y_1(t) + a_2 y_2(t) \overset{?}{=} a_1 \cdot 6\,x_1(t) + a_2 \cdot 6\,x_2(t)$$

or

$$a_1 \underbrace{\left[\frac{dy_1(t)}{dt} + y_1(t)\right]}_{=6x_1(t)} + a_2 \underbrace{\left[\frac{dy_2(t)}{dt} + y_2(t)\right]}_{=6x_2(t)} \overset{?}{=} a_1 \cdot \underbrace{6x_1(t)}_{=\frac{dy_1(t)}{dt} + y_1(t)} + a_2 \cdot \underbrace{6x_2(t)}_{=\frac{dy_2(t)}{dt} + y_2(t)}$$

From system equation

- Substituting on either left or right shows equality

$$\boxed{\Longrightarrow \text{Linear}}$$

Linear Systems Example 2

- Determine if linear system: $\frac{dy(t)}{dt} + 2\underbrace{y(t)}_{\text{Output}} = \underbrace{x^2(t)}_{\text{Input}}$

- **Solution:** Substitute:

$$\frac{d[a_1 y_1(t) + a_2 y_2(t)]}{dt} + 2[a_1 y_1(t) + a_2 y_2(t)] \stackrel{?}{=} [a_1 x_1(t) + a_2 x_2(t)]^2$$

or

$$\underbrace{a_1\left[\frac{dy_1(t)}{dt} + 2y_1(t)\right]}_{x_1^2(t)} + \underbrace{a_2\left[\frac{dy_2(t)}{dt} + 2y_2(t)\right]}_{x_2^2(t)}$$

$$\stackrel{?}{=} a_1^2 x_1^2(t) + 2a_1a_2 x_1(t) x_2(t) + a_2^2 x_2^2(t)$$

or

$$a_1 x_1^2(t) + a_2 x_2^2(t) \stackrel{?}{=} a_1^2 x_1^2(t) + 2a_1a_2 x_1(t) x_2(t) + a_2^2 x_2^2(t)$$

Not equal

$$\boxed{\Rightarrow \text{Non-Linear}}$$

Linear Systems Example 3

- Determine if linear system: $t^2 \cdot y(t) = 4 \cdot x(t)$

- Solution: Substitute

$$t^2 [a_1 y_1(t) + a_2 y_2(t)] \overset{?}{=} 4 [a_1 x_1(t) + a_2 x_2(t)]$$

or

$$a_1 \underbrace{t^2 y_1(t)}_{4 \cdot x_1(t)} + a_2 \underbrace{t^2 y_2(t)}_{4 \cdot x_2(t)} \overset{?}{=} a_1 \underbrace{4 x_1(t)}_{t^2 y_1(t)} + a_2 \underbrace{4 x_2(t)}_{t^2 y_2(t)}$$

- Next step either:

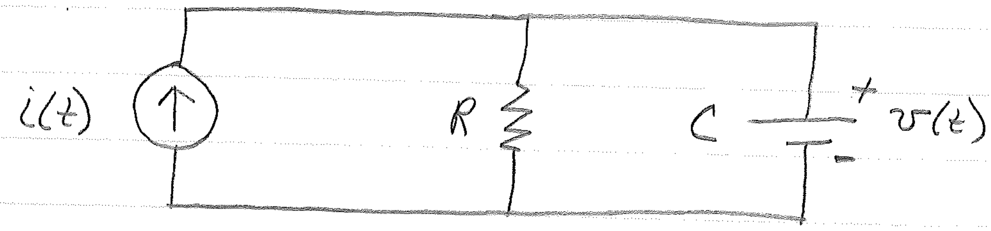
$$\underbrace{a_1 \cdot 4 \cdot x_1(t) + a_2 \cdot 4 \cdot x_2(t)}_{\text{Replaced}} \overset{\checkmark}{=} a_1 \cdot 4 \cdot x_1(t) + a_2 \cdot 4 \cdot x_2(t)$$

or,

$$a_1 t^2 y_1(t) + a_2 t^2 y_2(t) \overset{\checkmark}{=} \underbrace{a_1 t^2 y_1(t) + a_2 t^2 y_2(t)}_{\text{Replaced}}$$

$$\boxed{\Longrightarrow \text{Linear}}$$

Electrical Circuit Example



- If $i(t)$ is the input and $v(t)$ the output, is the system linear?

- Solution: By KCL:

$$i(t) = C\frac{dv(t)}{dt} + \frac{1}{R}v(t)$$

- Substituting:

$$a_1 i_1(t) + a_2 i_2(t) \stackrel{?}{=} C \cdot \frac{d}{dt}\left[a_1 v_1(t) + a_2 v_2(t)\right] + \frac{\left[a_1 v_1(t) + a_2 v_2(t)\right]}{R}$$

or

$$\stackrel{?}{=} a_1 \underbrace{\left[C\frac{dv_1(t)}{dt} + \frac{v_1(t)}{R}\right]}_{i_1(t)} + a_2 \underbrace{\left[C\frac{dv_2(t)}{dt} + \frac{v_2(t)}{R}\right]}_{i_2(t)}$$

So

$$a_1 i_1(t) + a_2 i_2(t) \stackrel{\checkmark}{=} a_1 i_1(t) + a_2 i_2(t)$$

$$\boxed{\Longrightarrow \text{ Linear System}}$$