Continuous-Time Signals and Systems

The Fourier Series

Edward (Ted) A. Clancy

Department of Electrical and Computer Engineering
Department of Biomedical Engineering
Worcester Polytechnic Institute
Worcester, MA U.S.A.

Periodic Signals

- **Signal periodic if:** $x(t) = x(t + T_o),\ \forall t$

$$T_o \equiv \text{Fundamental Period}$$

(Choose smallest possible T_o)

- **Then,** $\omega_o = \frac{2\pi}{T_o} \equiv \text{Fundamental Frequency}$

- **For most periodic signals *x(t)*, can re-write *x(t)* as sum of scaled complex sinusoids of frequency:** $0,\ \omega_o,\ 2\omega_o,\ 3\omega_o,\ \ldots$

DC term (Signal average value)

Fundamental frequency (First harmonic)

Second harmonic

Third harmonic

  - **No other frequencies required; just need frequency multiples of ω_o**

- **Sum can have finite or infinite number of terms**

- **Sum can represent real- or complex-valued signals**

- **Scaling coefficients, plotted at each corresponding frequency, give Fourier spectrum**

Periodic Expansion Example 1 — Real-Valued *x(t)*

- **Let:**

$$x(t) = \sum_{k=-2}^{2} d_k \, e^{jk2\pi t}$$

with $d_0 = 2,\ d_{\pm 1} = \frac{3}{2},\ d_{\pm 2} = 3$

Note: Complex conjugate coefficients (for $\pm k$) will yield real *x(t)* due to cancellation of imaginary parts. (Here, the complex conjugate of a real value is itself.)

- **Expanding:**

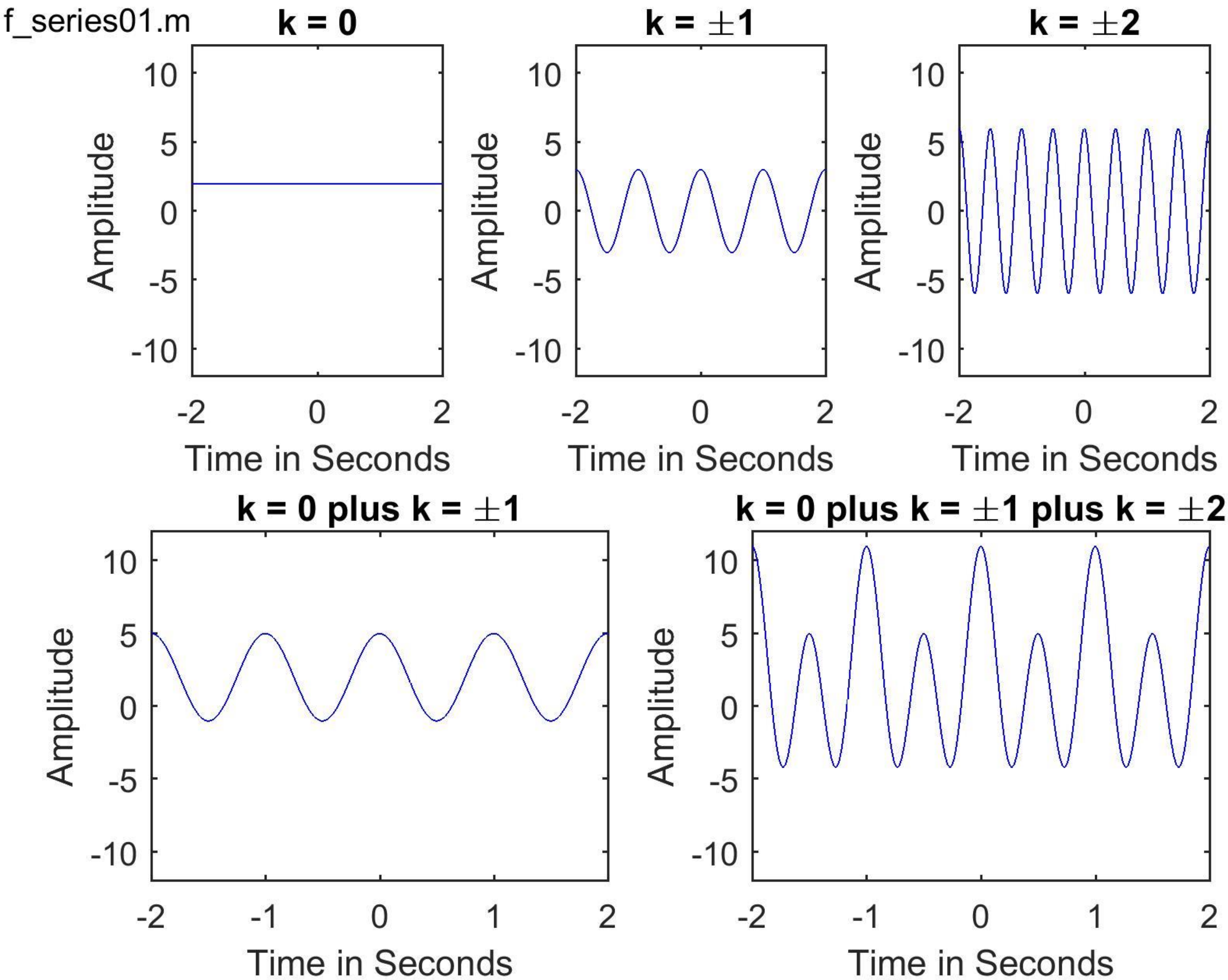
$$x(t) = \underbrace{2}_{k=0} + \frac{3}{2}\left(\underbrace{e^{j2\pi t}}_{k=1} + \underbrace{e^{-j2\pi t}}_{k=-1}\right) + 3\left(\underbrace{e^{j4\pi t}}_{k=2} + \underbrace{e^{-j4\pi t}}_{k=-2}\right)$$

- **Thus,**

$$x(t) = 2 + 3 \cdot \cos(2\pi t) + 6 \cdot \cos(4\pi t)$$

Continued

Periodic Expansion Example 1 — Plot



Periodic Expansion Example 2 — Real-Valued *x(t)*

- **Let:**

$$x(t) = \sum_{k=-100}^{100} d_k \, e^{jk2\pi t}$$

with $d_0 = 0$, else $d_{\pm k} = \sqrt{|k|} \cdot (1 \pm j)$

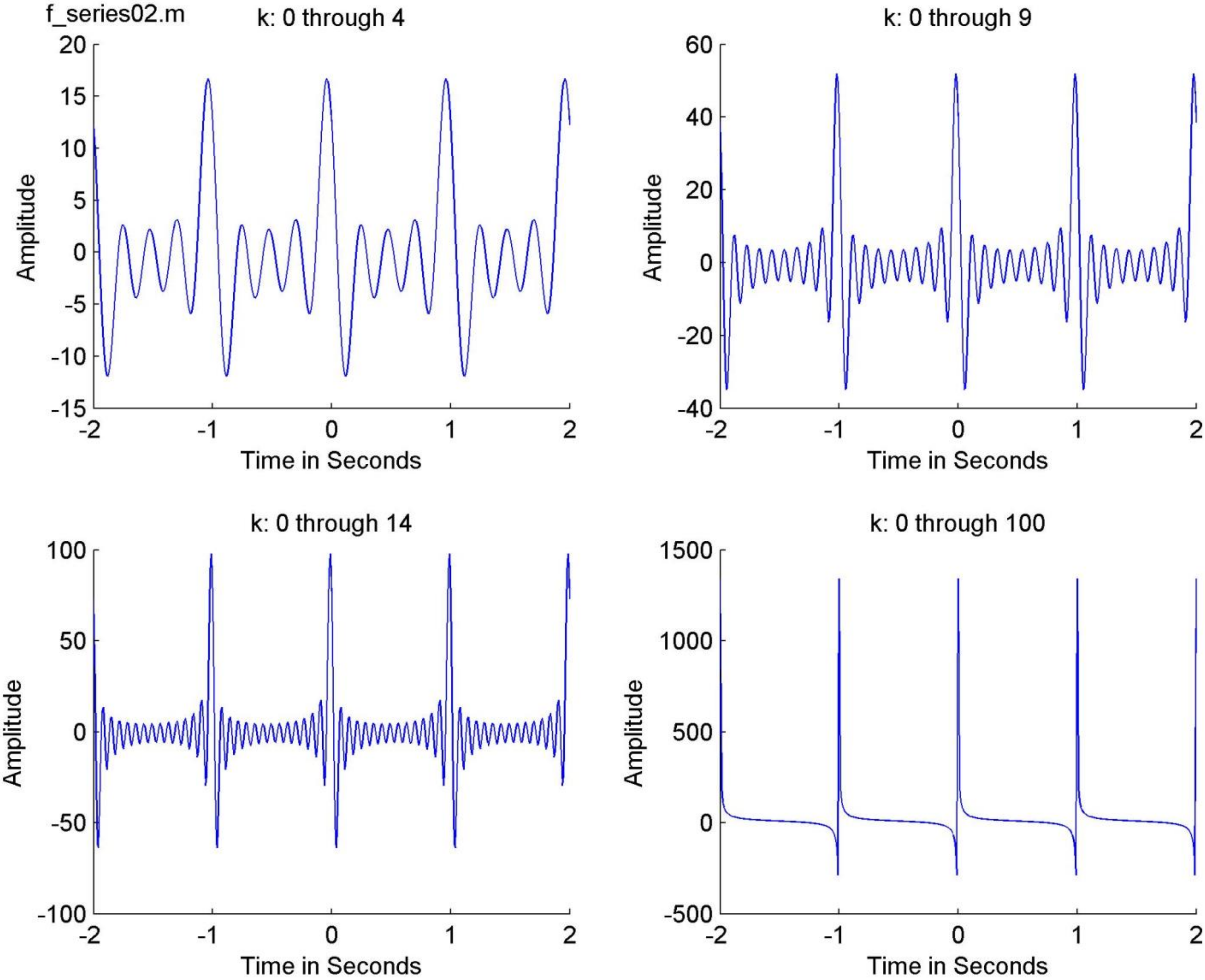
Note:	**Complex conjugate coefficients (for $\pm k$) will yield real *x(t)* due to cancellation of imaginary parts.**

- **Expanding and using Euler's identity:**

$$x(t) = \sum_{k=0}^{100} 2\sqrt{k}\,[\cos(2\pi kt) - \sin(2\pi kt)]$$

Continued

Periodic Expansion Example 2 — Plot



With enough exponential terms, nearly all periodic functions can be represented.

Q: Given a periodic function, how find the d_k coefficients?

Fourier Series Equations—Exponential Form

- Given periodic *x(t)*, how find spectral coefficients?

- Analysis Equation (forward transform):

$$d_k = \frac{1}{T_o} \int_{T_o} x(t)\, e^{-jk\omega_o t}\, dt = \frac{1}{T_o} \int_{T_o} x(t)\, e^{-jk2\pi f_o t}\, dt$$

where d_0 is the mean value of *x(t)*.

» Note that frequency is indexed with "$k\ \omega_o$". That is, the spectrum is only non-zero at multiples of the fundamental frequency ω_o.

- Synthesis Equation (inverse transform):

$$x(t) = \sum_{k=-\infty}^{\infty} d_k\ e^{jk\omega_o t} = \sum_{k=-\infty}^{\infty} d_k\ e^{jk2\pi f_o t}$$

Convergence

- Formal convergence can take many forms
  - Guaranteed for some input function forms
  - Can converge in the mean
    - Or, mean-squared
  - Etc.

- For practical engineering signals, Dirichlet conditions
  ➔ Convergence if *x(t)* …

1. … absolutely integrable over one period:

$$\int_{T_o} |x(t)|\, dt < \infty$$

2. … finite number of finite discontinuities in one period,

3. … finite number of maxima, minima in one period.

Fourier Series Equations—Alternative Forms

• **Trigonometric Form:**

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k \cos(k\omega_o t) + b_k \sin(k\omega_o t)]$$

$$a_0 = \frac{1}{T_o} \int_{T_o} x(t)\, dt$$

$$a_k = \frac{2}{T_o} \int_{T_o} x(t) \cos(k\omega_o t)\, dt, \qquad b_k = \frac{2}{T_o} \int_{T_o} x(t) \sin(k\omega_o t)\, dt$$

where,

$$a_0 = c_0 = d_0, \qquad \left.\begin{array}{l} a_k = d_k + d_{-k} \\ b_k = j\,(d_k - d_{-k}) \end{array}\right\} k \neq 0$$

and

$$d_k = \frac{a_k - j\, b_k}{2} = \frac{c_k\, e^{j\,\theta_k}}{2}, k \neq 0$$

• **Compact Trigonometric Form — *x(t)* MUST BE REAL-VALUED:**

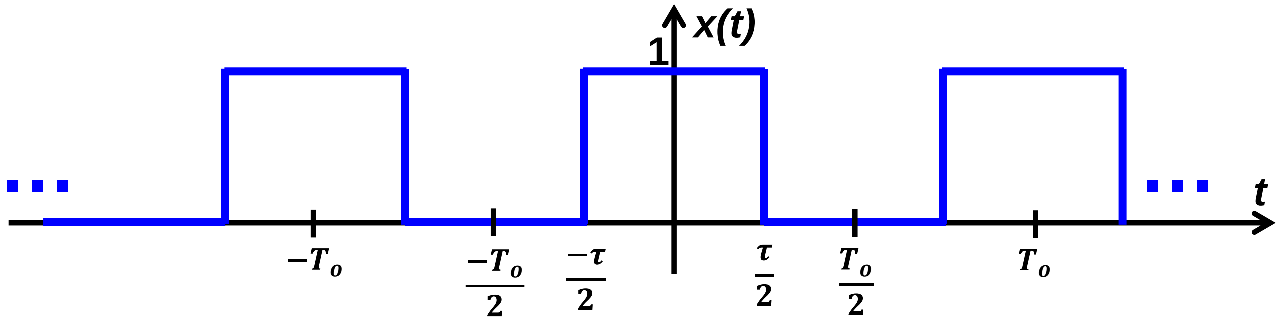
$$x(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t + \theta_k)$$

where,

$$c_0 = a_0 = d_0, \qquad c_k = \sqrt{a_k^2 + b_k^2} = \{2\,|d_k|, k \neq 0\}$$

$$\theta_k = \tan^{-1}\left(\frac{-b_k}{a_k}\right) = \angle d_k$$

Fourier Series of Square Wave — 1



- For period centered at $t = 0$: $x(t) = \begin{cases} 1, & |t| < \frac{\tau}{2} \\ 0, & \frac{\tau}{2} < |t| < \frac{T_o}{2} \end{cases}$

[Note: Some authors set $x(t) = ½$ at step locations.]

- **Fourier Series Coefficients (Exponential Form):**

$$d_k \equiv \frac{1}{T_o} \int_{t=\frac{-\tau}{2}}^{\frac{\tau}{2}} 1 \cdot e^{-jk\omega_o t}\, dt = \left. \frac{e^{-jk\omega_o t}}{-jk\omega_o T_o} \right|_{t=\frac{-\tau}{2}}^{\frac{\tau}{2}} = \frac{e^{-jk\omega_o \frac{\tau}{2}} - e^{-jk\omega_o \frac{-\tau}{2}}}{-jk\omega_o T_o}$$

$$= \frac{2}{k\omega_o T_o} \cdot \frac{e^{jk\omega_o \frac{\tau}{2}} - e^{-jk\omega_o \frac{\tau}{2}}}{2j} = \frac{2}{k\omega_o T_o} \cdot \sin\left(\frac{k\omega_o \tau}{2}\right)$$

– Using: $\omega_o T_o = 2\pi \Longrightarrow \quad d_k = \frac{1}{k\,\pi} \sin\left(\frac{k\,\omega_o \tau}{2}\right), \;\; k \neq 0$

- **For $k = 0$:**

$$d_0 \equiv \frac{1}{T_o} \int_{t=\frac{-\tau}{2}}^{\frac{\tau}{2}} e^{-j0\omega_o t}\, dt = \frac{1}{T_o} \int_{t=\frac{-\tau}{2}}^{\frac{\tau}{2}} dt = \left. \frac{t}{T_o} \right|_{t=\frac{-\tau}{2}}^{\frac{\tau}{2}} = \frac{\frac{\tau}{2} - \frac{-\tau}{2}}{T_o} = \frac{\tau}{T_o}$$

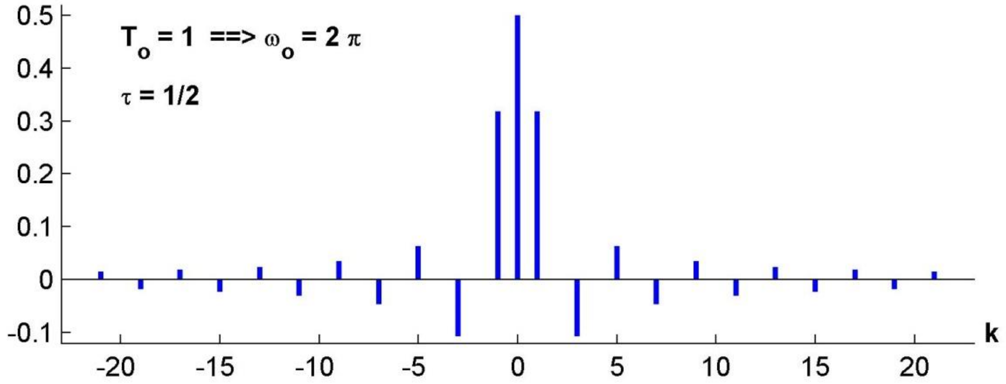
Continued

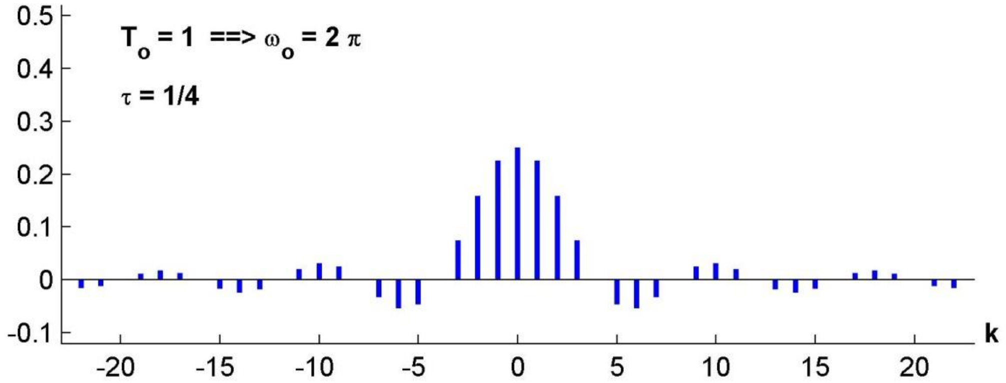
Fourier Series of Square Wave — 2

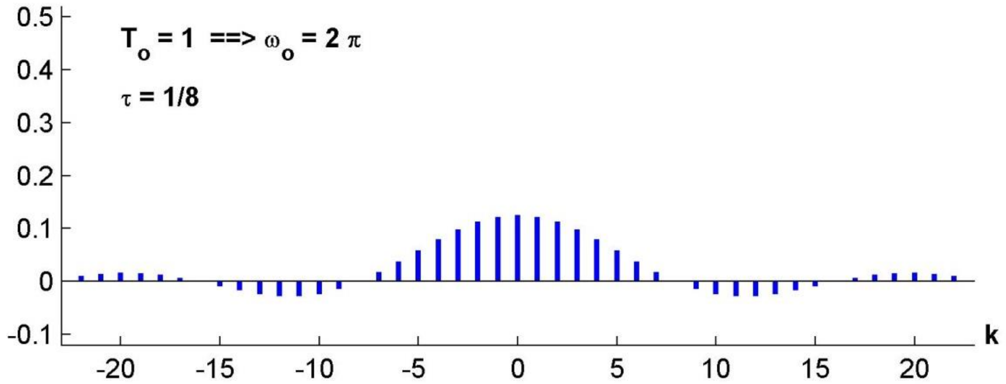
• **Full Fourier Series solution:**

$$d_k = \begin{cases} \dfrac{\tau}{T_o}, & k = 0 \\ \dfrac{\sin(k\omega_o\tau/2)}{k\pi}, & k \neq 0 \end{cases}$$

Note: Generally, d_k coefficients are complex-valued.



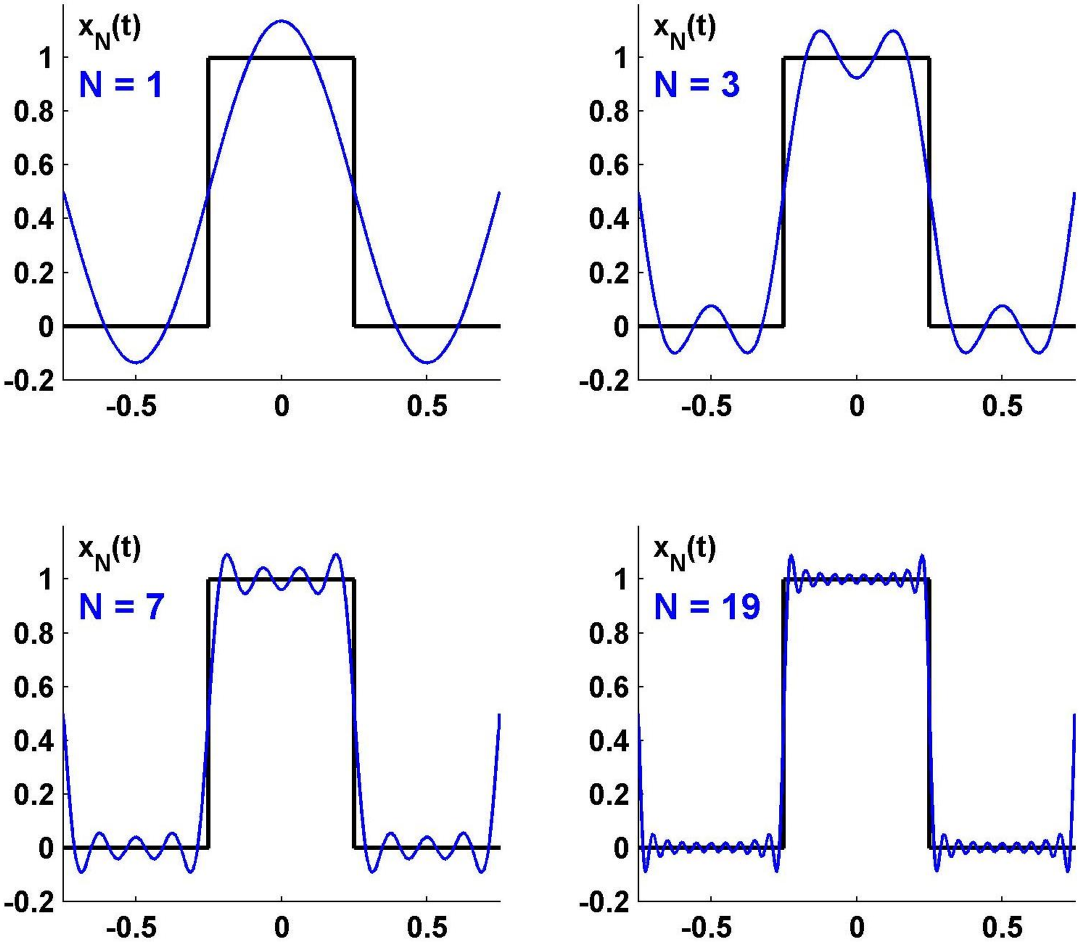


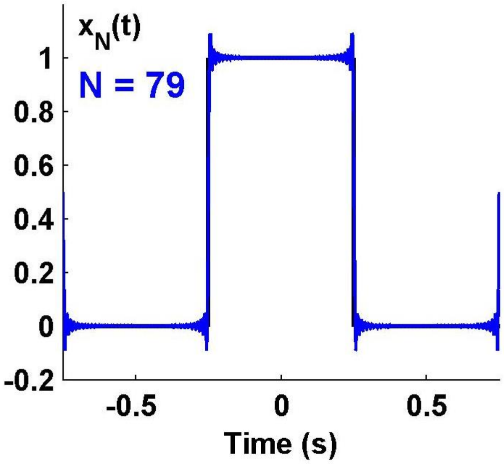


Continued

Fourier Series of Square Wave — 3

To = 1, Wo = 2 π, τ = 1/2





- **Approximation improves as more terms added**
- **Error/overshoot at discontinuities**
  - **Width of overshoot decreases with *N***
    - **Error area decreases**
  - **Height of overshoot remains the same !!!**

    ➔ Gibbs phenomenon

Fourier Series of Sine Wave — 1

- **Let:** $x(t) = \sin(2t)$
- **Find** Fourier Series:
  - Fundamental frequency: $\omega_o = 2 \text{ rad/sec}$

    $$\Rightarrow T_o = \frac{1}{f_o} = \frac{2\pi}{\omega_o} = \frac{2\pi}{2} = \pi \text{ seconds}$$
  - So,

$$d_k = \frac{1}{\pi} \int_{t=0}^{\pi} \left(\frac{e^{j2t} - e^{-j2t}}{2j} \right) e^{-jk2t}\, dt$$

$$= \frac{1}{2\pi j} \left[\frac{e^{j2t(1-k)}}{j2(1-k)} - \frac{e^{-j2t(1+k)}}{-j2(1+k)} \right] \Bigg|_{t=0}^{\pi}$$

$$= \frac{-1}{4\pi} \left[\frac{e^{j2\pi(1-k)} - 1}{1-k} + \frac{e^{-j2\pi(1+k)} - 1}{1+k} \right]$$

  - But, using that: $e^{j2\pi n} = 1$ for $n = 0, \pm 1, \pm 2, \ldots$

$$\Rightarrow d_k = 0 \text{ for } k \neq 1, k \neq -1$$

  - **At $k = 1$:**

$$d_1 = \frac{1}{\pi} \int_{t=0}^{\pi} \left(\frac{e^{j2t} - e^{-j2t}}{2j} \right) e^{-j2t}\, dt = \frac{1}{2\pi j} \int_{t=0}^{\pi} \left(1 - e^{-j4t} \right) dt$$

$$= \frac{1}{2\pi j} \left[t - \frac{e^{-j4t}}{-j4} \right] \Bigg|_{t=0}^{\pi} = \frac{1}{2\pi j} \left[\left(\pi - \frac{e^{j4\pi}}{-j4} \right) - \left(0 - \frac{e^{-j0}}{-j4} \right) \right]$$

$$= \frac{1}{2\pi j} \left[\pi - \frac{1}{-j4} + \frac{1}{-j4} \right] = \frac{\pi}{2\pi j} = -\frac{j}{2} = d_1$$

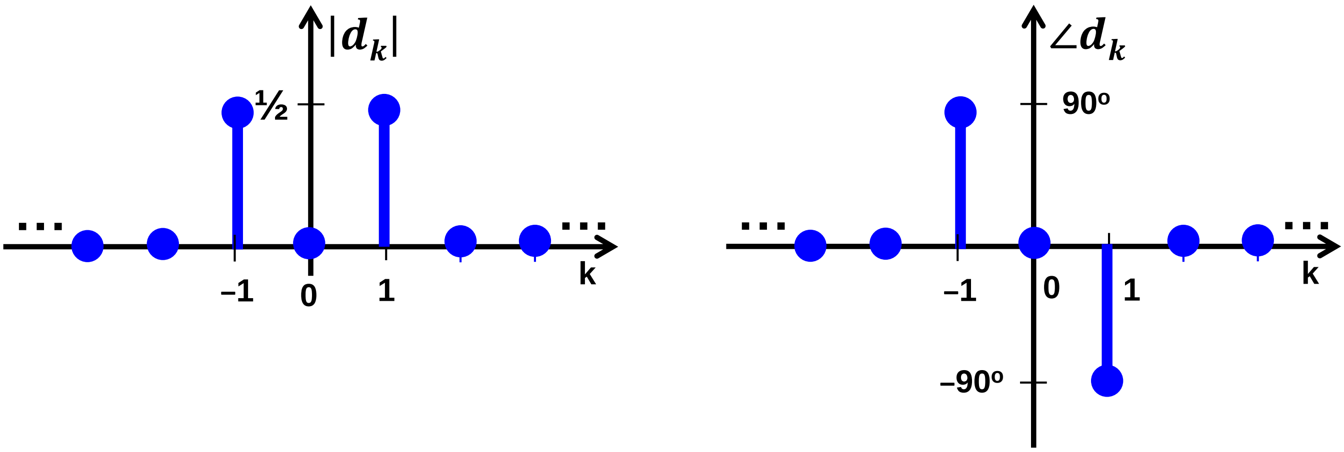
  - **Similarly at $k = -1$:** $d_{-1} = \frac{j}{2}$

Continued

Fourier Series of Sine Wave — 2

• **Thus:** $x(t) = \sin(2t) \iff d_k = \begin{cases} \frac{j}{2}, & k = -1 \\ -\frac{j}{2}, & k = 1 \\ 0, & \text{otherwise} \end{cases}$

– **Magnitude and phase plots:**



• **Alternative (EASIER !!) "trick" method for sinusoids:**

$$x(t) = \sin(2t) = \frac{1}{2j} \cdot e^{j2t} + \frac{-1}{2j} \cdot e^{-j2t}$$

– **Comparing to synthesis equation with $\omega_o = 2$:**

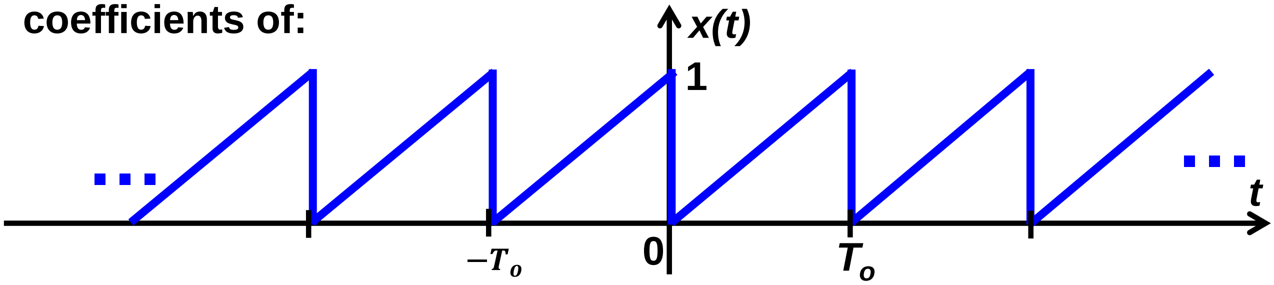
$$x(t) = \sum_{k=-\infty}^{\infty} d_k \, e^{jk2t}$$

– **Read coefficients by inspection:**

$$d_1 = \frac{1}{2j} = \frac{-j}{2}, \qquad d_{-1} = \frac{-1}{2j} = \frac{j}{2}, \qquad \text{All others} = 0$$

Exponential Fourier Series of Sawtooth Wave — 1

- Use direct integration to find the exponential-form Fourier Series coefficients of:



- **Solution:**

  – For $0 \le t \le T_o$: $x(t) = \frac{t}{T_o},\ 0 \le t \le T_o$

  – Thus,

$$d_k = \frac{1}{T_o}\int_{t=0}^{T_o} \frac{t}{T_o}\, e^{-jk\omega_o t}\, dt = \left.\frac{-e^{-jk\omega_o t}\,(jk\omega_o t + 1)}{T_o^2\,(jk\omega_o)^2}\right|_{t=0}^{T_o}$$

$$= \frac{e^{-jk\omega_o T_o}\,(jk\omega_o T_o + 1) - 1}{T_o^2\, k^2\, \omega_o^2}$$

  – Note 1: $\omega_o = 2\pi f_o = \frac{2\pi}{T_o} \Longrightarrow \omega_o T_o = 2\pi$

  – Note 2: $e^{-j2\pi k} = 1,\ \text{for } k = 0, \pm 1, \pm 2, \ldots$ (i.e., for k integer)

  – Giving,

$$d_k = \frac{jk \cdot 2\pi + 1 - 1}{(2\pi)^2\, k^2} = \frac{j}{2\pi \cdot k}, k \ne 0$$

Continued

Exponential Fourier Series of Sawtooth Wave — 2

- At $k = 0$:

$$d_0 = \frac{1}{T_o} \int_{t=0}^{T_o} \frac{t}{T_o}\, e^{-j0\omega_o t}\, dt = \frac{1}{T_o} \int_{t=0}^{T_o} \frac{t}{T_o}\, dt = \left. \frac{t^2}{2\, T_o^2} \right|_{t=0}^{T_o} = \frac{T_o^2}{2\, T_o^2} - \frac{0^2}{2\, T_o^2}$$
$$= \frac{1}{2}$$

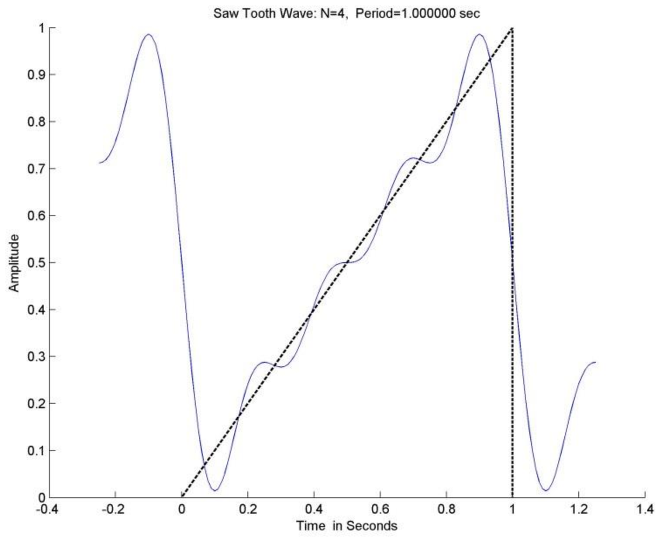
- Gathering the full solution:

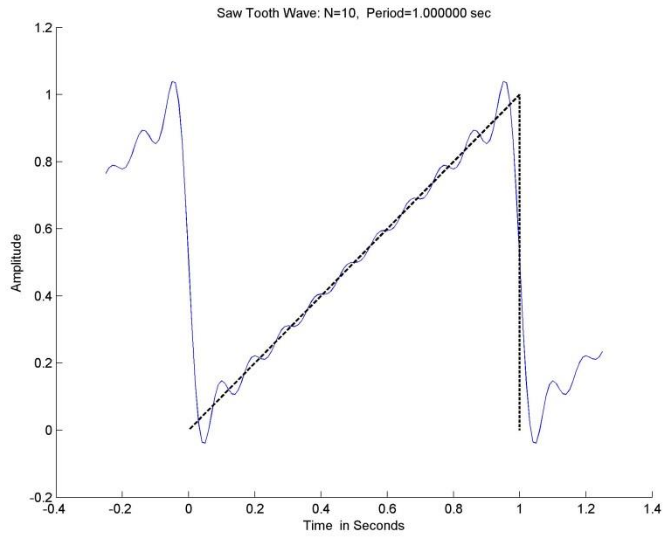
$$d_k = \begin{cases} \dfrac{1}{2}, & k = 0 \\ \dfrac{j}{2\pi \cdot k}, & k \neq 0 \end{cases}$$

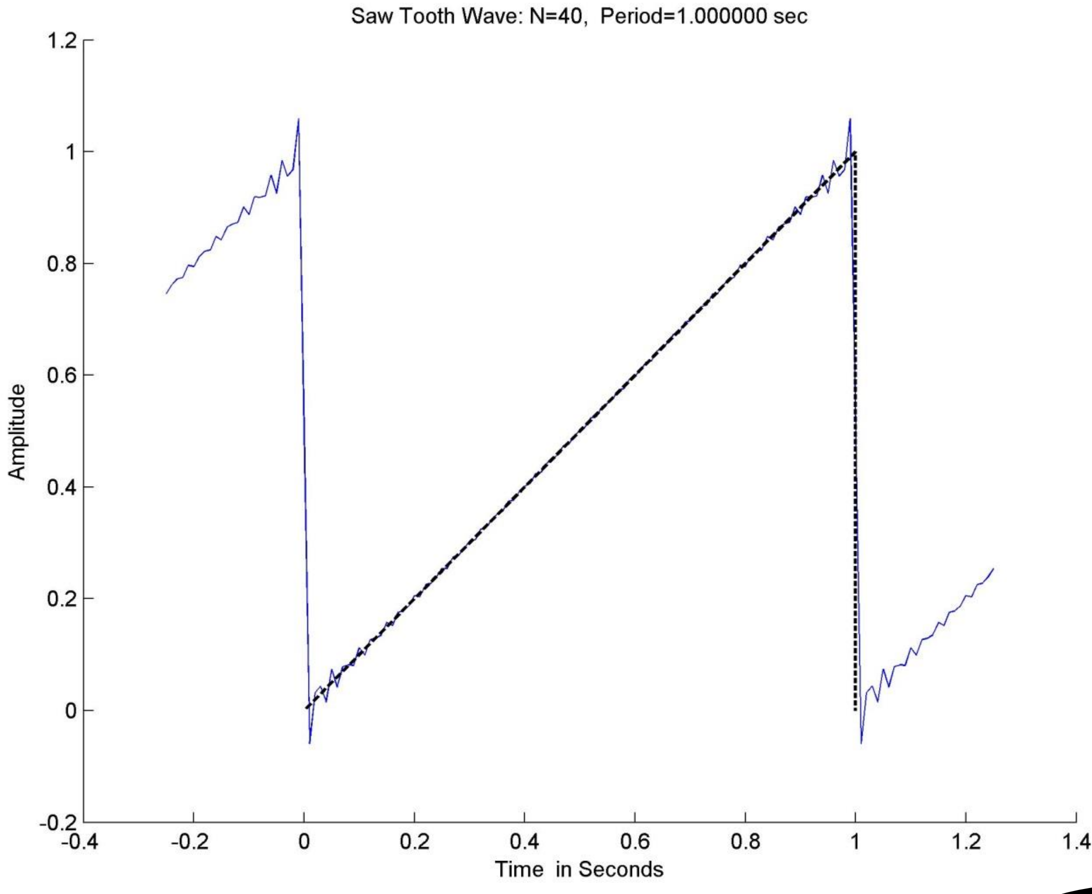
Continued

Exponential Fourier Series of Sawtooth Wave — 3

- **Synthesis (time) plots using 4, 10, 40 harmonics (T_o = 1 sec):**



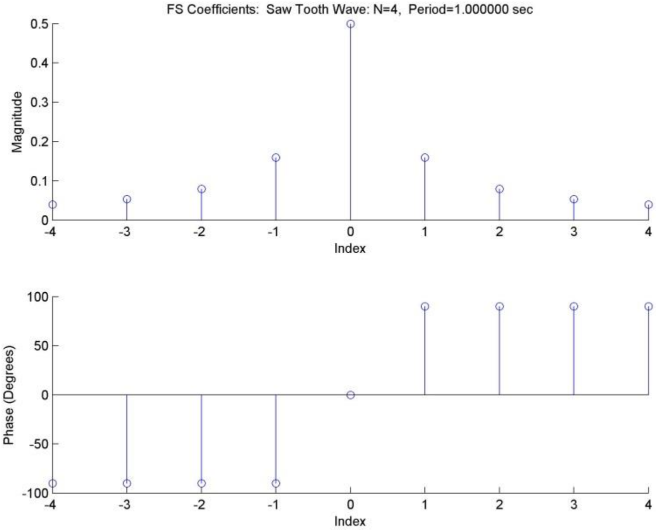


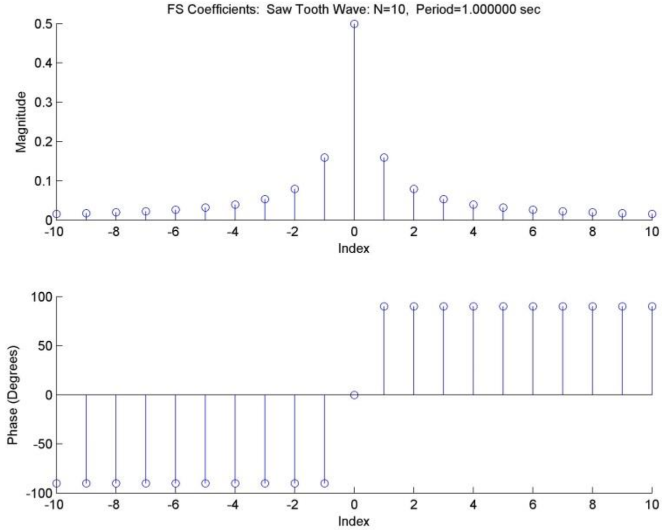


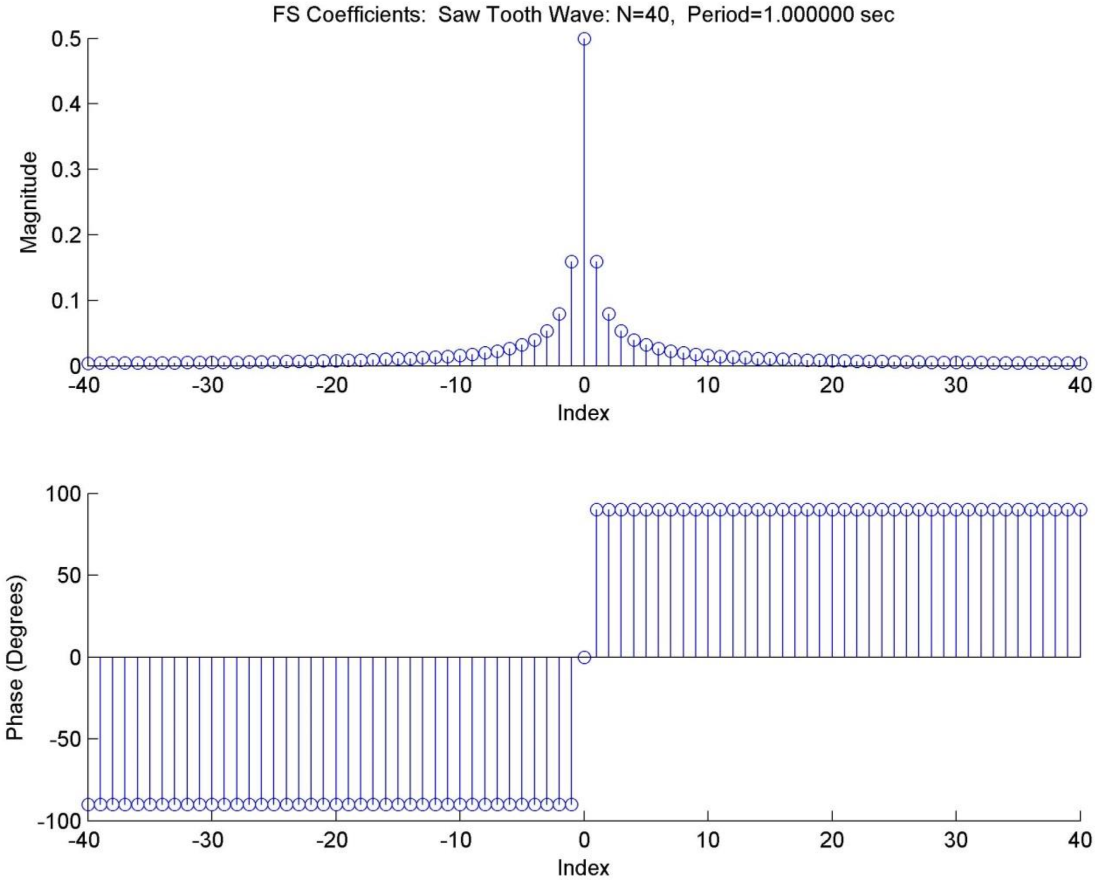
Continued

Exponential Fourier Series of Sawtooth Wave — 4

- **Analysis (frequency) plots using 4, 10, 40 harmonics (T_o = 1 sec)**







Trigonometric Fourier Series of Sawtooth Wave — A

- **Method A)** **Convert exponential form result.**

  Know: $d_0 = \frac{1}{2}, \quad d_{k\neq 0} = \frac{j}{2\pi \cdot k}$

- **Conversion:**

1. $a_0 = d_0 = \frac{1}{2}$

2. $k > 0$:

$$a_k = d_k + d_{-k} = \frac{j}{2\pi \cdot k} + \frac{j}{2\pi \cdot (-k)} = \frac{j}{2\pi \cdot k} + \frac{-j}{2\pi \cdot k} = 0$$

$$b_k = j\,(d_k - d_{-k}) = j\left[\frac{j}{2\pi \cdot k} - \frac{j}{2\pi \cdot (-k)}\right]$$

$$= j\left[\frac{j}{2\pi \cdot k} + \frac{j}{2\pi \cdot k}\right] = j \cdot \frac{j}{\pi\, k} = \frac{-1}{\pi\, k}$$

- **Thus:**

$$\boxed{a_0 = \frac{1}{2}, \qquad a_{k>0} = 0, \qquad b_{k>0} = \frac{-1}{\pi\, k}}$$

Continued

Trigonometric Fourier Series of Sawtooth Wave — B

- **Method B)** **Direct integration.**

- **Recall:**

a) $x(t) = \frac{t}{T_o},\ 0 \le t \le T_o$ b) $\int x \cdot \sin(ax)\, dx = \frac{\sin(ax) - ax\cos(ax)}{a^2} + C$

c) $\int x \cdot \cos(ax)\, dx = \frac{\cos(ax) + ax\sin(ax)}{a^2} + C$ d) $\omega_o T_o = 2\pi$

---

1)

$$a_0 = \frac{1}{T_o}\int_{t=0}^{T_o} \frac{t}{T_o}\, dt = \left.\frac{t^2}{2\,T_o^2}\right|_{t=0}^{T_o} = \frac{T_o^2}{2\,T_o^2} - \frac{0^2}{2\,T_o^2} = \frac{1}{2}$$

2)

$$a_{k>0} = \frac{2}{T_o}\int_{t=0}^{T_o} \frac{t}{T_o}\cos(k\omega_o t)\, dt$$

$$= \frac{2}{T_o^2}\left.\left[\frac{\cos(k\omega_o t) + k\omega_o t\sin(k\omega_o t)}{k^2\,\omega_o^2}\right]\right|_{t=0}^{T_o}$$

$$= \frac{2}{T_o^2\,k^2\,\omega_o^2}\left[\cos(k\omega_o T_o) + k\omega_o T_o \sin(k\omega_o T_o) - \cos(0) - 0\cdot\sin(0)\right]$$

$$= \frac{2}{T_o^2\,k^2\,\omega_o^2}\left[\cos(2\pi k) + 2\pi k\sin(2\pi k) - 1\right]$$

$$= \frac{2}{T_o^2\,k^2\,\omega_o^2}\left[1 + 0 - 1\right] = 0$$

Continued

Trigonometric Fourier Series of Sawtooth Wave — B(2)

3)

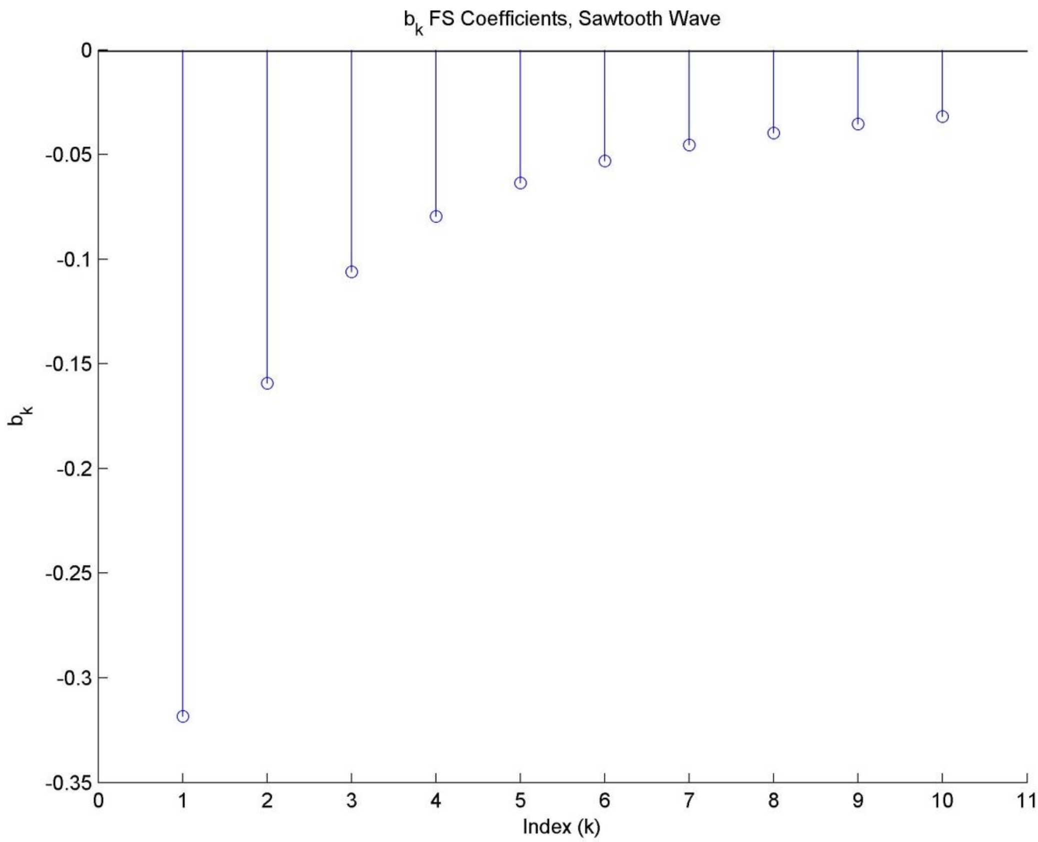
$$b_{k>0} = \frac{2}{T_o}\int_{t=0}^{T_o}\frac{t}{T_o}\sin(k\omega_o t)\ dt$$

$$= \frac{2}{T_o^2}\left[\frac{\sin(k\omega_o t) - k\omega_o t\cos(k\omega_o t)}{k^2\ \omega_o^2}\right]\Bigg|_{t=0}^{T_o}$$

$$= \frac{2}{T_o^2\ k^2\ \omega_o^2}\ [\sin(k\omega_o T_o) - k\omega_o T_o\cos(k\omega_o T_o) - \sin(0) + 0\cdot\cos(0)]$$

$$= \frac{2}{T_o^2\ k^2\ \omega_o^2}\ [\sin(2\pi k) - 2\pi k\cos(2\pi k)]$$

$$= \frac{2}{T_o^2\ k^2\ \omega_o^2}\ [0 - 2\pi k\cdot 1] = \frac{-4\pi k}{(2\pi)^2\ k^2} = \frac{-1}{\pi\ k}$$



Properties of the Fourier Series

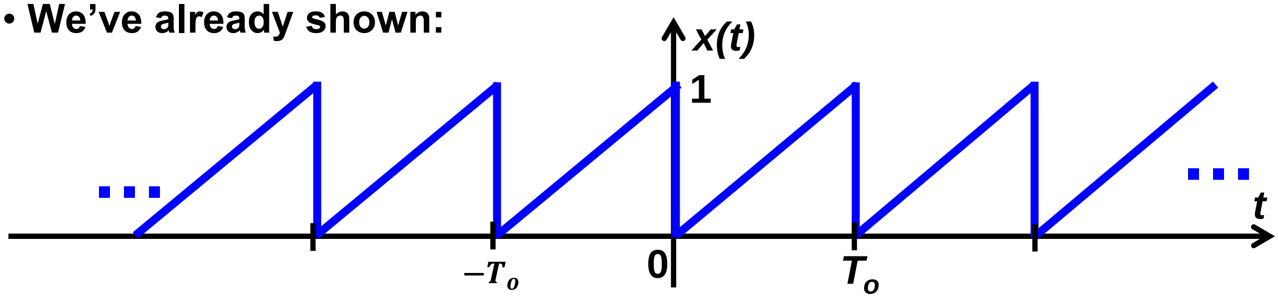
- **For periodic signals *x(t)* ←→ d_k, *y(t)* ←→ $d_{y,k}$ of period T_o, fundamental frequency $\omega_o = \frac{2\pi}{T_o}$:**

Property	Periodic Signal	Fourier Series Coefficients
Linearity	$a \cdot x(t) + b \cdot y(t)$	$a \cdot d_k + b \cdot d_{y,k}$
Time Shifting	$x(t-\tau)$	$d_k\, e^{-jk\omega_o \tau}$
Conjugation	$x^*(t)$	d^*_{-k}
Time Reversal	$x(-t)$	d_{-k}
Time Scaling	$x(at),\ a > 0$ (Becomes periodic in period $\frac{T_o}{a}$)	d_k
Periodic Convolution	$\int_{T_o} x(\tau) y(t-\tau) d\tau$	$T_o\, d_k\, d_{y,k}$
Multiplication	$x(t) y(t)$	$\sum_{l=-\infty}^{\infty} d_l\, d_{y,k-l}$
Differentiation	$\frac{d\,x(t)}{dt}$	$j k \omega_o d_k$
Integration	$\int_{t=-\infty}^{t} x(t) dt,\ d_0 = 0$	$\frac{d_k}{j k \omega_o}$
Conjugate Symmetry for Real Signals	$x(t)$ real-valued	$\begin{cases} d_k = d^*_{-k} \\ Re\{d_k\} = Re\{d_{-k}\} \\ Im\{d_k\} = -Im\{d_{-k}\} \\ \lvert d_k \rvert = \lvert d_{-k} \rvert \\ \angle d_k = -\angle d_{-k} \end{cases}$
Real and Even Signals	$x(t)$ real-valued and even	d_k real-valued and even
Real and Odd Signals	$x(t)$ real-valued and odd	d_k purely imaginary and odd
Even Decomposition of Real Signals	$x_e(t)$, real-valued	$Re\{d_k\}$
Odd Decomposition of Real Signals	$x_o(t)$, real-valued	$j\, Im\{d_k\}$

Parseval's Relation: $\frac{1}{T_o} \int_{T_o} |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |d_k|^2$

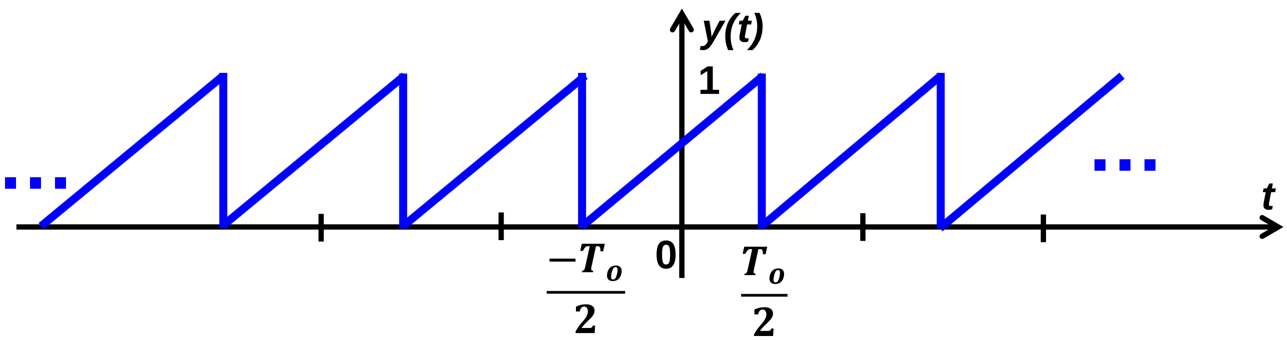
Example: Shift Property

- We've already shown:



$$x(t) = \frac{t}{T_o}, 0 \le t \le T_o \quad \Leftrightarrow \quad d_0 = \frac{1}{2}, d_{k \neq 0} = \frac{j}{2\pi \cdot k}$$

- **Find** exponential Fourier Series coefficients for:



– **Solution:**

$$y(t) = x\left(t - \frac{T_o}{2}\right)$$

- Thus, shift of $\frac{T_o}{2}$ implies (recalling that $\omega_o T_o = 2\pi$):

$$d_{y,k} = d_{x,k}\, e^{-j k \omega_o \frac{T_o}{2}} = d_{x,k}\, e^{-j k \pi} = d_{x,k}\, (-1)^k$$

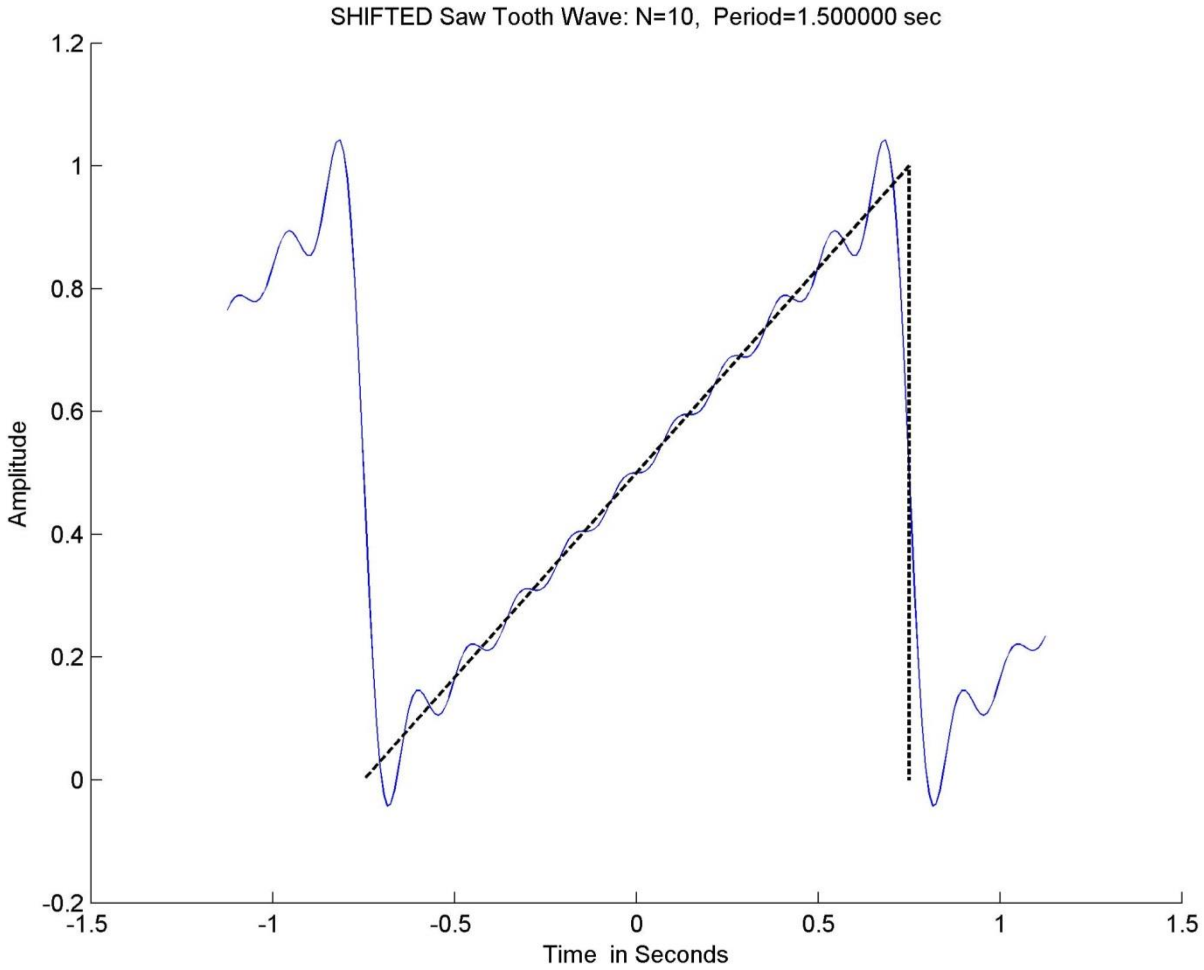
- Giving:

$$d_{y,0} = \frac{1}{2}, \qquad d_{y,k \neq 0} = \frac{(-1)^k \cdot j}{2\pi \cdot k}$$

Continued

Example: Shift Property (2)

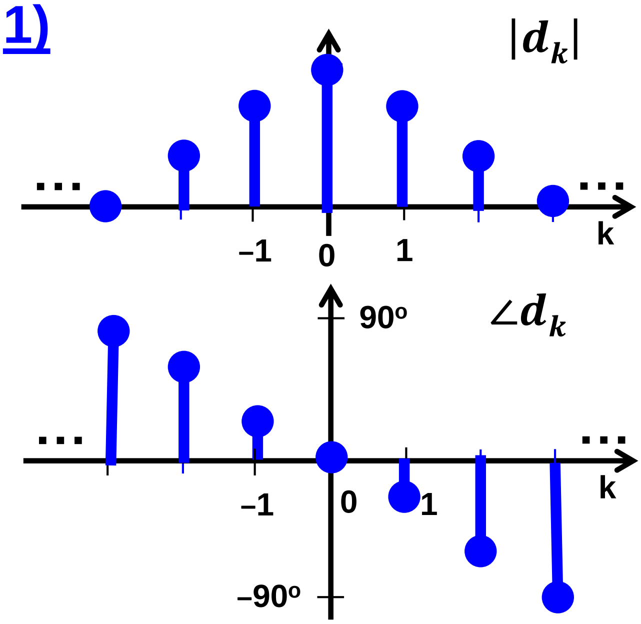
• **Synthesis (time) plot using 10 harmonics (T_o = 1.5 sec):**



Identifying Real- vs. Complex-Valued Signals

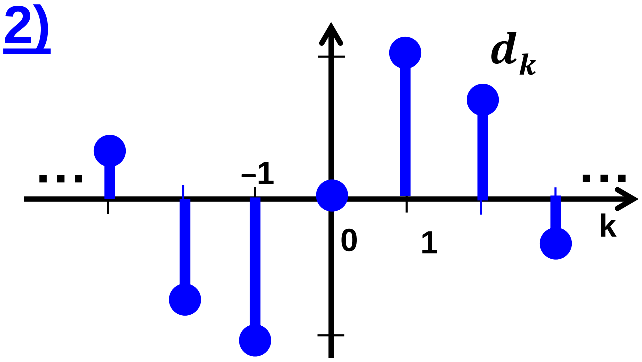
Exponential FS	x(t): Real- or Complex-Valued ?

1)



REAL

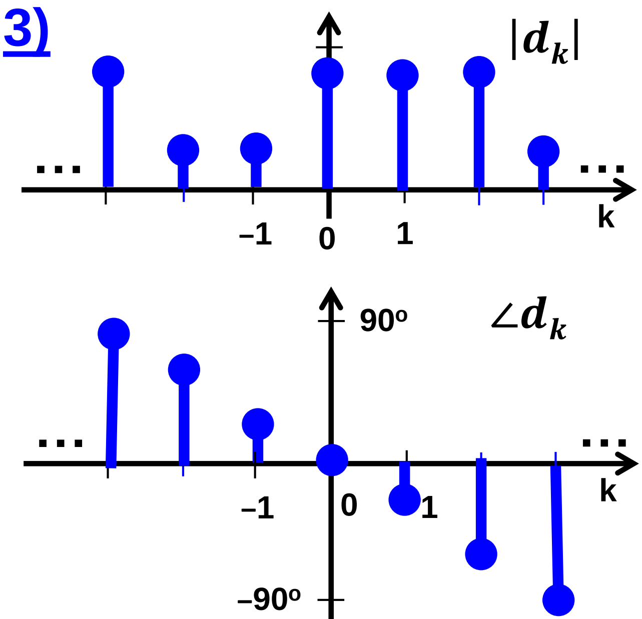
2)



COMPLEX

Phase not odd.
E.g., $\angle d_1 = 0^o$, $\angle d_{-1} = 180^o$

3)

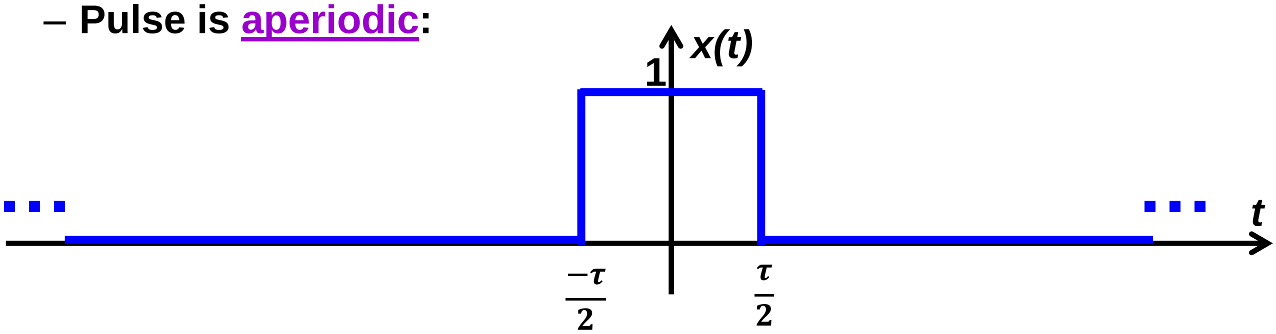


COMPLEX

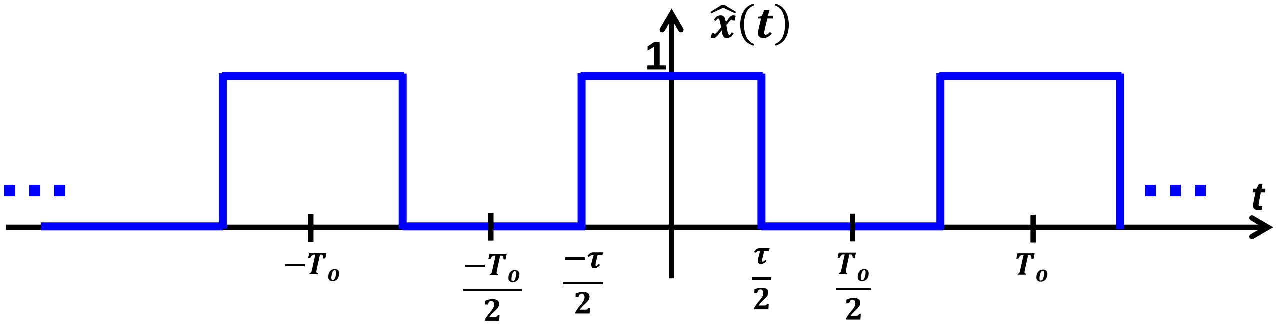
$|d_k| \neq |d_{-k}|$

From (Periodic) Fourier Series to (Aperiodic) Fourier Transform—1

- Use pulse as an example
  - Pulse is aperiodic:



  - Create periodic signal by replicating, shifting infinite copies:



- Previously showed, with $\omega_o = \frac{2\pi}{T_o}$:

$$\hat{d}_k = \begin{cases} \dfrac{\sin(k\,\omega_o\,\tau/2)}{k\,\pi} & k \neq 0 \\ \dfrac{\tau}{T_o}, & k = 0 \end{cases}$$

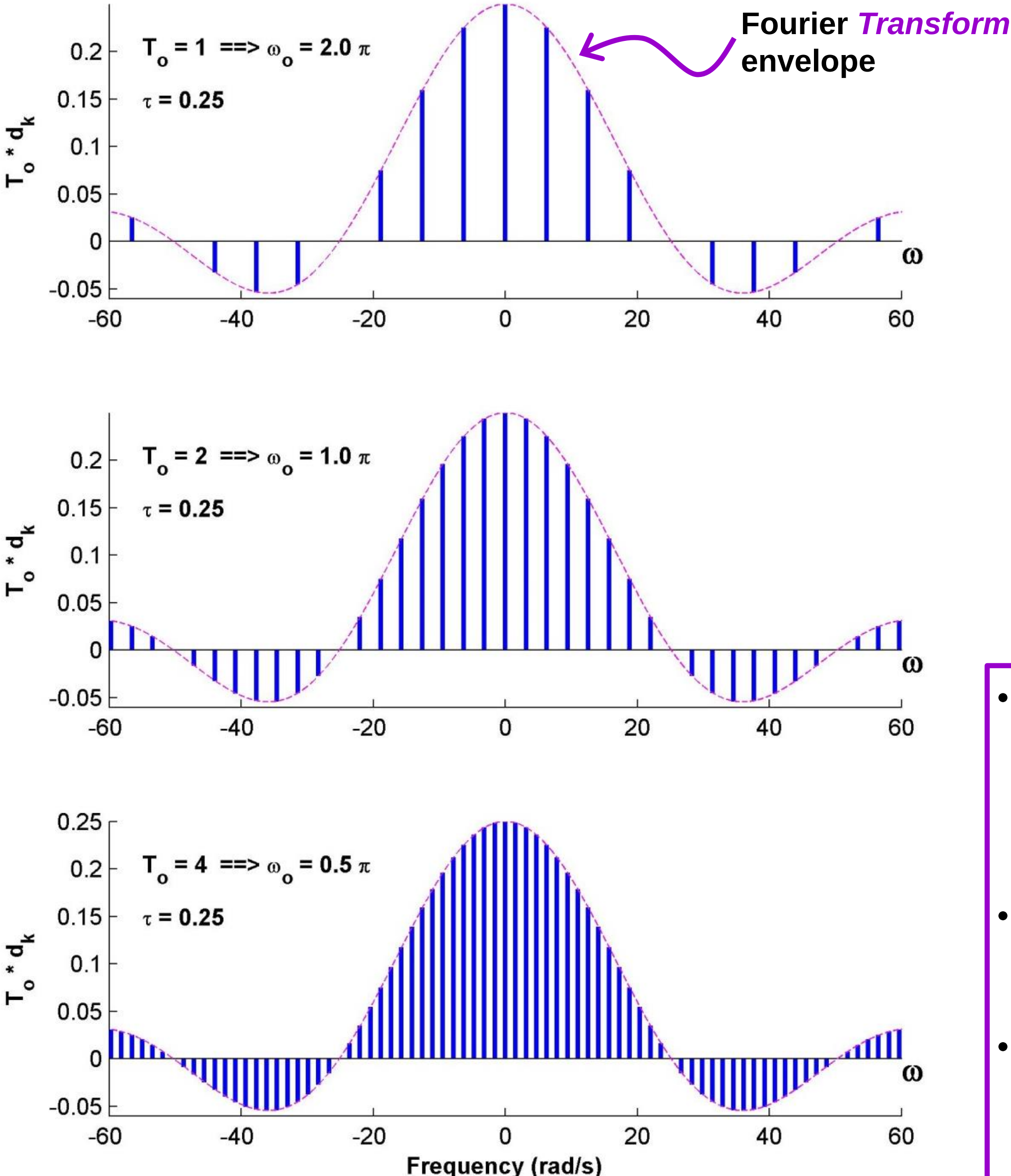
- If fix τ (must, so that aperiodic wave is represented near *t = 0*), then:

➔As T_o grows, $\hat{x}(t) \Longrightarrow x(t)$

Continued

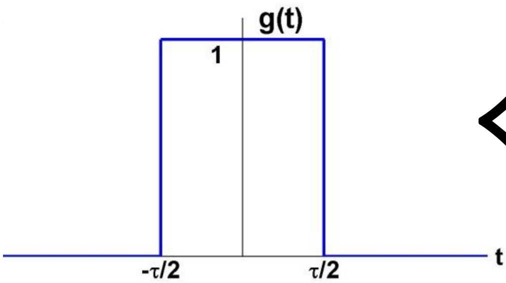
From (Periodic) Fourier *Series* to (Aperiodic) Fourier *Transform—2*

- Fourier Series : Plot T_o*d_k (instead of d_k) vs. ω (instead of k):

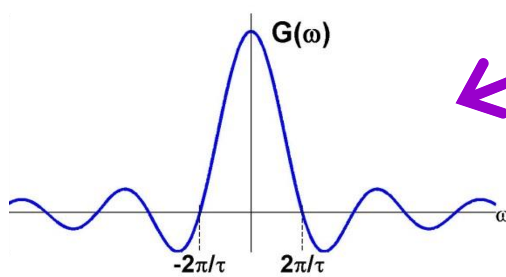


- As period $T_o \Rightarrow \infty$, then $\omega_o \Rightarrow 0$ and the coefficients trace an envelope.
- Envelope ⇒ Fourier Transform
- Fourier Transform of pulse (gate):

$$G(\omega) = \tau \cdot \sin\left(\frac{\omega\,\tau}{2}\right)$$



⇔



Fourier TRANSFORM of Periodic Signal — 1

- **Consider Fourier Transform:** $X(\omega) = 2\,\pi\,\delta(\omega - \omega_o)$
  - **Then, using the definition of the Inverse Fourier Transform:**

$$x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o)\, e^{j\omega t}\, d\omega = \frac{2\pi\, e^{j\omega_o t}}{2\pi} = e^{j\omega_o t}$$

- **In general:**

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, d_k\, \delta(\omega - k\,\omega_o)$$

$$\Updownarrow$$

$$x(t) = \sum_{k=-\infty}^{\infty} d_k\, e^{jk\omega_o t}$$

Fourier SERIES representation

$\Rightarrow$ **If *x(t)* periodic with Fourier Series coefficients d_k**

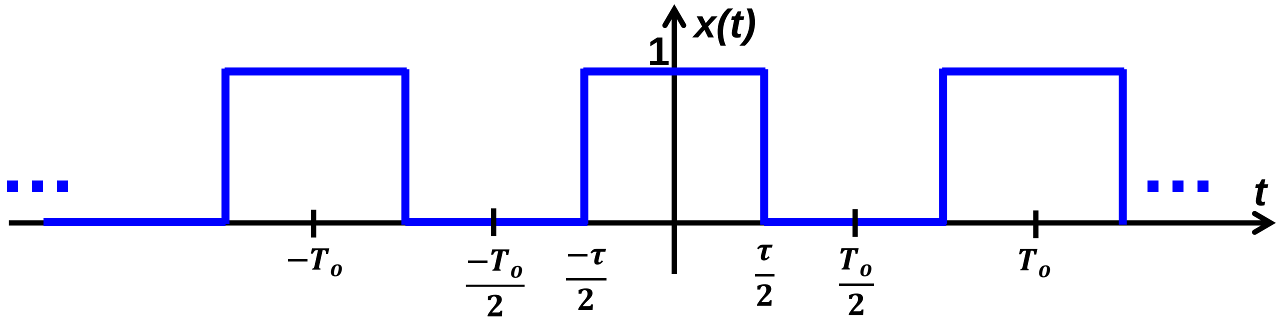
$$\Downarrow$$

Fourier Transform is impulses at corresponding harmonic frequencies, each impulse with area: $2\,\pi\,d_k$

Continued

Fourier TRANSFORM of Periodic Signal — 2

- **Example:** Periodic square wave

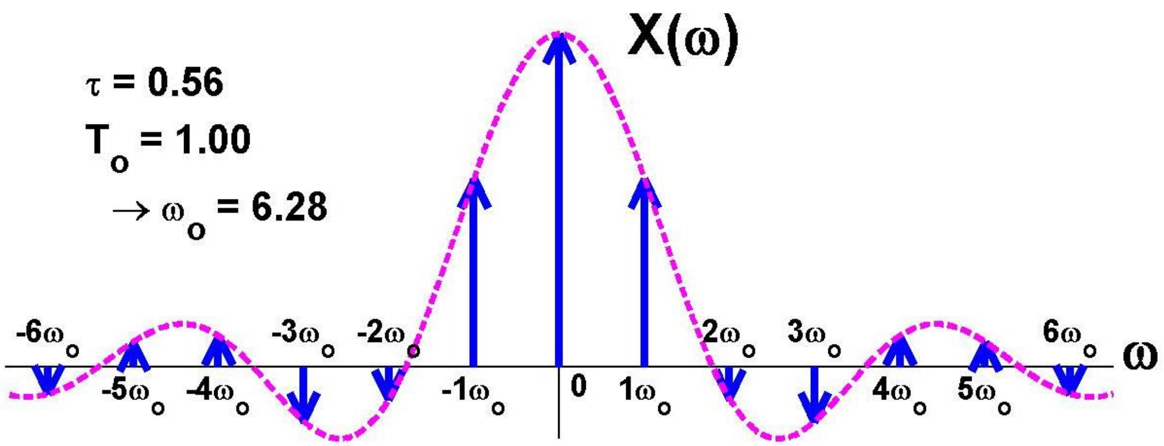


– Previously found Fourier Series:

$$d_k = \begin{cases} \dfrac{\tau}{T_o}, & k = 0 \\ \dfrac{\sin(k\omega_o\tau/2)}{k\pi}, & k \neq 0 \end{cases}$$

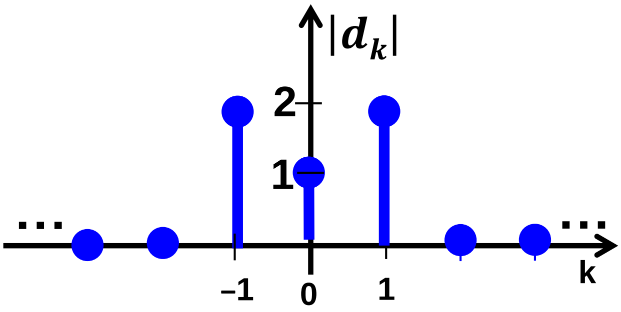
$$\Updownarrow$$

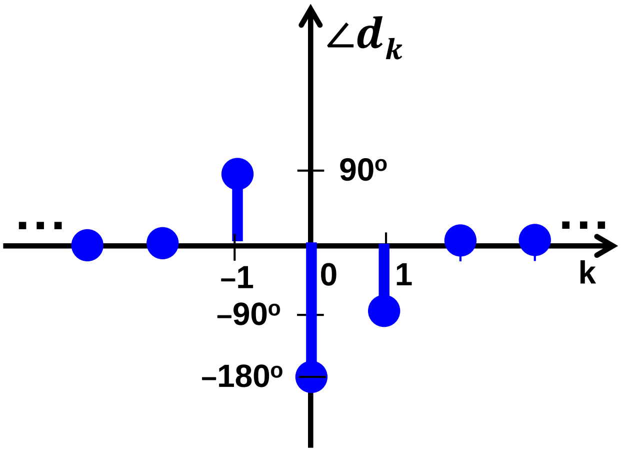
$$X(\omega) = \begin{cases} \dfrac{2\pi\,\tau}{T_o}\,\delta(\omega), & \omega = 0 \\ \displaystyle\sum_{k=-\infty}^{\infty} 2\,\dfrac{\sin(k\omega_o\tau/2)}{k}\,\delta(\omega - k\,\omega_o)\,, & \omega, k \neq 0 \end{cases}$$



Example: Fourier Transform from Fourier Series

- **Periodic *x(t)* has Fourier Series coefficients as shown below:**





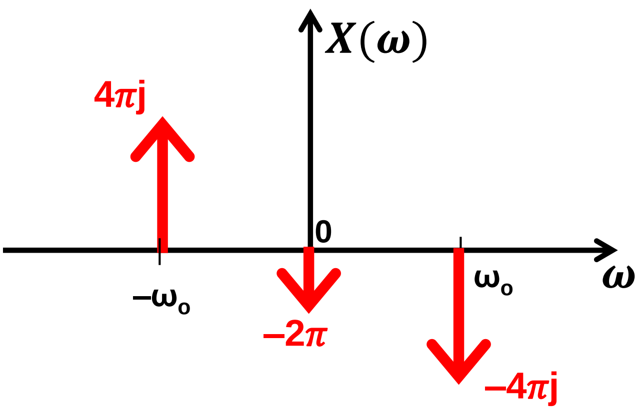
- **Find and draw Fourier Transform of *x(t)* ⇔ *X(ω)***

- **Solution: By inspection:**

$$d_k = \begin{cases} j2, & k = -1 \\ -1, & k = 0 \\ -j2, & k = 1 \\ 0, & \text{otherwise} \end{cases}$$

– **Thus, only three non-zero locations in *X(ω)***

$$\Rightarrow X(\omega) = 2\pi\left[2j\,\delta(\omega+\omega_o) - \delta(\omega) - 2j\,\delta(\omega-\omega_o)\right]$$



Fourier Series vs. Fourier Transform

- For periodic signals:
  - Fourier Series and Fourier Transform hold the same information
    - Fourier Series has coefficients d_k
    - Fourier Transform has impulses of area : $2\pi d_k$

- Fourier Series developed first historically
  - Coefficients (a.k.a. line spectrum) often easier to draw, visualize
  - Fourier Series coefficients usually easier to compute
    - Difficult to write, evaluate Fourier Transform integral for many periodic functions
  - More "intuitive" ?? $\Rightarrow$ sum of sinusoids
  - Discrete-time signal processing on the computer follows Fourier Series, not Fourier Transform!
  - Easy to study influence of removing high frequency components