Solution

Worcester Polytechnic Institute — Department of Electrical and Computer Engineering
ECE2311 — Continuous-Time Signal and System Analysis — Term B'17

Homework 5: Due Tuesday, 14 November 2017 (3:00 P.M.)

Write your name and ECE box at the top of each page.

General Reminders on Homework Assignments:

- Always complete the reading assignments *before* attempting the homework problems.
- Show all of your work. Use written English, where applicable, to provide a log or your steps in solving a problem. (For numerical homework problems, the writing can be brief.)
- A solution that requires physical units is *incorrect* unless the units are listed as part of the result.
- Get in the habit of underlining, circling or boxing your result.
- Always write neatly. Communication skills are essential in engineering and science. "If you didn't write it, you didn't do it!"

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1) Computing the Bilateral Laplace Transform:

a) Use direct integration to find the Bilateral Laplace Transform for: $x(t) = e^{-t}\,u(t) - e^{-2t}\,u(t)$. Draw the region of convergence on the s-plane.

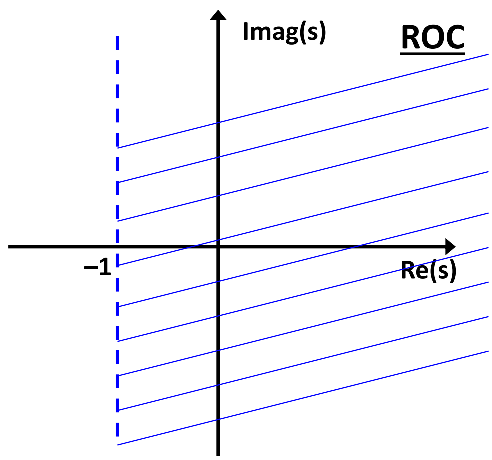
$$X(s) = \int_{t=-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{\infty} (e^{-t} - e^{-2t})\, e^{-st}\, dt = \int_{0}^{\infty} (e^{-t(s+1)}\, dt - \int_{0}^{\infty} e^{-t(s+2)}\, dt$$

$$= \left.\frac{e^{-t(s+1)}}{-(s+1)}\right|_{t=0}^{\infty} - \left.\frac{e^{-t(s+2)}}{-(s+2)}\right|_{t=0}^{\infty} = \frac{e^{-\infty(s+1)} - e^{-0(s+1)}}{-(s+1)} - \frac{e^{-\infty(s+2)} - e^{-0(s+2)}}{-(s+2)}$$

The term $e^{-\infty(s+1)}$ goes to zero for $\mathrm{Re}(s) > -1$. The term $e^{-\infty(s+2)}$ goes to zero for $\mathrm{Re}(s) > -2$. Both constraints are satisfied for $\mathrm{Re}(s) > -1$.

Noting that $e^{0} = 1$, the above simplifies to:

$$\boxed{X(s) = \frac{1}{s+1} - \frac{1}{s+2}, \quad \mathrm{Re}(s) > -1}$$



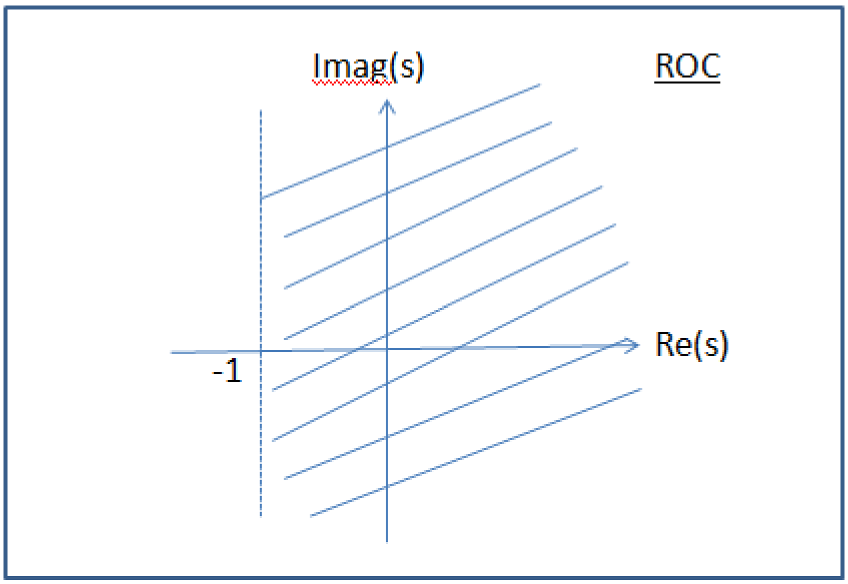
b) Use direct integration to find the Bilateral Laplace Transform for: $h(t) = t\, e^{-t}\, u(t)$. Draw the region of convergence on the s-plane. Note: $\int x\, e^{-ax}\, dx = \frac{-e^{-ax}(a\,x+1)}{a^2} + C$ and $\lim_{t \to \infty} (t^2 e^{-t}) = 0$.

$$H(s) = \int_{t=-\infty}^{\infty} h(t) e^{-st}\, dt = \int_{t=0}^{\infty} t\, e^{-t} e^{-st}\, dt = \int_{t=0}^{\infty} t\, e^{-t(s+1)} dt = \left. \frac{-e^{-t(s+1)}[(s+1)t+1]}{(s+1)^2} \right|_{t=0}^{\infty}$$

$$= \frac{-e^{-\infty(s+1)}[(s+1)\infty+1] - -e^{-0(s+1)}[(s+1)0+1]}{(s+1)^2}$$

As noted in the problem statement, the term $-e^{-\infty(s+1)}[(s+1)\infty+1]$ goes to zero for $\mathrm{Re}(s) > -1$. Also noting that $e^0 = 1$, the above simplifies to:

$$\boxed{H(s) = \frac{1}{(s+1)^2}, \quad \mathrm{Re}(s) > -1}$$



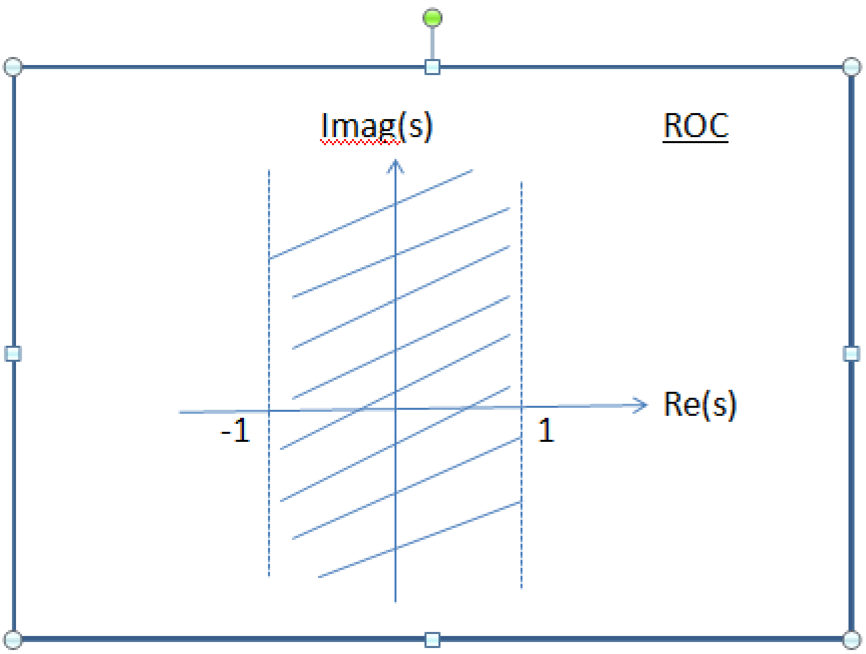
c) Use direct integration to <u>find</u> the Bilateral Laplace Transform for: $x(t) = e^{-|t|}$. <u>Draw</u> the region of convergence on the s-plane. *Hint: Re-write* $e^{-|t|}$ *without the absolute value operator over the range from* $-\infty<t<0$*, and separately over the range from* $0\le t<\infty$*. You can then break the Laplace integral into two ranges, summing the result.*

$$X(s)=\int_{t=-\infty}^{\infty} x(t)e^{-st}\,dt=\int_{t=-\infty}^{0} e^{t}\,e^{-st}dt+\int_{t=0}^{\infty} e^{-t}\,e^{-st}dt=\int_{t=-\infty}^{0} e^{-t(s-1)}\,dt+\int_{t=0}^{\infty} e^{-t(s+1)}dt$$

$$=\left.\frac{e^{-t(s-1)}}{-(s-1)}\right|_{t=-\infty}^{0}+\left.\frac{e^{-t(s+1)}}{-(s+1)}\right|_{t=0}^{\infty}=\frac{e^{-0(s-1)}-e^{--\infty(s-1)}}{-(s-1)}+\frac{e^{-\infty(s+1)}-e^{-0(s+1)}}{-(s+1)}$$

The term $e^{\infty(s-1)}$ goes to zero for $\mathrm{Re}(s)<1$. The term $e^{-\infty(s+1)}$ goes to zero for $\mathrm{Re}(s)>-1$. Both conditions are satisfied for $-1<\mathrm{Re}(s)<1$. Noting that $e^{0}=1$, the above becomes:

$$\boxed{X(s)=\frac{-1}{s-1}+\frac{1}{s+1},\quad -1<\mathrm{Re}(s)<1}.$$



d) Use the transform tables to find the Bilateral Laplace Transform for: $x(t)=7\cos(\pi t)u(t)$. You need only list the region of convergence.

From the transform table:

$$X(s)=\frac{7s}{s^{2}+\pi^{2}},\quad \mathrm{Re}(s)>0$$

e) Use the transform tables to find the Bilateral Laplace Transform for: $h(t) = t^3 u(t)$. You need only list the region of convergence.

From the transform table:

$$H(s) = \frac{6}{s^4}, \quad \text{Re}(s) > 0$$

2) Properties of the Bilateral Laplace Transform:

a) What do you know about the ROC of *X(s)* if *x(t)* is:

i) … of finite duration and absolutely integrable?

The ROC is the entire s-plane.

ii) … right-sided, with a non-null ROC?

ROC of the form: Re(s) > s_o

iii) … left-sided, with a non-null ROC?

ROC of the form: Re(s) < s_o

iv) … two-sided, with a non-null ROC?

ROC of the form: s_a <Re(s) < s_b

b) Use the transform tables and properties to find the Bilateral Laplace Transform (always include the ROC!) of: $h(t) = e^{-2t} u(t) - 5u(-t)$.

$$e^{-2t}\mu(t) \Leftrightarrow \frac{1}{s+2}, \quad \text{Re}(s) > -2$$

$$-\mu(-t) \Leftrightarrow \frac{1}{s}, \quad \text{Re}(s) < 0$$

From linearity: $\boxed{H(s) = \frac{1}{s+2} + \frac{5}{s}, \quad -2 < \text{Re}(s) < 0}$

c) Use the transform tables and properties to find the Bilateral Laplace Transform (always include the ROC!) of: $x(t) = e^{-(t-4)} u(t-4)$.

$e^{-t}\mu(t) \Leftrightarrow \frac{1}{s+1}, \quad \text{Re}(s) > -1$ and $x(t-\tau) \Leftrightarrow e^{-s\tau} X(s) \quad \text{(ROC unchanged)}$.

Here, $\tau = 4$. Applying this transform and property gives:

$$\boxed{X(s) = \frac{e^{-4s}}{s+1}, \quad \text{Re}(s) > -1}.$$

d) Use the transform tables and properties to find the Bilateral Laplace Transform (always include the ROC!) of: $x(t) = e^{-(t-5)} u(t)$.

$x(t) = e^{-(t-5)}\,\mu(t) = e^{5}\,e^{-t}\,\mu(t)$. Here, e^{5} serves as a scaling factor.
Thus,

$$\boxed{X(s) = \frac{e^{5}}{s+1}, \quad \mathrm{Re}(s) > -1}$$

e) Use the transform tables and properties to find the Bilateral Laplace Transform (always include the ROC!) of: $y(t) = e^{-6t}\,u(t) * t\,u(t)$.

$$y(t) = e^{-6t}\,\mu(t) * t\,\mu(t)$$

$$\Downarrow$$

$$Y(s) = L\{e^{-6t}\,\mu(t)\} \cdot L\{t\,\mu(t)\}$$

Since $L\{e^{-6t}\,\mu(t)\} = \dfrac{1}{s+6}, \quad \mathrm{Re}(s) > -6$ and $L\{t\,\mu(t)\} = \dfrac{1}{s^{2}}, \quad \mathrm{Re}(s) > 0$, the overall result is:

$$\boxed{Y(s) = \frac{1}{s+6} \cdot \frac{1}{s^{2}}, \quad \mathrm{Re}(s) > 0}$$

f) Use the transform tables and properties to find the Bilateral Laplace Transform (always include the ROC!) of: $h(t) = -u(-t) * u(t)$.

$$h(t) = -\mu(-t) * \mu(t)$$

$$\Downarrow$$

$$H(s) = L\{-\mu(-t)\} \cdot L\{\mu(t)\}$$

Since $L\{-\mu(-t)\} = \dfrac{1}{s}, \quad \mathrm{Re}(s) < 0$ and $L\{\mu(t)\} = \dfrac{1}{s}, \quad \mathrm{Re}(s) > 0$, the overall result is:

No Combined ROC. Thus, Bilateral Laplace Transform NOT EXIST.

g) Consider a LTI system with impulse response: $h(t) = e^{-7t}\,u(t)$. Let this system be in the zero-state condition. If the ***output*** to this system is: $y(t) = \cos(3\pi t)u(t)$, what is the Laplace transform [$X(s)$] of the system ***input***? (Always include the ROC.)

Since $\begin{array}{c} y(t) = x(t) * h(t) \\ \Updownarrow \\ Y(s) = X(s) \cdot H(s) \end{array}$, it follows that $X(s) = \dfrac{Y(s)}{H(s)}$.

Here $Y(s) = \dfrac{s}{s^{2} + 9\pi^{2}}, \quad \mathrm{Re}(s) > 0; \qquad H(s) = \dfrac{1}{s+7}, \quad \mathrm{Re}(s) > -7$.

Thus, $\boxed{X(s) = \dfrac{s(s+7)}{s^{2} + 9\pi^{2}}, \quad \mathrm{Re}(s) > 0}$