Rational Transforms and Linear Differential Equations

$$a_R \frac{d^R y(t)}{dt^R} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_P \frac{d^P x(t)}{dt^P} + \cdots + b_0 x(t)$$

- Take $\mathcal{L}\{\cdot\}$ — Recall: $\frac{d f(t)}{dt} \longleftrightarrow s \cdot F(s)$
  - Assume null initial conditions

$$a_R s^R Y(s) + \cdots + a_1 s Y(s) + a_0 Y(s) = b_P s^P X(s) + \cdots + b_0 X(s)$$

$$\boxed{Y(s)\left[a_R s^R + \cdots + a_1 s + a_0\right] = X(s)\left[b_P s^P + \cdots + b_1 s + b_0\right]}$$

1) Recall: $y(t) \longleftrightarrow Y(s)$
   a) Transform from time to frequency
   b) Solve for $Y(s)$ using (complex) algebra, rather than calculus
   c) Convert result back to time domain

   $\boxed{\text{Analogous to phasors}}$

2) Recall: $h(t) \longleftrightarrow H(s) = \frac{Y(s)}{X(s)}$

   - Can find system function in $\mathcal{L}$-domain, convert to time to determine $h(t)$

   - $H(s)$ has numerator, denominator polynomials
     $\Longrightarrow$ Rational Laplace Transform

Rational Laplace Transform

Inverse $\mathcal{L}$-Transform

- Formally:

$$X(s) = \int_{t=-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

and

$$x(t) = \frac{1}{2\pi j} \int_{s=\sigma-j\infty}^{\sigma+j\infty} X(s)\, e^{st}\, ds$$

- Contour integral
- Never evaluate directly

- Instead, write $X(s)$ as elementary functions with known $x_i(t) \longleftrightarrow X_i(s)$ transforms

  - Invert by inspection:

    - E.g. If $X(s) = \frac{1}{s}$, $Re\{s\} > 0$

$$\Downarrow$$

$$x(t) = u(t)$$

- For rational transforms $\rightarrow$ Partial fraction expansion

Rational Laplace Transform

Inverse $\mathcal{L}$-Tranform Example I (1)

- **Find** $\mathcal{L}^{-1}\{\ \}$ of $X(s) = \dfrac{7s+18}{(s+2)(s+3)}$, $Re(s) > -2$

- **Sol'n**

  Note: $Re(s) > \sigma_0 \rightarrow$ causal function

  - Partial fraction expansion: Guess:

$$\frac{7s+18}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

  - Multiply by $(s+2)(s+3)$:

$$7s+18 = A(s+3) + B(s+2)$$

1) At $s = -3$

$$7(-3)+18 = \cancel{A(-3+3)}^{\,0} + B(-3+2)$$

$$\Rightarrow B = \frac{-3}{-1} = 3$$

2) At $s = -2$

$$7(-2)+18 = A(-2+3) + \cancel{B(-2+2)}^{\,0}$$

$$\Rightarrow A = \frac{4}{1} = 4$$

(continued)

Inverse $\mathcal{L}$-Transform Example I (2)

Thus,

$$\frac{7s+18}{(s+2)(s+3)} = \frac{4}{s+2} + \frac{3}{s+3} = X(s)$$

For ROC of $Re(s) > \sigma_0$:

$$\frac{4}{s+2} = 4 \cdot \frac{1}{s+2} \longleftrightarrow 4 \cdot e^{-2t} u(t)$$

$$\frac{3}{s+3} = 3 \cdot \frac{1}{s+3} \longleftrightarrow 3 e^{-3t} u(t)$$

Thus,

$$\boxed{x(t) = \left[4 \cdot e^{-2t} + 3e^{-3t}\right] u(t)}$$

Rational Laplace Transform

Partial Fraction Expansion with MATLAB (1)

- Manual expansion tedious, prone to error
- Use MATLAB "residue()"

- Eg.: $\dfrac{7s+18}{(s+2)(s+3)}$, $Re(s) > -2$

1) Write numerator, denominator polynomials:

$$\frac{7s+18}{(s+2)(s+3)} = \frac{7s+18}{s^2+5s+6} \Rightarrow \begin{matrix} b=[7\ \ 18] \\ a=[1\ \ 5\ \ 6] \end{matrix}$$

2) Use residue():

```
>> [r, p, k] = residue([7 18], [1 5 6])
r =
    3.0000
    4.0000
p = -3.0000
    -2.0000
k = [ ]
```

In general, can be real or complex-valued

3) For non-repeated roots, assemble as

$$\frac{b(s)}{a(s)} = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \dots + \frac{r_N}{s-p_N} + k(s)$$

(continued)

Partial Fraction Expansion with MATLAB (2)

3 Continued)

Thus,

$$\frac{7s+18}{(s+2)(s+3)} = \frac{3}{(s+3)} + \frac{4}{(s+2)}$$

For causal sequences

$$\Downarrow$$

$$x(t) = \left[3e^{-3t} + 4e^{-2t}\right]u(t)$$

Rational Laplace Transform

Partial Fraction Expansion - Repeated Roots

$$\frac{b(s)}{(s+a)^m} = \frac{c_1}{s+a} + \frac{c_2}{(s+a)^2} + \cdots + \frac{c_m}{(s+a)^m}$$

Example: $\frac{5s+2}{(s+3)\ (s+3)}$, $Re(s) > -3$ $= X(s)$

```
>> [r, p, k] = residue([5 2], [1 6 9])
r =  5
   -13
p = -3
    -3
k = [ ]
```

So, $X(s) = \frac{5s+2}{(s+3)^2} = \frac{5}{s+3} + \frac{-13}{(s+3)^2}$

For causal sequence:

$$\boxed{x(t) = \left[5e^{-3t} - 13te^{-3t}\right] u(t)}$$

Rational Laplace Transform

Improper Fractions

- If numerator degree $\geq$ denominator degree $\Rightarrow$ improper
  - So, divide-out "direct terms"

E.g: $X(s) = \dfrac{(s-2)(s+3)}{(s+1)(s+2)}$, $Re(s) > -1$

So,

```
>> [r, p, k] = residue([1 1 -6], [1 3 2])
r = 4
   -6

p = -2
    -1

k = 1
```

$$\text{So, } X(s) = \frac{4}{s+2} + \frac{-6}{s+1} + 1$$

$$\Downarrow$$

$$\boxed{x(t) = \left[4e^{-2t} - 6e^{-t}\right]u(t) + \delta(t)}$$

Rational Laplace Transform

Inverse Transform Tips

- Laplace functions must match table **exactly**

  Write $\frac{10}{s}$ as $\underbrace{10}_{\text{Linearity}} \cdot \underbrace{\frac{1}{s}}_{\text{table}}$

- Region of convergence **needed** to relate to time function

- In real life, use automated tools (MATLAB) for partial fraction expansions.

- If time function is real-value, write it without "j" → E.g., complex conjugate roots

Rational Laplace Transform

Example: One-Sided Exponentials

Note: We solved this problem previously using time domain convolution.

- **For** zero-state LTI system with: $h(t) = e^{-2t} u(t)$

- **Find** $y(t)$ if $x(t) = e^{-t} u(t)$

$$h(t) = e^{-2t} u(t) \longleftrightarrow H(s) = \frac{1}{s+2}, \quad Re(s) > -2$$

$$x(t) = e^{-t} u(t) \longleftrightarrow X(s) = \frac{1}{s+1}, \quad Re(s) > -1$$

So,

$$Y(s) = H(s)\, X(s) = \frac{1}{s+2} \cdot \frac{1}{s+1}, \quad Re(s) > -1$$

$$Y(s) = \frac{1}{(s+2)(s+1)} = \frac{-1}{s+2} + \frac{1}{s+1}$$

$$\Updownarrow$$

$$\boxed{y(t) = \left[-e^{-2t} + e^{-t}\right] u(t)}$$

Another Example

Note: We solved this problem previously using time domain convolution.

- **For** zero-state LTI system with: $h(t) = u(t)$
- **Find** $y(t)$ if $x(t) = e^{-3t} u(t)$

Sol'n

$$h(t) = u(t) \longleftrightarrow H(s) = \frac{1}{s}, \quad Re(s) > 0$$

$$x(t) = e^{-3t} u(t) \longleftrightarrow X(s) = \frac{1}{s+3}, \quad Re(s) > -3$$

So,

$$Y(s) = H(s) \cdot X(s) = \frac{1}{s(s+3)}, \quad Re(s) > 0$$

$$Y(s) = \frac{1}{s(s+3)} = \frac{1/3}{s} + \frac{-1/3}{s+3}, \quad Re(s) > 0$$

$$\Updownarrow$$

$$\boxed{y(t) = \left(\frac{1 - e^{-3t}}{3}\right) u(t)}$$

Rational Laplace Transforms

Poles and Zeros

- Know:

$$Y(s)\left[a_R s^R + \cdots + a_1 s + a_0\right] = X(s)\left[b_P s^P + \cdots + b_1 s + b_0\right]$$

or

$$\frac{Y(s)}{X(s)} = H(s) = \frac{b_P s^P + \cdots + b_1 s + b_0}{a_R s^R + \cdots + a_1 s + a_0}$$

- Factor numerator, denominator polynomials

$$H(s) = \frac{b_P}{a_R} \cdot \frac{(s-z_1)(s-z_2)\cdots(s-z_P)}{(s-p_1)(s-p_2)\cdots(s-p_R)}$$

$$H(s) = G \frac{\prod_{m=1}^{P}(s-z_m)}{\prod_{m=1}^{R}(s-p_m)}$$

G: Gain

z_m: Locations where $H(s=z_m)=0$ "zeros"

p_m: Locations where $H(s=p_m)=\infty$ "poles"

Rational Laplace Transform

ROC for Rational Transforms

- ROC cannot contain a pole
- If $x(t)$ right-sided, ROC is region of s-plane to right of right-most pole
- If $x(t)$ left-sided, ROC is region of s-plane to left of left-most pole

Stability

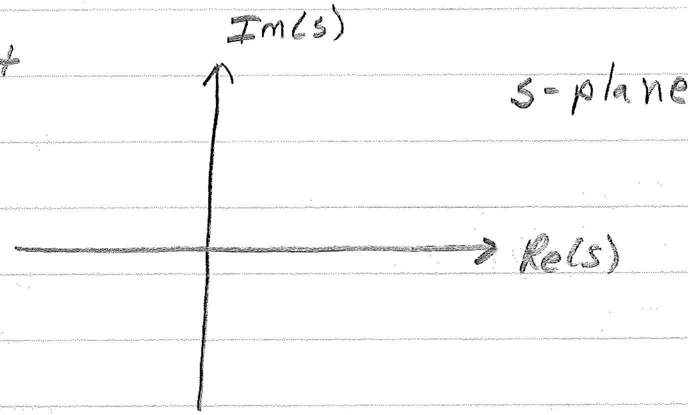
- Rational LTI system stable iff ROC of $H(s)$ includes entire $j\omega$ axis

  - For causal system $\Rightarrow$ all poles in Left Half Plane

Pole-zero Plots

- If a_i, b_i real $\Rightarrow$ real roots or complex conjugate pairs

- Pole-zero plot



- "X" identifies poles, "o" identifies zeros

- ROC can <u>not</u> include pole [since $X(s=p_n)=\infty$]

- Pole-zero plot reveals $X(s)$, absent gain factor

Pole-Zero Plot Example 1

- Draw pole-zero plot of system characterized by impulse response:

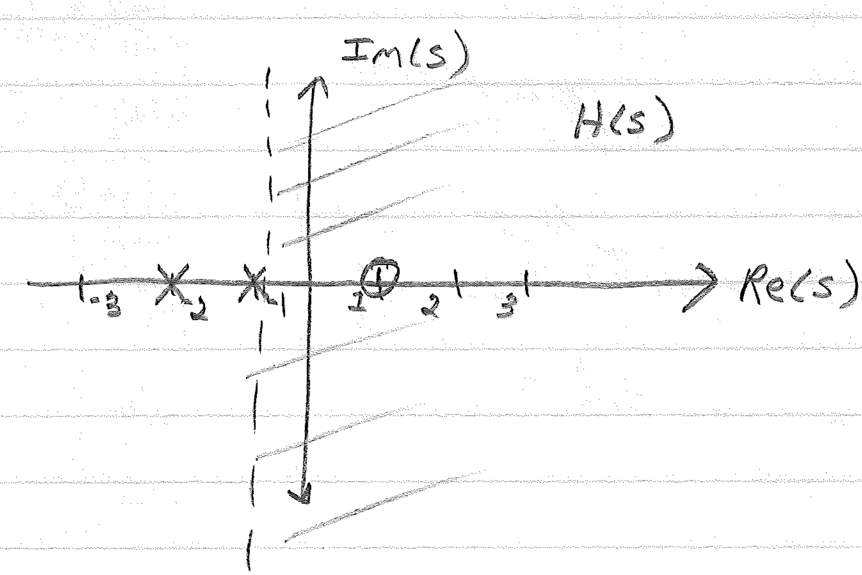
$$h(t) = \left[3e^{-2t} - 2e^{-t}\right] u(t)$$

Sol'n

$$H(s) = \frac{3}{s+2} - \frac{2}{s+1}, \quad Re(s) > -1$$

$$H(s) = \frac{(s-1)}{(s+1)(s+2)}, \quad Re(s) > -1$$

So,



This plot also shows ROC.

Rational Laplace Transform

Pole-Zero Plot Example 2

- Draw pole-zero plot of system characterized by impulse response:

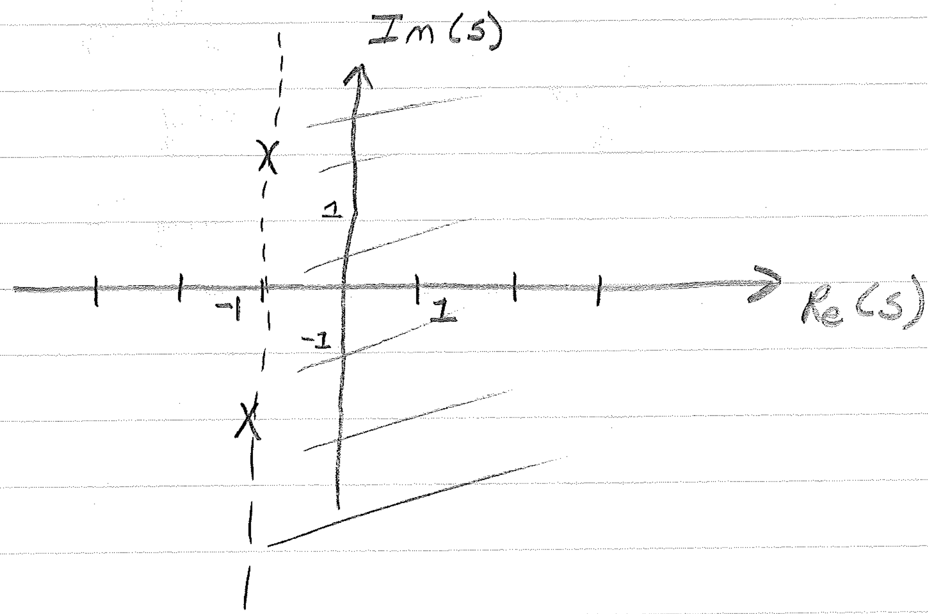
$$h(t) = e^{-t} \sin(2t)\, u(t)$$

Sol'n

$$H(s) = \frac{2}{(s+1)^2+4}, \quad Re(s) > -1$$

$$H(s) = \frac{2}{s^2+2s+5} = \frac{2}{(s+1+j2)(s+1-j2)}, \quad Re(s) > -1$$

So,



This plot also shows ROC.

Rational Laplace Transform

Pole-Zero Plot Example 3

- Draw pole-zero plot of system characterized by impulse response:

$$h(t) = 4\cdot\cos(\pi\cdot t)\,u(t)$$

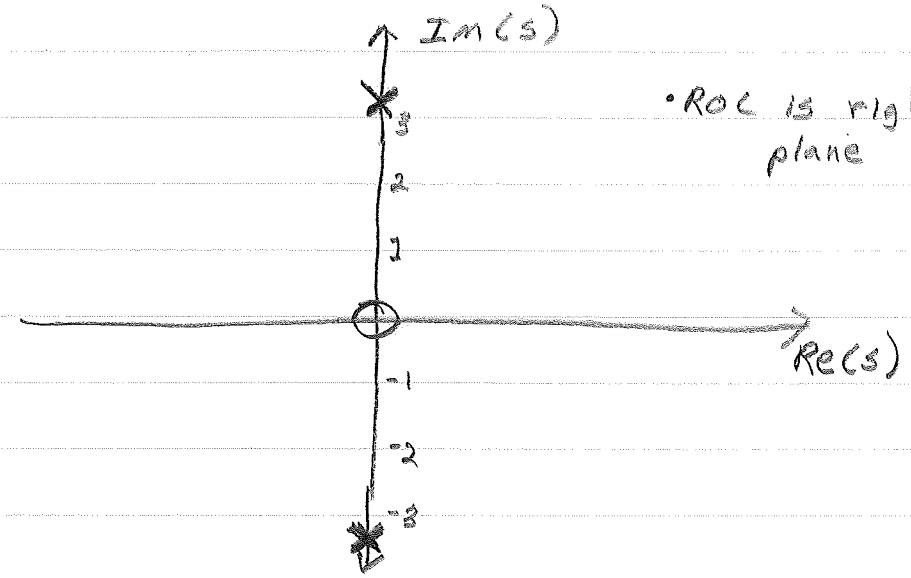
Sol'n

$$H(s) = \frac{4s}{s^2+\pi^2}, \quad Re(s)>0$$

or

$$H(s) = \frac{4s}{(s-j\pi)(s+j\pi)}, \quad Re(s)>0$$

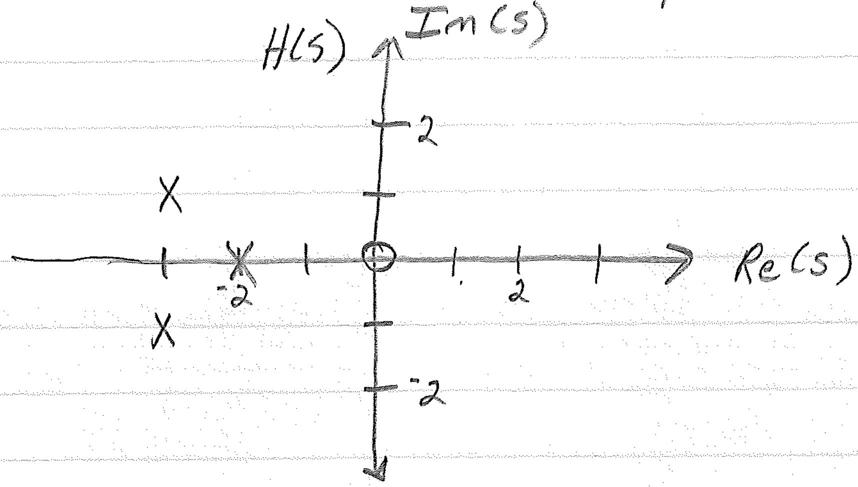
So,



- ROC is right-side plane

Determining H(s) from Pole-Zero Plot

- Find H(s) for a causal sequence if:



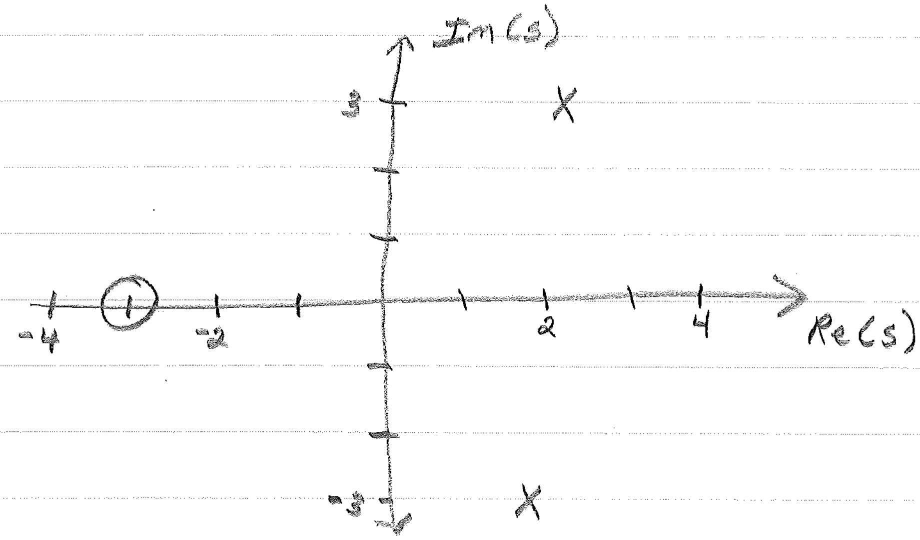
$$H(s) = \frac{G\,(s-0)}{(s+3-j)(s+3+j)(s+2)} = \boxed{G \cdot \frac{s}{(s+3-j)(s+3+j)(s+2)} \quad Re(s) > -2}$$

- Could determine $h(t) = \mathcal{L}^{-1}\{H(s)\}$

Also, need ROC!!

Example: Determine H(s) from Pole-Zero Plot

• Find $H(s)$ for an anti-causal sequence if:



Sol'n

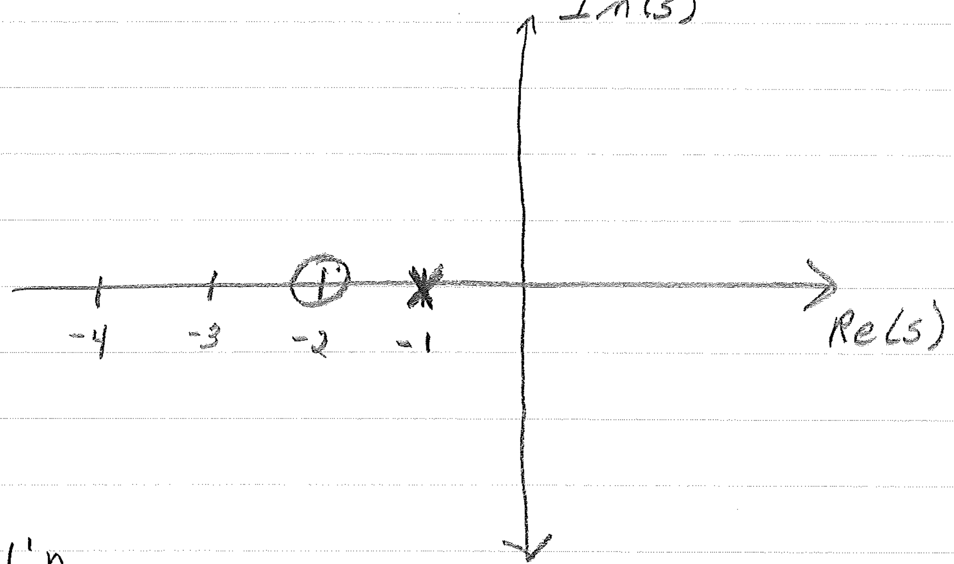
$$H(s) = \frac{G\ (s+3)}{(s-2+j3)(s-2-j3)}, \quad Re(s) < 2$$

Note ROC of anti-causal sequence

$$H(s) = G \cdot \frac{s+3}{s^2-4s+13}, \quad Re(s) < 2$$

Example: Find Impulse Response from Pole-Zero Plot

- Find impulse response of causal sequence if:



Sol'n

$$H(s) = G \cdot \frac{s+2}{s+1}, \quad Re(s) > -1$$

$$\begin{array}{r} 1 \;\; R1 \\ s+1 \overline{) \; s+2} \\ -(s+1) \\ \hline 1 \end{array}$$

$$H(s) = G + \frac{G}{s+1}, \quad Re(s) > -1$$

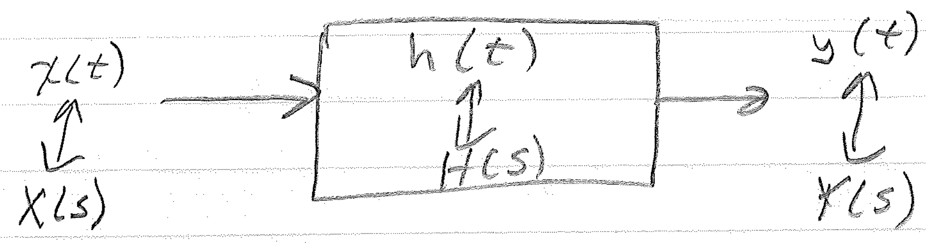
$$\Downarrow$$

$$\boxed{h(t) = G\delta(t) + Ge^{-t}u(t)}$$

Rational Laplace Transform

Bilateral $\mathcal{L}$-Transform Applications

- Bilateral $\mathcal{L}$-Transform characterizes <u>zero-state</u> response



- If LTI system, all initial conditions $= 0$:

$$y(t) = x(t) * h(t)$$

$$\updownarrow$$

$$Y(s) = X(s) \cdot H(s)$$

Rational Laplace Transform

Example - System Function

zero-state

- **Find** impulse response of causal, LTI, system satisfying:

$$\frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6\,y(t) = x(t)$$

Sol'n

1) Transform to Laplace domain:

$$s^2 Y(s) + 5s\,Y(s) + 6\,Y(s) = X(s)$$

2) Find system function:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+5s+6} = \frac{1}{(s+2)(s+3)}, \; Re(s) > -2$$

3) Transform back to time domain:

By partial fraction expansion: $H(s) = \dfrac{1}{s+2} + \dfrac{-1}{s+3}, \; Re(s) > -2$

Hence:

$$h(t) = \left[e^{-2t} - e^{-3t}\right]u(t)$$

Example - Step Response

zero-state

- Find step response of causal, LTI, system satisfy ing:

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t)$$

Sol'n

1) Transform to Laplace domain:

a) $x(t) = \mu(t) \longleftrightarrow X(s) = \frac{1}{s}$

b) $s^2Y(s) + 5sY(s) + 6Y(s) = X(s)$

2) Solve: $Y(s) = \underbrace{\frac{1}{s}}_{X(s)} \cdot \underbrace{\frac{1}{s^2+5s+6}}_{H(s)} = \frac{1}{s(s+2)(s+3)}, \; Re(s) > 0$

3) Transform back to time domain:

By partial fraction expansion: $Y(s) = \frac{1/6}{s} + \frac{-1/2}{s+2} + \frac{1/3}{s+3}, \; Re(s) > 0$

Hence:

$$y(t) = \left[\frac{1}{6} - \frac{e^{-2t}}{2} + \frac{e^{-3t}}{3}\right]\mu(t)$$

Example: General Response

- For an LTI system, input $5u(t)$ produces output $(10 - 10e^{-t})\,u(t)$

- Q: What output is produced if the input is $e^{-2t}u(t)$?

Sol'n

$$x_1(t) = 5u(t) \longleftrightarrow X_1(s) = \frac{5}{s}, \quad Re(s) > 0$$

$$y_1(t) = (10 - 10e^{-t})u(t) \Leftrightarrow Y_1(s) = \frac{10}{s} - \frac{10}{s+1}, \quad Re(s) > 0$$

So,

$$H(s) = \frac{Y_1(s)}{X_1(s)} = \left(\frac{10}{s} - \frac{10}{s+1}\right)\cdot\frac{s}{5} = 2 - \frac{2s}{s+1}, \quad Re(s) > -1$$

Then:

$$x_2(t) = e^{-2t}u(t) \longleftrightarrow X_2(s) = \frac{1}{s+2}, \quad Re(s) > -2$$

So,

$$Y_2(s) = H(s)\cdot X_2(s) = \left(2 - \frac{2s}{s+1}\right)\cdot\frac{1}{s+2} = \frac{2}{s+2} - \frac{2s}{(s+2)(s+1)}$$

$$Y_2(s) = \underbrace{\frac{2}{s+2} + \frac{-4}{s+2}} + \frac{2}{s+1}$$

$$\Downarrow$$

$$\boxed{y_2(t) = \left(2e^{-t} - 2e^{-2t}\right)u(t)}$$

Rational Laplace Transforms

$\mathcal{L}$-Transform and Impedance

R:

$$v_R(t) = i_R(t) \cdot R$$

$$\Big\updownarrow \mathcal{L}$$

$$V_R(s) = I_R(s) \cdot R \quad \text{or} \quad \frac{V_R(s)}{I_R(s)} = R$$

L:

$$v_L(t) = L \frac{d i_L(t)}{dt}$$

$$\Big\updownarrow \mathcal{L}$$

$$V_L(s) = L \cdot s I_L(s) \quad \text{or} \quad \frac{V_L(s)}{I_L(s)} = s \cdot L$$

C:

$$i_C(t) = C \frac{d v_C(t)}{dt}$$

unit $\equiv \Omega$

$$\Big\updownarrow \mathcal{L}$$

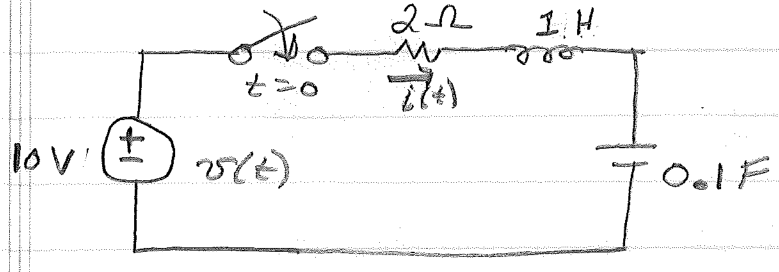
$$I_C(s) = C \cdot s V_C(s) \quad \text{or} \quad \frac{V_C(s)}{I_C(s)} = \frac{1}{sC}$$

$$\boxed{\text{Standard Impedances - BILATERAL } \mathcal{L}}$$

Rational Laplace Transforms

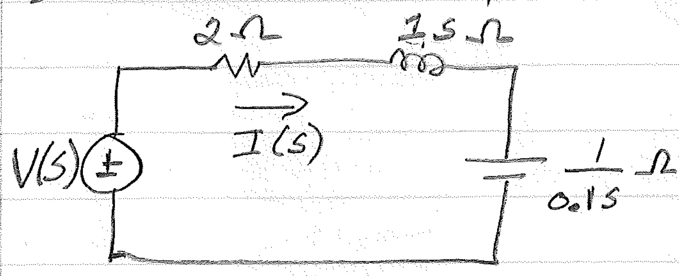
Example Electrical Circuit Zero-State Response

- Find $i(t)$ for $t>0$ in: [Note: Unrealistic component values]



Sol'n Problem asks for zero-state response

1) Transform to Laplace domain, $t>0$



Since $v(t)=10\mu(t)$

$$\Updownarrow$$

$$V(s)=\frac{10}{s}$$

2) Solve: By KVL: $V(s)=I(s)\left[2+s+\frac{10}{s}\right]$

$$\text{or } I(s)=\frac{10}{s}\left[\frac{1}{2+s+\frac{10}{s}}\right]=\frac{10}{s^2+2s+10}$$

3) Transform back to time domain (assume right-sided):

$$I(s)=\frac{10}{(s+1)^2+9}$$

$$\boxed{i(t)=\frac{10}{3}e^{-t}\sin(3t)\mu(t)\ A}$$

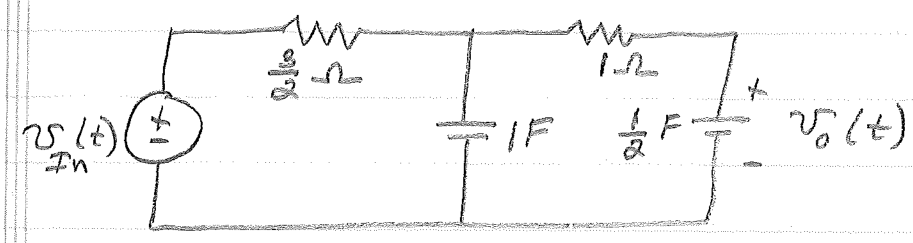
Unit

Example 2 (1)

- **Find** $v_o(t)$ if $v_{In}(t) = 8\cos(4t)\mu(t)$ V

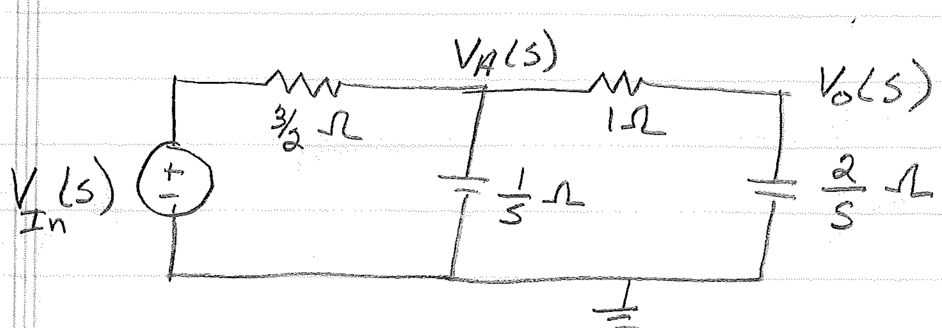
$\Downarrow$

$v_o(0) = 0V$



- **Sol'n**: Problem asks for zero-state response

1) Transform to Laplace domain, $t > 0$



2) Solve: Will use nodal analysis

a) $$\frac{V_A(s) - V_{In}(s)}{3/2} + \frac{V_A(s) - 0}{1/s} + \frac{V_A(s) - V_o(s)}{1} = 0$$

b) $$\frac{V_o(s) - V_A(s)}{1} + \frac{V_o(s) - 0}{2/s} = 0$$

also:

c) $$V_{In}(s) = \frac{8s}{s^2 + 16}$$

(continued)

Example 2 (2)

2 Continued) Solving for $V_o(s)$

$$V_o(s) = V_{In}(s)\left[\frac{4}{3s^2+11s+4}\right] = \frac{32s}{(s^2+16)(3s^2+11s+4)}$$

3) Transform back to time domain (assume right-sided)

MATLAB

$$V_o(s) = \frac{0.46}{s+3.26} - \frac{0.09}{s+0.41} + \underbrace{\frac{-0.18-j0.18}{s-j4} + \frac{-0.18+j0.18}{s+j4}}_{\text{These are the } (s^2+16) \text{ terms, Easiest to re-combine.}}$$

$$V_o(s) = \frac{0.46}{s+3.26} - \frac{0.09}{s+0.41} + \underbrace{\frac{-0.36s+1.45}{s^2+16}}_{2 \text{ terms}}$$

$$\boxed{v_o(t) = \underbrace{\left[0.46e^{-3.26t} - 0.09e^{-0.41t}\right]u(t)}_{\text{Transient response}} \quad \underbrace{\left[-0.36\cos(4t) + 0.36\sin(4t)\right]u(t)}_{\text{Steady-state response}}}$$