Definition of Bilateral Laplace Transform

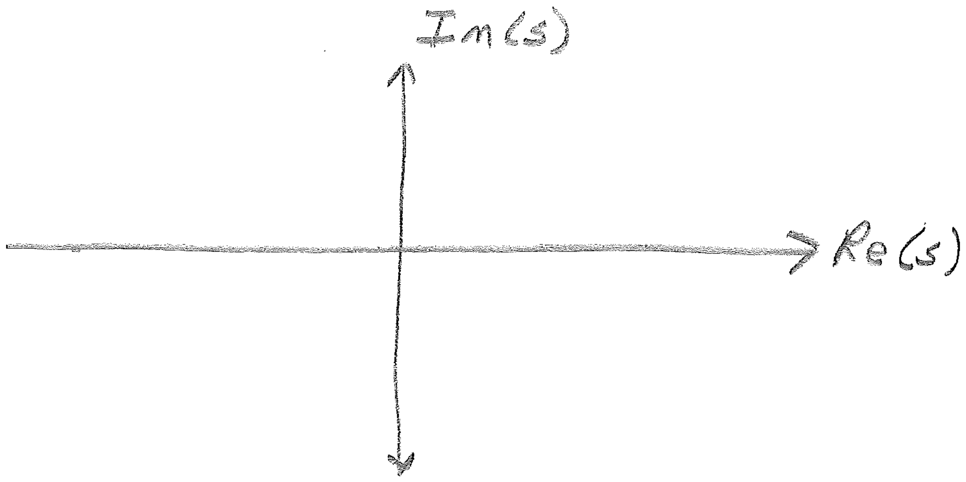
$$X(s) \equiv \int_{t=-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

- Notation: $X(s) = \mathcal{L}\{x(t)\}$

$$\underset{\text{Time domain}}{x(t)} \longleftrightarrow \underset{\text{Frequency domain}}{X(s)}$$

- $X(s)$ is an integral (sum) $\Rightarrow$ Must specify Region of Convergence (ROC)
  - Values of $s=\sigma+jw$ s.t. $X(s) < \infty$

- Since s is complex ($s=\sigma+jw$), show ROC in s-plane



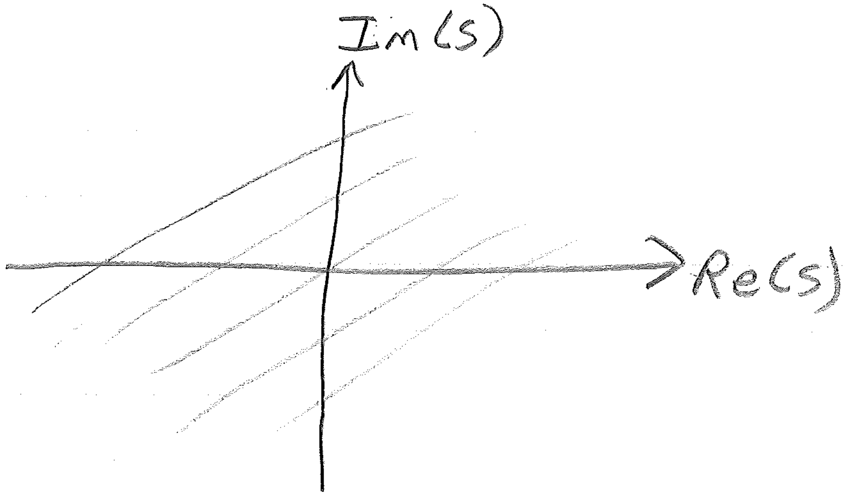
$\mathcal{L}$-Transform of $\delta(t)$

$$\mathcal{L}\{\delta(t)\} \equiv \int_{t=-\infty}^{\infty} \delta(t)\, e^{-st}\, dt$$

$$= \underbrace{\int_{-\infty}^{0^-} \delta(t) e^{-st} dt}_{\phi} + \underbrace{\int_{0^-}^{0^+} \delta(t) e^{-st} dt}_{\equiv 1} + \underbrace{\int_{0^+}^{\infty} \delta(t) e^{-st} dt}_{\phi}$$

So, $\boxed{\mathcal{L}\{\delta(t)\} = 1, \ \forall s}$

- ROC



$\mathcal{L}$-Transform of Step

$$\mathcal{L}\{u(t)\} = \int_{t=-\infty}^{\infty} u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-st}\, dt = \frac{-e^{-st}}{s}\Bigg|_{t=0}^{\infty} = \frac{-e^{-s\cdot\infty} - -e^{-s\cdot 0}}{s}$$

$$= \frac{-e^{-(\sigma+j\omega)\cdot\infty} + 1}{s} = \frac{-e^{-\sigma\cdot\infty}\, e^{-j\omega\cdot\infty} + 1}{s}$$

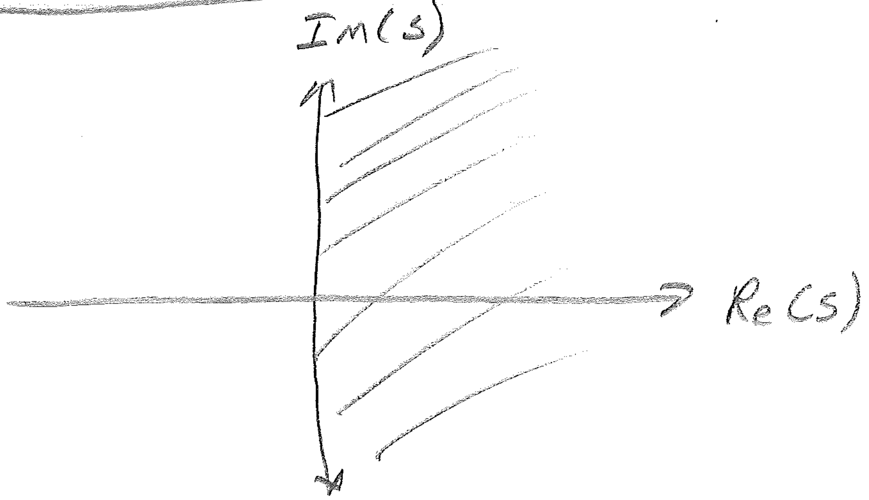
Notes: 1) $e^{-j\omega\infty} = j\sin(-\omega\infty) \rightarrow |e^{-j\omega\infty}| \leq 1$

2) $\underline{\underline{e^{-\sigma\infty} \rightarrow 0 \text{ for } \sigma > 0}}$; $e^{-\sigma\cdot\infty} \rightarrow \infty$ for $\sigma < 0$

So,

$$\boxed{\mathcal{L}\{u(t)\} = \frac{1}{s}, \quad Re(s) > 0}$$

∘ ROC



$\mathcal{L}$-Transform of Real Exponential

$$\mathcal{L}\{e^{-at}u(t)\} = \int_{t=-\infty}^{\infty} e^{-at}u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+s)t}\, dt = \frac{-e^{-(a+s)t}}{a+s}\Bigg|_{t=0}^{\infty} = \frac{-e^{-(a+s)\infty} - -e^{-(a+s)0}}{a+s}$$

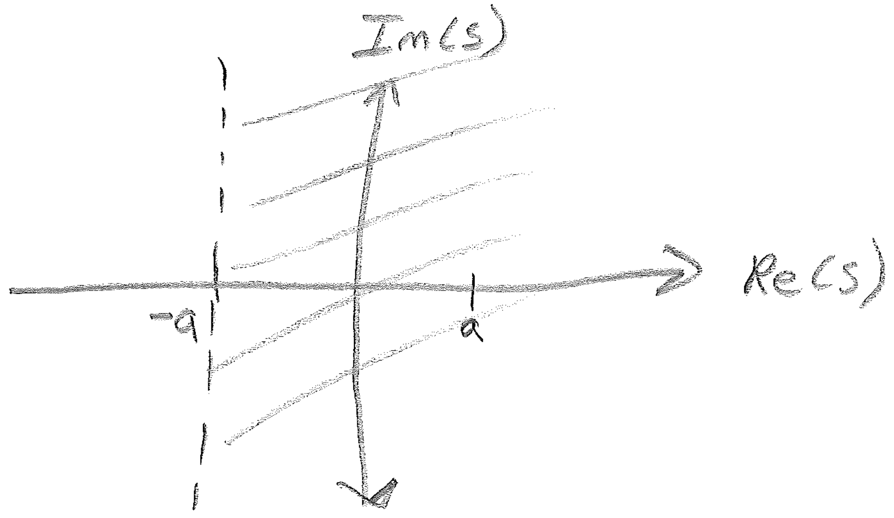
$$= \frac{-e^{-(a+\sigma)\infty}\, e^{-j\omega\infty} + 1}{a+s}$$

Notes: 1) $e^{-j\omega\infty} = j\sin(-\omega\infty) \rightarrow |e^{-j\omega\infty}| \leq 1$

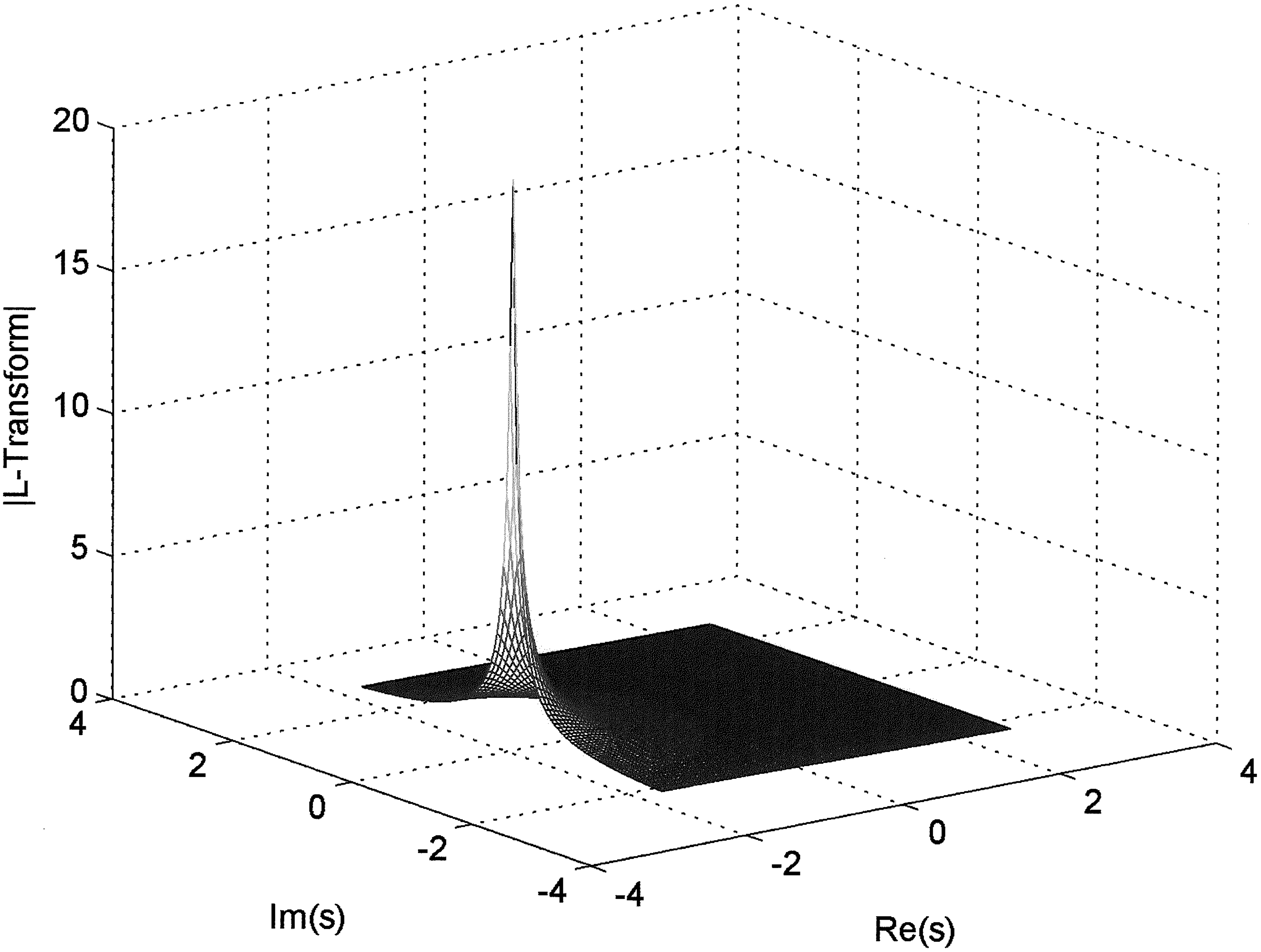
2) $e^{-(a+\sigma)\infty} \longrightarrow 0$ for $a+\sigma > 0$; $\rightarrow \infty$ for $a+\sigma < 0$

So, $\boxed{\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}, \quad Re(s) > -a}$

- ROC



x(t) = e^{-at} u(t) <==> L(s) = 1/(s+a), Re[s]>-a for a = 2



Laplace Transform

$\mathcal{L}$-Transform of Anti-Causal, Real Exponential

$$\mathcal{L}\{-e^{-at}u(-t)\} = \int_{t=-\infty}^{\infty} -e^{-at}u(-t)e^{-st}\,dt$$

$$= \int_{-\infty}^{0} -e^{-(a+s)t}\,dt = \frac{e^{-(a+s)t}}{a+s}\Bigg|_{t=-\infty}^{0} = \frac{e^{-(a+s)\cdot 0} - e^{-(a+s)(-\infty)}}{a+s}$$

$$= \frac{1 - e^{(a+\sigma)\infty}\, e^{j\omega\infty}}{a+s}$$

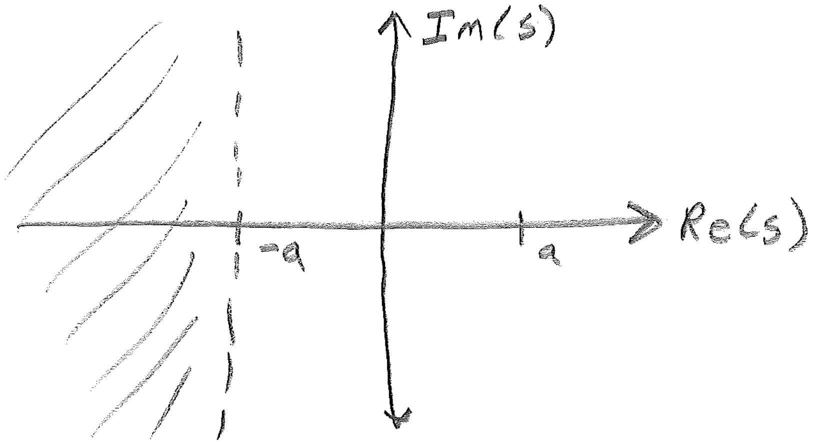
Notes: 1) $e^{j\omega\infty} = \cos(\omega\infty) + j\sin(\omega\infty) \rightarrow |e^{j\omega\infty}| \le 1$

2) $e^{(a+\sigma)\infty} \rightarrow 0$ for $a+\sigma < 0$; $\rightarrow \infty$ for $a+\sigma > 0$

So,

$$\boxed{\mathcal{L}\{-e^{-at}u(-t)\} = \frac{1}{s+a}, \quad Re(s) < -a}$$

◦ROC



$\mathcal{L}$-Transform of $\cos(w_0 t)u(t)$

$$\mathcal{L}\{\cos(w_0\cdot t)u(t)\} = \int_{t=-\infty}^{\infty} \cos(w_0\cdot t)u(t)e^{-st}\,dt$$

$$= \int_{t=0}^{\infty}\left[\frac{e^{jw_0t}+e^{-jw_0t}}{2}\right]e^{-st}\,dt = \int_{t=0}^{\infty}\frac{e^{t(jw_0-s)}}{2}\,dt + \int_{t=0}^{\infty}\frac{e^{-t(jw_0+s)}}{2}\,dt$$

$$= \frac{e^{t(jw_0-s)}}{2(jw_0-s)}\Bigg|_{t=0}^{\infty} - \frac{e^{-t(jw_0+s)}}{2(jw_0+s)}\Bigg|_{t=0}^{\infty}$$

$$= \underbrace{\frac{e^{\infty[-\sigma+j(w_0-w)]}}{2(jw_0-s)}}_{\to 0 \text{ for } \sigma>0} - \frac{e^{0}}{2(jw_0-s)} \underbrace{-\frac{e^{-\infty[\sigma+j(w_0+w)]}}{2(jw_0+s)}}_{\to 0 \text{ for } \sigma>0} + \frac{e^{0}}{2(jw_0+s)}$$

$$= \frac{-1}{2(jw_0-s)}\cdot\frac{-jw_0-s}{-jw_0-s} + \frac{1}{2(jw_0+s)}\cdot\frac{-jw_0+s}{-jw_0+s}$$

$$= \frac{jw_0+s \quad -jw_0+s}{2(w_0^2+s^2)} = \frac{s}{w_0^2+s^2}$$

$$\Rightarrow \boxed{\mathcal{L}\{\cos(w_0\cdot t)u(t)\} = \frac{s}{w_0^2+s^2}, \quad Re(s)>0}$$

Common Laplace Transforms

(Bilateral)

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos \omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12	$[\sin \omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13	$[e^{-\alpha t}\cos \omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin \omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

[From Oppenheim, Willsky, Nawab, Signals & Systems, 2nd ed, 1996, Prentice-Hall]

Laplace Transform

Some ROC Properties

If $x(t)$ is...	...Then ROC is...
Finite-duration, absolutely integrable	Entire s-plane
Right-sided, non-null ROC	$Re\{s\} > \sigma_0$
Left-sided, non-null ROC	$Re\{s\} < \sigma_0$
Two-sided, non-null ROC	$\sigma_a < Re\{s\} < \sigma_b$

Always:

ROC consists of bands parallel to $j\omega$ axis.

Linearity

- Let $x(t) = 2x_1(t) + 5x_2(t)$

Find $X(s) = \mathcal{L}\{x(t)\}$ wrt $X_1(s)$, $X_2(s)$

Sol'n

$$X(s) = \int_{t=-\infty}^{\infty} x(t)e^{-st}\,dt = \int_{t=-\infty}^{\infty} [2x_1(t) + 5x_2(t)]e^{-st}\,dt$$

$$= \underbrace{2\int_{t=-\infty}^{\infty} x_1(t)e^{-st}\,dt}_{X_1(s)} + \underbrace{5\int_{t=-\infty}^{\infty} x_2(t)e^{-st}\,dt}_{X_2(s)}$$

$X(s) = 2 \cdot X_1(s) + 5 \cdot X_2(s)$

In general: $\boxed{a_1 x_1(t) + a_2 x_2(t) \Longleftrightarrow a_1 X_1(s) + a_2 X_2(s)}$

- ROC is <u>at least</u> intersection

of ROC_{X_1} and ROC_{X_2}

Linearity Example

- Let $x_1(t) = e^{-2t}\mu(t) \longleftrightarrow X_1(s) = \dfrac{1}{s+2}$, $Re(s) > -2$

  $x_2(t) = \mu(t) \longleftrightarrow X_2(s) = \dfrac{1}{s}$, $Re(s) > 0$

- Find $Y(s)$ if $y(t) = x_1(t) + x_2(t)$

Sol'n

By linearity: $Y(s) = \dfrac{1}{s+2} + \dfrac{1}{s}$

$$= \frac{s+s+2}{s(s+2)} = \boxed{\frac{2(s+1)}{s(s+2)}, \ Re(s) > 0}$$

so-called "rational" Laplace transform. We'll discuss ROC later.

Time shifting

- Let $y(t) = x(t-T)$, then

$$Y(s) = \int_{t=-\infty}^{\infty} x(t-T)\, e^{-st}\, dt$$

- ~~Let~~ $\tau = t - T$ or $t = \tau + T$, $dt = d\tau$

$$Y(s) = \int_{\tau+T=-\infty}^{\infty} x(\tau)\, e^{-s(\tau+T)}\, d\tau$$

$$= e^{-sT} \underbrace{\int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-s\tau}\, d\tau}_{X(s)}$$

So, $\boxed{Y(s) = e^{-sT} X(s) \quad \text{with ROC unchanged}}$

$|e^{-sT}| = 1$

$\Rightarrow$ Pure phase shift

Convolution in Time

- Let $y(t) = x_1(t) * x_2(t) = \int_{\tau=-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$

$$Y(s) = \int_{t=-\infty}^{\infty} \left[\int_{\tau=-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \right] e^{-st}\, dt$$

$$= \int_{\tau=-\infty}^{\infty} x_1(\tau)\, d\tau \int_{t=-\infty}^{\infty} x_2(t-\tau)\, e^{-st}\, dt$$

- Let $l = t - \tau$ or $\begin{cases} dl = dt \\ t = l + \tau \end{cases}$

τ fixed while evaluating right-most integral

$$Y(s) = \int_{\tau=-\infty}^{\infty} x_1(\tau)\, d\tau \int_{l+\tau=-\infty}^{\infty} x_2(l)\, e^{-s(l+\tau)}\, dl$$

$$= \left[\int_{\tau=-\infty}^{\infty} x_1(\tau)\, e^{-s\tau}\, d\tau \right] \cdot \left[\int_{l=-\infty}^{\infty} x_2(l)\, e^{-sl}\, dl \right]$$

$$\underline{\underline{Y(s) = X_1(s) \cdot X_2(s)}}$$

Convolution in Time ↔ Multiplication in Frequency

Laplace Transform

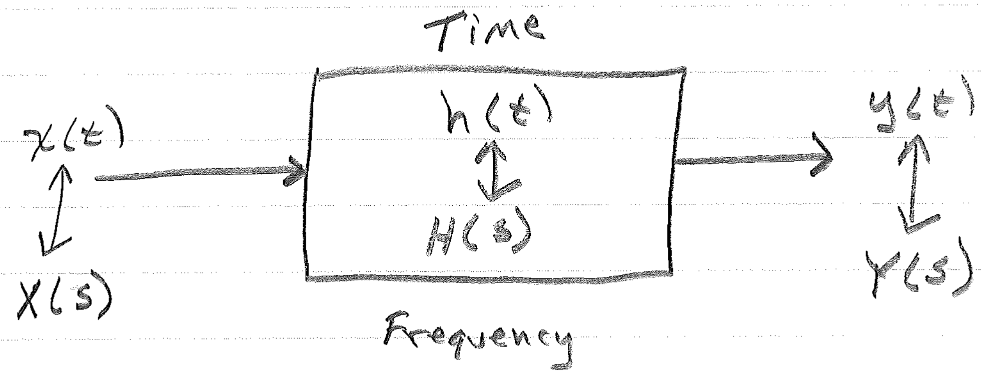
System Function

- For system with input $x(t) \longleftrightarrow X(s)$, output $y(t) \longleftrightarrow Y(s)$ and impulse response $h(t) \longleftrightarrow H(s)$:

$$y(t) = x(t) * h(t)$$

$$\Updownarrow$$

$$Y(s) = X(s) \cdot H(s)$$



$$\boxed{\begin{array}{c} H(s) \equiv \text{System Function} \\ \\ H(s) = \dfrac{Y(s)}{X(s)} \end{array}}$$

Showing Time Scaling Property (By Example)

- Let $y(t) = x(3t)$, then

$$Y(s) = \int_{t=-\infty}^{\infty} x(3t)\, e^{-st}\, dt$$

- Let $l = 3t \Rightarrow t = \frac{l}{3}$, $dl = 3\,dt$

$$Y(s) = \int_{\underset{\uparrow}{\frac{l}{3}}=-\infty}^{\infty} x(l)\, e^{-s\frac{l}{3}} \left(\frac{dl}{3}\right)$$

$\frac{1}{3}$ not change integral

$$Y(s) = \frac{1}{3} \underbrace{\int_{l=-\infty}^{\infty} x(l)\, e^{-\frac{s}{3}l}\, dl}_{X\left(\frac{s}{3}\right)}$$

So,

$$y(t) = x(3t)$$
$$\Downarrow$$
$$\boxed{Y(s) = \frac{X\left(\frac{s}{3}\right)}{3}, \quad s \text{ in } ROC_Y \text{ if } \frac{s}{3} \text{ in } ROC_X}$$

Time Scaling

- Can show:

$$x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

with new ROC = a · (old ROC)

- $\boxed{\text{KEY COURSE CONCEPT:}}$

$$\begin{tikzcd} x(at) \arrow[r, leftrightarrow, "\mathcal{L}"] & \sim X\left(\frac{s}{a}\right) \\ \text{multiply} \arrow[u] \arrow[r, leftrightarrow] & \text{divide} \arrow[u] \end{tikzcd}$$

- Spread in time ⟷ Shrink in frequency
- Shrink in time ⟷ Spread in frequency

Table of $\mathcal{L}$-Transform Properties

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM (Bilateral)

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$ $x_1(t)$ $x_2(t)$	$X(s)$ $X_1(s)$ $X_2(s)$	R R_1 R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{R}e\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to \infty} sX(s)$$

[From Oppenheim, Willsky, Nawab, Signals & Systems, 2nd ed, 1996, Prentice-Hall]

Why "Frequency Domain"?

Q: Why is s-domain referred to as the "Frequency Domain"?

A: $$e^{-st} = e^{-(\sigma+j\omega)t} = e^{-\sigma t} e^{-j\omega t}$$

$$= \underset{\substack{\uparrow \\ \text{Time-varying} \\ \text{scaling value}}}{e^{-\sigma t}} \underset{\substack{\uparrow \\ \text{complex sinusoid} \\ \text{of } \underline{\underline{\text{frequency}}} \ \omega !!!}}{\left[\cos(-\omega t) + j \sin(-\omega t)\right]}$$