Solution

Worcester Polytechnic Institute — Department of Electrical and Computer Engineering
ECE2311 — Continuous-Time Signal and System Analysis — Term B'17

Homework 3: Due Friday, 3 November 2017 (3:00 P.M.)

Write your name and ECE box at the top of each page.

General Reminders on Homework Assignments:

- Always complete the reading assignments *before* attempting the homework problems.
- Show all of your work. Use written English, where applicable, to provide a log or your steps in solving a problem. (For numerical homework problems, the writing can be brief.)
- A solution that requires physical units is *incorrect* unless the units are listed as part of the result.
- Get in the habit of underlining, circling or boxing your result.
- Always write neatly. Communication skills are essential in engineering and science. "If you didn't write it, you didn't do it!"

---

1) Systems: [Unless noted otherwise, *x(t)* is the system input and *y(t)* is the system output.]

a) Use the definition of time invariance to show if the following system is time-invariant or time-variant: $y(t) = 3x(t) - 4x^2(t)$.

1) Delay input; apply T

(a) $$x(t)\Big|_{t \to t-T} = x(t-T)$$

(b) $$y(t,T) = 3x(t-T) - 4x^2(t-T)$$

2) Delay output:

$$y(t)\Big|_{t \to t-T} = y(t-T) = 3x(t-T) - 4x^2(t-T)$$

Since $y(t,T) = y(t-T)$ ➔ **Time Invariant**

b) Use the definition of time invariance to show if the following system is time-invariant or time-variant: $y(t) = -13t^4\, x(t)$.

1) Delay input; apply T

(a) $$x(t)\Big|_{t \to t-T} = x(t-T)$$

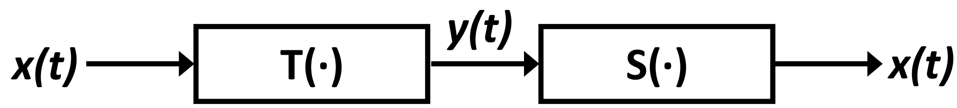
(b) $$y(t,T) = -13t^4\, x(t-T)$$

2) Delay output:

$$\begin{aligned}
y(t)\Big|_{t \to t-T} &= y(t-T) = -13(t-T)^4\, x(t-T) \\
&= -13\left(t^2 - 2Tt + T^2\right)\left(t^2 - 2Tt + T^2\right)x(t-T) \\
&= -13\left\{\left(t^4 - 2Tt^3 + T^2t^2\right) + \left(-2Tt^3 + 4T^2t^2 - 2T^3t\right) + \left(T^2t^2 - 2T^3t + T^4\right)\right\}x(t-T) \\
&= -13\left\{t^4 + t^3(-4T) + t^2\left(6T^2\right) + t\left(-4T^3\right) + \left(T^4\right)\right\}x(t-T)
\end{aligned}$$

Since $y(t,T) \neq y(t-T)$ ➔ **Time Varying**

c) A system $\mathcal{T}[\cdot]$ is invertible if a system $\mathcal{S}[\cdot]$ exists such that for input $x(t)$: $x(t) = \mathcal{S}\{\ \mathcal{T}[\ x(t)\ ]\ \}$. If it exists, find the inverse system of the system described by: $y(t) = \mathrm{T}[x(t)] = \dfrac{x(t)}{6}$.



The system T(·) just divides the input by 6. So, this operation can be inverted by a system that multiplies by 6.

$$\boxed{S[x(t)] = 6\,x(t)}$$

d) If it exists, find the inverse system of the system described by: $y(t) = x^2(t)$.

In general, **<u>an inverse system does not exist</u>**, because the output due to negative-valued inputs cannot be distinguished from the output due to the positive-valued inputs of the same magnitude.

e) A relaxed system is bounded-input bounded-output (BIBO) stable if and only if every bounded input produces a bounded output. Thus, a system can be proven *unstable* by listing a single bounded input that produces an unbounded output. The following system is BIBO *unstable*:

$y(t) = \displaystyle\int_{\tau=-\infty}^{t} x^2(\tau)\,d\tau$. Use the (bounded) step input to show that this system is unstable and derive the system output that leads to this conclusion.

Many possible inputs can be used to show that the system is unstable. Consider the bounded input $x(t) = u(t)$. Then,

$$y(t) = \int_{\tau=-\infty}^{t} u^2(\tau)\,d\tau.$$

But, $u^2(\tau) = u(\tau)$, so:

$$y(t) = \begin{cases} \displaystyle\int_{\tau=0}^{t} 1\cdot d\tau = \tau\Big|_{\tau=0}^{t} = t\ , t \geq 0 \\ 0\ , t < 0 \end{cases}.$$

We see that as $t \to \infty$, $y(t) \to \infty$. Thus, the system is **NOT BIBO STABLE**.

f) Use the definition of a linear system to show if the following system is linear:

$3\dfrac{d\,y(t)}{dt} - 2\,y(t) = x^2(t)$.

Substituting:

$$3\frac{d}{dt}\left[a_1 y_1(t)+a_2 y_2(t)\right]-2\left[a_1 y_1(t)+a_2 y_2(t)\right]\overset{?}{=}\left[a_1 x_1(t)+a_2 x_2(t)\right]^2$$

or

$$a_1\underbrace{\left[3\frac{dy_1(t)}{dt}-2y_1(t)\right]}_{\equiv x_1^2(t)}+a_2\underbrace{\left[3\frac{dy_2(t)}{dt}-2y_2(t)\right]}_{\equiv x_2^2(t)}\overset{?}{=}a_1^2 x_1^2(t)+2a_1a_2x_1(t)x_2(t)+a_2^2x_2^2(t)$$

or

$$a_1 x_1^2(t)+a_2 x_2^2(t)\overset{?}{=}a_1^2 x_1^2(t)+2a_1a_2x_1(t)x_2(t)+a_2^2x_2^2(t)$$

Since the two sides are not the same ➔ **Nonlinear System**

g) Use the definition of a linear system to show if the following system is linear: $3t^2\,y(t)=2\,x(t)$.

Substituting:

$$3t^2\left[a_1 y_1(t)+a_2 y_2(t)\right]\overset{?}{=}2\left[a_1 x_1(t)+a_2 x_2(t)\right]$$

or

$$a_1\underbrace{\left[3t^2 y_1(t)\right]}_{\equiv 2x_1(t)}+a_2\underbrace{\left[3t^2 y_2(t)\right]}_{\equiv 2x_2(t)}\overset{?}{=}a_1\underbrace{\left[2x_1(t)\right]}_{\equiv 3t^2 y_1(t)}+a_2\underbrace{\left[2x_2(t)\right]}_{\equiv 3t^2 y_2(t)}$$

Thus,

$$a_1 2x_1(t)+a_2 2x_2(t)\overset{?}{=}a_1 2x_1(t)+a_2 2x_2(t)$$

Since the two sides are the same ➔ **Linear System**

Could alternatively keep the left-hand side of the second equation, to achieve the same conclusion.

h) Use the definition of a linear system to show if the following system is linear: $-2\frac{d\,y(t)}{dt}+3\,y(t)=11\,x(t)$.

Substituting:

$$-2\frac{d}{dt}\left[a_1 y_1(t)+a_2 y_2(t)\right]+3\left[a_1 y_1(t)+a_2 y_2(t)\right]\overset{?}{=}11\left[a_1 x_1(t)+a_2 x_2(t)\right]$$

or

$$a_1\underbrace{\left[-2\frac{dy_1(t)}{dt}+3y_1(t)\right]}_{\equiv 11x_1(t)}+a_2\underbrace{\left[-2\frac{dy_2(t)}{dt}+3y_2(t)\right]}_{\equiv 11x_2(t)}\overset{?}{=}11a_1x_1(t)+11a_2x_2(t)$$

or

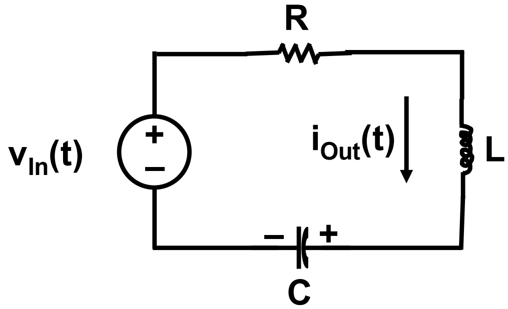
$$a_1 11x_1(t)+a_2 11x_2(t)\overset{?}{=}a_1 11x_1(t)+a_2 11x_2(t)$$

Since the two sides are the same ➔ **Linear System**

Could alternatively substitute into the right-hand side of the second equation, to achieve the same result.

2) Zero-Input Response (Time Domain):

Consider the following RLC circuit:



Assume that the capacitor voltage and inductor current are zero at time t=0. By KVL:

$$v_{in}(t) = i_{out}(t)R + L\frac{di_{out}(t)}{dt} + \frac{1}{C}\int_{\tau=0}^{t} i_{out}(\tau)d\tau + v_c(0)$$

Setting $v_c(0) = 0$ (given), differentiating with respect to time and collecting terms:

$$\frac{d^2 i_{out}(t)}{dt^2} + \frac{R}{L}\frac{di_{out}(t)}{dt} + \frac{1}{LC}i_{out}(t) = \frac{1}{L}\frac{dv_{in}(t)}{dt}$$

Note the use of Leibniz's Rule:

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)dt = f(u)\frac{du}{dx} - f(v)\frac{dv}{dx}$$

a) Let's consider the zero-input response of the circuit above, i.e. the case when: $\frac{d^2 i_{Out}(t)}{dt^2} + \frac{R}{L}\frac{d\,i_{Out}(t)}{dt} + \frac{1}{LC}i_{Out}(t) = 0$. This response relationship has the characteristic equation: $\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$, which is derived from the coefficients of the derivative terms. For this second-order system, the characteristic equation has two roots (r_1 and r_2). If the roots are **not** repeated and both are real, the zero-input response will be of the form: $i_{Out}(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$. Let $\frac{R}{L}$ = 6.6 Hz and $\frac{1}{LC}$ =10.08 Hz^2. Use the "roots()" command in MATLAB to determine the roots of the characteristic equation. (Use numeric calculation, ***not*** the symbolic engine.) As your solution, write the zero-input response with the correct root values (leave c_1 and c_2 as unknowns).

In MATLAB, using *"roots([1 6.6 10.08])"* leads to the solution:

$$i_{out}(t) = c_1 e^{-4.2t} + c_2 e^{-2.4t}$$

b) Continuing this example, consider the situation when the two roots (r_1 and r_2) are complex conjugates. The same zero-input response form results, although the symmetry of the roots would also allow the result to be written as a real-valued sinusoid. Let $\frac{R}{L}$ = 2.4 Hz and $\frac{1}{LC}$ =4.68 Hz^2. Use the "roots()" command in MATLAB to determine the roots of the characteristic equation. <u>As your solution,</u> write the zero-input response with the correct root values (leave c_1 and c_2 as unknowns).

In MATLAB, using *"roots([1 2.4 4.68])"* leads to the solution:

$$i_{out}(t) = c_1 e^{(-1.2+j1.8)t} + c_2 e^{(-1.2-j1.8)t}$$

c) Continuing this example, consider the situation when the two roots (r_1 and r_2) are repeated. (Note that this case is rather "trivial," since it only represents one case along a continuum. Nonetheless….) In this case, the form of the zero-input response is: $i_{Out}(t) = c_1\, e^{r_1 t} + c_2\, t\, e^{r_1 t}$.

Let $\frac{R}{L}$ = 10.9 Hz and $\frac{1}{LC}$ = 29.7025 Hz^2. Use the "roots()" command in MATLAB to determine the roots of the characteristic equation. <u>As your solution,</u> write the zero-input response with the correct root values (leave c_1 and c_2 as unknowns).

In MATLAB, using *"roots([1 10.9 29.7025])"* leads to the solution:

$$i_{out}(t) = c_1 e^{-5.45t} + c_2 t e^{-5.45t}$$

d) The root-finding results above used the numerical ("floating point") computation engine in MATLAB. For most applications, this approach is desired and leads to very accurate results. In some cases, however, we may want symbolic results — perhaps there is an important cancellation that can be lost due to round-off errors in the floating-point calculations. In class, you were shown how to use MATLAB's symbolic engine to determine roots symbolically, using the "syms()" and "solve()" commands. Use MATLAB's symbolic engine to find the roots of the three characteristic equations in parts a–c. For your convenience, the characteristic equations are written below:

i) $\lambda^2 + 6.6\,\lambda + 10.08 = 0$

ii) $\lambda^2 + 2.4\,\lambda + 4.68 = 0$

iii) $\lambda^2 + 10.9\,\lambda + 29.7025 = 0$

(*i*) Using *"syms x; solve(x^2+6.6*x+10.08)"* in MATLAB gives

$$r_1 = \frac{-12}{5}, \quad r_2 = \frac{-21}{5}$$

(*ii*) Using *"syms x; solve(x^2+2.4*x+4.68)"* in MATLAB gives

$$r_1 = \frac{-6+j9}{5}, \quad r_2 = \frac{-6-j9}{5}$$

(*iii*) Using *"syms x; solve(x^2+10.9*x+29.7025)"* in MATLAB gives

$$r_1 = r_2 = \frac{-109}{20}$$