Solution

Worcester Polytechnic Institute — Department of Electrical and Computer Engineering
ECE2311 — Continuous-Time Signal and System Analysis — Term B'17

Homework 4: Due Tuesday, 7 November 2017 (3:00 P.M.)

Write your name and ECE box at the top of each page.

General Reminders on Homework Assignments:

- Always complete the reading assignments *before* attempting the homework problems.
- Show all of your work. Use written English, where applicable, to provide a log or your steps in solving a problem. (For numerical homework problems, the writing can be brief.)
- A solution that requires physical units is *incorrect* unless the units are listed as part of the result.
- Get in the habit of underlining, circling or boxing your result.
- Always write neatly. Communication skills are essential in engineering and science. "If you didn't write it, you didn't do it!"

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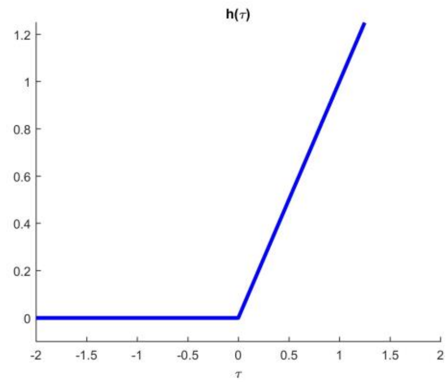
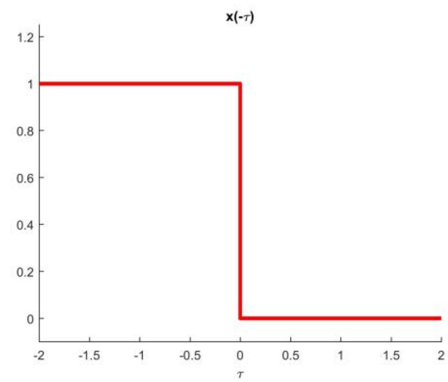
1) The Convolution Operation:

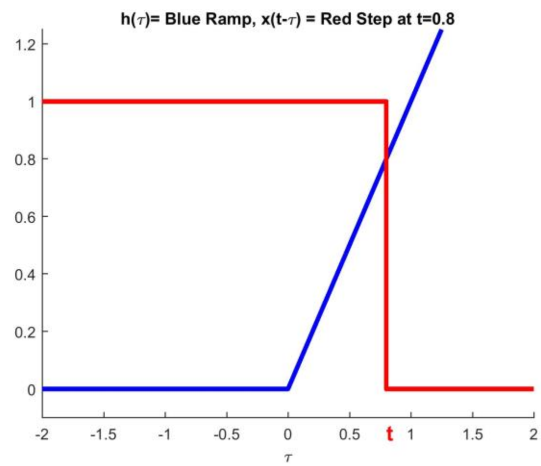
a) An LTI system has the ramp impulse response: $h(t) = t\,u(t)$. Use direct integration to find the system output if the input is: $x(t) = u(t)$.





So, $x(t) * h(t) = 0$ for $t < 0$

<u>For</u> $t \geq 0$

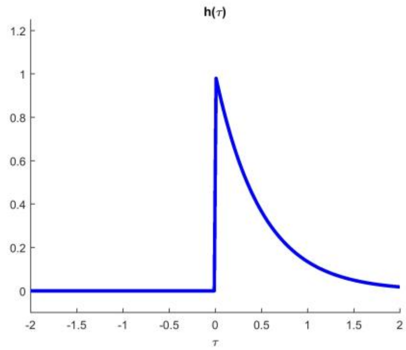


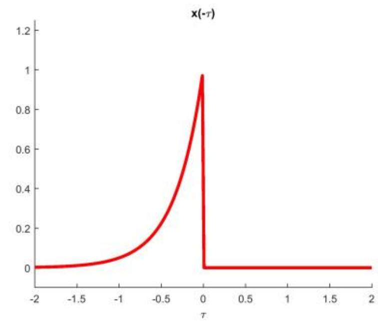
$$x(t) * h(t) = \int_{\tau=0}^{t} \tau \cdot d\tau = \frac{\tau^2}{2} \Big|_{\tau=0}^{t} = \frac{t^2}{2}$$

<u>Thus</u>

$$u(t) * t\,u(t) = \begin{cases} \dfrac{t^2}{2}, & t \geq 0 \\ 0, & t < 0 \end{cases} = \frac{t^2}{2} u(t)$$

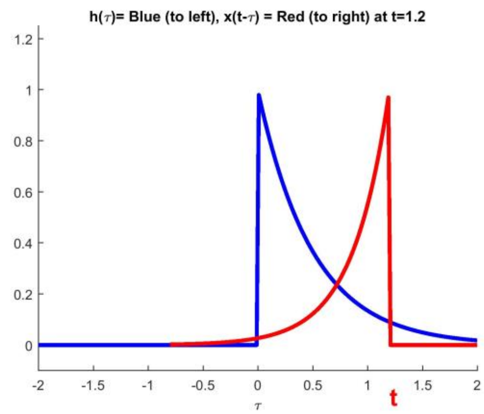
b) An LTI system has the ramp impulse response: $h(t) = e^{-2t} u(t)$. Use direct integration to find the system output if the input is: $x(t) = e^{-3t} u(t)$.





So, $x(t)*h(t)=0$ for $t<0$

For $t \geq 0$



$$x(t)*h(t)=\int_{\tau=0}^{t} e^{-2\tau} e^{-3(t-\tau)}\, d\tau = e^{-3t}\int_{\tau=0}^{t} e^{\tau}\, d\tau = e^{-3t} e^{\tau}\Big|_{\tau=0}^{t} = e^{-2t}-e^{-3t}$$

Thus

$$e^{-3t}u(t)*e^{-2t}u(t)=\begin{cases} e^{-2t}-e^{-3t}, & t\geq 0 \\ 0, & t<0 \end{cases} = \left(e^{-2t}-e^{-3t}\right)u(t)$$

c) Use the "Convolution Table" on the class notesheet to determine the convolution: $5u(t)*12u(t)$.

From the Convolution Table:

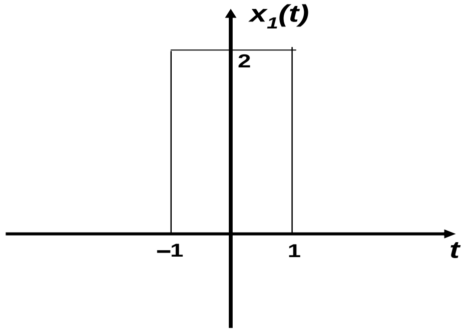
$$5u(t)*12u(t)=60\,t\,u(t)$$

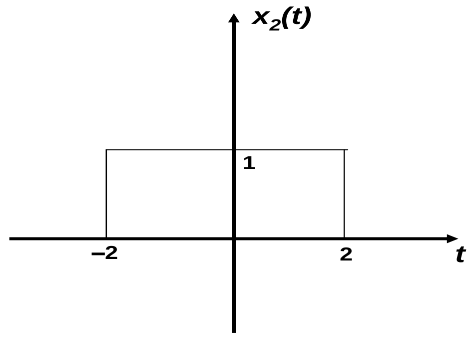
d) Use the "Convolution Table" on the class notesheet to determine the convolution: $12t^2e^{-4t}u(t)*t^3e^{-4t}u(t)$.

From the Convolution Table:

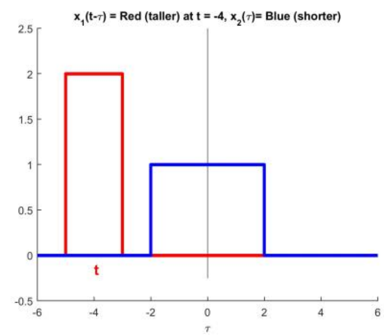
$$12t^2e^{-4t}u(t)*t^3e^{-4t}u(t)=12\frac{2!\,3!}{(2+3+1)!}t^{2+3+1}e^{-4t}u(t)=12\frac{t^6e^{-4t}}{60}u(t)=\frac{t^6e^{-4t}}{5}u(t)$$

e) Use the concept of graphical convolution (and any associated computation required) to convolve $x_1(t)$ and $x_2(t)$ shown below. Be sure to show a drawing for each region where a distinct integral range is required in the convolution. Sketch the result as well and provide the result via a mathematical expression.



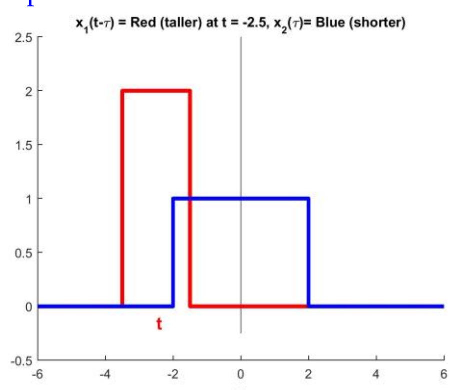


When $t < -3$ or $t > 3$, no overlap; so $x_1(t) * x_2(t) = 0$:



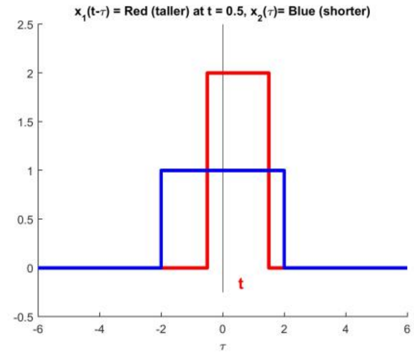
$$x_1(t) * x_2(t) = 0$$

When $-3 \le t \le -1$, partial overlap from left:



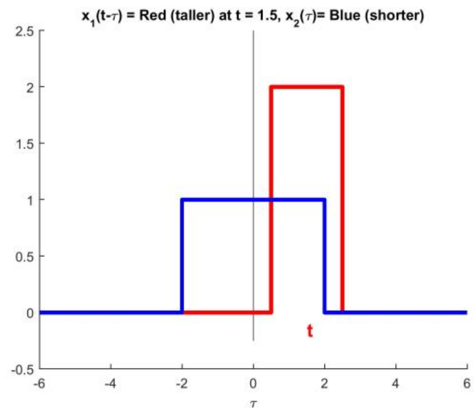
$$x_1(t) * x_2(t) = \int_{\tau=-2}^{t+1} 1 \cdot 2\, d\tau = 2\tau \Big|_{\tau=-2}^{t+1} = 2(t+1) - (-4) = 2t + 6$$

When $-1 \le t \le 1$, full overlap:



$$x_1(t) * x_2(t) = \int_{\tau=t-1}^{t+1} 1 \cdot 2\, d\tau = 2\tau \Big|_{\tau=t-1}^{t+1} = 2(t+1) - 2(t-1) = 4$$

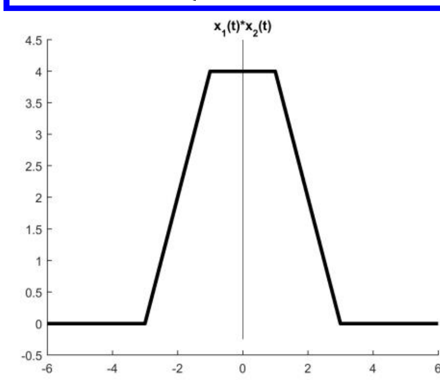
<u>When</u> $1 \le t \le 3$, partial overlap from right:



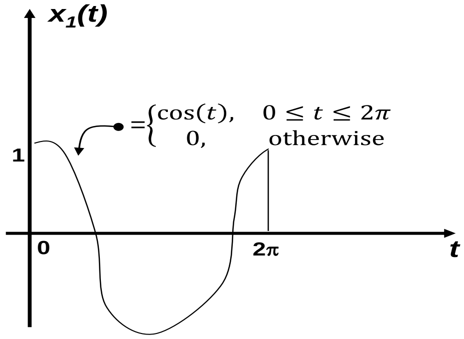
$$x_1(t) * x_2(t) = \int_{\tau=t-1}^{2} 1 \cdot 2\, d\tau = 2\tau \Big|_{\tau=t-1}^{2} = 4 - 2(t-1) = 6 - 2t$$

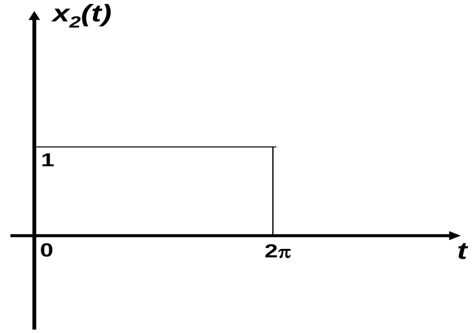
<u>Full Solution:</u>

$$\boxed{x_1(t) * x_2(t) = \begin{cases} 0, & t < -3 \\ 2t + 6, & -3 \le t \le -1 \\ 4, & -1 \le t \le 1 \\ 6 - 2t, & 1 \le t \le 3 \\ 0, & t > 3 \end{cases}}$$

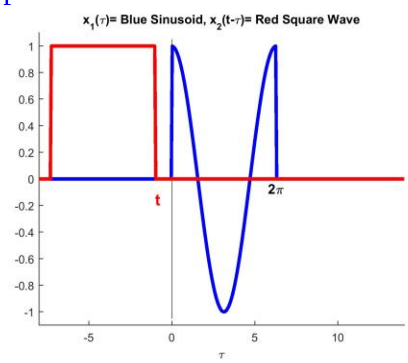


f) Use the concept of graphical convolution (and any associated computation required) to convolve $x_1(t)$ and $x_2(t)$ shown below. Be sure to show a drawing for each region where a distinct integral range is required in the convolution. Sketch the result as well and provide the result via a mathematical expression.



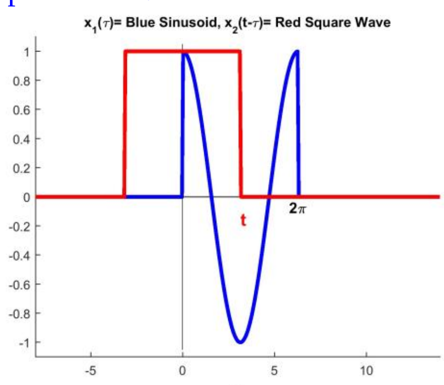


When $t < 0$ or $t > 4\pi$, no overlap:



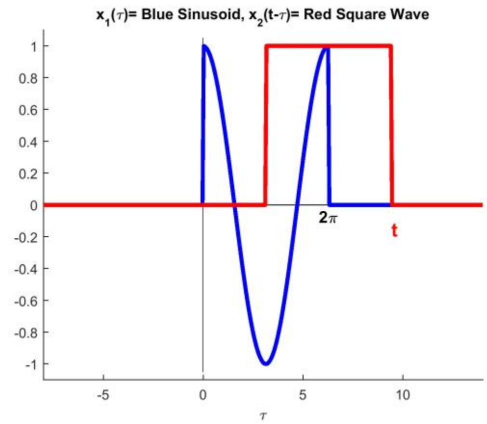
$$x_1(t) * x_2(t) = 0$$

When $0 \le t \le 2\pi$, partial overlap from left:



$$x_1(t) * x_2(t) = \int_{\tau=0}^{t} 1 \cdot \cos(\tau)\, d\tau = \sin(\tau)\Big|_{\tau=0}^{t} = \sin(t) - \sin(0) = \sin(t)$$

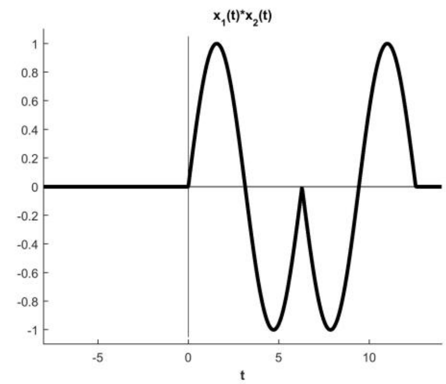
When $2\pi \le t \le 4\pi$, partial overlap from right:



$$x_1(t) * x_2(t) = \int_{\tau=t-2\pi}^{2\pi} 1\cdot\cos(\tau)\,d\tau = \sin(\tau)\Big|_{\tau=t-2\pi}^{2\pi} = \sin(2\pi) - \sin(t-2\pi) = -\sin(t)$$

Full Solution:

$$x_1(t) * x_2(t) = \begin{cases} 0, & t < 0 \\ \sin(t), & 0 \le t \le 2\pi \\ -\sin(t), & 2\pi \le t \le 4\pi \\ 0, & t > 4\pi \end{cases}$$



g) The impulse response of an LTI system is: $h(t) = 3e^{-2t}u(t)$. What is the step response of this system?

We know that the step response is: $s(t) = \int_{\tau=-\infty}^{t} h(\tau)\,d\tau$. So,

$$s(t) = \int_{\tau=-\infty}^{t} 3e^{-2\tau} u(\tau)\,d\tau .$$

For $t < 0$, this integral equals zero (due to the step function). For $t > 0$:

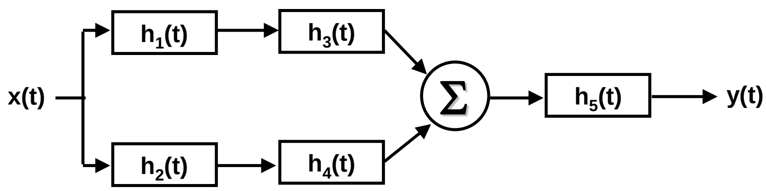
$$s(t) = 3\int_{\tau=0}^{t} e^{-2\tau}\,d\tau = 3\left.\frac{e^{-2\tau}}{-2}\right|_{\tau=0}^{t} = \frac{3e^{-2t}-3}{-2}, \quad t > 0$$

Thus,

$$\boxed{s(t) = \frac{3-3e^{-2t}}{2}\mu(t)}$$

2) Convolution Properties:

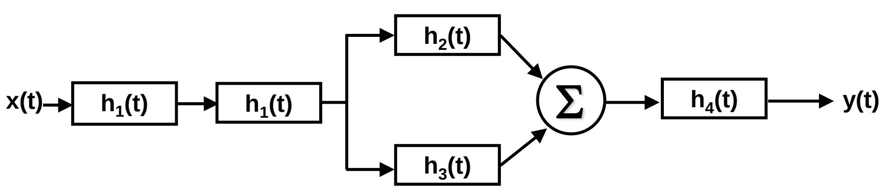
a) Given the interconnected LTI system below, write the overall system response *y(t)* as a function of the impulse responses [$h_i(t)$] of the individual systems and the input *x(t)*.



Observing the block diagram:

$$y(t) = x(t) * [h_1(t) * h_3(t) + h_2(t) * h_4(t)] * h_5(t)$$

b) Given the interconnected LTI system below, write the overall system response *y(t)* as a function of the impulse responses [$h_i(t)$] of the individual systems and the input *x(t)*. [Note that $h_1(t)$ is repeated.]



Observing the block diagram:

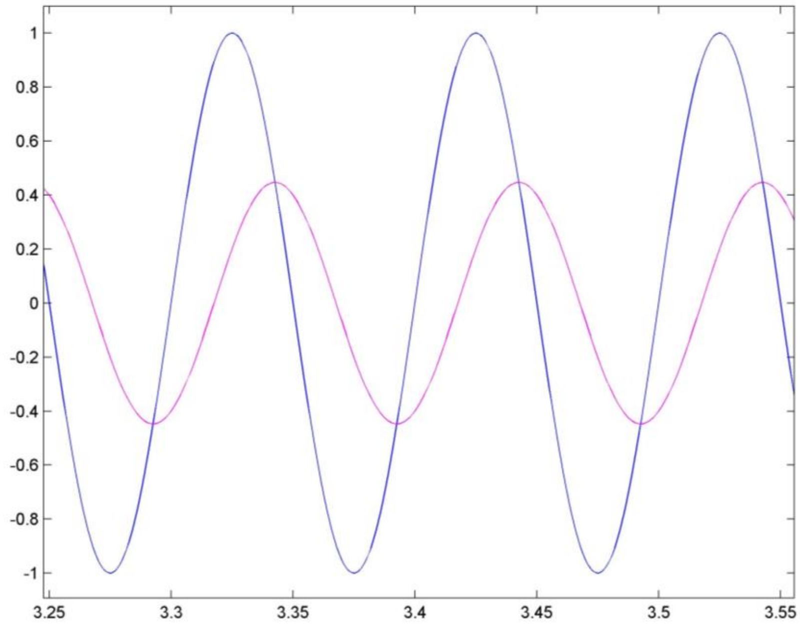
$$y(t) = x(t) * h_1(t) * h_1(t) * [h_2(t) + h_3(t)] * h_4(t)$$

3) Linear System Concept with MATLAB:

a) One overriding principle of a linear system is that if you input a sinusoid to a linear system, the output of that linear system (after transients have decayed) will be a sinusoid of the same frequency, with a modified gain and phase. For causal systems, the phase is always lagging (or zero). We can investigate this behavior using MATLAB's *numerical* calculations. Technically, we will be performing discrete-time signal processing — but we can simulate continuous-time processing quite well if we are careful. (I won't explain all of the details of the differences between the domains, as they are not relevant to this course. For now, just follow the instructions carefully.)

Begin a fresh session using MATLAB. (If MATLAB is currently open, close all ancillary figure windows, etc. Also, type "clear all" to eliminate all current variables.) First, we will create a low-pass filter by typing: "[b, a] = butter(1, 0.01);". The vectors "b" and "a" actually hold the low-pass filter "information." In the next few steps, we'll examine the shape of this filter. Second,, create a time vector "t" by typing: "t = 0: 0.001: 10;". Third, create an input sinusoid "x" by typing: "f = 10; x = sin(2*pi*f*t);". Here, "f" is the sine wave frequency in Hertz and "pi" is the constant π (already known to MATLAB). You can visualize this sine wave by typing: "plot(t, x)". This command will open a new window with the sinusoid. The time axis will be in seconds. [*Note that too many periods of the sine wave are displayed, thus successive sine wave cycles overlap. One way to manage this issue is to manually use the zoom feature within the plotting window.]* Use the zoom feature within the plot window to isolate 5–10 sine wave cycles. You should be able to verify that you have a sine wave with amplitude one, zero phase and 10 Hz frequency.

We can pass this sine wave through the low pass filter to create the output "y" with the command: "y = filter(b, a, x);". Then, we can plot the input and output sine waves in one plot with the commands: "plot(t, x, 'b'), hold on, plot(t, y, 'm'), hold off". *[**Note:** MATLAB is very specific about what quotation characters are required. I found that when I cut and pasted the above text, I had to manually re-type the single quotations before MATLAB would accept the statement.]* The input sine wave will plot in blue and the output sine wave in magenta. Use the zoom feature to examine the two waveforms near zero time. You should notice that the output waveform is not perfectly periodic at the beginning. This variation is the startup transient due to the initial conditions. For now, ignore this problem by zooming on a later portion of the waveform, after the transient has extinguished, as shown in the figure below. After the transient, the output should be a sinusoid. Manually measure its amplitude with respect to the amplitude of the input sinusoid, to determine the gain. Also, manually measure its phase (in degrees) with respect to the input sinusoid. The phase should be a lag, thus it will be a negative number.



Gain:

$$\frac{\text{Output Magnitude}}{\text{Input Magnitude}} =$$

$$\frac{0.447}{1} = 0.447$$

Phase:

$$\frac{-Delay}{Period} \bullet 360^{o}$$

$$= \frac{-(3.3425s - 3.325s)}{3.425 - 3.325s} \bullet 360^{o}$$

$$= -63^{o}$$

To plot the full frequency response, we need only repeat this filtering and measurement process for sinusoids of different frequencies. Do so for several sinusoid frequencies, including at least 1 Hz, 5, Hz, 10 Hz (already completed), 15 Hz, 20 Hz and 25 Hz. **DO NOT EXCEED A SINUSOIDAL FREQUENCY OF 50 HZ!!** *[Note: use the "up arrow" key on the keyboard to repeat MATLAB commands. You can use the "up arrow" key, edit the command, then hit "enter" to execute the command.]* **For your results, submit a plot of the amplitude response (gain vs. frequency), a plot of the phase response (phase delay vs. frequency, in degrees) and your MATLAB code. You can draw these plots by hand from your measurements (or, use MATLAB to plot these data.)**

Here are the exact plots. Your results should have measured values, at least, at: 1, 5, 10, 15, 20 and 25 Hz.