lect 03

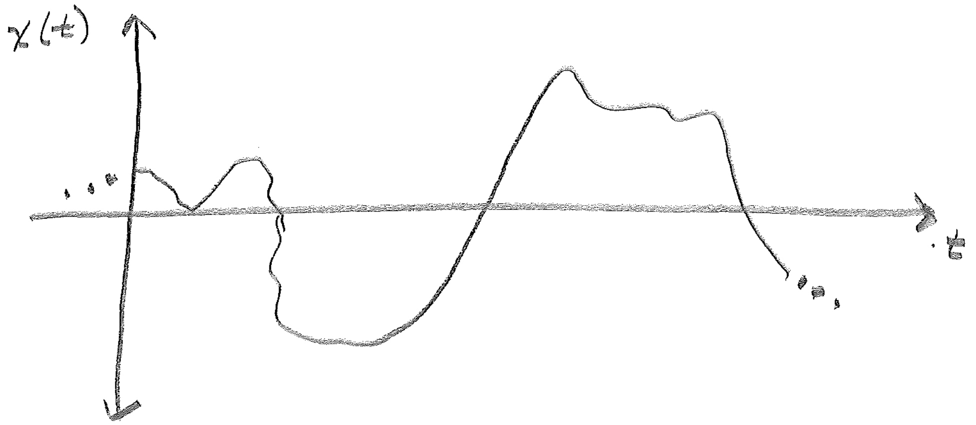
What is a Signal?

- Signal: Data set, usually function of independent variable ⇒ eg., time, position

Ex

- Circuit voltage as function of time
- Circuit current as function of time
- Temperature as function of altitude
- Temperature as function of altitude, latitude, longitude ⇒ 3 dimensional

- Can be real-valued or complex



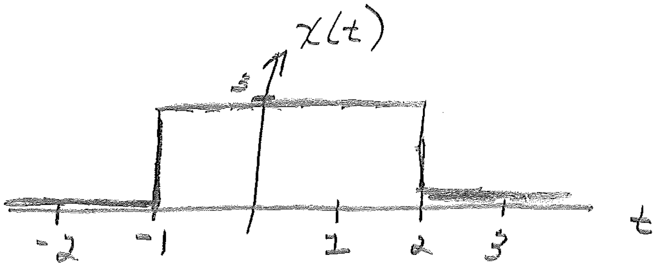
Signal Energy

$$\mathcal{E}_x = \int_{t=-\infty}^{\infty} |x(t)|^2 \, dt$$

- Measure of overall signal "size"
- Squaring counts positive & negative-value portions
- $\mathcal{E}_x$ not always finite [e.g., $x(t) = \sin(\omega t)$]

Ex

$$x(t) = \begin{cases} 3, & -1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$



$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-1}^{2} (3)^2 \, dt = 9t \Big|_{x=-1}^{2}$$

$$= 9(2 - -1) = 9 \cdot 3 = 27$$

If $x(t)$ units → V, then $\underline{\underline{\mathcal{E}_x = 27 \ V^2 \cdot s}}$

"Signal Energy," **not** electrical circuit energy

Signal Power

$\boxed{\text{General:}}$

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{t=-T/2}^{T/2} |x(t)|^2 \, dt$$

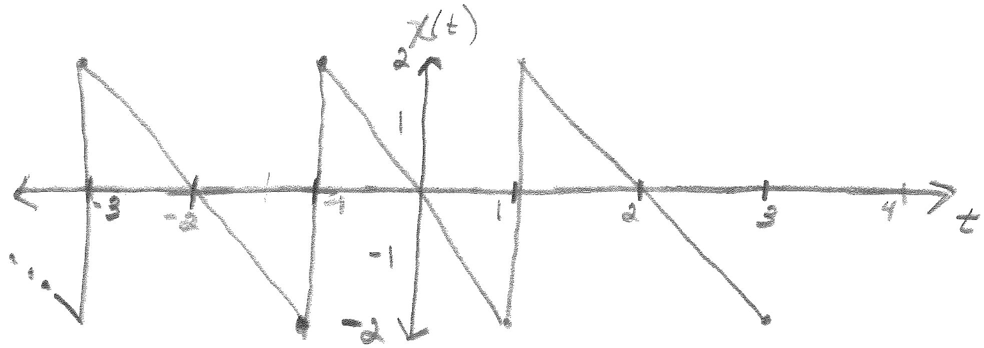
- Time average of energy
- Useful when $\mathcal{E}_x = \infty$
  - E.g., periodic signals

For periodic signal:

$$P_x = \frac{1}{T} \int_{T} |x(t)|^2 \, dt$$

- If $x(t)$ periodic $\rightarrow |x(t)|^2$ periodic
  $\rightarrow P_x$ is average over one period

Ex



Periodic ∴ $P_x = \frac{1}{2} \int_{-1}^{1} |-2t|^2 \, dt = \frac{1}{2} \int_{-1}^{1} 4t^2 \, dt$

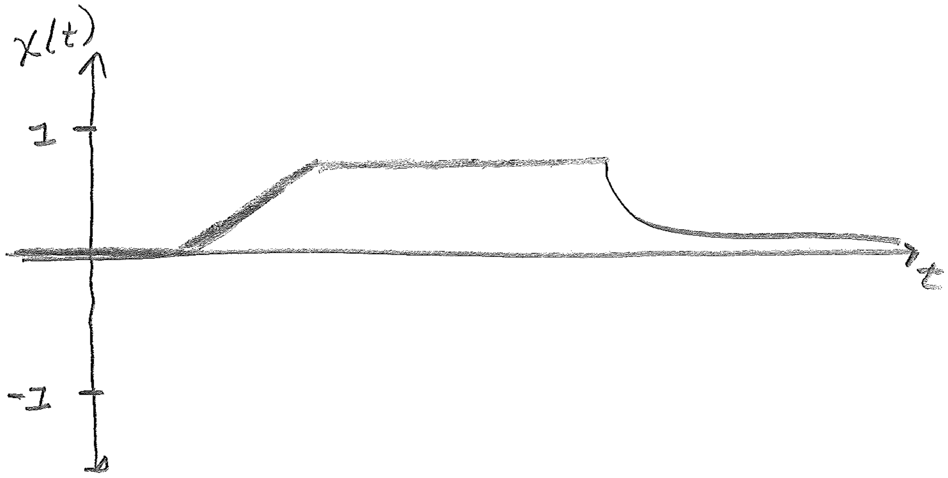
$$= \frac{4}{2} \cdot \frac{t^3}{3} \Big|_{t=-1}^{1} = \frac{2}{3}\left(1 - -1\right) = \frac{4}{3}$$

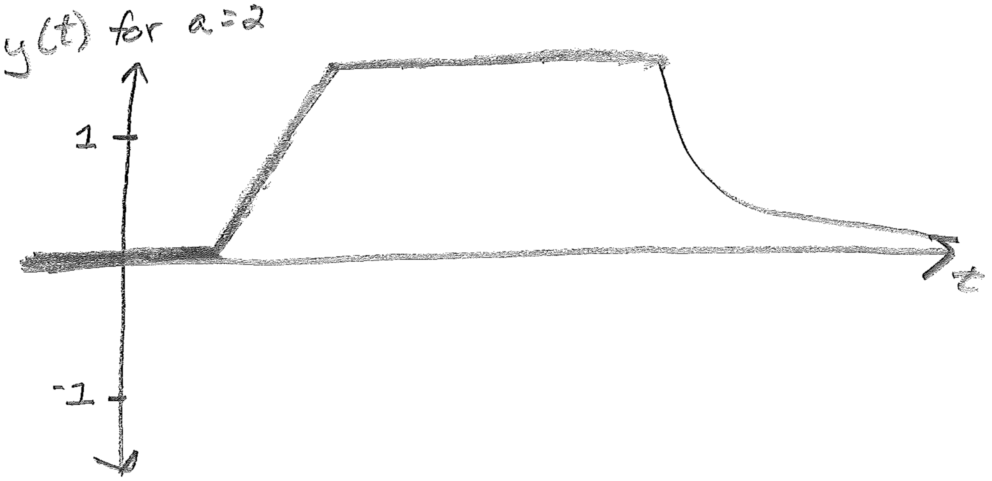
If $x(t)$ units → V, then $P_x = \frac{4}{3} \; V^2$

Not electrical circuit power

Signal Operations: Scalar Multiplication

- Signal $x(t)$ multiplied by constant a gives new signal: $y(t) = a \cdot x(t)$
    - Scaled copy of $x(t)$

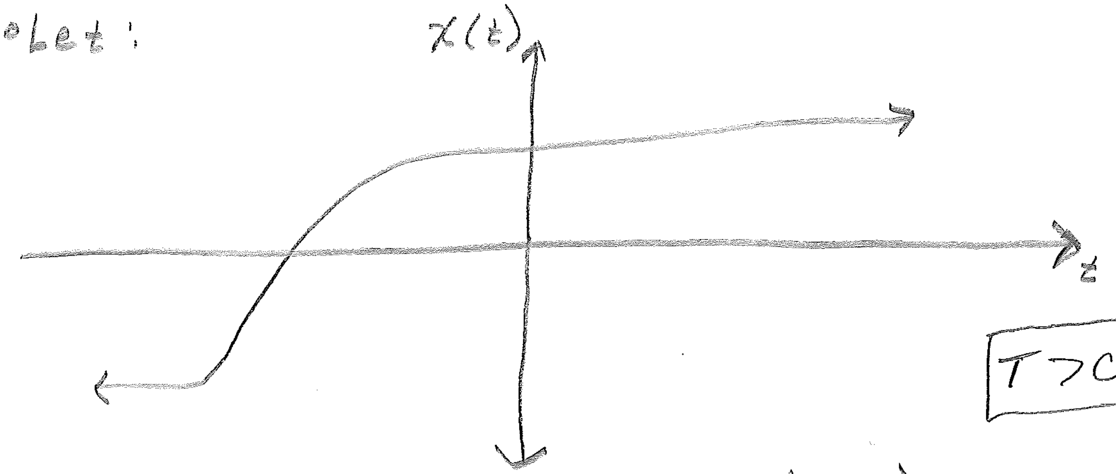




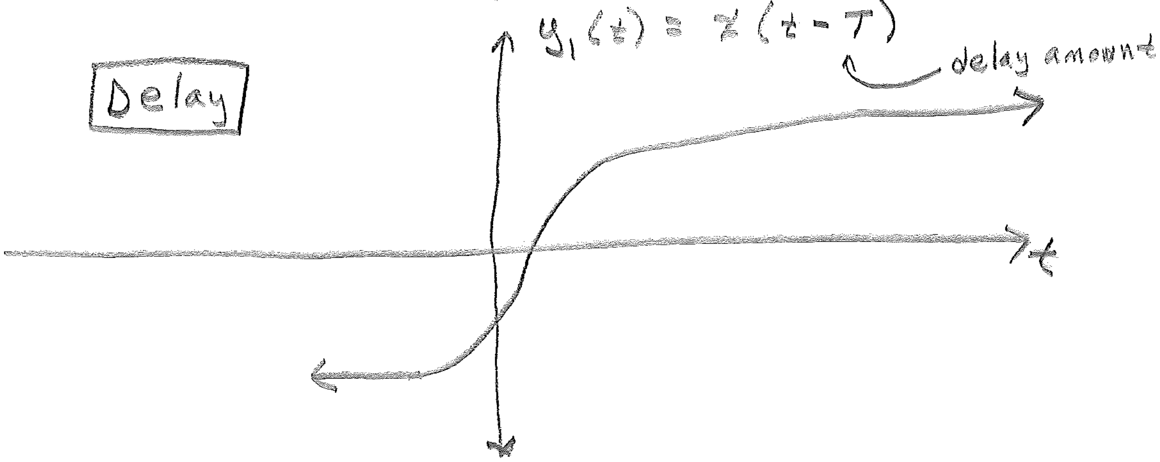
Signal Operations: Time Shifting

- Time-delay or advance signal

- Let:

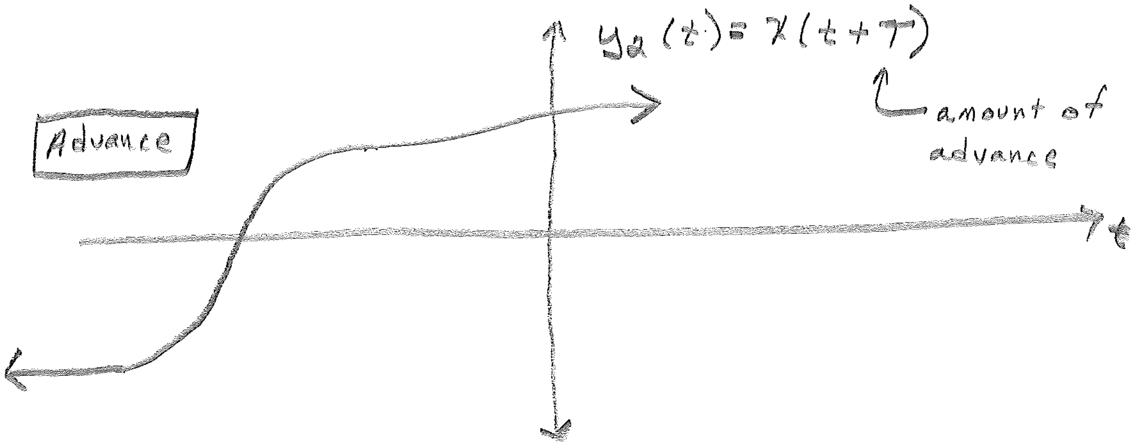


$\boxed{T>0}$



Delay

$y_1(t) = x(t-T)$ delay amount

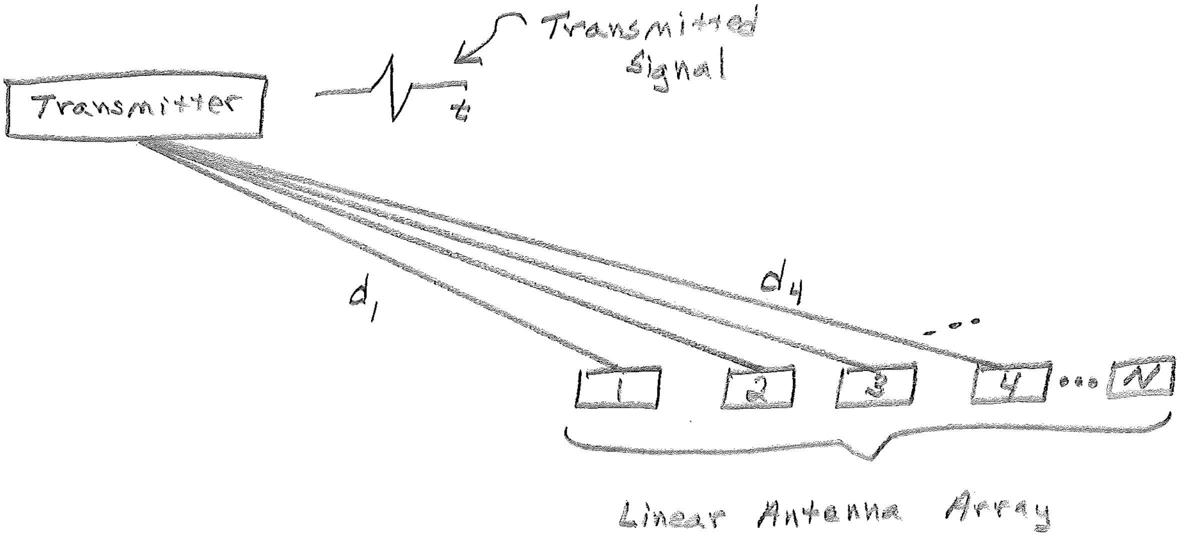


Advance

$y_2(t) = x(t+T)$ amount of advance

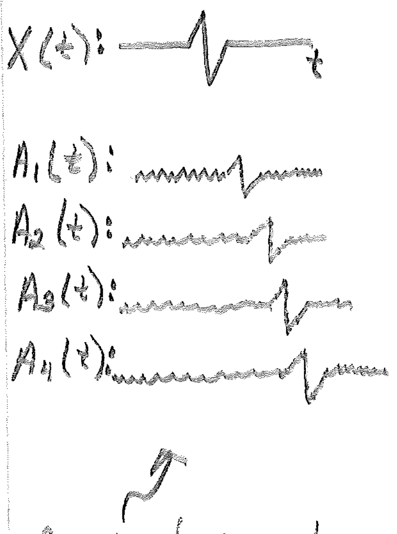
- Same concept if independent variable $\neq$ time

Time-Shifting Example: Phased-Array Antenna



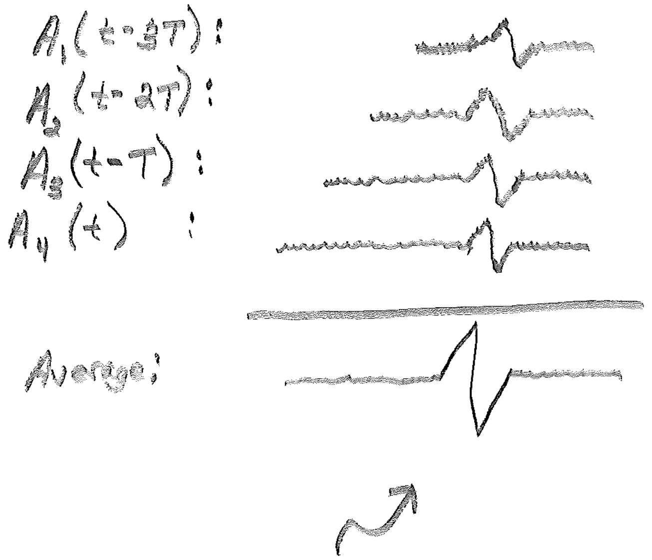
- Distance varies to each antenna element

- **Received:**



- Received signal includes noise
- Often, xmit signal buried in noise

Aligned:



- Average of noise → 0
- Average of signal → signal

$\boxed{\text{Focus array by changing } T}$

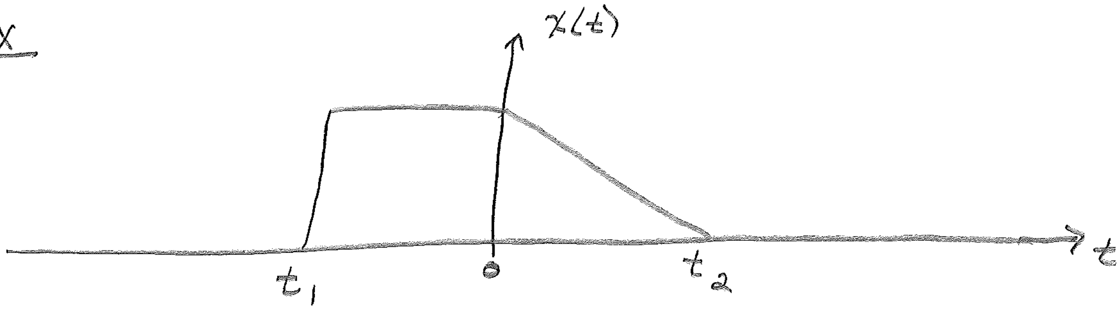
Signal Operations: Time Scaling

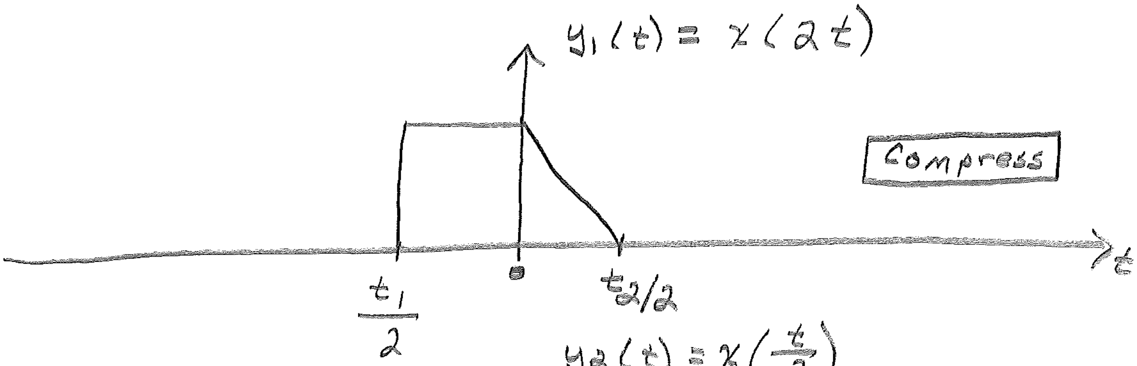
- Compress or expand (stretch) signal along independent variable (usually time).

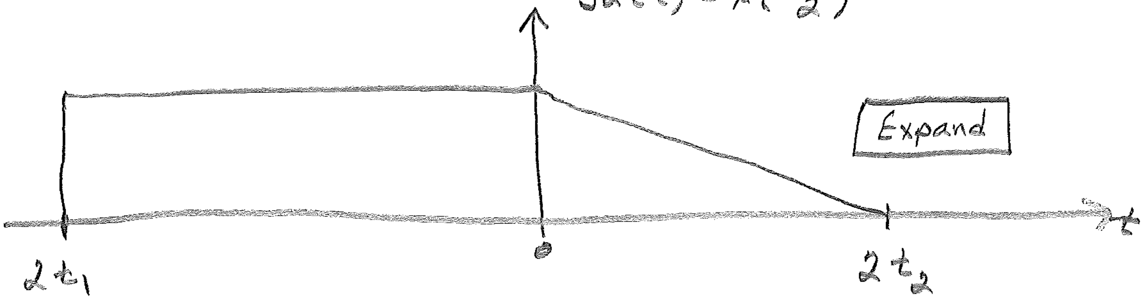
$$y(t) = x(a \cdot t) \begin{cases} a > 1 \Rightarrow \text{compression} \\ a < 1 \Rightarrow \text{expansion} \end{cases}$$

- Function value of $y(\cdot)$ at time t equals function value of $x(\cdot)$ at time $a \cdot t$

Ex



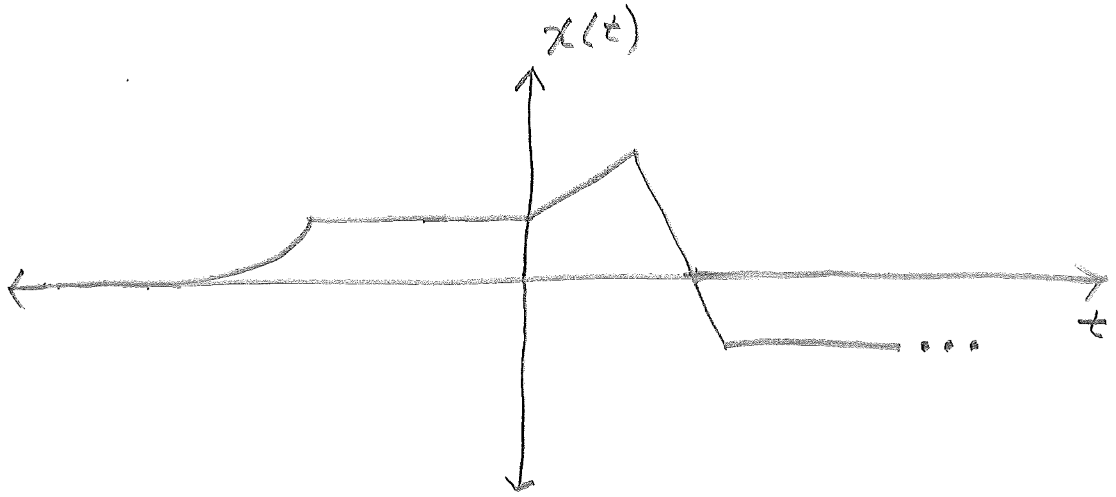


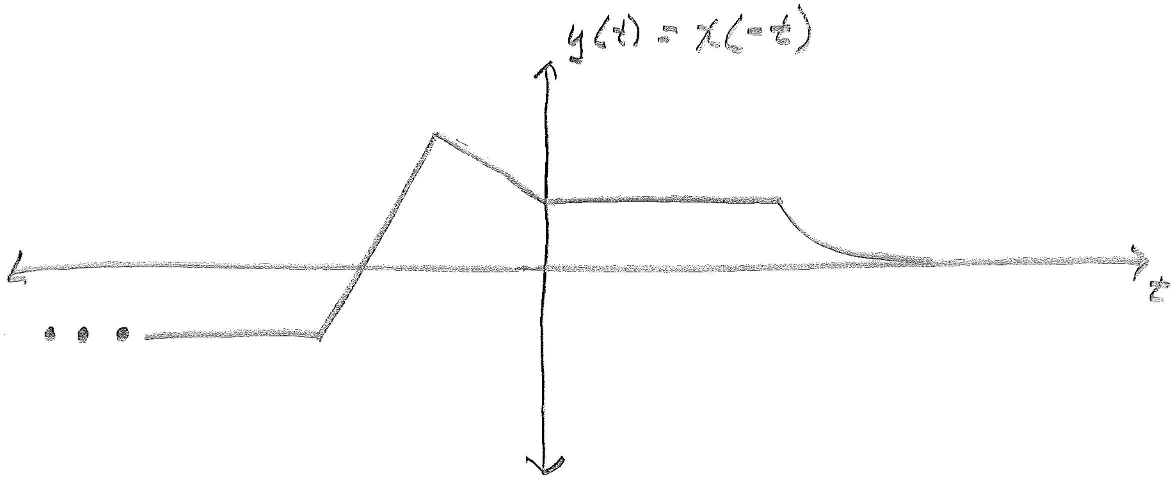


Signal Operations: Time Reversal

- Reflect / hinge function around vertical axis
  - Time runs backwards

$$y(t) = x(-t)$$

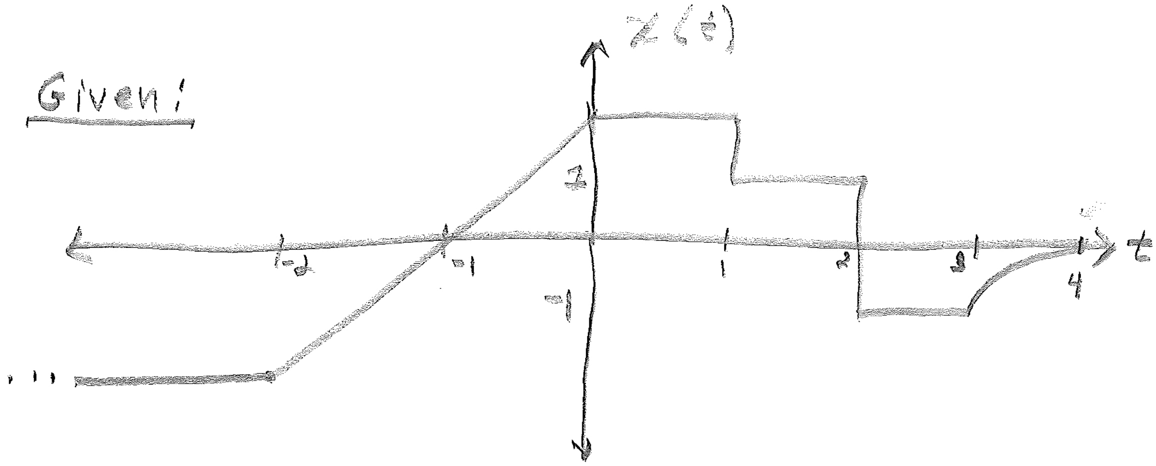




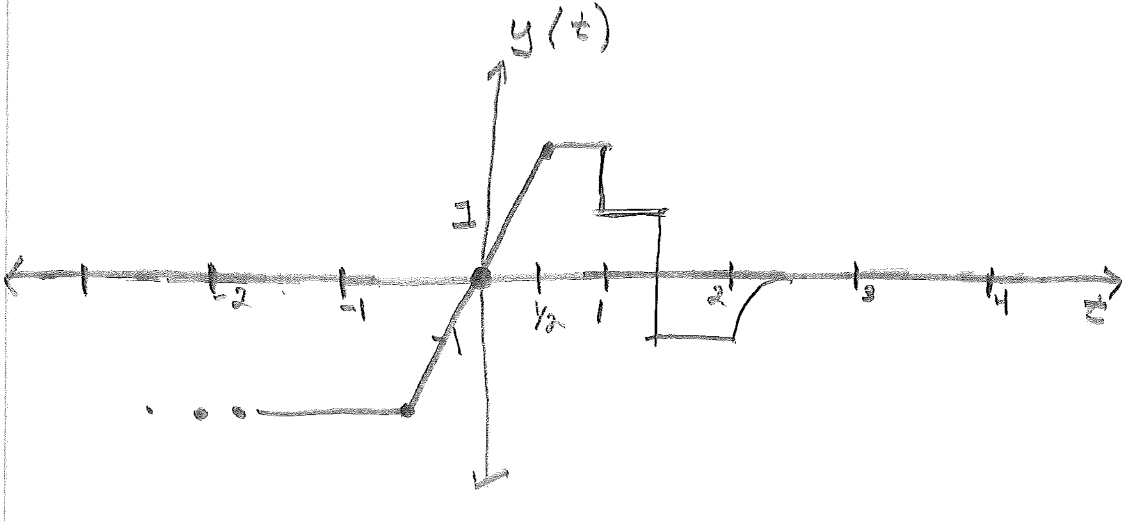
- Recall: Other independent variables, e.g., space

Signal Operations: Combined Example

Given:



Draw: $y(t) = x(2t-1)$



Example: Signal Energy

Q Find signal energy for

$$x(t) = \begin{cases} e^{-2t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Sol'n

$$E_x = \int_{t=-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_0^{\infty} |e^{-2t}|^2 dt = \int_0^{\infty} (e^{-2t})^2 dt = \int_0^{\infty} e^{-4t} \, dt$$

$$= \frac{e^{-4t}}{-4}\Bigg|_{t=0}^{\infty} = \frac{\cancelto{0}{e^{-4\infty}} - \cancelto{1}{e^{-4\cdot 0}}}{-4} = \boxed{\frac{1}{4} \text{ units}^2 \cdot s}$$

Example: Signal Power

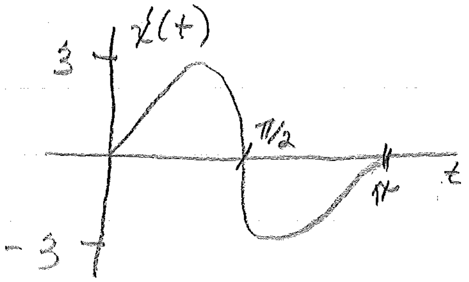
Q Find signal power in $x(t) = 3\sin(2t + 20°)$.

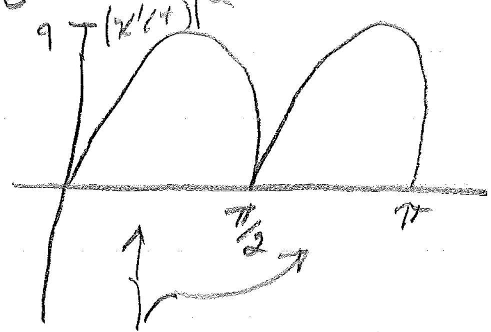
Sol'n

Periodic with $\omega = 2 \frac{rad}{s} \Rightarrow T = \frac{2\pi}{\omega} = \pi \, s$

Note that phase shift is irrelevant to shape over one period. Thus,

$$P_x = \frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{\pi}\int_0^{\pi} |3 \sin(2t)|^2 dt$$





will be same shape & area

So,

$$P_x = 2 \cdot \left[\frac{1}{\pi}\int_0^{\pi/2} 9\sin^2(2t)\, dt\right]$$

Recall:

$$\int \sin^2(ax)\, dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

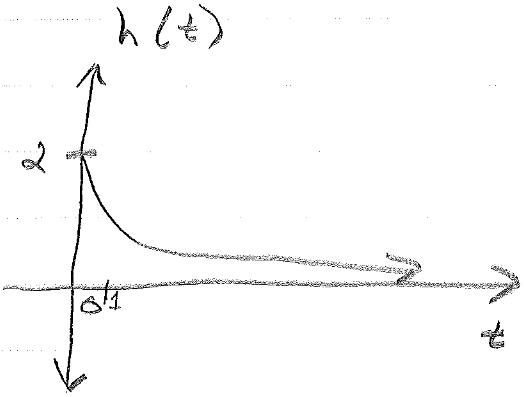
$$= \frac{18}{\pi}\left[\frac{t}{2} - \frac{\sin(4t)}{8}\right]\Bigg|_{t=0}^{\pi/2} = \frac{18}{\pi}\left[\frac{\pi}{4} - 0 - 0 + 0\right]$$

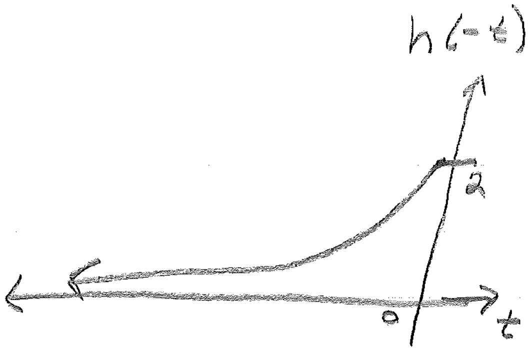
$$= \boxed{\frac{9}{2} \text{ units}^2}$$

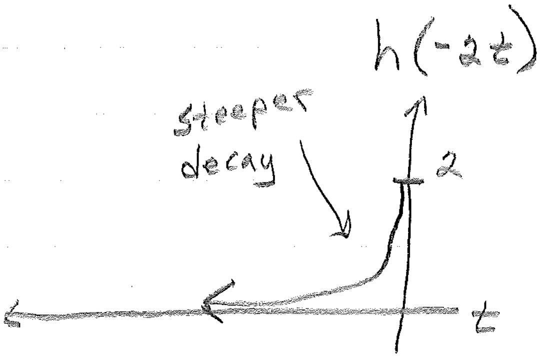
Example: Signal Operations

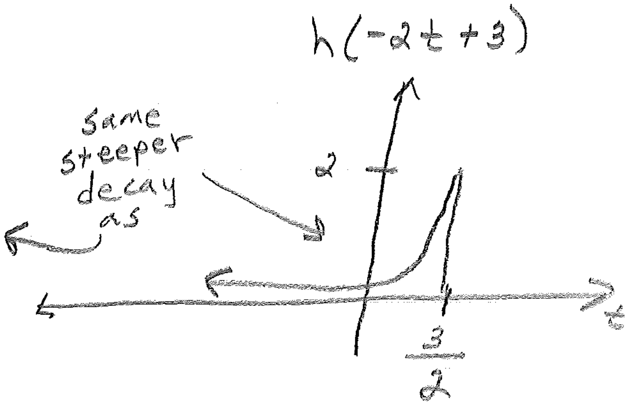
Let $$h(t) = \begin{cases} 2e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Draw $h(t)$, $h(-t)$, $h(-2t)$, $h(-2t+3)$



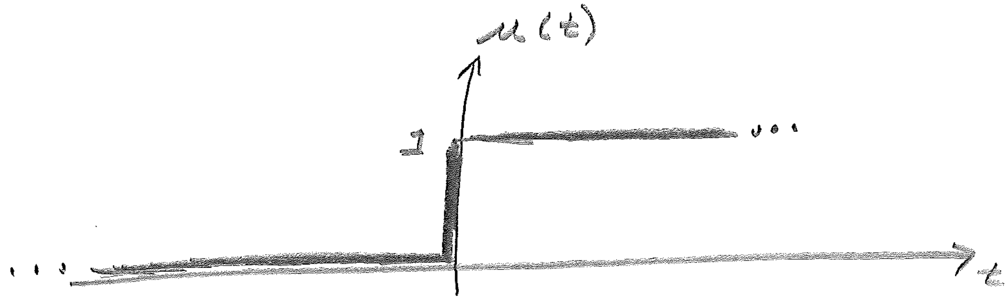




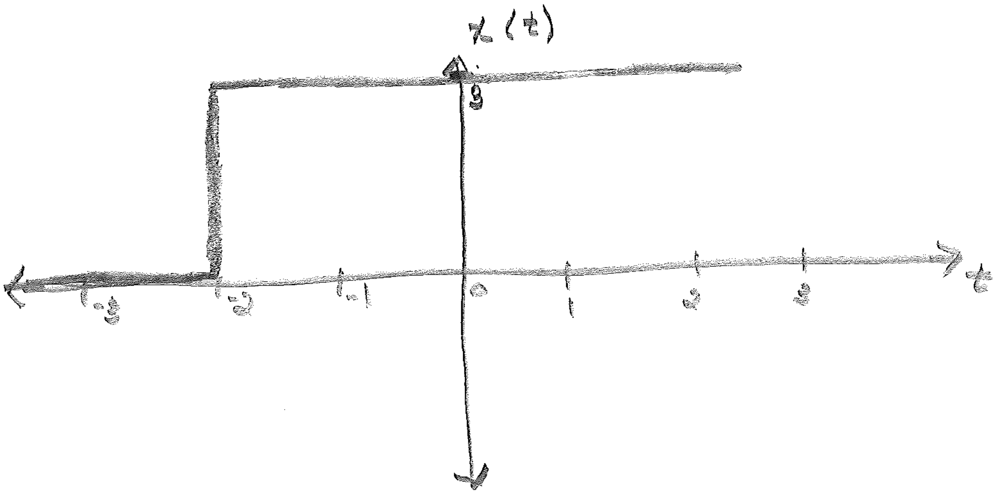


Basic Signals: Unit Step

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

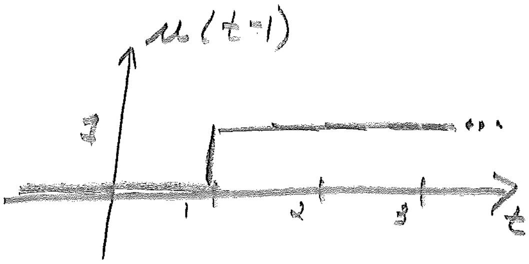


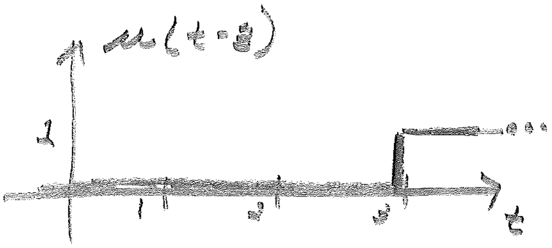
Draw: $x(t) = 3\,u(t+2)$

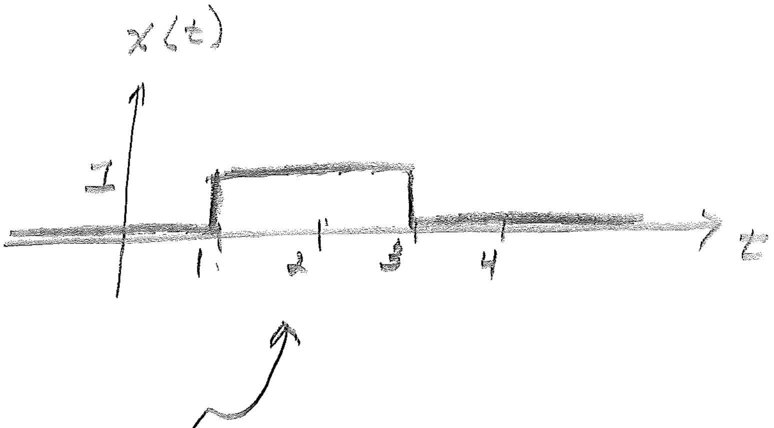


Basic Signals: Step Function Example

Draw: $x(t) = u(t-1) - u(t-3)$



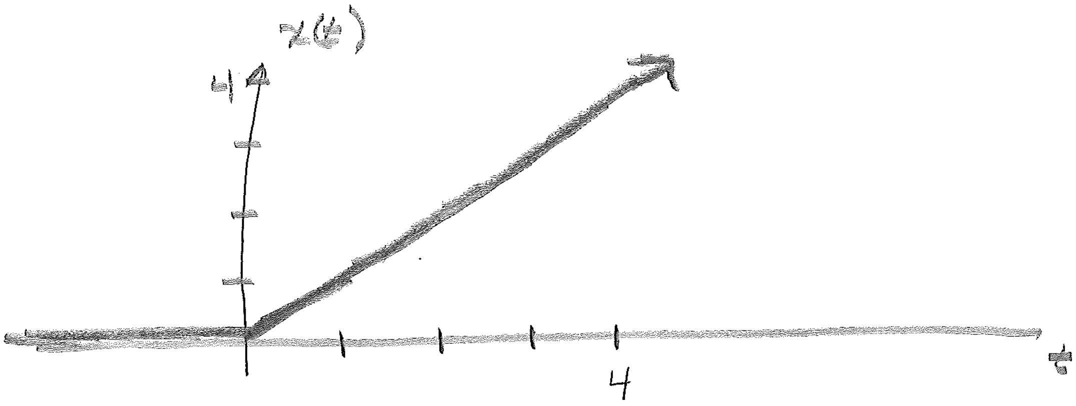




Rectangular Pulse

Basic Signals: Unit Ramp

$$x(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



$\boxed{\text{Note: As } t \to \infty, \quad x(t) \to \infty}$

Often written as $r(t)$

or $r(t)\,u(t)$

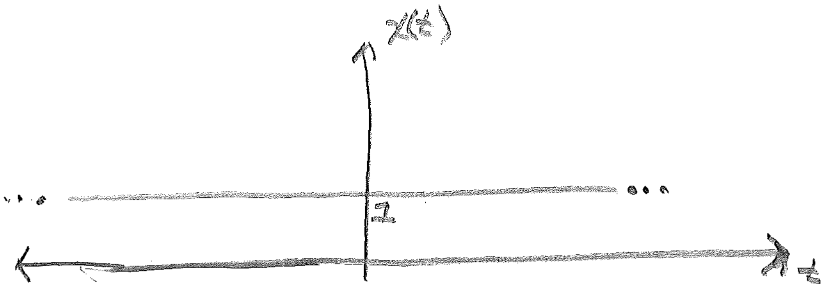
Basic Signals: Exponential Function

$$x(t) = e^{st}, \qquad -\infty \le t \le \infty$$

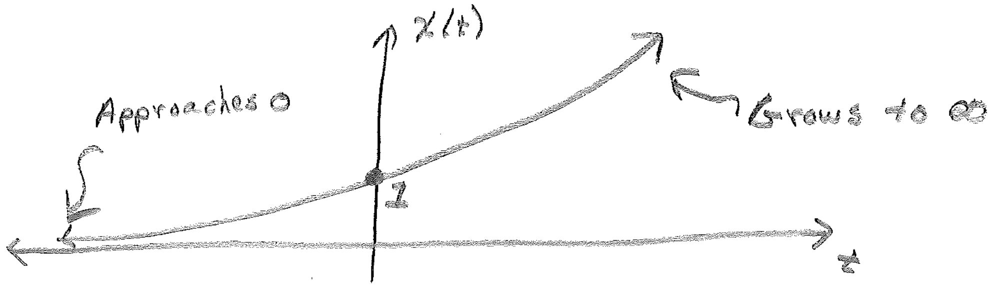
- Parameter s real or complex ($s = \sigma + j\omega$)

- For real $s \longrightarrow \omega = 0$

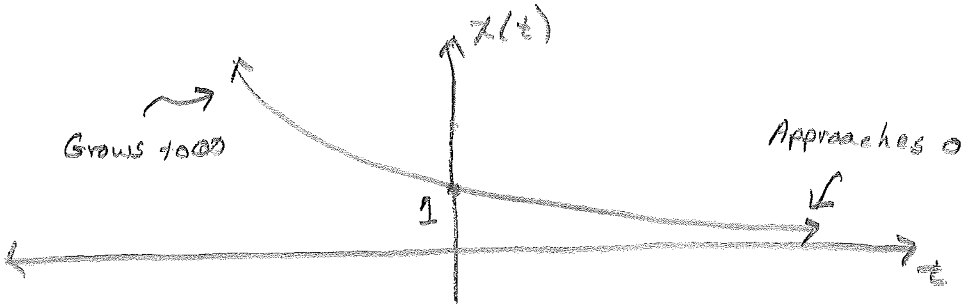
  - Case: $\sigma = 0 \rightarrow x(t) = e^{0} = 1$



  - Case: $\sigma > 0 \longrightarrow x(t) = e^{\sigma t}$



  - Case: $\sigma < 0 \longrightarrow x(t) = e^{-|\sigma| t}$



Basic Signals: Complex Exponential Function (1)

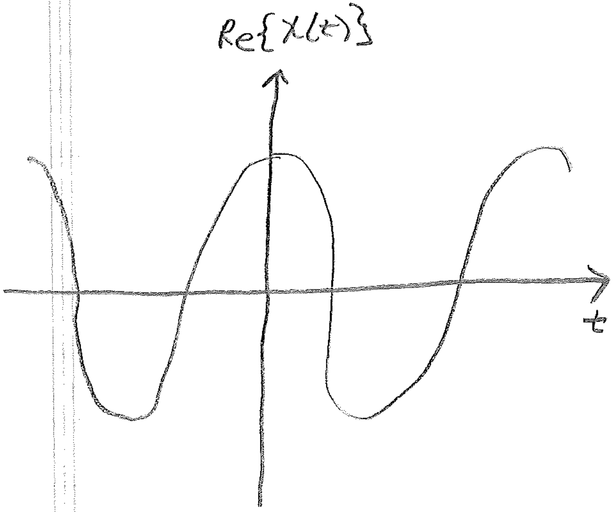
- For complex $s \rightarrow s = \sigma + j\omega$

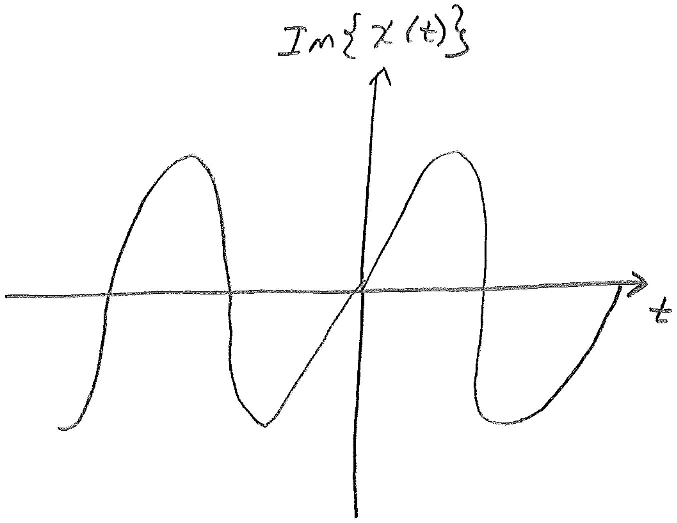
$$x(t) = e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t}\, e^{j\omega t} = e^{\sigma t}\underbrace{\left[\cos(\omega t) + j\sin(\omega t)\right]}$$

- Real exponential function
- Complex exponential function
- Oscillates at frequency ω

- Variable "s" ≡ "Complex frequency"

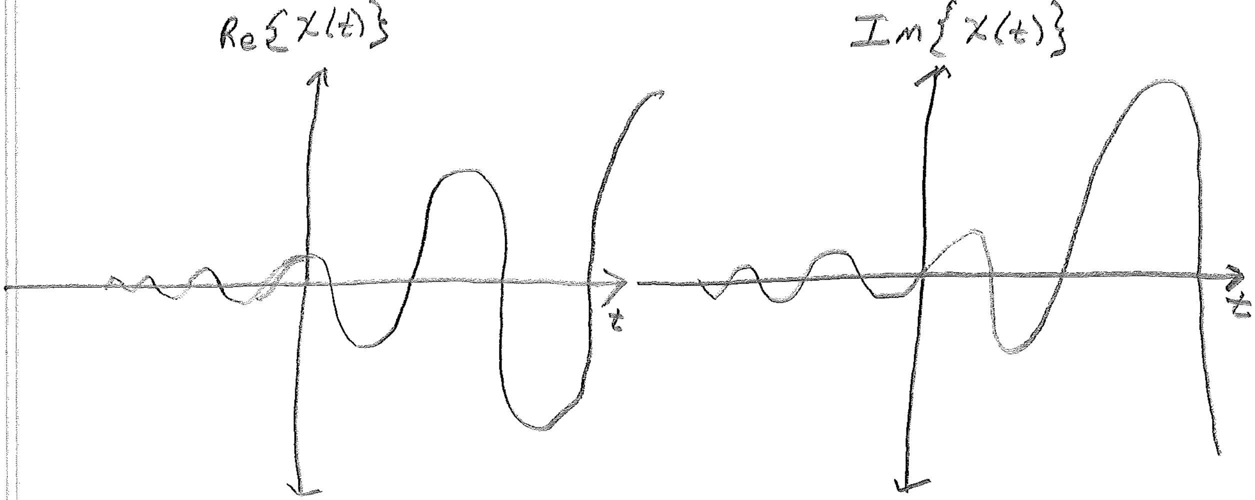
- Case $\sigma = 0 \rightarrow x(t) = e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$



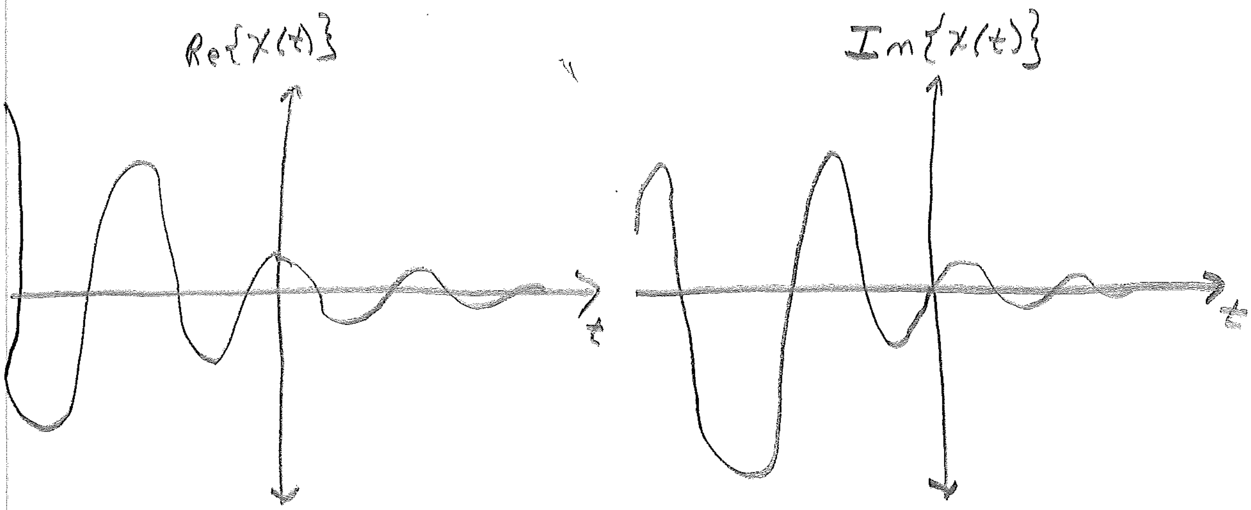


Basic Signals: Complex Exponential Function (2)

- Case $\sigma > 0$ → Sinusoids grow with increasing time



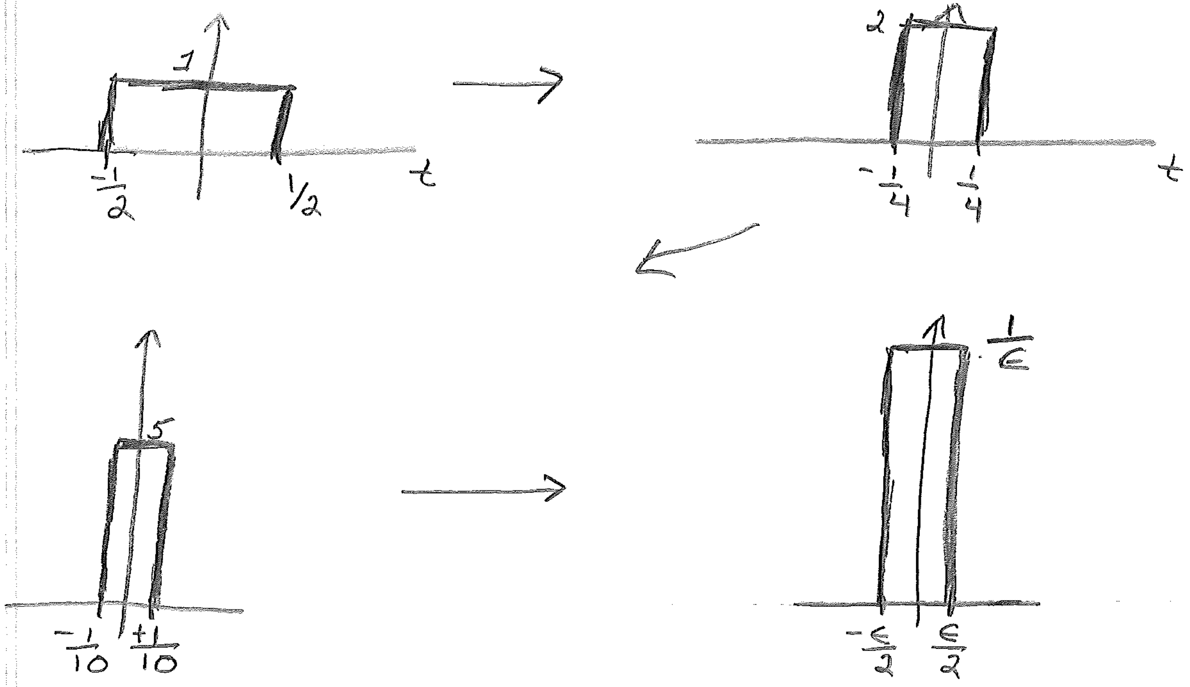
- Case $\sigma < 0$ → Sinusoids decay with increasing time



Concept of Impulse Function

- Impulse function defined by its effect on other functions, not its value at each time instant.

- Consider:
  - Rectangle of area = 1 unit^2
  - Centered at t = zero
  - Progressively narrowed



- Eventually:
  - Width → 0
  - Height → ∞
  - ALWAYS: Define Area = 1

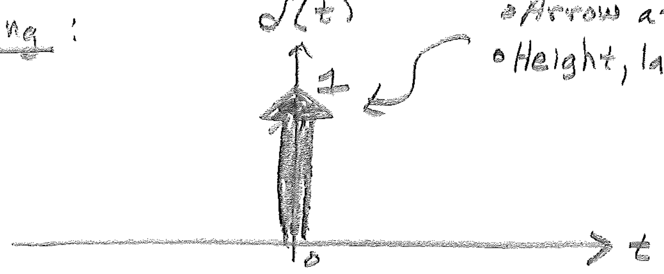
"Unit Impulse"

Definition of Unit Impulse

$$\begin{cases} \delta(t) = 0, & t \neq 0 \\ \int_{t=-\infty}^{\infty} \delta(t)\, dt = 1 \end{cases}$$

Symbol: Lower case delta

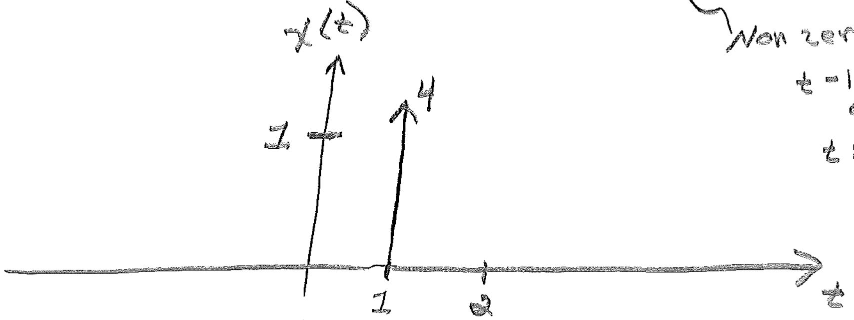
Drawing:



- Arrow at impulse location
- Height, label indicate area

E.g. Draw $x(t) = 4\delta(t-1)$

Non zero at: $t-1=0$ or $t=1$



- Relative heights of impulses → relative areas
- Drawing height often match y axis
  - But, units not same

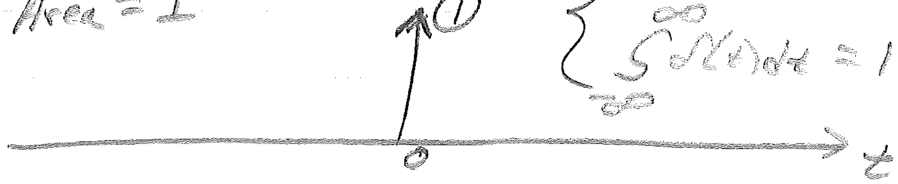
- Impulse fundamental to linear systems

Derivatives of Impulses

- $\delta(t)$:

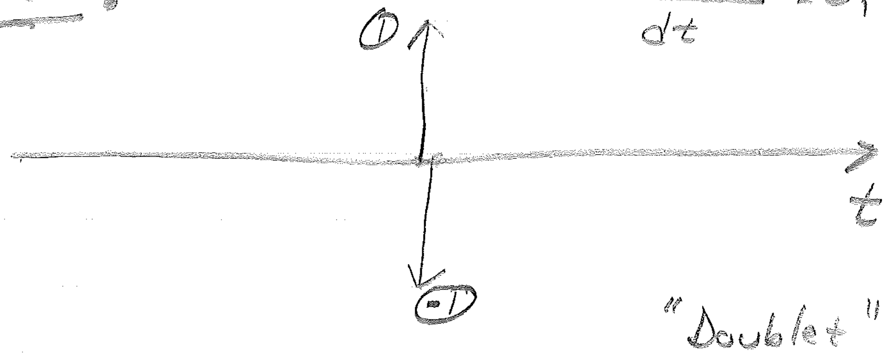
Area = 1

$$\begin{cases}\delta(t)=0, & t\neq 0\\ \int_{-\infty}^{\infty}\delta(t)\,dt=1\end{cases}$$



- $\frac{d\,\delta(t)}{dt}$:

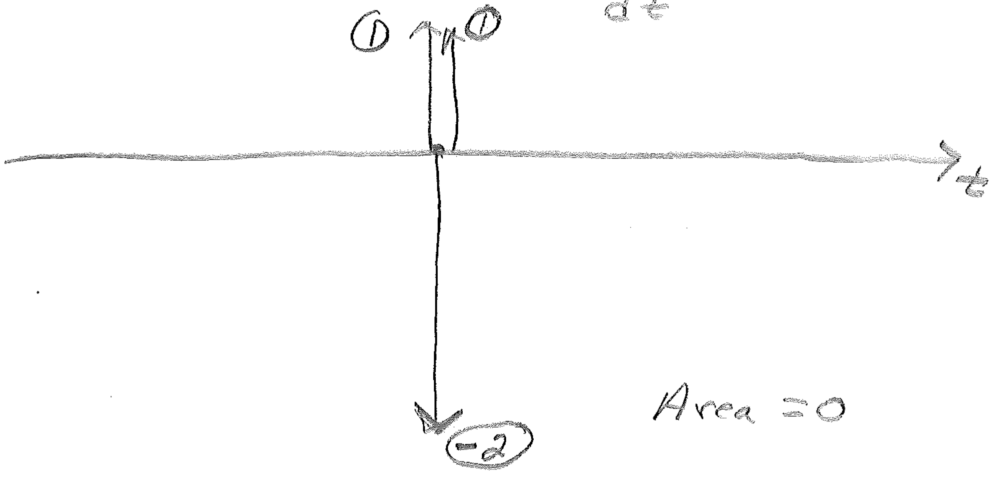
$$\frac{d\,\delta(t)}{dt}=0,\ t\neq 0$$



"Doublet"
Area = 0

- $\frac{d^2\delta(t)}{dt^2}$:

$$\frac{d^2\delta(t)}{dt^2}=0,\ t\neq 0$$

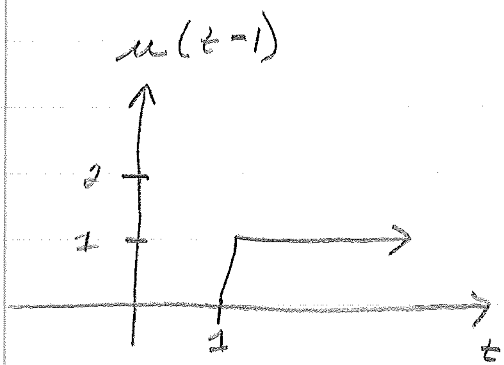


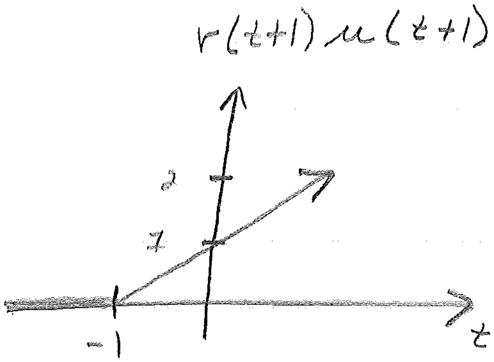
Area = 0

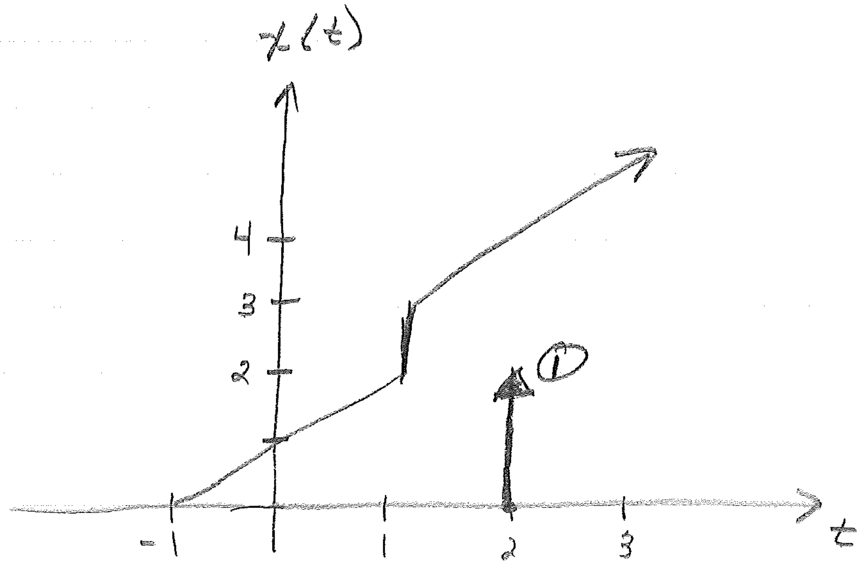
Example: Signal Operations

Q: Draw $x(t) = u(t-1) + r(t+1)u(t+1) + \delta(t-2)$.

Sol'n:

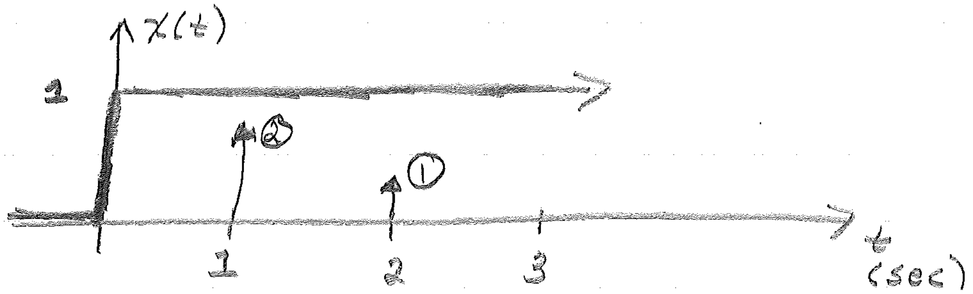






Example: Integration with Impulses

Q Let $x(t) = \mu(t) + 2\delta(t-1) + \delta(t-2)$



Find the area of $x(t)$ between $t=0$ and $t=3$.

Sol'n:

$$A = \int_0^3 x(t)\,dt = \int_0^3 1\,dt + \int_{1-\epsilon}^{1+\epsilon} 2\delta(t-1)\,dt + \int_{2-\epsilon}^{2+\epsilon} \delta(t-2)\,dt$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$t\Big|_{t=0}^{3} \quad + \quad 2\cdot 1 \quad + \quad 1$$

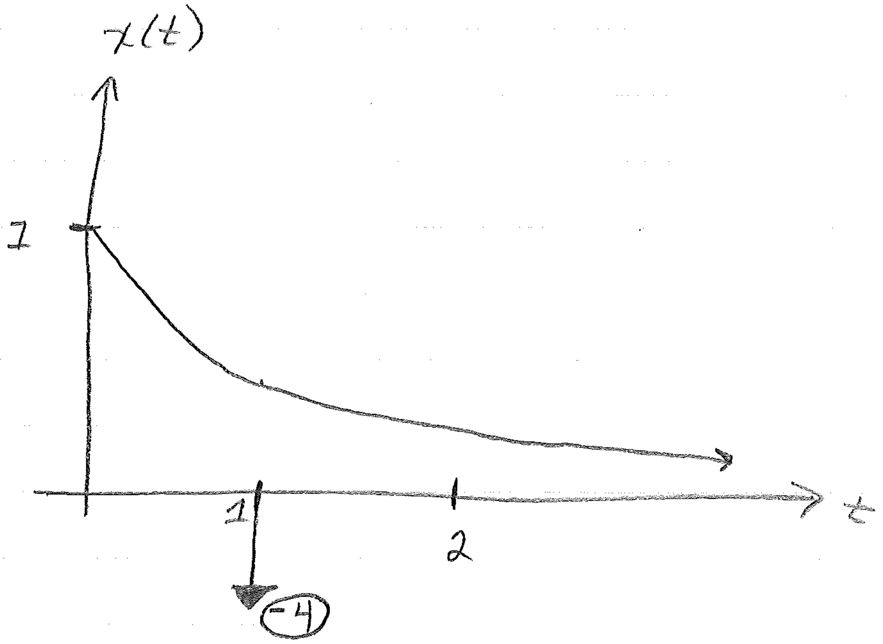
$$= (3-0) + 2 + 1 = \boxed{6 = A}$$

Units: Time X Units of $x(t)$

Example: Signal Operations

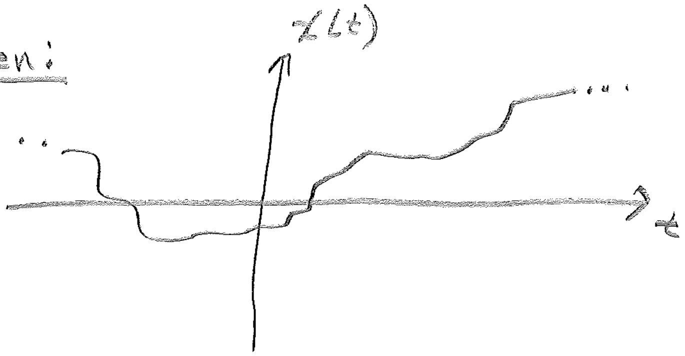
Q: Draw $x(t) = e^{-t} u(t) - 4\delta(t-1)$

Sol'n



Impulse Multiplied by a Function

Given:



Find: $y_1(t) = x(t)\delta(t)$

a) For $t \neq 0$, $\delta(t) = 0 \rightarrow y_1(t) = 0$
b) For $t = 0$, $x(0)$ scales impulse
$\rightarrow y_1(0) = x(0) \cdot \delta(0)$

$$\Longrightarrow y_1(t) = x(0) \cdot \delta(t)$$

Find: $y_2(t) = x(t) \cdot \delta(t+3)$

a) For $t \neq -3$, $\delta(t+3) = 0 \rightarrow y_2(t) = 0$
b) For $t = -3$, $x(-3)$ scales impulse
$\longrightarrow y_2(-3) = x(-3)\,\delta(t+3)$

$$\Longrightarrow y_2(t) = x(-3) \cdot \delta(t+3)$$

$$\boxed{\text{General: } x(t) \cdot \delta(t-T) = x(T) \cdot \delta(t-T)}$$

Sampling Property of Unit Impulse

For function $x(t)$, find

$$\int_{t=-\infty}^{\infty} x(t)\cdot\delta(t)\, dt$$

- Know: $\begin{cases} \delta(t)=0, & t\neq 0 \\ x(t)\cdot\delta(t) = x(0)\cdot\delta(t) & \end{cases}$

$$\Rightarrow = \int_{-\infty}^{\infty} \underset{\uparrow \text{Constant}}{x(0)}\cdot\delta(t)\, dt \quad = x(0)\underset{\uparrow \text{See impulse definition}}{\int_{-\infty}^{\infty}\delta(t)\, dt}$$

$$= x(0)\cdot 1 = \underline{\underline{x(0)}}$$

- Only requires $x(t)$ function value at location of impulse

$$\boxed{\text{General: } \int_{t=-\infty}^{\infty} x(t)\cdot\delta(t-T)\,dt = x(T)}$$

Example: Impulse Function

Find:

$$\int_{-\infty}^{\infty} 3\delta(t-4)\sin\left(\frac{\pi t}{8}\right) dt$$

Sol'n

Here, $x(t) = 3\sin\left(\frac{\pi t}{8}\right)$, $T = 4$

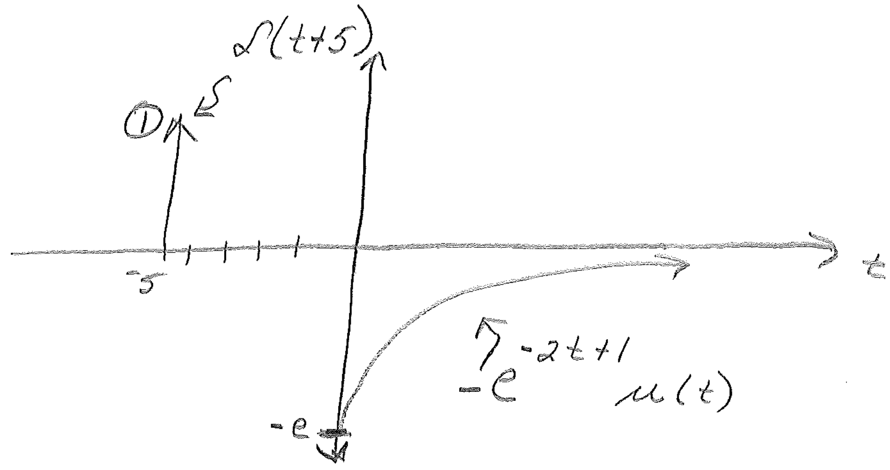
$$\Rightarrow = x(4) = 3\sin\left(\frac{\pi \cdot 4}{8}\right) = 3\sin\left(\frac{\pi}{2}\right) = 3\cdot 1 = \underline{\underline{3}}$$

Example: Impulse Function

Evaluate:

$$x(t) = -e^{-2t+1}\,\delta(t+5)\,u(t)$$

Sol'n:



So,

$$x(t) = \left\{ -e^{-2t+1} \cdot u(t) \Big|_{t=-5} \right\} \cdot \delta(t+5)$$

↖ $= 0$

⇓

$$\boxed{x(t) = 0}$$

Note:

If $y(t) = -e^{-2t+1}\,\delta(t-5)\,u(t)$

Then $y(t) = \left(-e^{-2\cdot 5+1}\right)\cdot\delta(t-5)\cdot u(5)$

or

$$\boxed{y(t) = -e^{-9}\cdot\delta(t-5)}$$

Signal Classification: Periodic vs. Aperiodic

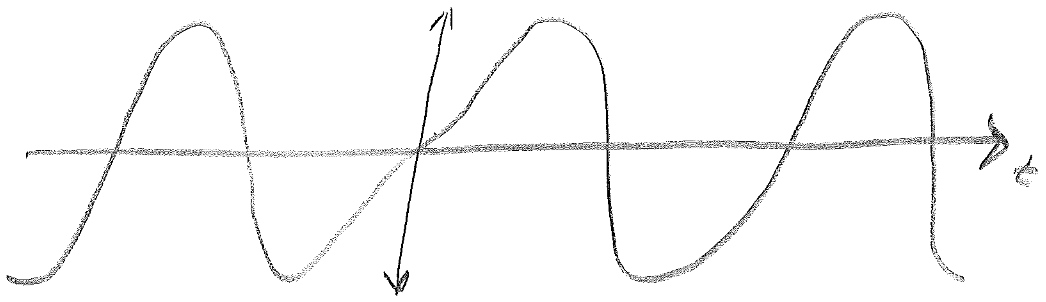
$x(t)$ periodic if, for some T

$$x(t) = x(t+T), \quad \forall t$$

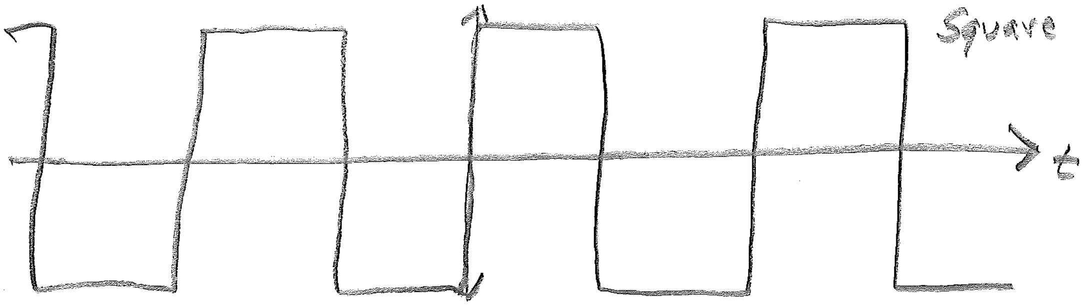
- Period ≡ smallest value of T that satisfies definition
- If not periodic → Aperiodic

Periodic Signal Examples

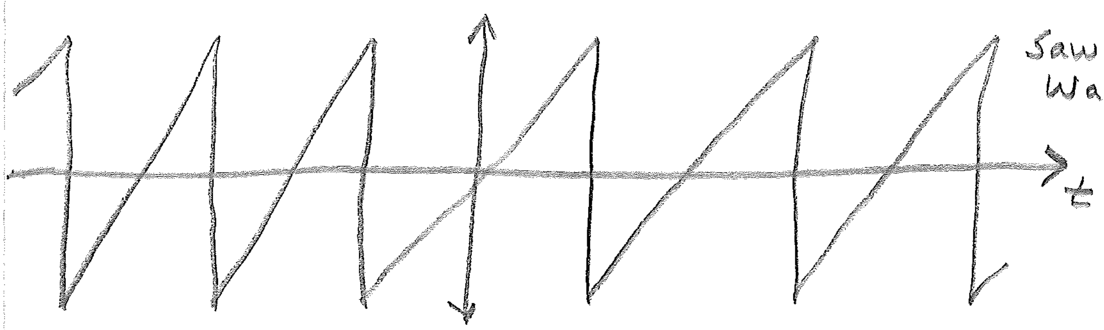
Sine Wave



Square Wave



Sawtooth Wave



Properties of Periodic Signals

- Signal unchanged if time-shift by multiple of one period
- Signal must extend from $t=-\infty$ to $t=+\infty$
- Can form periodic signal from infinite replications of finite-duration signal
- Area under $x(t)$ over any interval of duration T is same

Finite-duration signal $\equiv$ Non-zero over finite range of t.

vs.

Infinite-duration (everlasting) signal

Deterministic vs. Random (Stochastic) Signals

- Deterministic signals: Value known (determined) for all times

- Random signals: Function only defines signal probabilities ← a.k.a. "stochastic"
  - E.g.: Noise

- We will study deterministic signals

  - Stochastic signals studied as extension to deterministic signals

Energy & Power Signals

- Energy signal: Signal with finite energy
  - All real-life signals.

- Power signal: Signal with finite (non-zero) power
  - Recall: Power is time average of energy

Causal & Non-Causal Signals

- Causal signal: $x(t)=0, \quad t<0$

- Non-causal signal: $x(t)\neq 0$ for some $t<0$

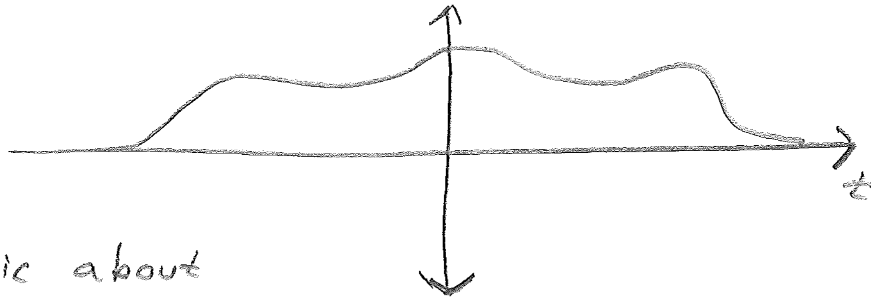
- Anti-causal signal: $x(t)=0, \quad t\geq 0$

Even, Odd Functions

- Real function $x_e(t)$ is even function of t

if $$x_e(t) = x_e(-t)$$

E.g.

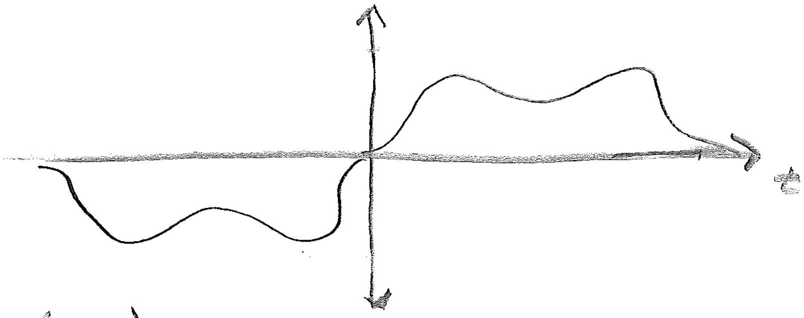


- Symmetric about y axis
- E.g.: $\cos(\omega t)$

- Real function $x_o(t)$ is odd function of t

if $$x_o(t) = -x_o(-t)$$

E.g.



- E.g.: $\sin(\omega t)$

Note: At $t=0$:

$$x_o(0) = -x_o(0)$$

$$\Rightarrow \underline{\underline{x_o(0) \equiv 0}}$$

°A.K.A. even/odd symmetry

Signals

Even, Odd Signal Components

- Every signal can be written as sum of even, odd components:

$$x(t) = x_e(t) + x_o(t) \quad , \quad \text{where}$$

$$x_e(t) = \frac{x(t) + x(-t)}{2}, \qquad x_o(t) = \frac{x(t) - x(-t)}{2}$$

Prove: $x_e(t)$ above is even.

Sol'n:

$$x_e(-t) = \frac{x(-t) + x(--t)}{2}$$

$$= \frac{x(-t) + x(t)}{2}$$

$$= \frac{x(t) + x(-t)}{2}$$

$$\overset{\checkmark}{=} x_e(t)$$

Even, Odd Component Example

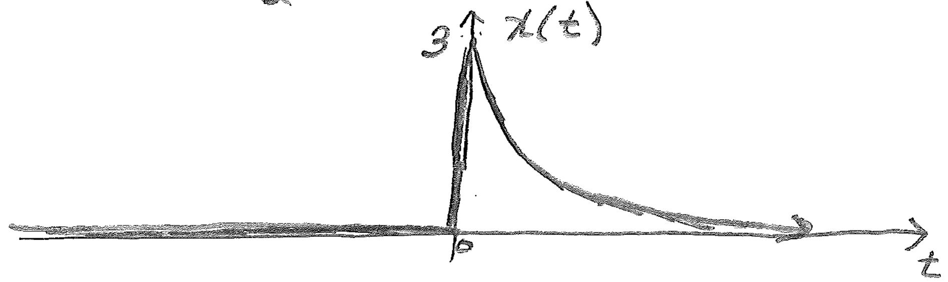
- **Find & Draw** even & odd components of

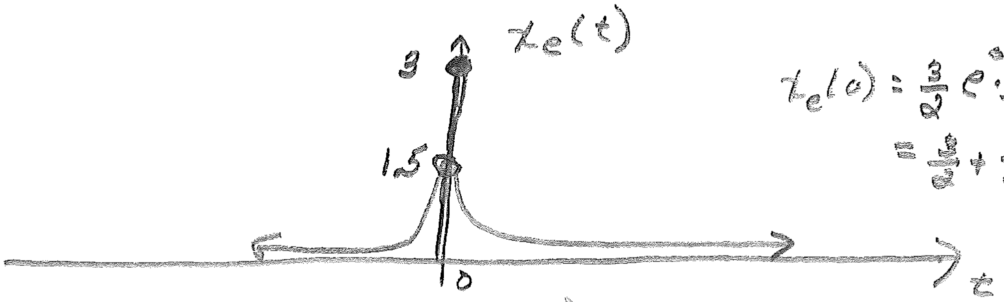
$$x(t) = 3\,e^{-2t}\,u(t)$$

Sol'n

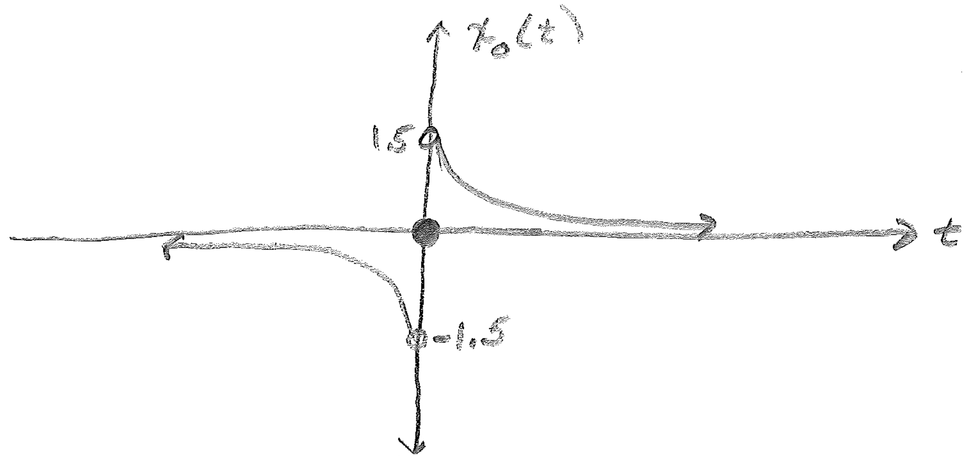
$$x_e(t) = \underset{\uparrow \text{Causal signal}}{\frac{3}{2}}\,e^{-2t}\,u(t) + \underset{\uparrow \text{Noncausal signal}}{\frac{3}{2}}\,e^{2t}\,u(-t)$$

$$x_o(t) = \frac{3}{2}\,e^{-2t}\,u(t) - \frac{3}{2}\,e^{2t}\,u(-t)$$





$$x_e(0) = \frac{3}{2}e^{0}\cdot 1 + \frac{3}{2}e^{0}\cdot 1 = \frac{3}{2} + \frac{3}{2} = \underline{\underline{3}}$$



Example: Even & Odd Components

- Find & Draw even & odd components of

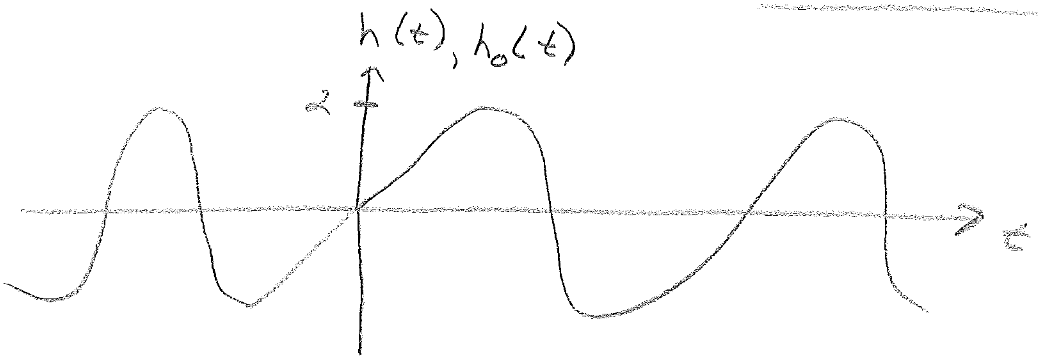
$$h(t) = 2 \sin(3t)$$

Sol'n

$$h_e(t) = \frac{2\sin(3t) + 2\sin(-3t)}{2} = \sin(3t) - \sin(3t) = 0$$

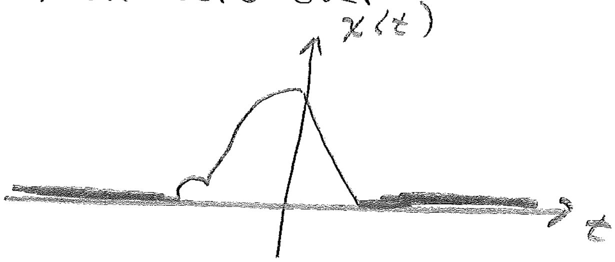
$$h_o(t) = \frac{2\sin(3t) - 2\sin(-3t)}{2} = 2\sin(3t)$$

$\Longrightarrow$ sin(ωt) is an odd function

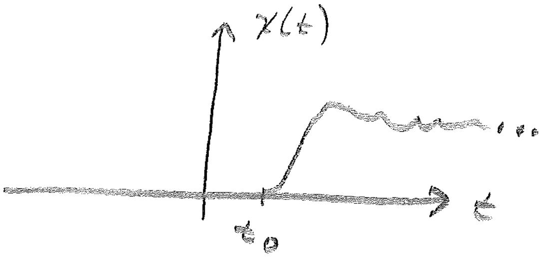


Signal Duration

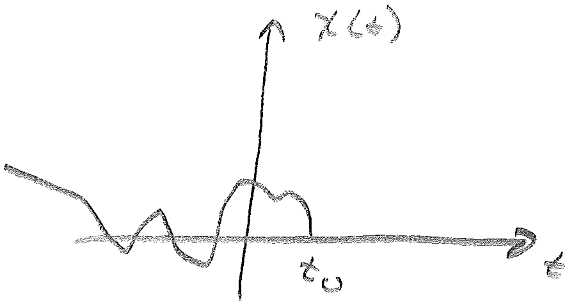
- Finite duration signal: Non-zero over a finite range



- Right-sided signal: Non-zero for $t > t_0$



- Left-sided signal: Non-zero for $t < t_0$



- Two-sided sequence: Non-zero over all time range