Bayesian Restricted Likelihood Methods: Conditioning on Insufficient Statistics in Bayesian Regression

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Introduction

- Formal Bayesian inference relies on the prior, likelihood, loss
- Each require assumptions that should be questioned
- We focus this paper on imperfections in the Likeihood:
    - Start with a full model as if it is correct
    - Sumarise the data with a summary statistic
    - The prior is updated with the summary statistic rather than the complete data

Resticted Likelihood Examples

- Outliers
  - Known subset of bad data are removed prior to analysis.

$$L(\boldsymbol{\theta}|\boldsymbol{y}) = \left(\prod_{i=1}^{n-k} f(y_i|\boldsymbol{\theta})\right)\left(\prod_{i=n-k+1}^{n} f(y_i|\boldsymbol{\theta})\right)$$

Resticted Likelihood Examples

- Censoring
  - Reaction Time Experiments (e.g., Ratcliff, 1993)
  - Some reactions are too fast or too slow to be believable
- $c(y_i) = t_1$ if $y_i \leq t_1$, $c(y_i) = t_2$ if $y_i \geq t_2$, and $c(y_i) = y_i$ otherwise.

$$L(\boldsymbol{\theta}|\boldsymbol{y}) = \left(\prod_{i=1}^{n} g(c(y_i)|\boldsymbol{\theta})\right)\left(\prod_{i=1}^{n} f(y_i|\boldsymbol{\theta}, c(y_i)).\right)$$

Resticted Likelihood Generalization

- Conditioning statistic $T(\boldsymbol{y})$
  - Example 1: $T(\boldsymbol{y}) = (y_1, \ldots, y_{n-k})$
  - Example 2: $T(\boldsymbol{y}) = (c(y_1), \ldots, c(y_n))$

$$L(\boldsymbol{\theta}|\boldsymbol{y}) \quad = \quad f(T(\boldsymbol{y})|\boldsymbol{\theta}) \; f(\boldsymbol{y}|\boldsymbol{\theta}, T(\boldsymbol{y}))$$

- Restricted Likelihood Posterior

$$\pi(\boldsymbol{\theta}|T(\boldsymbol{y})) \quad = \quad \frac{\pi(\boldsymbol{\theta})f(T(\boldsymbol{y})|\boldsymbol{\theta})}{m(T(\boldsymbol{y}))}$$

- Predictive Density

$$f(y_{n+1}|T(\boldsymbol{y})) = \int f(y_{n+1}|\boldsymbol{\theta})\pi(\boldsymbol{\theta}|T(\boldsymbol{y})) \; d\boldsymbol{\theta}$$

Literature Review

- Rank Likelihoods: Savage (1969), Pettitt (1983, 1982), Hoff et al. (2013).
- Order Statistics: Lewis et al. (2012)
- Asymptotics: Doksum and Lo (1990), Clarke and Ghosh (1995), Yuan and Clarke (2004), and Hwang et al. (2005)
  - Often, posterior distribution resembles the asymptotic sampling distribution of the conditioning statistic
- Approximate sufficiency of mean/sd: Pratt (1965)

Literature Review: Approximate Bayesian Computation

- Posterior approximation with success in many applications: (Tavaré et al., 1997; Pritchard et al., 1999; Beaumont et al., 2002; Marjoram et al., 2003; Fearnhead and Prangle, 2012; Drovandi et al., 2015).
- $L(\boldsymbol{\theta}|\mathcal{B}(\boldsymbol{y}))$, where $\mathcal{B}(\boldsymbol{y}) = \{\boldsymbol{y}^*|\rho(T(\boldsymbol{y}), T(\boldsymbol{y}^*)) \leq \epsilon\}$
  - metric ρ, tolerance level ϵ
- Goal is often to approximate the full posterior
- Choose an approximately sufficient (Joyce and Marjoram, 2008) $T(\boldsymbol{y})$ and small ϵ
- Sampling Methods: Standard reject (Pritchard et al., 1999), Extensions: Beaumont et al. (2009); Turner and Van Zandt (2012, 2014)

Application to the Linear Model

$$
\begin{aligned}
\boldsymbol{\theta} &= (\boldsymbol{\beta}, \sigma^2) \sim \pi(\boldsymbol{\theta}) \\
y_i &= x_i^\top \boldsymbol{\beta} + \epsilon_i, \text{ for } i = 1, \ldots, n
\end{aligned}
$$

- $T(\boldsymbol{y}) = (\boldsymbol{b}(X, \boldsymbol{y}), s(X, \boldsymbol{y}))$
  - $\boldsymbol{b}(X, \boldsymbol{y}) = (b_1(X, \boldsymbol{y}), \ldots, b_p(X, \boldsymbol{y}))^\top \in \mathbb{R}^p$
  - $s(X, \boldsymbol{y}) \in \{0\} \cup \mathbb{R}^+$
- E.g. M-estimators Huber (1964), Least Median Squares, Least Trimmed Squares

Computational Strategy

- Direct Sampling (Small Dimensions): Relies on KDE of $L(\theta|T(y))$ Lewis (2014)
- MCMC: Data augmented Gibbs Sampler targeting $f(\boldsymbol{\theta}, \boldsymbol{y}|T(\boldsymbol{y}) = T(\boldsymbol{y}_{obs}))$
  1. $\pi(\boldsymbol{\theta}|\boldsymbol{y}, T(\boldsymbol{y}) = T(\boldsymbol{y}_{obs})) = \pi(\boldsymbol{\theta}|\boldsymbol{y})$ (full poserior)
  2. $f(\boldsymbol{y}|\boldsymbol{\theta}, T(\boldsymbol{y}) = T(\boldsymbol{y}_{obs}))$
- For 2, prosose a Metropolis-Hastings sampler.
- accept/reject a sample full data $\boldsymbol{y} \in \mathcal{A} := \{\boldsymbol{y} \in \mathbb{R}^n | T(\boldsymbol{y}) = T(\boldsymbol{y}_{obs})\}$ from a well defined distribution with support $\mathcal{A}$.

Computational Strategy

- Difficult to sample from $\mathcal{A}$ directly
- Can sample $\boldsymbol{z}^* \in \mathbb{R}^n$ and transform:

$$\boldsymbol{y} = h(\boldsymbol{z}^*) := \frac{s(X, \boldsymbol{y}_{obs})}{s(X, \boldsymbol{z}^*)}\boldsymbol{z}^* + X\left(\boldsymbol{b}(X, \boldsymbol{y}_{obs}) - \boldsymbol{b}(X, \frac{s(X, \boldsymbol{y}_{obs})}{s(X, \boldsymbol{z}^*)}\boldsymbol{z}^*)\right)$$

- Under certain conditions on regression/scale estimators (C3-C8 in the paper), $T(y) = T(y_{obs})$
- Idea:
  - sample $\boldsymbol{z}^* \sim p(\boldsymbol{z}^*)$, transform via h
  - proposal $p(\boldsymbol{y}|\boldsymbol{\theta})$ is then a change-of-variables adjustment on $p(\boldsymbol{z}^*)$.

Computational Strategy

- h is not 1-1/onto. (Change of variables difficult)
- Can restrict sample space of z^*, so that it is
  - $\mathcal{A}$ is an $n - p - 1$ space
  - Sample space for z^*: $\mathbb{S} := \{\boldsymbol{z}^* \in \mathcal{C}^\perp(X) \mid ||\boldsymbol{z}^*|| = 1\}$
    - i.e. the unit space in the orthogonal complement of the column space of the design matrix.
  - $h : \mathbb{S} \to \mathcal{A}$ is then 1-1/onto.
  - easier to figure out the Jacobian of the transformation from $p(\boldsymbol{z}^*)$ to $p(\boldsymbol{y}|\boldsymbol{\theta})$

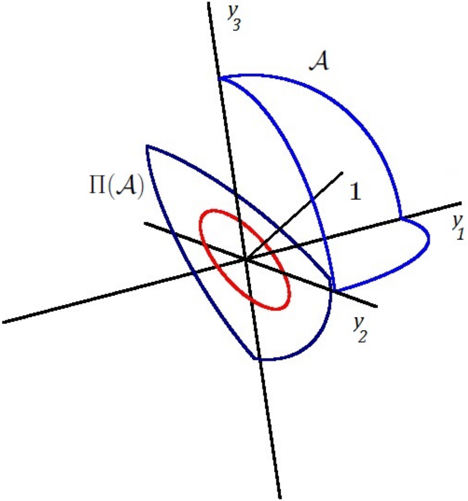
Computational Strategy

For the proposal distribrution.

1. Sample $\boldsymbol{z}^*$ from a distribution with known density whose support is the entirety of $\mathbb{S}$.
2. Set $\boldsymbol{y} = h(\boldsymbol{z}^*)$
3. Compute the Jacobian (think about it in 2 steps)
   - $\boldsymbol{z} = \frac{s(X,\boldsymbol{y}_{obs})}{s(X,\boldsymbol{z}^*)}\boldsymbol{z}^*$. Scale from $\mathbb{S}$ to the set $\Pi(\mathcal{A}) := \{\boldsymbol{z} \in \mathbb{R}^n | \ \exists \ \boldsymbol{y} \in \mathcal{A} \ s.t. \ \boldsymbol{z} = Q\boldsymbol{y}\}$ with $Q = I - XX^\top$.
   - $\boldsymbol{y} = \boldsymbol{z} + X\left(\boldsymbol{b}(X, \boldsymbol{y}_{obs}) - \boldsymbol{b}(X, \boldsymbol{z})\right)$. Shift from $\Pi(\mathcal{A})$ to $\mathcal{A}$ along $\mathcal{C}(X)$

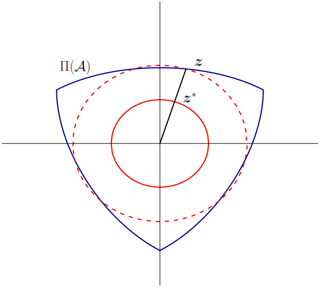
Computational Strategy (Visualization, $n = 3, p = 1$)

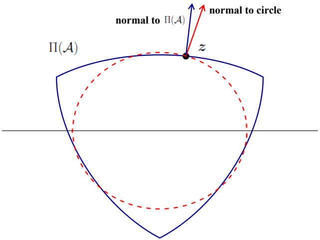
$$T(\boldsymbol{y}) = (\min(\boldsymbol{y}), \sum(y_i - \min(\boldsymbol{y}))^2), \; T(\boldsymbol{y}_{obs}) = (0, 1)$$



Computational Strategy (Visualization, $n = 3, p = 1$)

Scaling step





- resize sphere: $r^{-(n-p-1)}$
- deformation onto $\Pi(\mathcal{A})$: $\cos(\gamma) = \dfrac{\nabla s(X,\boldsymbol{y})^\top \boldsymbol{z}}{\|\nabla s(X,\boldsymbol{y})\| \|\boldsymbol{z}\|}$

Computational Strategy (Visualization, $n = 3, p = 1$)

Shifting step of $\boldsymbol{z}$ to $\boldsymbol{y}$ along the column space of X

- Contribution is the ratio of the infinitesimal volumes along $\Pi(\mathcal{A})$ at $\boldsymbol{z}$ to the corresponding volume along $\mathcal{A}$ at $\boldsymbol{y}$.
- $\mathrm{Vol}(P) := \sqrt{\det(P^\top P)} = \prod_{i=1}^{r} \sigma_i$
  - $P = QA$,
  - columns of A form an orthonormal basis for the tangent space to $\mathcal{A}$ at $\boldsymbol{y}$. Can be found from $\nabla s(X, \boldsymbol{y}), \nabla b_1(X, \boldsymbol{y}), \ldots, \nabla b_p(X, \boldsymbol{y})$
  - σ_i are the singular values of P

Full Jacobian: $p(\boldsymbol{y}) = p(\boldsymbol{z}^*) r^{-(n-p-1)} \cos(\gamma) \mathrm{Vol}(P)$

Simulation Example 1

- hierarchical setting with outliers.

$$\theta_i \sim N(0,1),\ i = 1, 2, \ldots, 90$$
$$y_{ij} \sim (1 - p_i)N(\theta_i, 4) + p_i N(\theta_i, 4m_i),\ j = 1, 2, ..., n_i$$

- $p_i = .1, .2, .3,\ m_i = 9, 25,$ and $n_i = 25, 50, 100$
- 5 groups for each combination, 90 groups total
- Base model for fitting:

$$\theta_i \sim N(\mu, \tau^2),\ \sigma_i^2 \sim IG(a_s, b_s),\ i = 1, 2, \ldots, 90,$$
$$y_{ij} \sim N(\theta_i, \sigma_i^2),\ j = 1, 2, \ldots, n_i.$$

- Restricted likihood fit: Robust M-estimators for each group: $T_i(y_{i1}, \ldots, y_{in_i}) = (\hat{\theta}_i, \hat{\sigma}_i^2), i = 1, 2, ..., 90.$

K = 30 simulations, M indexes the method, MSE

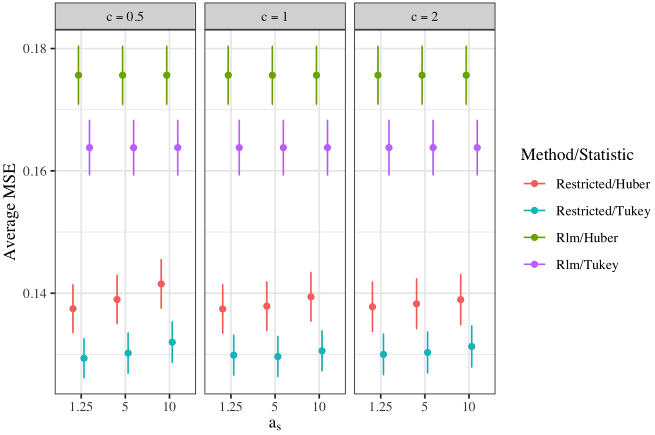


Figure: Average MSE plus/minus one standard error for each value of a_s and c. Smaller values represent better fits. The panels correspond to $c = 0.5$ (left), $c = 1$ (middle), and $c = 2$ (right), with the values of a_s on the horizontal axis. The average MSE for the normal theory model ranges from 0.24 to 0.25 and is left out of the figure.

Simulation Example 2

Data: several correlated covariates, only a few govern data

- $y = \boldsymbol{\beta}^\top x + \epsilon$
- $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)^\top$
- $\epsilon \sim N(0, \sigma^2)$ with probability 0.8 and
  $\epsilon \sim$ Half-Normal$(0, 25\sigma^2)$ with probability 0.2
- $x_1 \sim N(0, 1)$ and $x_j = x_1 + \eta_j$ with $\eta_j \sim N(0, 4)$ for $j = 2, 3$
- additional covariates: x^* 27 additional covariates
- 21 generated independently, 6 are x_1, x_2, and x_3 with random noise.

Model used for fitting:

- $y = \beta^\top x + \beta^{*\top} x^* + \epsilon$
- $\boldsymbol{\beta}_{all} \sim N_{20}(\mathbf{0}, \sigma_\beta^2 I)$ with $\boldsymbol{\beta}_{all} = (\boldsymbol{\beta}, \boldsymbol{\beta}^*)^\top$ and $\sigma^2 \sim IG(5, 8)$

$K = 30$ simulations, n $= 500$, MSE

$$MSE = (||\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}||^2 + ||\hat{\boldsymbol{\beta}}^*||^2)/30$$

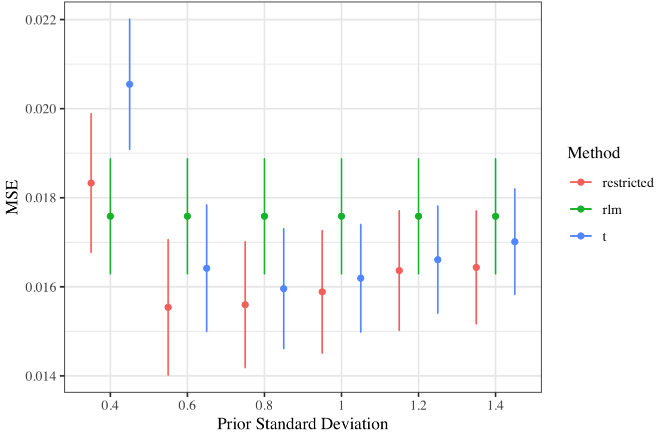


Figure: Average MSE plus/minus one standard error over the $K = 30$ simulations for each value of the prior standard deviation (σ_β) and each of the fitting methods.

Prediction of non-outlying data

- $MNLL = -\frac{1}{N}\sum \log f(y_i|\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\beta}}^*, \hat{\sigma})$, f is the assumed likelihood, average over non-outlying points

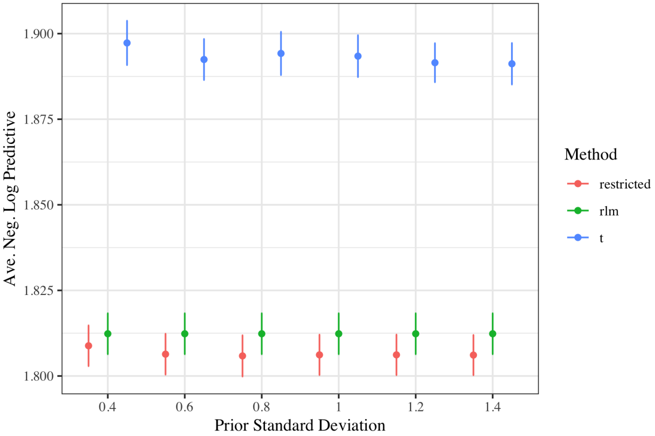


Figure: Average MNLL plus/minus one standard error for each value of the prior standard deviation (σ_β)

Real Data: Insurance Data

- Interested in future performance of agencies that have varying contractual agreements

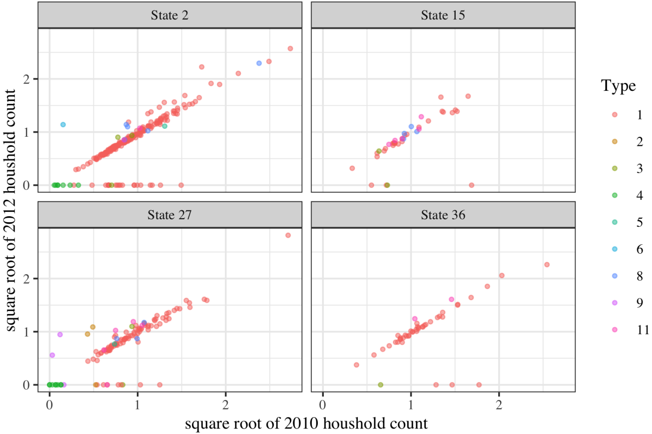


Figure: The square root of (scaled) count in 2012 versus that in 2010 for four states.

Real Data: State Level Regressions

Regression fit seperately within each state

$$\boldsymbol{\beta} \sim N(\boldsymbol{\mu}_0, \Sigma_0); \quad \sigma^2 \sim IG(a_0, b_0); \quad y_i = \boldsymbol{x}_i^\top \boldsymbol{\beta} + \epsilon_i, \quad \epsilon_i \overset{iid}{\sim} N(0, \sigma^2)$$

- Covariates: square root of household count in 2010, two different size/experience measures
- response: square root of household count in 2012
- hyper-parameters fixed via regression of data in time-period two years before.
- model misspecification: omitting contract type, closure information
- causes many cases to appear "outlying"

Method of Model Comparison

- both training and validation data will contain outliers
- timming approach with log predictive density (Jung et al., 2014)
- case i in holdout set: $log(f(y_i))$
- Scoring a procedure:
  - Choose a base method (e.g. Student-t model) and trimming fractions α
  - Order holdout sample by $log(f_b(y_i))$
  - Denote ordering by: $y^b_{(1)}, y^b_{(2)}, \ldots, y^b_{(M)}$
  - score each method with "mean trimmed log margainal psuedo likelihood"

$$TLM_b(A) = (M - [\alpha M])^{-1} \sum_{i=[\alpha M]+1}^{M} \log(f_A(y^b_{(i)})),$$

- f_A - predictive distribution under the method "A" being scored.

Predictive Performance: 50 training/holdouts sets

Evaluation of 'Type 1' agencies (of special interest to the company)

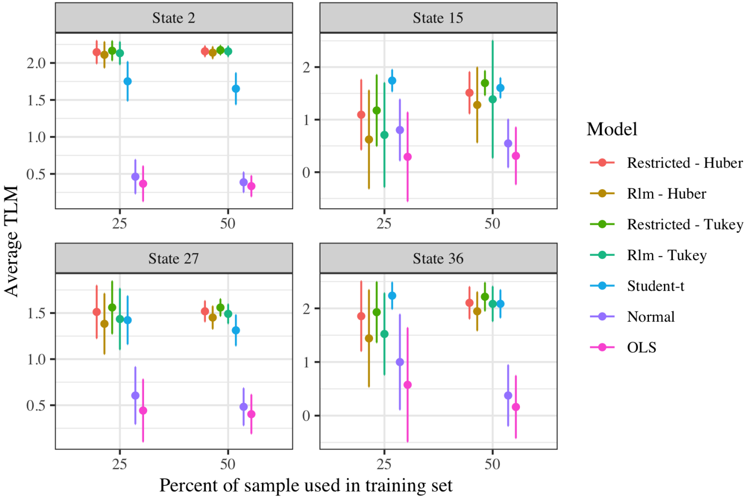
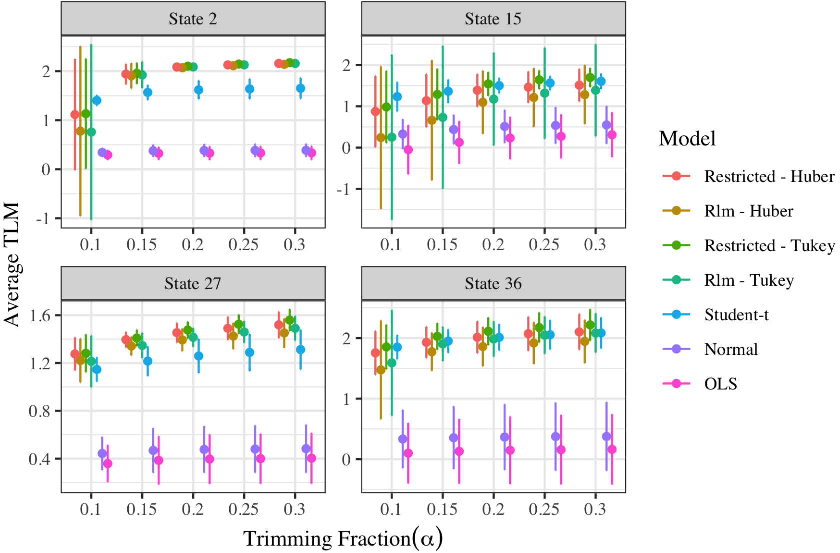


Figure: $\alpha = 0.3$. States 2, 15, 27, and 36, have $n = 222, 40, 117$, and 46

Predictive Performance: changing trimming fraction

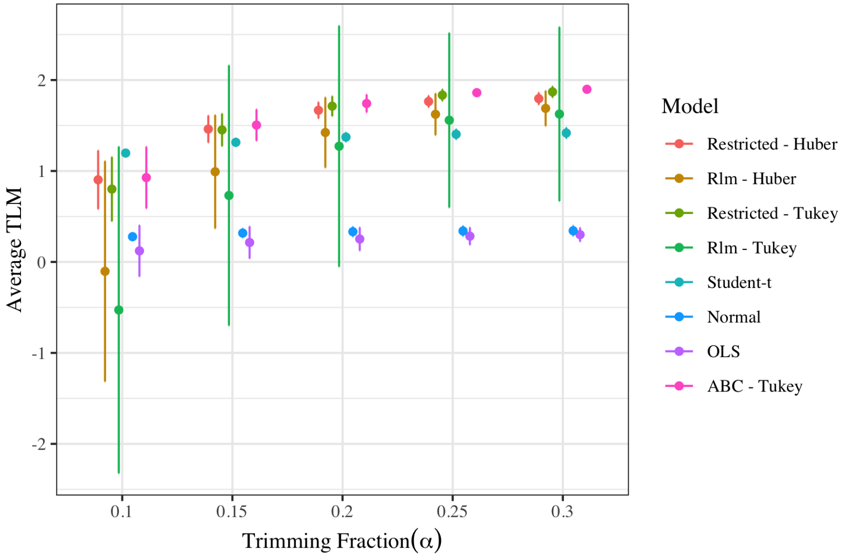


Real Data: Hierarchical Regression

$$\boldsymbol{\beta} \sim N_p(\boldsymbol{\mu}_0, a\Sigma_0); \quad \boldsymbol{\beta}_j \overset{iid}{\sim} N_p(\boldsymbol{\beta}, b\Sigma_0); \quad \sigma_j^2 \sim IG(a_0, b_0);$$

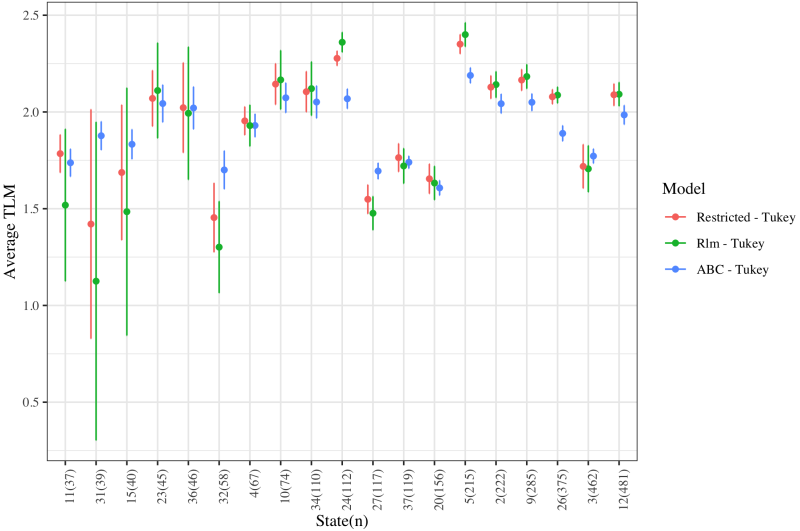
$$y_{ij} = \boldsymbol{x}_{ij}^\top \boldsymbol{\beta}_j + \epsilon_{ij}, \quad \epsilon_{ij} \overset{iid}{\sim} N(0, \sigma_j^2), \; i = 1, \ldots, n_j, \; j = 1, \ldots, J$$

Predictive Performance



Predictive Performance by State

- restricted likelihood average TLM is larger than ABC in 14 of the 20 states, median difference of 0.04



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