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New valid inequalities and formulations for the static joint Chance-constrained Lot-sizing problem

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Abstract

We study the static joint chance-constrained lot-sizing problem, in which production decisions over a planning horizon are made before knowing random future demands, and the inventory variables are then determined by the demand realizations. The joint chance constraint imposes a service level requirement that the probability that all demands are met on time be above a threshold. We model uncertain outcomes with a finite set of scenarios and begin by applying existing results about chance-constrained programming to obtain an initial extended mixed-integer programming formulation. We further strengthen this formulation with a new class of valid inequalities that generalizes the classical (ℓ, S) inequalities for the deterministic uncapacitated lot-sizing problem. In addition, we prove an optimality condition of the solutions under a modified Wagner-Whitin condition, and based on this derive a new extended mixed-integer programming formulation. This formulation is further extended to the case with constant capacities. We conduct a thorough computational study demonstrating the effectiveness of the new valid inequalities and extended formulation.

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1 Introduction

The lot-sizing problem is a classic production planning problem in which production and inventory levels are planned over a finite set of discrete time periods. In the

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deterministic uncapacitated lot-sizing problem (ULS) (without backlogging) [31], the problem is to determine a production plan for a product to satisfy demands over a finite time horizon while minimizing the sum of setup, production, and inventory holding costs. In the ULS problem, the demand in every period is assumed to be known (deterministic). However, in many realistic settings future demand is predicted by a forecast which inevitably has errors, and the actual realized demands may therefore be modeled as random variables. We consider the static stochastic lot-sizing (SLS) problem, in which it is assumed that all the production amounts are chosen before observing any demand realizations, and the inventory levels over time adjust to random demands. We use a joint chance constraint to require that the probability that the chosen production schedule meets all demands over time is at least $1 - \epsilon$, where $\epsilon \in (0, 1)$ is a given risk tolerance.

Chance-constrained programming (CCP) dates back to [9, 10]. The CCP formulation that arises in the SLS problem is a special case of the CCP problem with stochastic right-hand side under a finite discrete distribution. This special case has been studied extensively in the literature on CCP. Given the discrete scenarios, a deterministic equivalent formulation using binary variables to indicate for which scenarios the constraints are satisfied can be constructed. In [23] it was observed that this formulation can be strengthened using mixing inequalities [5, 15]. Further valid inequalities were investigated in [1, 20, 33].

For the deterministic ULS problem, an explicit convex hull description is given by [6] utilizing the so-called (ℓ, S) inequalities. The first polyhedral study of the deterministic ULS problem with backlogging (ULSB) is performed by [24], in which the authors reformulate the structure of the original formulation by several methods to obtain extended formulations. The complete linear description of the convex hull of ULSB is provided by [21] by generalizing the valid inequalities of [24]. In addition, [26] conduct a polyhedral study of the lot-sizing problem in several cases with Wagner-Whitin costs.

When considering the uncertainty of demands, and penalizing the expected cost of shortages, a stochastic uncapacitated lot-sizing problem (SULS) is proposed. In [2] and [3] the stochastic capacity expansion problem is studied, which includes SULS as a submodel. A polyhedral study of the SULS problem based on a scenario tree is conducted in [14]. They provide several kinds of valid inequalities, and give a sufficient condition under which those inequalities are facet-defining. Afterwards, [13] propose an efficient dynamic programming algorithm for SULS, and similar algorithms can be generalized to SULS with random lead times [16, 17]. None of these papers consider chance constraints.

The static (joint) chance-constrained lot-sizing problem was first proposed in [7] as an application of general CCP. However, they did not consider the inventory cost in the objective function. In [1, 20, 33] a model that includes inventory cost is solved by a branch-and-cut algorithm, but the model in these works is used as a test case for general-purpose methods without investigating the particular structure of the SLS. The work [22] is the most closely related to our work. They provide the first polyhedral study exploiting the lot-sizing structure to identify valid inequalities for the SLS. Gicquel and Cheng [12] also study SLS with a joint chance constraint and production capacity constraints. They propose an inner approximation approach for obtaining feasible solutions based on partial sample-average approximation and an approximation of the holding cost. Finally, in [18] generalizations of SLS with joint chance constraints are considered, including incorporating pricing decisions.

There is also significant work on the *dynamic* stochastic lot-sizing problem, in which the production decisions may be postponed until observing all demands up the time the production is done. In this literature, service level constraints are typically imposed as separate chance constraints for each period that impose a bound on the probability of a stockout for that period, see, e.g., [8, 19, 28, 29]. One exception to this is [32] who use a scenario tree model to solve dynamic problem with a joint chance constraint.

We investigate valid inequalities for the joint chance-constrained SLS that exploit the structure of both CCP and the lot-sizing problem. We derive an initial strong extended formulation using the CCP results in [23], which we call E-SLS. This derivation was first done in [12]. Next, we propose a new class of valid inequalities, the CC-(ℓ, S) inequalities, to strengthen the formulation E-SLS by exploiting the characterizations of lot-sizing problem. We also derive a property of optimal solutions under a modified Wagner-Whitin (WW) cost condition, and use this to construct a new extended formulation NE-SLS, which is valid under this condition. The modified WW condition is a slightly relaxed version of the classical WW condition [31], and hence is at least as broadly applicable. While our focus in this paper is on the uncapacitated case, we also derive one formulation for the case with constant capacities, as this is a natural generalization of formulation NE-SLS that we obtain for the uncapacitated case. We conduct a computational study to compare the performance between the valid inequalities in [22] and our CC-(ℓ, S) inequalities, and find that our inequalities lead to significantly better relaxations which translates into faster solve times. Additional experiments demonstrate the potential value of the new extended formulations in some cases.

This paper is organized as follows. In Sect. 2.1, we write the original mathematical formulation for SLS. In Sect. 2.2, we construct a stronger extended formulation E-SLS. In Sect. 2.3, we present our new CC-(ℓ, S) inequalities. In Sect. 3, we derive the new extended formulation NE-SLS. In Sect. 4, we consider a version of the problem with constant capacities and present a generalization of formulation NE-SLS to this case. We present results of our computational study in Sect. 5 and make concluding remarks in Sect. 6.

Notation. For integers $a \leq b$, define $[a, b] = \{a, a+1, \ldots, b-1, b\}$. For a set $Y \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p\}$ define $\text{Proj}_x(Y) = \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^p, (x, y) \in Y\}$.

2 Formulations

2.1 SLS formulation

We consider a planning horizon with length T. We assume the demand in period $i \in [1, T]$ is a random variable ξ_i and the joint distribution of $\xi = (\xi_1, \ldots, \xi_T)$ has known finite support. The decision variables are the production levels x_t for $t \in [1, T]$ and the production setup variables y_t for $t \in [1, T]$, where $y_t = 1$ indicates a setup is done in period t and $y_t = 0$ otherwise. We assume all these decisions must be made at the beginning of the planning horizon before observing the values of the random demands. Let $\alpha = (\alpha_1, \ldots, \alpha_T)$ and $\beta = (\beta_1, \ldots, \beta_T)$ be the unit production cost vector and fixed setup cost vector, respectively; and $\varepsilon \in (0, 1)$ be the given risk tolerance. With initial inventory $s_0 = 0$, the corresponding static joint chance-constrained stochastic lot-sizing (SLS) problem is formulated as follows:

$$\min \quad \alpha^\top x + \beta^\top y + \mathbb{E}_\xi(\Theta(x, \xi)) \tag{1}$$

$$\mathbb{P}\left(\sum_{t=1}^{i} x_t \geq \sum_{t=1}^{i} \xi_t,\ i \in [1, T]\right) \geq 1 - \varepsilon \tag{2}$$

$$x_i \leq M_i y_i, \quad i \in [1, T] \tag{3}$$

$$x \in \mathbb{R}_+^T, \quad y \in \{0, 1\}^T. \tag{4}$$

Constraints (2) ensure that the probability of meeting all demands in all time periods is at least $1 - \epsilon$. Constraints (3) ensure that production is zero in periods where no setup is done; in these constraints, for each $i \in [1, T]$ the constant M_i is large enough so that when $y_i = 1$ the corresponding constraint is redundant. For a given $x = (x_1, \ldots, x_T)$ and observation of the random demands ξ, $\Theta(x, \xi)$ calculates the holding costs over the periods $1, \ldots, T$:

$$\Theta(x, \xi) = \min h^\top s$$

$$s_i \geq \sum_{t=1}^{i} (x_t - \xi_t), \quad i \in [1, T] \tag{5}$$

$$s \in \mathbb{R}_+^T. \tag{6}$$

Here $s = (s_1, \ldots, s_T)$ is the vector of inventory variables in periods $[1, T]$ and $h = (h_1, \cdots, h_T)$ is the vector of (nonnegative) holding costs. Constraints (5) and (6) ensure that s_i is equal to the inventory level in period i, if it is positive, and s_i is equal to zero in period i, otherwise. If the expression appearing in the right-hand-side of (5) is negative for a period $i \in [1, T]$, this represent a shortfall of cumulative production relative to cumulative demand up to that period in this scenario, and hence the demand is implicitly backlogged in that period. While alternative models introduce backlog variables to record such shortfalls and penalize them in the objective, our model does not do so because the chance constraint (2) instead limits the likelihood of shortfalls occurring.

Let $\Omega = [1, m]$ be the index set of demand scenarios and let π_ω be the probability of scenario ω, for all $\omega \in \Omega$. In addition, let $d_{\omega i}$ be the demand for period i under scenario ω, for all $i \in [1, T]$ and $\omega \in \Omega$ (i.e., $\xi_i = d_{\omega i}$ under scenario $\omega \in \Omega$).

Under our assumption of a finite set of demand scenarios, problem (1)–(4) can be reformulated as an explicit deterministic mixed-integer program (refer to [22]). Specifically, for each scenario $\omega \in \Omega$ let z_ω be a binary variable, which equals to 0 if the demand in all time periods is satisfied under scenario ω, and 1 otherwise. In addition, let $s_{\omega i}$ be the inventory at the end of time period $i \in [1, T]$ in scenario $\omega \in \Omega$. Then the deterministic equivalent formulation is:

$$\min \quad \alpha^\top x + \beta^\top y + \sum_{\omega=1}^{m} \pi_\omega h^\top s_\omega \tag{7}$$

$$\sum_{t=1}^{i} x_t \geq \sum_{t=1}^{i} d_{\omega t}(1 - z_\omega), \quad i \in [1, T], \quad \omega \in \Omega \tag{8}$$

$$\sum_{\omega=1}^{m} \pi_\omega z_\omega \leq \varepsilon \tag{9}$$

$$s_{\omega i} \geq \sum_{t=1}^{i} (x_t - d_{\omega t}), \quad i \in [1, T], \quad \omega \in \Omega \tag{10}$$

$$x_i \leq M_i y_i, \quad i \in [1, T] \tag{11}$$

$$x \in \mathbb{R}_+^T, \quad y \in \{0, 1\}^T, \quad z \in \{0, 1\}^m \tag{12}$$

$$s_\omega \in \mathbb{R}_+^T, \quad \omega \in \Omega \tag{13}$$

With the given scenario model on demands, a large enough value for M_i can be calculated as $M_i = \max_{\omega \in \Omega}\{\sum_{t=i}^{T} d_{\omega t}\}$, for $i \in [1, T]$.

In the following sections we explore how this formulation can be improved so that the linear programming (LP) relaxation is a closer approximation to the optimal value. In Sect. 2.2, we exploit the chance constraint structure to obtain a stronger extended formulation. In Sect. 2.3, we further use the lot-sizing structure of the problem to propose a class of valid inequalities.

2.2 An extended formulation of SLS

In this section we derive an extended formulation for SLS, first introduced in [12], following the general approach in [23]. Define $D_{\omega i} = \sum_{t=1}^{i} d_{\omega t}$, for every $i \in [1, T]$ and $\omega \in \Omega$, i.e., $D_{\omega i}$ is the cumulative demand from period 1 to period i in scenario ω. For each $i \in [1, T]$, let $\{\sigma_1^i, \sigma_2^i \ldots, \sigma_m^i\}$ be a permutation of index set Ω, which satisfies $D_{\sigma_1^i i} \geq D_{\sigma_2^i i} \geq \cdots \geq D_{\sigma_m^i i}$. As a notation simplification, we define $D_{\sigma_j^i} = D_{\sigma_j^i i}$, for each $i \in [1, T]$ and $j \in [1, m]$. Furthermore, let $q_i^* = \min\{q \in [1, m] : \sum_{j=1}^{q} \pi_{\sigma_j^i} > \varepsilon\}$. For each $i \in [1, T]$ we introduce a set of binary variables w_j^i for $j = 1, \ldots, q_i^* - 1$, where if $w_j^i = 0$ then the production up to and including time period i must be enough to meet the cumulative demand $D_{\sigma_j^i}$. The application of the technique in [23] then yields the formulation:

$$\min \quad \alpha^\top x + \beta^\top y + \sum_{\omega=1}^{m} \pi_\omega h^\top s_\omega$$

$$\sum_{t=1}^{i} x_t + \sum_{j=1}^{q_i^*-1} \left(D_{\sigma_j^i} - D_{\sigma_{j+1}^i}\right) w_j^i \geq D_{\sigma_1^i}, \quad i \in [1, T] \tag{14}$$

$$w_j^i - w_{j+1}^i \geq 0, \quad j \in [1, q_i^* - 2], \quad i \in [1, T] \tag{15}$$

$$z_{\sigma_j^i} - w_j^i \geq 0, \quad j \in [1, q_i^* - 1], \quad i \in [1, T] \tag{16}$$

$$\sum_{\omega=1}^{m} \pi_\omega z_\omega \leq \varepsilon \tag{17}$$

$$s_{\omega i} \geq \sum_{t=1}^{i} x_t - D_{\omega i}, \quad i \in [1, T], \quad \omega \in \Omega \tag{18}$$

$$x_i \leq M_i y_i, \quad i \in [1, T] \tag{19}$$

$$x \in \mathbb{R}_+^T, \; y \in \{0, 1\}^T, \; z \in \{0, 1\}^m, \tag{20}$$

$$s_\omega \in \mathbb{R}_+^T, \; \omega \in \Omega, \; w^i \in \{0, 1\}^{q_i^* - 1}, \; i \in [1, T] \tag{21}$$

We denote this extended formulation as E-SLS. In this formulation, constraints (16) enforce the logical relationship that if $z_{\sigma_j^i} = 0$ (so all demand in scenario σ_j^i are satisfied in all periods) then $w_j^i = 0$ (so there is enough cumulative production to period i to cover the cumulative demand in scenario σ_j^i). Constraints (15) enforce the logic that in each time period $i \in [1, T]$, if there is enough cumulative production to exceed demand level $D_{\sigma_j^i}$, then there is certainly enough to exceed demand level $D_{\sigma_{j+1}^i} \leq D_{\sigma_j^i}$. Finally, constraints (14) replace (8) to enforce that the cumulative production up to time period $i \in [1, T]$ is at least as large as required according to the values of the w_j^i variables. Let $P = \{(x, s, y, z) \in \mathbb{R}_+^{T+mT} \times \{0, 1\}^{T+m} : (8)\text{–}(13)\}$ be the feasible region of SLS and let

$$Q = \{(x, s, y, z, w) : (14) - (21)\},$$

be the feasible region of E-SLS. Then the results in [23] imply $\text{Proj}_{(x,s,y,z)}\, Q = P$, and therefore, E-SLS is a valid model for SLS. In [23] it is demonstrated that for chance-constrained *linear* programs the LP relaxation of the extended formulation derived this way can be significantly closer to the optimal value than the LP relaxation of the original formulation. We find similar improvement from using E-SLS in our results, but given the presence of the binary setup variables, we explore how to further strengthen the LP relaxation of E-SLS in the next subsection.

2.3 CC-(ℓ, S) inequalities for E-SLS

In the deterministic ULS problem, the class of inequalities known as the (ℓ, S) inequalities are sufficient to give the complete linear description of convex hull. In this section, we propose an extension of the (ℓ, S) inequalities to the static joint chance-constrained lot-sizing problem. We remind the reader of our assumption that $s_0 = 0$. (Extension of all results to $s_0 > 0$ is trivial.)

Let $D_{i\ell}^\omega = \sum_{t=i}^{\ell} d_{\omega t}$ for $1 \leq i \leq \ell \leq T$, $\omega \in \Omega$ denote the cumulative demand from period i to period ℓ in scenario ω, and let $\bar{D}_{i\ell} = \max_{\omega \in \Omega}\{D_{i\ell}^\omega\}$ be the maximum cumulative demand from period i to period ℓ over all scenarios. Then we define the CC-(ℓ, S) inequalities as the following.

Definition 1 For each $\ell \in [1, T]$ and $S \subseteq [1, \ell]$, the inequality

$$\sum_{t \in \bar{S}} x_t + \sum_{t \in S} \bar{D}_{t\ell} y_t + \sum_{j=1}^{q_\ell^*-1} \left(D_{\sigma_j^\ell} - D_{\sigma_{j+1}^\ell} \right) w_j^\ell \geq D_{\sigma_1^\ell} \tag{22}$$

is called a CC-(ℓ, S) inequality for E-SLS, where $\bar{S} = [1, \ell] \setminus S$.

We define CC-(ℓ, S) to be the formulation defined by the E-SLS model with the addition of the CC-(ℓ, S) inequalities for all $\ell \in [1, T]$ and $S \subseteq [1, \ell]$. We note that the proposed CC-(ℓ, S) inequalities are not *valid inequalities* for the formulation E-SLS in the classical sense, as it is possible for there to be solutions to E-SLS that are not feasible to CC-(ℓ, S). However, we next argue that the formulation CC-(ℓ, S) is a valid formulation of SLS, which is the underlying model we wish to solve, and thus we may safely add the CC-(ℓ, S) inequalities to formulation E-SLS.

One direction of this argument is straightforward. If (x, s, y, z, w) is a feasible solution of CC-(ℓ, S), then it is also a feasible solution of E-SLS, and because E-SLS is a valid extended formulation of SLS, this implies (x, s, y, z) is a feasible solution of SLS. Hence, we only need to show that for any feasible solution (x, s, y, z) of SLS, there exists w such that (x, s, y, z, w) is a feasible solution of CC-(ℓ, S).

Theorem 1 *For any feasible solution (x, s, y, z) of SLS, there exists $w \in \{0, 1\}^{\sum_{i=1}^T (q_i^*-1)}$ such that (x, s, y, z, w) is also a feasible solution of CC-(ℓ, S), and therefore, CC-(ℓ, S) is a valid model for SLS.*

Proof For each $i \in [1, T]$, define $\bar{j}(i) = \min\{j \in [1, q_i^*] : z_{\sigma_j^i} = 0\}$, and observe that by definition $z_{\sigma_1^i} = z_{\sigma_2^i} = \cdots = z_{\sigma_{\bar{j}(i)-1}^i} = 1$. Now, for $i \in [1, T]$ define

$$w_j^i = \begin{cases} 1 & \text{for } j \in [1, \bar{j}(i) - 1] \\ 0 & \text{for } j \in [\bar{j}(i), q_i^*]. \end{cases}$$

Then, by construction, w and z satisfy (15) and (16). Next, for any $i \in [1, T]$, using (8) and $z_{\sigma_{\bar{j}(i)}^i} = 0$, we have

$$\sum_{t=1}^{i} x_t \geq \sum_{t=1}^{i} d_{\sigma_{\bar{j}(i)}^i, t} = D_{\sigma_{\bar{j}(i)}^i}. \tag{23}$$

Using the definition of w_j^i we have

$$\sum_{j=1}^{q_i^*-1} \left(D_{\sigma_j^i} - D_{\sigma_{j+1}^i} \right) w_j^i = \sum_{j=1}^{\bar{j}(i)-1} \left(D_{\sigma_j^i} - D_{\sigma_{j+1}^i} \right) = D_{\sigma_1^i} - D_{\sigma_{\bar{j}(i)}^i}. \tag{24}$$

Combining (23) and (24) yields

$$\sum_{t=1}^{i} x_t + \sum_{j=1}^{q_i^*-1} \left(D_{\sigma_j^i} - D_{\sigma_{j+1}^i}\right) w_j^i \geq D_{\sigma_1^i}$$

thus showing that (14) is satisfied and hence (x, s, y, z, w) is a feasible solution of E-SLS.

It remains to show that (x, s, y, z, w) satisfies (22) for each $\ell \in [1, T]$ and $S \subseteq [1, \ell]$. Thus, fix $\ell \in [1, T]$ and $S \subseteq [1, \ell]$.

Suppose first $y_t = 0$ for all $t \in S$. This implies $x_t = 0$ for $t \in S$. Then,

$$\begin{aligned}
&\sum_{t \in \bar{S}} x_t + \sum_{t \in S} \bar{D}_{t\ell} y_t + \sum_{j=1}^{q_\ell^*-1} \left(D_{\sigma_j^\ell} - D_{\sigma_{j+1}^\ell}\right) w_j^\ell \\
&\quad = \sum_{t=1}^{\ell} x_t + \sum_{j=1}^{q_\ell^*-1} \left(D_{\sigma_j^\ell} - D_{\sigma_{j+1}^\ell}\right) w_j^\ell \geq D_{\sigma_1^\ell}
\end{aligned}$$

where the inequality holds because we have already shown (14) holds.

Next, assume $y_t = 1$ for some $t \in S$, then let $t^* = \min\{t \in S : y_t = 1\}$. We then have

$$\begin{aligned}
&\sum_{t \in \bar{S}} x_t + \sum_{t \in S} \bar{D}_{t\ell} y_t + \sum_{j=1}^{q_\ell^*-1} \left(D_{\sigma_j^\ell} - D_{\sigma_{j+1}^\ell}\right) w_j^\ell \\
&\quad \geq \sum_{t=1}^{t^*-1} x_t + \bar{D}_{t^*\ell} + \sum_{j=1}^{q_\ell^*-1} \left(D_{\sigma_j^\ell} - D_{\sigma_{j+1}^\ell}\right) w_j^\ell && (25)\\
&\quad = \sum_{t=1}^{t^*-1} x_t + \bar{D}_{t^*\ell} + D_{\sigma_1^\ell} - D_{\sigma_{\bar{j}(\ell)}^\ell} && (26)\\
&\quad \geq D_{\sigma_{\bar{j}(\ell)}^\ell, t^*-1} + \bar{D}_{t^*\ell} + D_{\sigma_1^\ell} - D_{\sigma_{\bar{j}(\ell)}^\ell} && (27)\\
&\quad \geq D_{\sigma_1^\ell} && (28)
\end{aligned}$$

where (25) follows from the definition of t^*, (26) follows from (24), (27) follows from (23) because $z_{\sigma_{\bar{j}(\ell)}} = 0$, and (28) follows because

$$D_{\sigma_{\bar{j}(\ell)}^\ell} = \sum_{t=1}^{\ell} d_{\sigma_{\bar{j}(\ell)}^\ell, t} \leq \sum_{t=1}^{t^*-1} d_{\sigma_{\bar{j}(\ell)}^\ell, t} + \max_{\omega \in \Omega} \sum_{t=t^*}^{\ell} d_{\omega t} = D_{\sigma_{\bar{j}(\ell)}^\ell, t^*-1} + \bar{D}_{t^*, \ell}.$$

□

As the number of CC-(ℓ, S) inequalities grows exponentially with T, we require a separation algorithm for identifying violated inequalities from this class. Let $(\bar{x}, \bar{s}, \bar{y}, \bar{z}, \bar{w}) \in \mathbb{R}_+^{(m+1)T} \times [0, 1]^{T+m+\sum_{i=1}^{T}(q_i^*-1)}$ be a given solution of the continuous relaxation. Algorithm 1 provides a natural generalization of the separation algorithm for the traditional (ℓ, S) inequalities to the inequalities of the form (22). The running time of this algorithm is $\mathcal{O}(T \log T)$.

Algorithm 1 Separation Algorithm for CC-(ℓ, S) Inequalities

1: **for** $t = 1, \ldots, T$ **do**
2:     Determine $\ell(t) \in [t, T]$ such that $\bar{D}_{t,\ell(t)-1}\bar{y}_t < \bar{x}_t \leq \bar{D}_{t,\ell(t)}\bar{y}_t$ by bisection.
3: **end for**
4: Let $\Delta_0 = 0$,
5: **for** $\ell = 1, \ldots, T$ **do**
6:     Select sets $Y_\ell = \{t \in [1, \ell] : \ell(t) > \ell\}$, $X_\ell = \{t \in [1, \ell] : \ell(t) = \ell\}$.
7:     Calculate $\Delta_\ell = \Delta_{\ell-1} + (\bar{D}_{t,\ell} - \bar{D}_{t,\ell-1})(\sum_{t \in Y_\ell} \bar{y}_t) + \sum_{t \in X_\ell} (\bar{x}_t - \bar{D}_{t,\ell-1}\bar{y}_t)$.
8:     If $\Delta_\ell < D_{\sigma_1^\ell} - \sum_{j=1}^{q_\ell^*-1} (D_{\sigma_j^\ell} - D_{\sigma_{j+1}^\ell})\bar{w}_j^\ell$, output ℓ and $S = Y_\ell$.
9: **end for**

At each iteration ℓ where an (ℓ,S) pair is output in line 8, the corresponding inequality (22) is violated by the current solution. In case that there are no such outputs, there are no violated inequalities. The proof of correctness of this algorithm follows exactly that of the separation algorithm for the traditional (ℓ, S) inequalities (see [27,Page 219]).

In the deterministic ULS problem, adding the (ℓ, S) inequalities to the LP relaxation is sufficient to define the convex hull of feasible solutions [6]. Unfortunately, this is not true for the CC-(ℓ, S) inequalities and the feasible region of the SLS problem, even if one considers a fixed binary $\bar{z}$. Indeed, for a fixed $\bar{z}$ the feasible region reduces to a deterministic ULS problem in which for each time period t, the total production up to period t should be enough to satisfy the maximum cumulative demand up to time period t, where the maximum is taken over scenarios ω with $\bar{z}_\omega = 0$. These maximum demands depend on the chosen scenarios as indicated by $\bar{z}$, and thus the (ℓ, S) inequalities that would be necessary to define the convex hull for different choices of $\bar{z}$ would need coefficients on the y_t variables that vary based on the choice of $\bar{z}$ variables. The coefficients $\bar{D}_{t\ell}$ used in the CC-(ℓ, S) inequalities are independent of z, which indicates that these inequalities cannot be sufficient to define the convex hull.

If the production and holding costs in the deterministic ULS satisfy the Wagner-Whitin cost condition [31] then adding only the (ℓ, S) inequalities with $S = [k, \ell]$ for $k = 1, \ldots, \ell$ and $\ell \in [1, T]$ to the LP relaxation is sufficient to solve the deterministic ULS [26]. Unfortunately, this again does not hold for the SLS and the CC-(ℓ, S) inequalities, for the same reason as discussed in the last paragraph. However, as the CC-(ℓ, S) inequalities are not a necessary part of the formulation of SLS as long as the binary restrictions are enforced on y and z, this result for deterministic ULS still motivates using this same subset of CC-(ℓ, S) inequalities for the SLS. We thus define CC-(ℓ, S)-WW to be the formulation defined by the E-SLS model with the addition of the CC-(ℓ, S) inequalities with $S = [k, \ell]$ for all $k \in [1, \ell]$ and $\ell \in [1, T]$. While this formulation is motivated by a result that holds for deterministic ULS under the Wagner-Whitin condition, the formulation is valid and can be used whether this condition holds or not. Empirical evidence presented in Sect. 5 confirms that using this subset of inequalities improves performance, thanks to avoiding the need to separate and add CC-(ℓ, S) inequalities.

3 A new extended formulation of SLS under a modified Wagner-Whitin condition

In this section we derive a stronger formulation in a lifted variable space. This formulation can be seen as an extension of the classic shortest path formulation for the deterministic ULS problem of Eppen and Martin [11] to SLS. The shortest path formulation of Eppen and Martin is valid in general for the deterministic ULS problem. In contrast, our generalization for the SLS requires the following modified Wagner-Whitin cost condition.

Assumption 1 *(Modified Wagner-Whitin condition)* The SLS problem satisfies the following conditions:

$$\alpha_i + (1 - \varepsilon)h_i \geq \alpha_{i+1}, \quad \text{for } i \in [1, T - 1].$$

Assumption 1 is a slightly relaxed version of the classical Wagner-Whitin condition [31] which state $\alpha_i + h_i \geq \alpha_{i+1}$ for each i. This condition states that if setups occur in both periods i and $i + 1$, then it is not more expensive to produce in period $i + 1$ than to produce in period i and then pay the holding cost. This assumption is always satisfied when production costs are constant over time, and the vast literature on this special case for the deterministic lot-sizing problem indicates that this assumption is very often satisfied more generally. The modified condition relaxes this slightly, as the relaxed version is sufficient in the chance-constrained setting to obtain the following special property of the optimal solution of the SLS, which we subsequently use to derive the new formulation.

Lemma 1 *Suppose Assumption 1 holds, let* (x, y, s, z) *be a feasible solution to the SLS problem* (8)–(13)*, and let* $I = \{i_1, i_2, \ldots, i_r\} = \{i \in [1, T] : y_i = 1\}$ *where* $1 = i_1 < i_2 < \ldots < i_r \leq T$*. Then, there exists* $\bar{s}$ *such that the solution* $(\bar{x}, y, \bar{s}, z)$ *with production levels* $\bar{x}$ *defined as*

$$\bar{x}_{i_k} = \delta_{i_{k+1}}(z) - \delta_{i_k}(z), \quad k \in [1, r] \tag{29}$$

and $\bar{x}_i = 0$ *for* $i \in [1, T] \setminus I$ *is feasible to* (8)–(13) *and has cost not more than the cost of* (x, y, s, z)*, where* $i_{r+1} := T + 1$*,* $\delta_i(z) := \max_{\omega \in J_z}\{\sum_{t=1}^{i-1} d_{\omega t}\}$*, and* $J_z = \{\omega \in \Omega : z_\omega = 0\}$*.*

Proof Let

$$f(x,y,s,z)=\alpha^\top x+\beta^\top y+\sum_{\omega=1}^{m}\pi_\omega h^\top s_\omega$$

be the cost of solution (x,y,s,z) in (7).

If there is some $k\in[1,r]$ such that $x_{i_k}\neq\delta_{i_{k+1}}(z)-\delta_{i_k}(z)$, then let $k^*=\min\{k\in[1,r]:\ x_{i_k}\neq\delta_{i_{k+1}}(z)-\delta_{i_k}(z)\}$. If $x_{i_{k^*}}<\delta_{i_{k^*+1}}(z)-\delta_{i_{k^*}}(z)$, then there must be some scenario $\omega\in J_z$ whose demand in period $(i_{k^*+1}-1)$ can not be satisfied, so we may assume $x_{i_{k^*}}>\delta_{i_{k^*+1}}(z)-\delta_{i_{k^*}}(z)$.

Define

$$g(x,y,s,z)=\sum_{i\in[1,T]\setminus[i_{k^*},i_{k^*+1}]}(\alpha_i x_i+\beta_i y_i)+\sum_{\omega=1}^{m}\left(\pi_\omega\sum_{i\in[1,T]\setminus[i_{k^*},i_{k^*+1}-1]}h_i s_{\omega i}\right).$$

Then

$$\begin{aligned}f(x,y,s,z)=\,&g(x,y,s,z)+\left(\alpha_{i_{k^*}}x_{i_{k^*}}+\beta_{i_{k^*}}y_{i_{k^*}}\right)\\&+\sum_{\omega=1}^{m}\left(\pi_\omega\sum_{i=i_{k^*}}^{i_{k^*+1}-1}h_i s_{\omega i}\right)+\left(\alpha_{i_{k^*+1}}x_{i_{k^*+1}}+\beta_{i_{k^*+1}}y_{i_{k^*+1}}\right).\end{aligned}$$

Let $\bar{x}$ and $\bar{s}$ be defined by $\bar{x}_{i_{k^*}}=x_{i_{k^*}}-\eta$, $\bar{s}_{\omega i}=s_{\omega i}-\eta$, for $i\in[i_{k^*},i_{k^*+1}-1]$ and $\omega\in J_z$, $\bar{s}_{\omega i}=[s_{\omega i}-\eta]^+$, for $i\in[i_{k^*},i_{k^*+1}-1]$ and $\omega\in\Omega\setminus J_z$, $\bar{x}_{i_{k^*+1}}=x_{i_{k^*+1}}+\eta$, and other components are the same as x and s. Then

$$\begin{aligned}f(\bar{x},y,\bar{s},z)=\,&g(x,y,s,z)+\left(\alpha_{i_{k^*}}(x_{i_{k^*}}-\eta)+\beta_{i_{k^*}}y_{i_{k^*}}\right)\\&+\sum_{\omega\in J_z}\left(\pi_\omega\sum_{i=i_{k^*}}^{i_{k^*+1}-1}h_i(s_{\omega i}-\eta)\right)+\sum_{\omega\in\Omega\setminus J_z}\left(\pi_\omega\sum_{i=i_{k^*}}^{i_{k^*+1}-1}h_i[s_{\omega i}-\eta]^+\right)\\&+\left(\alpha_{i_{k^*+1}}(x_{i_{k^*+1}}+\eta)+\beta_{i_{k^*+1}}y_{i_{k^*+1}}\right)\\=\,&f(x,y,s,z)-\left(\alpha_{i_{k^*}}+\sum_{i=i_{k^*}}^{i_{k^*+1}-1}\left(\sum_{\omega\in J_z}\pi_\omega\right)h_i-\alpha_{i_{k^*}+1}\right)\eta\\&+\sum_{\omega\in\Omega\setminus J_z}\left(\pi_\omega\sum_{i=i_{k^*}}^{i_{k^*+1}-1}h_i([s_{\omega i}-\eta]^+-s_{\omega i})\right)\\\leq\,&f(x,y,s,z)-\left(\alpha_{i_{k^*}}+\sum_{i=i_{k^*}}^{i_{k^*+1}-1}\left(\sum_{\omega\in J_z}\pi_\omega\right)h_i-\alpha_{i_{k^*}+1}\right)\eta\\=\,&f(x,y,s,z)-\sum_{i=i_{k^*}}^{i_{k^*+1}-1}\left(\alpha_i+\left(\sum_{\omega\in J_z}\pi_\omega\right)h_i-\alpha_{i+1}\right)\eta.\end{aligned}$$

By Assumption 1, $\sum_{i=i_{k^*}}^{i_{k^*+1}-1}(\alpha_i + (\sum_{\omega \in J_z} \pi_\omega)h_i - \alpha_{i+1}) \geq \sum_{i=i_{k^*}}^{i_{k^*+1}-1}(\alpha_i + (1-\varepsilon)h_i - \alpha_{i+1}) \geq 0$, then setting $\eta = x_{i_{k^*}} - (\delta_{i_{k^*+1}}(z) - \delta_{i_{k^*}}(z))$, $(\bar{x}, y, \bar{s}, z)$ is also a feasible solution of SLS with $f(\bar{x}, y, \bar{s}, z) \leq f(x, s, y, z)$, and satisfies $\bar{x}_{i_{k^*}} = \delta_{i_{k^*+1}}(z) - \delta_{i_{k^*}}(z)$.

This process can now be repeated with the solution $(\bar{x}, y, \bar{s}, z)$ as long as there is an index i that does not satisfy (29), eventually yielding a solution that does satisfy (29) and has cost no worse than the cost of (x, y, s, z). □

Motivated by Lemma 1, we introduce new decision variables $\phi_{it}^{jk} = 1$ if an amount $D_{\sigma_k^t} - D_{\sigma_j^{i-1}}$ is produced in period i, for $1 \leq i \leq t \leq T$, $j \in [1, q_{i-1}^*]$, and $k \in [1, q_t^*]$, where we define $q_0^* := 1$ and $D_{\sigma_1^0} := 0$. We call the production plan determined by setting $\phi_{it}^{jk} = 1$ (i.e., producing $D_{\sigma_k^t} - D_{\sigma_j^{i-1}}$ in period i to meet demand in periods $i, \ldots, t$) a subplan.

We then obtain the following new extended formulation(NE-SLS):

$$\min \quad \alpha^\top x + \beta^\top y + \sum_{\omega=1}^{m} \pi_\omega h^\top s_\omega$$

$$\sum_{\tau=1}^{T} \sum_{k=1}^{q_\tau^*} \phi_{1\tau}^{1k} = 1 \tag{30}$$

$$\sum_{i=1}^{t-1} \sum_{j=1}^{q_{i-1}^*} \sum_{k=1}^{q_{t-1}^*} \phi_{i,t-1}^{jk} - \sum_{\tau=t}^{T} \sum_{j=1}^{q_{t-1}^*} \sum_{k=1}^{q_\tau^*} \phi_{t\tau}^{jk} = 0, \quad t \in [2, T] \tag{31}$$

$$\sum_{i=1}^{T} \sum_{j=1}^{q_{i-1}^*} \sum_{k=1}^{q_T^*} \phi_{iT}^{jk} = 1 \tag{32}$$

$$\sum_{\tau=t}^{T} \sum_{j=1}^{q_{t-1}^*} \sum_{k=1}^{q_\tau^*} \phi_{t\tau}^{jk} = y_t, \quad t \in [1, T] \tag{33}$$

$$\sum_{\tau=t}^{T} \sum_{j=1}^{q_{t-1}^*} \sum_{k=1}^{q_\tau^*} \left(D_{\sigma_k^\tau} - D_{\sigma_j^{t-1}}\right) \phi_{t\tau}^{jk} = x_t, \quad t \in [1, T] \tag{34}$$

$$\sum_{t=i}^{T} \sum_{k=1}^{q_t^*} \phi_{it}^{jk} = w_{j-1}^{i-1} - w_j^{i-1}, \quad j \in [1, q_{i-1}^* - 1], \quad i \in [2, T] \tag{35}$$

$$\sum_{i=1}^{t} \sum_{j=1}^{q_{i-1}^*} \phi_{it}^{jk} = w_{k-1}^t - w_k^t, \quad k \in [1, q_t^* - 1], \quad t \in [1, T] \tag{36}$$

$$w_0^i = y_{i+1}, \quad i \in [1, T] \tag{37}$$

$$w_0^i \geq w_1^i, \quad i \in [1, T]$$

$$\phi_{it}^{jk} \in \{0,1\}, \quad j \in [1, q_{i-1}^*], \quad k \in [1, q_t^*], \quad 1 \le i \le t \le T$$
$$w_0^i \in \{0,1\}, \quad i \in [1, T]$$
$$(x, y, s, z, w) \text{ satisfy } (15)-(18), (20)-(21) \tag{38}$$

where we define $y_{T+1} := 1$. Constraints (30)–(32) are the flow conservation constraints that model a production plan as a sequence of subplans. Constraints (33) ensure that if $y_t = 0$ then no subplan can start in period t, for $t \in [1, T]$. Constraints (34) calculate the production levels in each period. Constraints (35)–(38) together with (15)–(17) ensure that the production plan determined by the subplans satisfies the chance constraint. The auxiliary variables $(w_0^i, w_1^i, \ldots, w_{q_i^*-1}^i)$ for $i \in [1, T]$ play a similar role as in E-SLS, except that here they are extended to include w_0^i, which by (37) is equal to y_{i+1}. (Although w_0^i can be eliminated we use it because it simplifies presentation of constraints (35) and (36).) This modification ensures that if production is not done in period $i+1$ ($y_{i+1} = 0$) then $w_j^i = 0$ for all $j \in [1, q_j^* - 1]$, so that in this case these variables do not impact the z_ω variables determining which scenarios are satisfied via (16). For each $i \in [1, T]$, constraints (38) and (15) ensure that there is at most one $j \in [1, q_i^* - 1]$ such that $w_{j-1}^i - w_j^i = 1$, and $w_{j-1}^i - w_j^i = 0$ otherwise. As in E-SLS, $w_{j-1}^i - w_j^i = 1$ is an indication that there is sufficient inventory available in period i to meet scenarios with demand up to $D_{\sigma_j^i}$. Thus, constraints (35) enforce consistency between this in period $i-1$ and the subplan determined by the ϕ_{it}^{jk} variables. Likewise, constraints (36) enforce consistency between the indication of the available inventory in period t (as determined by the expression $w_{k-1}^t - w_k^t$) and the subplan determined by the ϕ_{it}^{jk} variables. As in the E-SLS formulation constraints (38), (15) and (16) enforce consistency between the w_j^i variables and the z_ω variables used in the chance constraint (17).

In the shortest path formulation of Eppen and Martin [11], variables ϕ'_{it} for $1 \le i \le t \le T$ are introduced to model the selection of a path from node 1 to node T with the interpretation that if $\phi'_{it} = 1$ then in period i the production amount is equal to the cumulative demand between periods i and t. The variables ϕ_{it}^{jk} in NE-SLS are analogous, except that for each $1 \le i \le t \le T$, there are multiple variables (the j and k indices) corresponding to the different possible production levels that may be needed according to the scenarios that are chosen to be satisfied for the chance constraint. Thus, constraints (30)–(33) of NE-SLS follow directly from the Eppen and Martin formulation. Constraint (34) is the natural generalization for determining the production levels. The remaining constraints link the variables for this formulation with the w and z variables required for modeling the chance constraint.

Proposition 1 *Under Assumption 1, the new extended formulation NE-SLS is a valid model for SLS.*

Proof Let (x, y, s, z) be a feasible solution of SLS where x satisfies (29). Let $I = \{i_1, i_2, \ldots, i_r\} = \{i \in [1, T] : y_i = 1\}$ where $1 = i_1 < i_2 < \ldots < i_r \le T$. For each $i \in [1, T]$, define $\bar{j}(i) = \min\{j \in [1, q_i^*] : z_{\sigma_j^i} = 0\}$. Now, for $t \in [1, T]$, $k \in [1, q_t^*]$, define

$$\phi_{1t}^{1k} = \begin{cases} 1 & \text{if } t = i_2 - 1, \quad k = \bar{j}(i_2 - 1) \\ 0 & \text{otherwise}\ , \end{cases}$$

and for $2 \leq i \leq t \leq T$, $j \in [1, q_{i-1}^*]$, $k \in [1, q_t^*]$, define

$$\phi_{it}^{jk} = \begin{cases} 1 & \text{if } i = i_u,\ t = i_{u+1} - 1,\ j = \bar{j}(i_u - 1),\ k = \bar{j}(i_{u+1} - 1),\ u \in [2, r] \\ 0 & \text{otherwise}\ , \end{cases}$$

and for $i \in [1, T]$, $j \in [0, q_i^* - 1]$ define

$$w_j^i = \begin{cases} 1 & \text{if } i + 1 \in I \cup \{T + 1\},\ j \in [0, \bar{j}(i) - 1] \\ 0 & \text{otherwise}\ . \end{cases}$$

Direct calculations verify that (x, y, s, z, w, ϕ) satisfies all the constraints of NE-SLS.

Let (x, y, s, z, w, ϕ) be a feasible solution of the NE-SLS. For $i \in [1, T]$, if $y_i = 0$, then (33)–(34) imply $x_i = 0$, which indicates that (x, y, s, z) satisfies constraints (11). Obviously, (x, y, s, z) satisfies (9) and (10) as they are contained in the formulation of NE-SLS. Let $\hat{I} = \{1\} \cup \{i \in [2, T] : \exists t \in [i, T], j \in [1, q_{i-1}^*], k \in [1, q_t^*], \text{ s.t. } \phi_{it}^{jk} = 1\} = \{\hat{i}_1, \hat{i}_2, \ldots, \hat{i}_r\}$ where $1 = \hat{i}_1 < \hat{i}_2 < \ldots < \hat{i}_r \leq T < \hat{i}_{r+1} := T + 1$. Then by (30)–(32) there exists exactly one $j_u \in [1, q_{\hat{i}_u - 1}^*]$ and one $k_u \in [1, q_{\hat{i}_{u+1} - 1}^*]$, such that $\phi_{\hat{i}_u, \hat{i}_{u+1}-1}^{j_u k_u} = 1$ for $u \in [1, r]$. For $u \in [1, r - 1]$, if $\phi_{\hat{i}_u, \hat{i}_{u+1}-1}^{j_u k_u} = \phi_{\hat{i}_{u+1}, \hat{i}_{u+2}-1}^{j_{u+1} k_{u+1}} = 1$, then we conclude that $k_u = j_{u+1}$. If $k_u \neq q_{\hat{i}_{u+1}}^*$, then by (36) there is $w_{k_u - 1}^{\hat{i}_{u+1}-1} - w_{k_u}^{\hat{i}_{u+1}-1} = \phi_{\hat{i}_u, \hat{i}_{u+1}-1}^{j_u k_u} = 1 = \phi_{\hat{i}_{u+1}, \hat{i}_{u+2}-1}^{j_{u+1} k_{u+1}}$, therefore, by (35) and the flow conservation constraints (30)–(32) we can obtain $k_u = j_{u+1}$; if $k_u = q_{\hat{i}_{u+1}}^*$, then by (36)–(37) we have $w_0^{\hat{i}_{u+1}-1} = w_1^{\hat{i}_{u+1}-1} = \cdots = w_{q_{\hat{i}_{u+1}-1}^*}^{\hat{i}_{u+1}-1} = 1$, also, by (35) and the flow conservation constraints (30)–(32) we can obtain $j_{u+1} = q_{\hat{i}_{u+1}}^* = k_u$. Thus, for any $i \in [\hat{i}_u, \hat{i}_{u+1} - 1]$, $u \in [1, r]$, we have

$$\begin{aligned} \sum_{t=1}^{i} x_t &= \sum_{v=1}^{u} \left(D_{\sigma_{k_v}^{\hat{i}_{v+1}-1}} - D_{\sigma_{j_v}^{\hat{i}_v - 1}} \right) \phi_{\hat{i}_v, \hat{i}_{v+1}-1}^{j_v k_v} \\ &= \sum_{v=1}^{u} \left(D_{\sigma_{k_v}^{\hat{i}_{v+1}-1}} - D_{\sigma_{j_v}^{\hat{i}_v - 1}} \right) \\ &= D_{\sigma_{k_u}^{\hat{i}_{u+1}-1}} \\ &\geq \begin{cases} D_{\sigma_j^{\hat{i}_{u+1}-1}} \left(1 - z_{\sigma_j^{\hat{i}_{u+1}-1}} \right) & \text{for } j \in [1, k_u - 1] \\ D_{\sigma_j^{\hat{i}_{u+1}-1}} & \text{for } j \in [k_u, m] \end{cases} \end{aligned}$$

$$\geq \begin{cases} D_{\sigma_j^{\hat{i}_{u+1}-1}i}\left(1 - z_{\sigma_j^{\hat{i}_{u+1}-1}}\right) & \text{for } j \in [1, k_u - 1] \\ D_{\sigma_j^{\hat{i}_{u+1}-1}i} & \text{for } j \in [k_u, m] \end{cases}$$

$$\geq D_{\omega i}(1 - z_\omega) \quad \text{for } \omega \in \Omega.$$

Thus, (x, y, s, z) satisfies (8) and hence is a feasible solution of SLS. □

The new model NE-SLS potentially has a tighter linear programming relaxation than other models, which can lead to better root relaxation gaps and fewer nodes explored when used withing a branch-and-bound algorithm.

Remark 1 The shortest path formulation of Eppen and Martin for the deterministic lot-sizing problem does not require the Wagner-Whiten condition to be valid. The reason our proof of validity requires the modified Wagner-Whitin condition is that the SLS allows shortfall of demand in some scenarios, which can lead to an asymmetric impact on the holding cost when analyzing perturbations of solutions obtained when increasing or decreasing production amounts. Although NE-SLS is only proved to be valid under the modified Wagner-Whitin condition, in our computational experience we have observed that the optimal solution of NE-SLS is usually equal to the optimal solution of SLS even when this condition does not hold. Thus, in cases where solving NE-SLS is more efficient, it may be useful computationally as an approximation even when the modified Wagner-Whitin condition does not hold.

A significant potential limitation of NE-SLS is its large size. To reduce its size, we present in the following lemma conditions under which decision variables can be fixed to zero.

Lemma 2 *Let (x, y, s, z, w, ϕ) be a feasible solution of the NE-SLS, and suppose for some $1 \leq i \leq t \leq T$, $j \in [1, q^*_{i-1}]$, and $k \in [1, q^*_t]$, one of the following two conditions holds:*

1. $\sum_{\tau \in J^{jk}_{it}} \pi_\tau > \varepsilon$, *where* $J^{jk}_{it} = \{\sigma^{i-1}_\tau, \tau \in [1, j-1]\} \cup \{\sigma^t_\tau, \tau \in [1, k-1]\}$, *or*
2. $D_{\sigma^{i-1}_j} < D_{\sigma^t_k, i-1}$ *or* $D_{\sigma^{i-1}_j, t} > D_{\sigma^t_k}$.

Then $\phi^{jk}_{it} = 0$.

Proof If $\phi^{jk}_{it} = 1$, then by constraints (35)–(36) and (15)–(16), it follows that

i. $z_{\sigma^{i-1}_j} = 0$, $z_{\sigma^{i-1}_\tau} = 1$ for $\tau \in [1, j-1]$ and $z_{\sigma^t_k} = 0$, $z_{\sigma^t_\tau} = 1$ for $\tau \in [1, k-1]$,

and hence

ii. $\sigma^{i-1}_j = \arg\max_{\omega \in J_z}\{D_{\omega, i-1}\}$ and $\sigma^t_k = \arg\max_{\omega \in J_z}\{D_{\omega t}\}$.

Hence, we can conclude that $\sum_{\tau \in J^{jk}_{it}} \pi_\tau \leq \varepsilon$ (by i and constraint (17)) and $D_{\sigma^{i-1}_j} \geq D_{\sigma^t_k, i-1}$ and $D_{\sigma^{i-1}_j, t} \leq D_{\sigma^t_k}$ (by ii). Thus, if either of these conditions is violated (i.e., 1 or 2 in the statement of the Lemma is satisfied) then $\phi^{jk}_{it} = 0$. □

In our experiments, we find that approximately 60-70% of the ϕ_{it}^{jk} variables can be eliminated by applying Lemma 2, which improves the computational performance of NE-SLS significantly.

4 Static chance-constrained capacitated lot-sizing problem

In this section, we briefly explore the static chance-constrained capacitated lot-sizing problem (SCLS). The only difference between SCLS and SLS is that the capacity of each period is limited, i.e., constraints (10) change to $x_i \leq C_i y_i,\ i \in [1, T]$ in SCLS, where $C_i \leq M_i$ for $i \in [1, T]$ are given constants. Because the feasible region of SCLS is a subset of the feasible region of SLS, all results derived in the previous sections can be directly applied to SCLS.

There is a large literature on valid inequalities and extended formulations for the deterministic capacitated lot-sizing (CLS) problem – see, e.g., [27] and references therein. It would be interesting to explore extensions of many of these results to SCLS, but this is largely outside the scope of this paper. However, since the shortest path formulation of Eppen and Martin [11] for the deterministic ULS problem has a generalization to the CLS problem with constant capacities [25], we explore here an analogous extension of the NE-SLS formulation to this case.

In the deterministic CLS, the concept of a *regeneration interval* plays a crucial role in the polyhedral study. We adapt this concept to SCLS as follows, where for a fixed z we use the notation $\omega_i := \arg\max_{\omega \in J_z}\{D_{\omega i}\}$, for $i \in [1, T]$.

Definition 2 The *interval* $[i, t]$ is a regeneration interval with respect to z for a feasible solution (x, y, s, z) of SCLS if and only if $s_{\omega_{i-1}, i-1} = s_{\omega_t t} = 0$, but $s_{\omega_p p} > 0$ for $p \in [i, t-1]$. Furthermore, $i - 1$ and t are called regeneration points with respect to z and $X_{it} = \{x_p,\ p \in [i, t]\}$ is called a *production sequence* with respect to z.

When z is fixed, the optimal solution of SCLS must consist of regeneration intervals with respect to z. Indeed, since the optimal cumulative production should be equal to the largest cumulative demand among scenarios of J_z for the whole plan horizon, i.e., $\sum_{t=1}^{T} x_t = \max_{\omega \in J_z}\{D_{\omega T}\}$, an optimal solution consists of at least one regeneration interval $[1, T]$ with respect to z. With this observation, we derive a characterization of the optimal solutions for SCLS as follows.

Definition 3 A production sequence, X_{it} $(1 \leq i \leq t \leq T)$, is *capacity constrained with respect to* z if the production level in at most one period $p \in [i, t]$ is positive but less than capacity, i.e., $0 < x_p < C_p$. Such period p is called the fractional period, and all other production levels in the production sequence are either zero or at their capacities.

Lemma 3 *Suppose Assumption 1 holds. Let (x, y, s, z) be a feasible solution to the SCLS problem and let $I = \{[i_1, t_1], [i_2, t_2], \ldots, [i_r, t_r]\}$ be the set of all regeneration intervals, where $1 = i_1 \leq t_1 < i_2 \leq t_2 < \ldots < i_r \leq t_r = T$ and $i_{p+1} = t_p + 1$ for $p \in [1, r-1]$. Then, there exists $\bar{x}$ which consists of capacity constrained production sequences and $\bar{s}$ such that the solution $(\bar{x}, y, \bar{s}, z)$ is feasible to the SCLS problem and has cost not more than the cost of (x, y, s, z).*

Proof Let

$$f(x, y, s, z) = \alpha^\top x + \beta^\top y + \sum_{\omega=1}^{m} \pi_\omega h^\top s_\omega$$

be the cost of solution (x, y, s, z) in (7).

If there is some $k \in [1, r]$ such that at least two periods in $[i_k, t_k]$ are fractional, then let p_1 and p_2 be the first two consecutive fractional periods, i.e., $0 < x_{p_1} < C_{p_1}$, $0 < x_{p_2} < C_{p_2}$ and $x_i = C_i y_i$ for $p_1 < i < p_2$.

Define

$$g(x, y, s, z) = \sum_{i \in [1,T] \setminus \{p_1, p_2\}} (\alpha_i x_i + \beta_i y_i) + \sum_{\omega=1}^{m} \left(\pi_\omega \sum_{i \in [1,T] \setminus [p_1, p_2 - 1]} h_i s_{\omega i} \right).$$

Then

$$\begin{aligned} f(x, y, s, z) = g(x, y, s, z) &+ (\alpha_{p_1} x_{p_1} + \beta_{p_1} y_{p_1}) \\ &+ \sum_{\omega=1}^{m} \left(\pi_\omega \sum_{i=p_1}^{p_2-1} h_i s_{\omega i} \right) + (\alpha_{p_2} x_{p_2} + \beta_{p_2} y_{p_2}). \end{aligned}$$

Let $\eta = \min\{x_{p_1}, s_{\omega_{p_2-1}, p_2-1}, C_{p_2} - x_{p_2}\} > 0$ by the definition of regeneration interval. Let $\bar{x}$ and $\bar{s}$ be defined by $\bar{x}_{p_1} = x_{p_1} - \eta$, $\bar{y}_{p_1} = 0$ if $\eta = x_{p_1}$, $\bar{y}_{p_1} = 1$ otherwise, $\bar{s}_{\omega i} = s_{\omega i} - \eta$, for $i \in [p_1, p_2 - 1]$ and $\omega \in J_z$, $\bar{s}_{\omega i} = [s_{\omega i} - \eta]^+$, for $i \in [p_1, p_2 - 1]$ and $\omega \in \Omega \setminus J_z$, $\bar{x}_{p_2} = x_{p_2} + \eta$, and other components are the same as x, y and s. Then

$$\begin{aligned} f(\bar{x}, \bar{y}, \bar{s}, z) &= g(x, y, s, z) + (\alpha_{p_1}(x_{p_1} - \eta) + \beta_{p_1} \bar{y}_{p_1}) \\ &\quad + \sum_{\omega \in J_z} \left(\pi_\omega \sum_{i=p_1}^{p_2-1} h_i (s_{\omega i} - \eta) \right) + \sum_{\omega \in \Omega \setminus J_z} \left(\pi_\omega \sum_{i=p_1}^{p_2-1} h_i [s_{\omega i} - \eta]^+ \right) \\ &\quad + (\alpha_{p_2}(x_{p_2} + \eta) + \beta_{p_2} y_{p_2}) \\ &\le f(x, y, s, z) - \left(\alpha_{p_1} + \sum_{i=p_1}^{p_2-1} \left(\sum_{\omega \in J_z} \pi_\omega \right) h_i - \alpha_{p_2} \right) \eta \\ &\quad + \sum_{\omega \in \Omega \setminus J_z} \left(\pi_\omega \sum_{i=p_1}^{p_2-1} h_i ([s_{\omega i} - \eta]^+ - s_{\omega i}) \right) \\ &\le f(x, y, s, z) - \left(\alpha_{p_1} + \sum_{i=p_1}^{p_2-1} \left(\sum_{\omega \in J_z} \pi_\omega \right) h_i - \alpha_{p_2} \right) \eta \\ &= f(x, y, s, z) - \sum_{i=p_1}^{p_2-1} \left(\alpha_i + \left(\sum_{\omega \in J_z} \pi_\omega \right) h_i - \alpha_{i+1} \right) \eta. \end{aligned}$$

By Assumption 1, $\sum_{i=p_1}^{p_2-1}(\alpha_i + (\sum_{\omega \in J_z} \pi_\omega)h_i - \alpha_{i+1}) \geq \sum_{i=p_1}^{p_2-1}(\alpha_i + (1-\varepsilon)h_i - \alpha_{i+1}) \geq 0$, and hence $(\bar{x}, \bar{y}, \bar{s}, z)$ is a feasible solution of SCLS with $f(\bar{x}, \bar{y}, \bar{s}, z) \leq f(x, s, y, z)$. Furthermore,

(i) if $\eta = x_{p_1}$, then $\bar{x}_{p_1} = 0$, and the production sequence $\bar{X}_{i_k t_k}$ has one less fractional period than $X_{i_k t_k}$.

(ii) if $\eta = C_{p_2} - x_{p_2}$, then $\bar{x}_{p_2} = C_{p_2}$, and the production sequence $\bar{X}_{i_k t_k}$ has one less fractional period than $X_{i_k t_k}$.

(iii) if $\eta = s_{\omega_{p_2-1}, p_2-1}$, then $\bar{s}_{j_{p_2-1}, p_2-1} = 0$, $\bar{X}_{i_k, p_2-1}$ and $\bar{X}_{p_2 t_k}$ are both production sequences of $(\bar{x}, \bar{y}, \bar{s}, z)$, in which $\bar{X}_{i_k p_2-1}$ is capacity constrained with respect to z and $\bar{X}_{p_2 t_k}$ has one less fractional period than $X_{i_k t_k}$.

This process can be repeated with the solution $(\bar{x}, \bar{y}, \bar{s}, z)$ until finding a solution whose production levels are all made up of capacity constrained production sequences with respect to z and its cost is not more than the cost of (x, y, s, z). □

When $C_k = C$ for $k \in [1, T]$, given a feasible z and regeneration interval $[i, t]$, we can write $D_{\omega_t t} - D_{\omega_{i-1}, i-1} = \kappa C + \epsilon$, where κ is a nonnegative integer and $0 \leq \epsilon < C$. Then we have an immediate consequence of Lemma 3.

Corollary 1 *If $C_k = C$, $k \in [1, T]$, given an optimal z, then an optimal production sequence X_{it} with respect to z has $\kappa \in \mathbb{Z}_+$ periods in which the production levels are equal to C, at most one period with production level ϵ, and the remaining periods have zero production levels.*

We now construct a new formulation by introducing the following decision variables that take advantage of the structure of production levels according to Corollary 1 for constant capacity SLS (CCSLS):

- $\phi_{it}^{jk} = 1$ if $[i, t]$ is a regeneration interval with respect to z, and $\sigma_j^{i-1} = \arg\max_{\omega \in J_z}\{D_{\omega, i-1}\}$ and $\sigma_k^t = \arg\max_{\omega \in J_z}\{D_{\omega t}\}$, and production sequence X_{it} is capacity constrained with respect to z, for $1 \leq i \leq t \leq T$, $j \in [1, q_{i-1}^*]$, and $k \in [1, q_t^*]$,
- $\psi_{ipt}^{j\tau k} = 1$ if $\phi_{it}^{jk} = 1$ and $\sigma_\tau^p = \arg\max_{\omega \in J_z}\{D_{\omega p}\}$, for $1 \leq i \leq p < t \leq T$, $j \in [1, q_{i-1}^*]$, $k \in [1, q_t^*]$, and $\tau \in [1, q_p^*]$,
- $\lambda_{ipt}^{jk} = 1$ if $\phi_{it}^{jk} = 1$ and an amount C is produced in period p, for $1 \leq i \leq p \leq t \leq T$, $j \in [1, q_{i-1}^*]$, and $k \in [1, q_t^*]$,
- $\theta_{ipt}^{jk} = 1$ if $\phi_{it}^{jk} = 1$ and an amount r_{it}^{jk} is produced in the fractional period p, where $r_{it}^{jk} = (D_{\sigma_k^t} - D_{\sigma_j^{i-1}}) - C\lfloor (D_{\sigma_k^t} - D_{\sigma_j^{i-1}})/C \rfloor$, for $1 \leq i \leq p \leq t \leq T$, $j \in [1, q_{i-1}^*]$, and $k \in [1, q_t^*]$,

where we define $q_0^* := 1$ and $D_{\sigma_1^0} := 0$. We call the production plan determined by setting $\phi_{it}^{jk} = 1$ a subplan.

Define $I_{ip}^{j\tau} = \lceil (D_{\sigma_\tau^p} - D_{\sigma_j^{i-1}})/C \rceil$, for $1 \le i \le p \le T$, $j \in [1, q_{i-1}^*]$, and $\tau \in [1, q_p^*]$; $\bar{I}_{ipt}^{j\tau k} = \lceil (D_{\sigma_\tau^p} - D_{\sigma_j^{i-1}} - r_{it}^{jk})/C \rceil$, for $1 \le i \le p \le t \le T$, $j \in [1, q_{i-1}^*]$, $k \in [1, q_t^*]$, and $\tau \in [1, q_p^*]$.

We then obtain the following new extended formulation (NE-CCSLS):

$$\min \quad \alpha^\top x + \beta^\top y + \sum_{\omega=1}^{m} \pi_\omega h^\top s_\omega$$

$$\sum_{\tau=1}^{T} \sum_{k=1}^{q_\tau^*} \phi_{1\tau}^{1k} = 1 \tag{39}$$

$$\sum_{i=1}^{t-1} \sum_{j=1}^{q_{i-1}^*} \sum_{k=1}^{q_{t-1}^*} \phi_{i,t-1}^{jk} - \sum_{\tau=t}^{T} \sum_{j=1}^{q_{t-1}^*} \sum_{k=1}^{q_\tau^*} \phi_{t\tau}^{jk} = 0, \quad t \in [2, T] \tag{40}$$

$$\sum_{i=1}^{T} \sum_{j=1}^{q_{i-1}^*} \sum_{k=1}^{q_T^*} \phi_{iT}^{jk} = 1 \tag{41}$$

$$\sum_{i=1}^{p} \sum_{t=p}^{T} \sum_{j=1}^{q_{i-1}^*} \sum_{k=1}^{q_t^*} \left(\lambda_{ipt}^{jk} + \theta_{ipt}^{jk} \right) = y_p, \quad p \in [1, T] \tag{42}$$

$$\sum_{i=1}^{p} \sum_{t=p}^{T} \sum_{j=1}^{q_{i-1}^*} \sum_{k=1}^{q_t^*} \left(C\lambda_{ipt}^{jk} + r_{it}^{jk} \theta_{ipt}^{jk} \right) = x_p, \quad p \in [1, T] \tag{43}$$

$$\sum_{t=p+1}^{T} \sum_{k=1}^{q_t^*} \phi_{p+1,t}^{\tau k} + \sum_{i=1}^{p} \sum_{t=p+1}^{T} \sum_{j=1}^{q_{i-1}^*} \sum_{k=1}^{q_t^*} \psi_{ipt}^{j\tau k} = w_{\tau-1}^p - w_\tau^p, \quad \tau \in [1, q_p^* - 1], \quad p \in [1, T-1] \tag{44}$$

$$\sum_{i=1}^{p} \sum_{j=1}^{q_{i-1}^*} \phi_{ip}^{j\tau} + \sum_{i=1}^{p} \sum_{t=p+1}^{T} \sum_{j=1}^{q_{i-1}^*} \sum_{k=1}^{q_t^*} \psi_{ipt}^{j\tau k} = w_{\tau-1}^p - w_\tau^p, \quad \tau \in [1, q_p^* - 1], \quad p \in [1, T] \tag{45}$$

$$\sum_{\tau=1}^{q_p^*} \psi_{ipt}^{j\tau k} = \phi_{it}^{jk}, \quad j \in [1, q_{i-1}^*], \quad k \in [1, q_t^*], \quad 1 \le i \le p < t \le T \tag{46}$$

$$\sum_{l=i}^{p} \left(\lambda_{ilt}^{jk} + \theta_{ilt}^{jk} \right) \ge \sum_{\tau=1}^{q_p^*} I_{ip}^{j\tau} \psi_{ipt}^{j\tau k}, \quad j \in [1, q_{i-1}^*], \quad k \in [1, q_t^*], \quad 1 \le i \le p < t \le T \tag{47}$$

$$\sum_{l=i}^{t} \left(\lambda_{ilt}^{jk} + \theta_{ilt}^{jk} \right) = I_{it}^{jk} \phi_{it}^{jk}, \quad j \in [1, q_{i-1}^*], \quad k \in [1, q_t^*], \quad 1 \le i \le t \le T \tag{48}$$

$$\sum_{l=i}^{p} \lambda_{ilt}^{jk} \geq \sum_{\tau=1}^{q_p^*} \bar{I}_{ipt}^{j\tau k} \psi_{ipt}^{j\tau k}, \quad j \in [1, q_{i-1}^*], \quad k \in [1, q_t^*],$$
$$1 \leq i \leq p < t \leq T \tag{49}$$

$$\sum_{l=i}^{t} \lambda_{ilt}^{jk} = \bar{I}_{itt}^{jkk} \phi_{it}^{jk}, \quad j \in [1, q_{i-1}^*], \quad k \in [1, q_t^*], \quad 1 \leq i \leq t \leq T \tag{50}$$

$$\lambda_{ipt}^{jk} + \theta_{ipt}^{jk} \leq \phi_{it}^{jk}, \quad j \in [1, q_{i-1}^*], \quad k \in [1, q_t^*], \quad 1 \leq i \leq p \leq t \leq T$$
$$\phi_{it}^{jk} \in \{0, 1\}, \quad j \in [1, q_{i-1}^*], \quad k \in [1, q_t^*], \quad 1 \leq i \leq t \leq T$$
$$\psi_{ipt}^{j\tau k} \in \{0, 1\}, \ j \in [1, q_{i-1}^*], \ \tau \in [1, q_p^*], \ k \in [1, q_t^*], \ 1 \leq i \leq p < t \leq T$$
$$\lambda_{ipt}^{jk} \in \{0, 1\}, \theta_{ipt}^{jk} \in \{0, 1\}, \ j \in [1, q_{i-1}^*], \ k \in [1, q_t^*], \ 1 \leq i \leq p \leq t \leq T$$
$$(x, y, s, z, w) \text{ satisfy } (15) - (18), (20) - (21) \tag{51}$$

where we define $w_0^p := 1$, for $p \in [1, T]$. Constraints (39)–(41) are the flow conservation constraints that model a production plan as a sequence of subplans. Constraints (42) indicate when the production happens. Constraints (43) calculate the production levels in each period. Constraints (44)–(45) together with (15)–(17) ensure that the production plan determined by the subplans satisfies the chance constraint. The auxiliary variables $(w_1^p, \ldots, w_{q_p^*-1}^p)$ for $p \in [1, T]$ play a similar role as in E-SLS and NE-SLS. For each $p \in [1, T]$, (15) ensures that there is at most one $\tau \in [1, q_p^*-1]$ such that $w_{\tau-1}^p - w_\tau^p = 1$, and $w_{\tau-1}^p - w_\tau^p = 0$ otherwise. As in NE-SLS, $w_{\tau-1}^p - w_\tau^p = 1$ is an indication that the largest cumulative demand from the beginning to period p needs to be satisfied is that of scenario σ_τ^p. Thus, constraints (44) (or (45)) enforce consistency between this in period p and the subplan next to period p determined by the $\phi_{p+1,t}^{\tau k}$ (or $\phi_{ip}^{j\tau}$) variables or containing period p determined by the $\psi_{ipt}^{j\tau k}$ variables. Constraints (46) ensure that the auxiliary variables associated with subplan ϕ_{it}^{jk} can be nonzero only when that subplan is selected. Constraints (47)–(51) are to depict the full-capacity production and fractional production when ϕ_{it}^{jk} is selected as a subplan.

Proposition 2 *Under Assumption 1, formulation NE-CCSLS is a valid model for CCSLS.*

Proof Let (x, y, s, z) be a feasible solution of CCSLS where x consists of capacity constrained production sequences with respect to z. Let $I = \{[i_1, t_1], [i_2, t_2], \ldots, [i_r, t_r]\}$ be the set of all regeneration intervals, where $1 = i_1 \leq t_1 < i_2 \leq t_2 < \ldots < i_r \leq t_r = T$ and $i_{u+1} = t_u + 1$ for $u \in [1, r-1]$. For each $i \in [1, T]$, define $\bar{j}(i) = \min\{j \in [1, q_i^*] : z_{\sigma_j^i} = 0\}$, and let $\bar{j}(0) = 1$. Now, for $1 \leq i \leq t \leq T$, $j \in [1, q_{i-1}^*], k \in [1, q_t^*]$, define

$$\phi_{it}^{jk} = \begin{cases} 1 & \text{if } i = i_u, \ t = t_u, \ j = \bar{j}(i_u - 1), \ k = \bar{j}(t_u), \ u \in [1, r] \\ 0 & \text{otherwise.} \end{cases}$$

For $1 \leq i \leq p \leq t \leq T$, $j \in [1, q^*_{i-1}]$, $k \in [1, q^*_t]$, define

$$\lambda^{jk}_{ipt} = 1 \text{ if } (\phi^{jk}_{it} = 1 \ \& \ y_p = 1 \ \& \ x_p = C) \text{ and } \lambda^{jk}_{ipt} = 0 \text{ otherwise, and}$$
$$\theta^{jk}_{ipt} = 1 \text{ if } (\phi^{jk}_{it} = 1 \ \& \ y_p = 1 \ \& \ 0 < x_p < C) \text{ and } \theta^{jk}_{ipt} = 0 \text{ otherwise.}$$

For $1 \leq i \leq p < t \leq T$, $j \in [1, q^*_{i-1}]$, $k \in [1, q^*_t]$, $\tau \in [1, q^*_p]$, define

$$\psi^{j\tau k}_{ipt} = 1 \text{ if } (\phi^{jk}_{it} = 1 \ \& \ \bar{j}(p) = \tau), \text{ and } \psi^{j\tau k}_{ipt} = 0 \text{ otherwise.}$$

Finally, for $i \in [1, T]$, $j \in [1, q^*_i]$ define

$$w^i_j = 1 \text{ if } j \in [1, \bar{j}(i) - 1], \text{ and } w^i_j = 0 \text{ otherwise.}$$

Direct calculations verify that $(x, y, s, z, w, \phi, \lambda, \theta, \psi)$ satisfies all the constraints of NE-CCSLS.

Now let $(x, y, s, z, w, \phi, \lambda, \theta, \psi)$ be a feasible solution of the NE-CCSLS. For $i \in [1, T]$, (42)–(43) imply that when $y_i = 0$, then $x_i = 0$, and when $y_i = 1$, then $x_i \leq C$, which indicates that (x, y, s, z) satisfies the constant capacity constraints. Trivially, (x, y, s, z) satisfies (9) and (10) as they are contained in the formulation of NE-CCSLS. Let $\hat{I} = \{i \in [1, T] : \exists t \in [i, T],\ j \in [1, q^*_{i-1}], k \in [1, q^*_t], \text{ s.t. } \phi^{jk}_{it} = 1\} = \{\hat{i}_1, \hat{i}_2, \ldots, \hat{i}_r\}$, where $1 = \hat{i}_1 < \hat{i}_2 < \ldots < \hat{i}_r \leq T < \hat{i}_{r+1} := T + 1$, and define $\hat{t}_u = \hat{i}_{u+1} - 1$, for $u \in [1, r]$. Similar to the proof of Proposition 1, we can conclude that there exists exactly one $j_u \in [1, q^*_{\hat{i}_u - 1}]$ and one $k_u \in [1, q^*_{\hat{t}_u}]$, such that $\phi^{j_u k_u}_{\hat{i}_u \hat{t}_u} = 1$, for $u \in [1, r]$, and $k_u = j_{u+1}$, for $u \in [1, r-1]$. By (48) and (46), when $\phi^{jk}_{it} = 0$, $\lambda^{jk}_{ipt} = \theta^{jk}_{ipt} = 0$, for $i \leq p \leq t$, and $\psi^{j\tau k}_{ipt} = 0$, for $i \leq p \leq t$, $\tau \in [1, q^*_p]$. Thus, for any $p \in [\hat{i}_u, \hat{t}_u]$, $u \in [1, r]$, by (43) we have

$$\sum_{l=1}^{p} x_l = I + II \tag{52}$$

where

$$I = \sum_{v=1}^{u-1} \sum_{l=\hat{i}_v}^{\hat{t}_v} \left(C\lambda^{j_v k_v}_{\hat{i}_v l \hat{t}_v} + r^{j_v k_v}_{\hat{i}_v \hat{t}_v} \theta^{j_v k_v}_{\hat{i}_v l \hat{t}_v} \right),$$

$$II = \sum_{l=\hat{i}_u}^{p} \left(C\lambda^{j_u k_u}_{\hat{i}_u l \hat{t}_u} + r^{j_u k_u}_{\hat{i}_u \hat{t}_u} \theta^{j_u k_u}_{\hat{i}_u l \hat{t}_u} \right).$$

Our main claim is that

$$I + II \geq D_{\sigma^p_{\tau_p}}. \tag{53}$$

First, we compute

$$
\begin{aligned}
I &= \sum_{v=1}^{u-1} \left(C \sum_{l=\hat{i}_v}^{\hat{t}_v} \lambda_{\hat{i}_v l \hat{t}_v}^{j_v k_v} + r_{\hat{i}_v \hat{t}_v}^{j_v k_v} \sum_{l=\hat{i}_v}^{\hat{t}_v} \theta_{\hat{i}_v l \hat{t}_v}^{j_v k_v} \right) \\
&= \sum_{v=1}^{u-1} \left(C \bar{I}_{\hat{i}_v \hat{t}_v \hat{t}_v}^{j_v k_v k_v} \phi_{\hat{i}_v \hat{t}_v}^{j_v k_v} + r_{\hat{i}_v \hat{t}_v}^{j_v k_v} (I_{\hat{i}_v \hat{t}_v}^{j_v k_v} - \bar{I}_{\hat{i}_v \hat{t}_v \hat{t}_v}^{j_v k_v k_v}) \phi_{\hat{i}_v \hat{t}_v}^{j_v k_v} \right) \text{ (by (48), (50))} \\
&= \sum_{v=1}^{u-1} \left(C \bar{I}_{\hat{i}_v \hat{t}_v \hat{t}_v}^{j_v k_v k_v} + r_{\hat{i}_v \hat{t}_v}^{j_v k_v} \left(I_{\hat{i}_v \hat{t}_v}^{j_v k_v} - \bar{I}_{\hat{i}_v \hat{t}_v \hat{t}_v}^{j_v k_v k_v} \right) \right) \\
&= \sum_{v=1}^{u-1} \left(D_{\sigma_{k_v}^{\hat{t}_v}} - D_{\sigma_{j_v}^{\hat{i}_v - 1}} \right) = D_{\sigma_{k_{u-1}}^{\hat{t}_{u-1}}}.
\end{aligned}
$$

If $p = \hat{t}_u$, then the calculation identical to that for I yields $I + II = D_{\sigma_{\hat{t}_u}^{\hat{t}_u}}$ and hence (53) follows. Thus, assume now that $p < \hat{t}_u$. By (46), there is $\tau_p \in [1, q_p^*]$ such that $\psi_{\hat{i}_u p \hat{t}_u}^{j_u \tau_p k_u} = 1$. According to (49), we know $\sum_{l=\hat{i}_u}^{p} \lambda_{\hat{i}_u l \hat{t}_u}^{j_u k_u} \geq \bar{I}_{\hat{i}_u p \hat{t}_u}^{j_u k_u}$. If $\sum_{l=\hat{i}_u}^{p} \lambda_{\hat{i}_u l \hat{t}_u}^{j_u k_u} \geq \bar{I}_{\hat{i}_u p \hat{t}_u}^{j_u \tau_p k_u} + 1$, then

$$
II \geq C \left(\bar{I}_{\hat{i}_u p \hat{t}_u}^{j_u \tau_p k_u} + 1 \right) \geq C I_{\hat{i}_u p}^{j_u \tau_p} \geq D_{\sigma_{\tau_p}^{p}} - D_{\sigma_{j_u}^{\hat{i}_u - 1}} = D_{\sigma_{\tau_p}^{p}} - D_{\sigma_{k_{u-1}}^{\hat{t}_{u-1}}}.
$$

If $\sum_{l=\hat{i}_u}^{p} \lambda_{\hat{i}_u l \hat{t}_u}^{j_u k_u} = \bar{I}_{\hat{i}_u p \hat{t}_u}^{j_u \tau_p k_u}$, then

$$
\begin{aligned}
II &\geq C \bar{I}_{\hat{i}_u p \hat{t}_u}^{j_u \tau_p k_u} + r_{\hat{i}_u \hat{t}_u}^{j_u k_u} (I_{\hat{i}_u p}^{j_u \tau_p} - \bar{I}_{\hat{i}_u p \hat{t}_u}^{j_u \tau_p k_u}) \text{ (by (47))} \\
&\geq \begin{cases} C I_{\hat{i}_u p}^{j_u \tau_p} & \text{if } I_{\hat{i}_u p}^{j_u \tau_p} = \bar{I}_{\hat{i}_u p \hat{t}_u}^{j_u \tau_p k_u} \\ D_{\sigma_{\tau_p}^{p}} - D_{\sigma_{j_u}^{\hat{i}_u - 1}} - r_{\hat{i}_u \hat{t}_u}^{j_u k_u} + r_{\hat{i}_u \hat{t}_u}^{j_u k_u} & \text{if } I_{\hat{i}_u p}^{j_u \tau_p} = \bar{I}_{\hat{i}_u p \hat{t}_u}^{j_u \tau_p k_u} + 1 \end{cases} \\
&\geq D_{\sigma_{\tau_p}^{p}} - D_{\sigma_{k_{u-1}}^{\hat{t}_{u-1}}}.
\end{aligned}
$$

In either case combining this with $I = D_{\sigma_{k_{u-1}}^{\hat{t}_{u-1}}}$ we obtain the claim (53). By (44) and (45), we have $w_j^p = 1$ for $j \in [1, \tau_p - 1]$ and $z_{\sigma_j^p} = 1$ for $j \in [1, \tau_p - 1]$. Combining (52) with (53) and using this observation yields

$$\sum_{l=1}^{p} x_l \geq D_{\sigma^p_{\tau_p}} \geq \begin{cases} D_{\sigma^p_j}(1 - z_{\sigma^p_j}) & \text{for } j \in [1, \tau_p - 1] \\ D_{\sigma^p_j} & \text{for } j \in [\tau_p, m] \end{cases}$$
$$\geq D_{\omega i}(1 - z_\omega) \quad \text{for } \omega \in \Omega.$$

Thus, (x, y, s, z) satisfies (8) and hence is a feasible solution of CCSLS. □

The size of NE-CCSLS can also be significantly reduced by an analogous result to Lemma 2. However, in any case, with $\mathcal{O}(T^3(\varepsilon m)^2)$ constraints the size of NE-CCSLS increases rapidly with the planning horizon T and number of scenarios m, thus potentially limiting its direct use to small instances.

5 Computational experiments

In this section we report results from our experiments with the formulations introduced in this paper. All the experiments were executed on a Windows 10 Home workstation with 1.80GHz Intel(R) Core(TM) i7-8550U CPU and 16.0 GB RAM. The algorithms tested in the computational experiment were implemented using Python programming language, with Python 3.7 and Gurobi 8.1.1 as the MIP solver. A time limit of one hour is enforced for all experiments.

For formulation CC-(ℓ, S) we use Algorithm 1 to separate the CC-(ℓ, S) inequalities and add them to the LP relaxation iteratively in a cutting plane loop. We keep adding cuts until the LP relaxation is not improved for five consecutive iterations. After that, the formulation with these cuts included is given to the solver Gurobi and solved. We do not test the use of CC-(ℓ, S) inequalities at nodes within the branch-and-bound process. Although it is possible that such use may yield improved performance, we see in the computational results that adding the cuts at the root node only already significantly reduces the number of branch-and-bound nodes explored, and hence the additional benefit from such an implementation is likely to be minimal.

As an additional comparison, we investigate the use of the valid inequalities of Proposition 3.2 in [22] (which we refer to as LK–Cuts) added to formulation E-SLS, which we refer to as E–SLS+LK. We separate LK–Cuts using the heuristic separation algorithm proposed in [22]. We found that adding these cuts within a cutting plane loop led to a large number of LK-Cuts, which after a while had very marginal improvement on the LP relaxation. To achieve the most benefit from these cuts while also limiting the number added we implemented a stopping condition for the cut generation loop. Let $objval1$ be the LP relaxation objective value of E-SLS after adding the LK-Cuts for some iterations, and $objval2$ be the LP relaxation objective value of E-SLS after one more iteration. Then, we stop the process of generating LK-Cuts once $\frac{objval2-objval1}{objval1} \leq \beta$ occurs in five consecutive iterations of the cutting plane loop. We tested the performance of E–SLS+LK with $\beta = 0.0001$ and $\beta = 0.02$, and found that generally E–SLS+LK with $\beta = 0.02$ performs better in terms of total computation time. Thus, we use $\beta = 0.02$ in our experiments using the E–SLS+LK method.

In our test instances we set $\pi_\omega = \frac{1}{m}$ for all $\omega \in \Omega$. The data we used is adapted from [4], in which the integrality gap of the lot-sizing problem instances is influenced by the ratio between the setup cost and inventory holding cost. Therefore, the instances are generated for varying setup/holding cost ratios $f \in \{100, 200, 500, 1000\}$. For all instances, holding cost h_i equals to 10. We generate cost coefficients in two ways. For instances with CoefType=RND the unit production cost α_i and setup cost β_i are drawn from a discrete uniform distribution over $[81, 119]$ and $[9f, 11f]$, respectively, for all $i \in [1, T]$. For instances with CoefType=Const we set $\alpha_i = 100$ and $\beta_i = 10f$ for all $i \in [1, T]$. In addition, the demand $d_{\omega i}$ follows a discrete uniform distribution $[1, 19]$, for each period $i \in [1, T]$ and scenario $\omega \in \Omega$.

We next discuss the metrics we report in tables and figures. When "Time" is reported it refers to the average solution time, in seconds, for the instances that are solved to optimality within the time limit. When present, the number in the brackets "[ ]" next to a time entry indicates how many instances among those averaged are solved to optimality within the time limit, and the "*" symbol indicates that none of the three instances are to optimality within the time limit. "Nodes" refers to the average number of nodes explored during the branch-and-bound process, and the ">" sign preceding a nodes quantity indicates that the reported number of nodes is a lower bound, because not all the instances were solved within the time limit and the average is computed using the number of nodes processed up to the limit. "Ending Gap" ("Gap" for short) refers to the average percentage optimality gap at the time limit, i.e., Ending Gap $= \frac{ubval - lbval}{ubval}$ where $ubval$ is the value of the best feasible solution found by that method within the time limit and $lbval$ is the lower bound obtained within the time limit. "LP Gap" and "Root Gap" refer to the average percentage gap of the LP relaxation and root node, respectively, i.e., LP Gap $= \frac{bestubval - lpval}{bestubval} \times 100$, and Root Gap $= \frac{bestubval - rootval}{bestubval} \times 100$, where $bestubval$, $lpval$ and $rootval$ are the objective function values of the best feasible solution (the best solution found among all the comparing formulations), the LP relaxation after the addition of user cuts (CC-(ℓ, S) inequalities or LK cuts), and the LP relaxation after all Gurobi cuts are added before branching, respectively. "Cuts" refers to the average number of user cuts added to the formulation.

First, we compare the performance of the original formulation SLS and the extended formulation E-SLS in Table 1 on instances with $\varepsilon = 0.1$, $T = 30$, and $m = 100$. The size of the instances we used here are small enough so that both formulations can solve all the instances to optimality. These instances can be solved significantly faster using formulation E-SLS than with SLS, in particular due to significantly better LP and root relaxations. Therefore, we do not report results with SLS in further experiments.

We next present results comparing formulations E-SLS, LK, CC-(ℓ, S), and CC-(ℓ, S)-WW on larger instances of SLS. For this comparison we use a relatively large value of $\varepsilon = 0.1$ in all these instances, as smaller values of ε lead to easier to solve instances. We create instances having $T \in \{30, 60\}$, CoefType $\in$ {RND,Const.}, $f \in \{100, 200, 500, 1000\}$ and $m \in \{500, 1000, 2000\}$. We generate three test instances for each case for a total of 144 test instances.

In Fig. 1 we present results separately for instances having a fixed combination of T and CoefType (36 in each). In each such plot, the x-axis is time in seconds, the

Table 1 Comparison between SLS and E-SLS on instances with $\varepsilon = 0.1$, $T = 30$, and $m = 100$

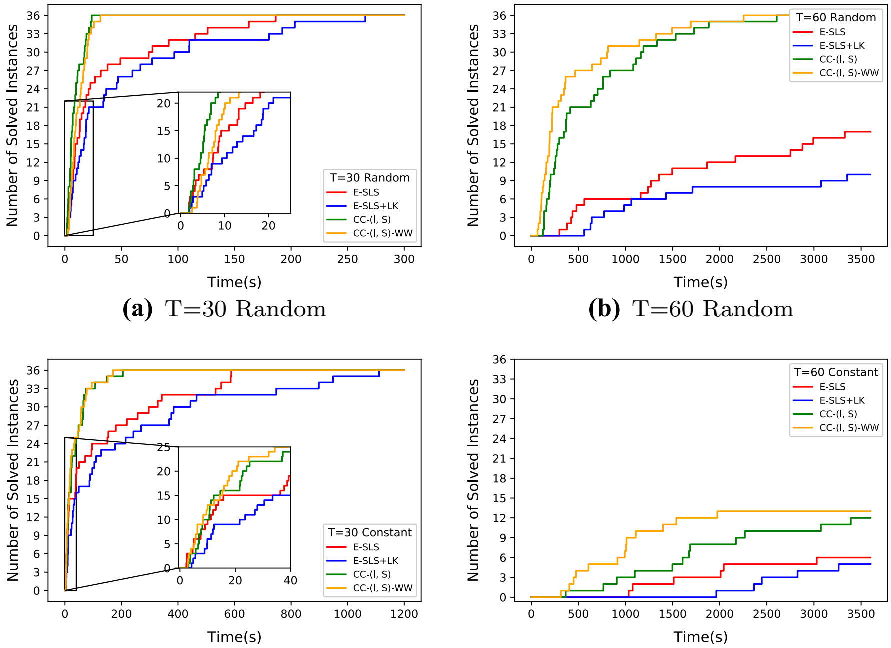
	SLS				E-SLS			
f	Time	Nodes	LP Gap (%)	Root Gap (%)	Time	Nodes	LP Gap (%)	Root Gap (%)
100	506.7	70721	41.06	14.28	0.9	387	12.26	2.38
200	159.1	22076	41.30	14.63	0.7	338	14.70	2.54
500	103.8	15097	41.01	19.01	0.7	217	15.69	3.30
1000	46.3	5968	36.30	16.40	0.5	100	14.65	0.72

y-axis is number of instances, and the plot for each method is the number instances solved over time. These results indicate that the two methods that use the proposed CC-(ℓ, S) inequalities significantly outperform the other methods, and that they can solve all instances within the time limit, except for the case where $T = 60$ and CoefType=Const. The results also indicate that using the fixed subset of CC-(ℓ, S) inequalities in the CC-(ℓ, S)-WW method yields somewhat better results than separating them as needed, even for the instances with CoefType=RND, which do not necessarily satisfy the modified Wagner-Whitin cost condition.

Table 2 provides results of other metrics, averaged over instances for each combination of T and CoefType tested. From this table we observe that the methods that use CC-(ℓ, S) inequalities yield significantly smaller ending optimality gaps on the instances with $T = 60$ and CoefType=Const. Most of the instances with $T = 60$ are not solved within the time limit using E-SLS and E-SLS+LK. We observe that E-SLS+LK does close significant gap at the root node compared to E-SLS, but this does not generally translate to improved computation time or ending gap. As the LK-Cuts do not improve the computational performance in our test instances, we do not display the results of E-SLS+LK in our following experiments.

In Table 3 we investigate the impact of varying $\varepsilon \in \{0.05, 0.1, 0.2\}$ for instances with fixed $T = 60$,$m = 1000$, and $f = 200$. We find that instances with higher ε are more difficult to solve, but otherwise the observations are consistent with those seen in the previous results, in particular with CC-(ℓ, S)-WW having the best performance.

In Tables 4–5, we present results comparing the performance of E-SLS, CC-(ℓ, S)-WW, and NE-SLS on instances having more time periods ($T = 90$) but fewer scenarios ($m \in \{100, 200, 300\}$) and ε =0.05 and CoefType=RND. We assure these instances satisfy the modified Wagner-Whitin condition by generating α_1 according to a discrete uniform distribution over [81,119] and α_{i+1} according to a discrete uniform distribution over [81, $\min\{119, \alpha_i + (1-\varepsilon)h_i\}$] for $i \in [2, T]$. Each entry in these tables is an average over three instances. We observe that both CC-(ℓ, S)-WW and NE-SLS perform much better than E-SLS, so we do not display the detailed results for E-SLS in Table 5. NE-SLS performs better than CC-(ℓ, S)-WW for all instances in this regime (relatively small m and ε). In Table 5, we see that NE-SLS has smaller root gap and explores many fewer nodes to reach optimality than CC-(ℓ, S)-WW. Due to its size, formulation NE-SLS becomes less effective for instances with more scenarios or



(a) T=30 Random **(b)** T=60 Random

(c) T=30 Constant **(d)** T=60 Constant

Fig. 1 Number of instances solved over time by formulations E-SLS, E-SLS+LK, CC-(ℓ, S) and CC-(ℓ, S)-WW

larger ε. Thus, we conclude that NE-SLS and CC-(ℓ, S) are complementary: NE-SLS generally performs better if the number of scenarios and ε are relatively small, and CC-(ℓ, S)-WW is better otherwise.

In our final computational experiment we consider the capacitated problem CCSLS. We compare three methods: (i) E-CCSLS, which is exactly formulation E-SLS but with the constraints $x_i \le M_i y_i$ replaced by $x_i \le C y_i$ for $i \in [1, T]$; (ii) CC-(ℓ, S)-WW, the same as E-CCLS but with the limited set of CC-(ℓ, S) inequalities added a priori to the reformulation, and (iii) NE-CCSLS. Since our primary goal with this experiment is to investigate the strength of the NE-CCSLS formulation, we limit the size of the test instances in this experiment to $T = 30$ and $m = 100$. We use a fixed constant capacity of $C = 40$ and vary $\varepsilon \in \{0.05, 0.1\}$ and $f \in \{100, 200, 500, 1000\}$. The results of this experiment are presented in Tables 6 and 7. We find that for these instances the CC-(ℓ, S) inequalities once again reduce the LP relaxation and root gaps, and this translates into a reduction in the number of branch-and-bound nodes explored. However, this reduction does not translate into a reduction of the computation time relative to E-CCLS, although we observe that instances of this size are solved quickly by both formulations. On the other hand, formulation NE-CCSLS has an extremely strong LP relaxation leading to all instances being solved with no branching required. However, due to the very large size of NE-CCSLS, the computation time of this formulation is much higher than the other methods. Thus, we conclude that an interesting direction

Table 2 Comparison among E-SLS, E-SLS+LK, CC-(ℓ, S) and CC-(ℓ, S)-WW

Metric	T	Costs	E-SLS	LK	CC-(ℓ, S)	CC-(ℓ, S)-WW
Ending	30	RND	0.00	0.00	0.00	0.00
Gap (%)	30	Const.	0.00	0.00	0.00	0.00
	60	RND	1.45	2.25	0.00	0.00
	60	Const.	5.35	5.60	1.31	1.12
LP Gap	30	RND	16.41	9.27	4.47	4.47
(%)	30	Const.	16.38	9.92	4.16	4.16
	60	RND	20.64	15.29	5.05	5.05
	60	Const.	22.13	18.23	4.96	4.96
Root Gap	30	RND	5.48	4.95	1.44	1.50
(%)	30	Const.	6.49	5.63	1.84	1.88
	60	RND	11.14	9.91	3.29	3.33
	60	Const.	13.62	12.35	3.81	3.74
Nodes	30	RND	2615	2262	403	383
	30	Const.	8645	7256	1812	1525
	60	RND	$>$ 50960	$>$ 21562	7195	6630
	60	Const.	$>$ 29559	$>$ 18367	$>$ 11186	$>$ 13814
Cuts	30	RND	–	289	414	465
	30	Const.	–	274	434	465
	60	RND	–	554	2314	1830
	60	Const.	–	421	2842	1830

Table 3 Comparison among E-SLS, CC-(ℓ, S) and CC-(ℓ, S)-WW with (T, m, f)= (60, 1000, 200)

Instances		E-SLS		CC-(ℓ, S)		CC-(ℓ, S)-WW	
Costs	ε	Time	Gap	Time	Gap	Time	Gap
Random	0.05	1953.4	0.00	111.0	0.00	98.7	0.00
	0.1	*	2.93	526.3	0.00	355.4	0.00
	0.2	*	5.41	[1]377.5	0.71	1811.7	0.00
Constant	0.05	*	3.79	*	0.89	[1]2964.9	0.50
	0.1	*	7.83	*	1.98	*	1.87
	0.2	*	10.95	*	3.06	*	3.08

for future research for CCSLS is to investigate formulations or valid inequalities whose relaxation can be solved more quickly than NE-CCLS, but which have a better relaxation value than can be obtained with the CC-(ℓ, S) inequalities.

Table 4 Comparison of the average time and ending optimality gap among E-SLS, CC-(ℓ, S)-WW, and NE-SLS with (T, ε)= (90, 0.05)

Instances		E-SLS		CC-(ℓ, S)-WW		NE-SLS	
m	f	Time	Gap	Time	Gap	Time	Gap
100	100	1370.2	0.00	181.5	0.00	28.4	0.00
	200	[1]1824.0	0.46	745.1	0.00	69.4	0.00
	500	[2]1900.3	0.21	247.4	0.00	21.7	0.00
	1000	1133.1	0.00	165.7	0.00	8.6	0.00
200	100	[1]1681.8	1.28	755.9	0.00	391.2	0.00
	200	*	2.38	1836.4	0.00	585.5	0.00
	500	*	3.69	1975.6	0.00	383.1	0.00
	1000	*	4.07	1024.5	0.00	297.9	0.00
300	100	*	2.06	[2]1291.9	0.34	[2]2135.3	0.30
	200	*	3.50	*	0.66	[2]2793.7	0.14
	500	*	5.36	1744.0	0.00	1470.0	0.00
	1000	*	4.33	969.3	0.00	867.6	0.00

Table 5 Comparison between CC-(ℓ, S)-WW and NE-SLS with (T, ε) = (90, 0.05)

Instances		CC-(ℓ, S)-WW			NE-SLS		
m	f	Nodes	LP Gap (%)	Root Gap (%)	Nodes	LP Gap (%)	Root Gap (%)
100	100	4674	3.24	0.97	28	1.11	0.15
	200	23920	3.99	1.73	267	1.47	0.44
	500	3928	4.30	1.94	13	0.98	0.01
	1000	1738	4.31	1.79	1	0.34	0.00
200	100	11744	3.57	1.70	2155	2.44	1.73
	200	26309	4.28	2.37	2421	2.53	1.59
	500	43660	5.39	3.42	2428	2.22	1.35
	1000	12384	5.80	3.81	605	1.92	0.56
300	100	$>$ 19369	3.79	1.83	$>$ 8552	2.69	2.08
	200	$>$ 34605	4.40	2.51	$>$ 10912	2.90	2.24
	500	22917	5.15	3.02	2416	2.19	1.47
	1000	9799	5.46	2.96	696	1.61	0.47

6 Conclusions

There are several directions that could be explored related to this work. In our experience, we found that for most of the instances we tested, formulation NE-SLS has the same optimal solutions as SLS, even when Assumption 1 does not hold. Thus an interesting direction to explore is to see if it can be shown that NE-SLS is a valid formulation under weaker assumptions. We found that formulations NE-SLS and NE-CCSLS both

Table 6 Comparison between E-CCSLS and NE-CCSLS with constant capacity $C = 40$

Instances		E-CCSLS		CC-(ℓ, S)-WW		NE-CCSLS	
(ε, T, m)	f	Time	Nodes	Time	Nodes	Time	Nodes
(0.05, 30, 100)	100	0.9	796	2.5	336	75.0	1
	200	0.5	226	2.2	69	47.2	1
	500	0.5	309	1.8	137	87.1	1
	1000	0.3	50	1.4	39	95.0	1
(0.1, 30, 100)	100	4.6	4899	5.2	1220	1529.6	1
	200	1.6	860	3.5	661	2803.8	1
	500	0.9	839	2.0	157	810.1	1
	1000	0.6	279	2.0	173	1554.5	1

Table 7 Comparison between E-CCSLS and NE-CCSLS with constant capacity $C = 40$

Instances		E-CCSLS		CC-(ℓ, S)-WW		NE-CCSLS	
(ε, T, m)	f	LP Gap (%)	Root Gap (%)	LP Gap (%)	Root Gap (%)	LP Gap (%)	Root Gap (%)
(0.05, 30, 100)	100	7.11	1.62	3.59	0.92	0.13	0.00
	200	7.72	0.84	5.03	0.32	0.00	0.00
	500	7.11	0.83	5.03	0.83	0.10	0.00
	1000	6.39	0.49	5.33	0.52	0.09	0.00
(0.1, 30, 100)	100	7.86	2.84	4.38	2.25	0.58	0.00
	200	7.39	2.24	4.66	1.96	0.21	0.00
	500	6.74	1.98	4.71	1.25	0.30	0.00
	1000	8.16	0.78	7.00	0.72	0.05	0.00

provide very tight linear programming relaxations, but these formulations (particularly NE-CCSLS) are limited in scalability due to their large size. Thus, it would be interesting to explore either decomposition algorithms for solving these formulations or the use of approximate extended formulations [30] to obtain a better trade-off between formulation size and relaxation strength. Alternatively, it would be interesting to explore generalizations of known valid inequalities for the deterministic CLS problem to the joint chance-constrained stochastic version.

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