1 Options (derivatives) of stocks and how to price them

Here we will start with our Binomial Lattice Model (BLM) for a risky asset (stock), introduce options (derivatives) of the stock, and learn how to price them. We will in passing obtain the famous Black-Scholes-Merton option pricing formula in discrete time.

Recall that the price per share of our risky asset at any time can be defined recursively via

$$S_{n+1} = S_n Y_{n+1},\ n \geq 0,$$

where the $\{Y_i\}$ are iid with distribution $P(Y = u) = p,\ P(Y = d) = 1 - p$. d, r, u are constants (parameters) satisfying

$$0 < d < 1 + r < u, \tag{1}$$

with r the *risk-free interest rate*: $x_0(1 + r)$ is the payoff you would receive one unit of time later if you placed x_0 in a bank account at fixed rate r at time $n = 0$. $x_0(1 + r)^n$ would be your payoff n time units from now.

r is also the interest rate you would pay the bank if your took out a loan: If you borrowed x_0 dollars now (time $n = 0$), then you would owe the bank $x_0(1 + r)^n$ n time units later.

For an amount of money x promised to you at time n in the future, the *present value* of x is given by $x_0 = (1 + r)^{-n}x$. It represents how much money x promised you at time n in the future is worth now. The idea is that by placing the *discounted* amount $x_0 = (1 + r)^{-n}x$ in the bank at interest rate r now, you would have payoff x at time n.

Given the value S_n, at any time $n \geq 0$,

$$S_{n+1} = \begin{cases} uS_n & \text{with probability } p\ , \\ dS_n & \text{with probability } 1 - p. \end{cases}$$

independent of the past. Thus the stock price either goes up (u) or down (d) in each time period, and the randomness is due to iid Bernoulli (p) rvs (flips of a coin so to speak) where we can view "up=success", and "down=failure".

Expanding the recursion yields

$$S_n = S_0 \times Y_1 \times \cdots \times Y_n,\ n \geq 1, \tag{2}$$

where S_0 is the initial price per share.

Equation (1) must hold due to basic economic considerations. For example, suppose instead that

$$0 < 1 + r < d < u.$$

If so, then the stock would be a better investment with certainty than placing money in the bank, so nobody would place any money in the bank, which of course is nonsense!

In fact this is related to the notion of *arbitrage*, meaning that it would be possible for you to make an arbitrarily large amount of money from nothing!

To see this: You could borrow from the bank the amount of money S_0 now at interest rate r, and immediately use it to buy a share of the stock. Then, when it is time to pay back the loan at time (say) n in the future, since $S_n \geq d^n S_0 > (1+r)^n S_0$, you would make a profit of $P = S_n - (1+r)^n > 0$ for free! You could do even better by borrowing (say) $1000S_0$ and immediately buying 1000 shares, and so on.

1.1 Options

Options or *derivatives* of the stock, are financial instruments created from the stock and based on the values of the stock over time. The simplest example is the *European call option*, which works as follows: There is a given fixed *strike price* $K > 0$, and a fixed *expiration date (time)* $T \geq 1$ in the future, at which time a payoff is given out. The holder of the option has the right to buy a share of stock at time T at price K.

On the one hand, if at time T, the market price $S_T > K$, then the holder will exercise the option to get payoff $S_T - K$ (imagine that the holder buys at the lower price K, then immediately sells at the higher market price S_T).

On the other hand, if $S_T < K$, then the holder would not exercise the option because K is higher than the available market price; the payoff is thus zero. In any case, the option expires at time T.

Therefore, the payoff from this option at time T is the random variable

$$C_T = (S_T - K)^+,$$

where $x^+ = \max\{0, x\}$ denotes the positive part of x: $x^+ = x$ if $x > 0$, $x^+ = 0$ if $x \leq 0$.

We say that the owner of such an option is 'in the money' at time T if $S_T > K$. The buyer of such an option is betting on the stock price going up above K.

The alternative European *put* option has payoff

$$C_T = (K - S_T)^+,$$

which is the option allowing the holder to *sell* a share of stock at the fixed time T at price K. In this case the holder would sell only if the price $S_T < K$.

The buyer of a put option is thus betting on the stock going down below K.

Further Examples:

1. *Asian call option:* The payoff C_T at time T is based on the average value of the stock over the T time units:
$$C_T = \left(\frac{1}{T}\sum_{i=1}^{T} S_i - K\right)^+.$$

2. *Lookback options:* Letting $m_T = \min\{S_0, \ldots, S_T\}$ denote the minimum value of the stock over the time period, the payoff at time T is given by
$$C_T = S_T - m_T.$$

One can also incorporate a strike price K; letting $M_T = \max\{S_0, \ldots, S_T\}$ denote the maximum value of the stock over the time period, the payoff at time T is given by

$$C_T = (M_T - K)^+.$$

These options are referred to as *lookback* options since at the termination time, one must look back at all the previous values to determine the payoff. (There are many variations on this.)

3. *Barrier options:* For any event A, let $I\{A\}$ demote the *indicator* random variable defined by

$$I\{A\} = \begin{cases} 1 & \text{if } A \text{ occurs,} \\ 0 & \text{if } A \text{ dos not occur.} \end{cases}$$

   $I\{A\}$ is actually just a Bernoulli (p) rv in which $p = P(A)$, the probability that the event (outcome) A occurs.

   Consider a European call option (with T, K), but with an additional given fixed barrier constant $b > 0$, and given fixed k times $0 < n_1 < n_2 < \cdots < n_k < T$, such that the payoff at time T is given by

$$C_T = (S_T - K)^+ I\{S_{n_1} \geq b,\ S_{n_2} \geq b, \ldots, S_{n_k} \geq b\},$$

   ($S_0 < b < K$.) The payoff requires that the stock be above level b at all the specified times n_i; otherwise the option becomes worthless. If the stock indeed never goes below b at any of those times n_i, then $C_T = (S_T - K)^+$ like a European call. This particular option is called a *down and out* barrier option, because if it does go down below b at any one of those times n_i, it is knocked out and becomes worthless.

4. *Digital options:* Starting with the notion of a European call option with payoff $(S_T - K)^+$, the payoff of a digital option is

$$C_T = \begin{cases} 1 & \text{if } S_T > K\ , \\ 0 & \text{if } S_T \leq K. \end{cases}$$

   Using indicator random variable notation, $I\{A\}$, we can re-write this as $C_T = I\{S_T > K\}$, the event $A = \{S_T > K\}$, the event that at time T, it happens that $S_T > K$. In other words, you get a payoff of 1 dollar if the European call option is 'in the money' and 0 dollars if not.
   Note that $E(C_T) = 1P(S_T > K) + (0)P(S_T \leq K) = P(S_T > K)$: The expected payoff is the probability that you are in the money for the European call option.

We point out that there are also options which allow the owner to exercise it (and receive a payoff) before the expiration date T. For example, *American* options are the kind in which the option can be exercised at any time n in the interval $(0, T]$.

Bermuda options have a pre-specified set of time points $0 < n_1 < \cdots < n_k \leq T$ at which times the option can be exercised if the owner so wishes.

Our presentation and theory in these notes, however, will cover only options that are exercised at the expiration time T, that is when the payoff is received; the other types of options are more difficult to price. For example, the American style options involve finding an optimal time to exercise, so as to maximize the payoff; that itself can be a complicated mathematical optimization problem.

1.2 Pricing options

Whereas the price of the stock is known at time 0 (now)—it is simply S_0 (the market price), and hence that is how much we would pay for a share, and αS_0 is how much we would pay to buy α such shares, the price of an option is not a priori known. Note that we can view the stock itself as an option, namely, at time T it has payoff $C_T = S_T$, the price at that time; so its price at time 0 is known, namely S_0; but beyond that, we need some further theory to determine prices of more complicated options.

For example, the European call option with payoff $C_T = (S_T - K)^+$ has the obvious property that $C_T \leq S_T$; the payoff is less than that for the stock itself, hence its price (at time 0) must be less too. But what should that price be? We will demote the price of an option by C_0.

The key to determining this price C_0 to also recall that we can invest by putting money in the bank at interest rate r. If we put β dollars in the bank now at time $n = 0$, then it has payoff $(1 + r)^T \beta$ at time T; so it too has a known price β at time 0.

1.2.1 Portfolios of stock and money

Putting stock and money in the bank jointly as an investment at time 0 is called a *portfolio* $\mathcal{P}$ of stock and money. We can describe it by the pair (α, β), which means we have placed $\$\beta$ in the bank and bought α shares of the stock. The price is thus $C_0(\mathcal{P}) = \alpha S_0 + \beta$. Moreover, its payoff at any time T is given by $C_T(\mathcal{P}) = \alpha S_T + \beta(1 + r)^T$.

1.2.2 Pricing an option when $T = 1$: The matching portfolio method

When $T = 1$, the payoff of our option C_1 at time $T = 1$ can take on two known values which we denote by $C_{1,u}$ if the stock goes up, $S_1 = uS_0$, and $C_{1,d}$ if the stock goes down, $S_1 = dS_0$.

For the European call option, $C_1 = (S_1 - K)^+$ and so

$$\begin{aligned} C_{1,u} &= (uS_0 - K)^+ && (3) \\ C_{1,d} &= (dS_0 - K)^+. && (4) \end{aligned}$$

Both of these values are known since they depend only on the known values u, d, S_0, and K.

This allows us to compute the expected value (average) $E(C_1)$ of the payoff random variable C_1:

$$E(C_1) = pC_{1,u} + (1 - p)C_{1,d}, \tag{5}$$

where p is the up/down probability for the stock within the BLM.

Similarly, if we buy a portfolio (α, β) at time 0, then we also know its two payoff values at time $T = 1$:

$$\begin{aligned} C_{1,u}(\mathcal{P}) &= \alpha u S_0 + \beta(1 + r) && (6) \\ C_{1,d}(\mathcal{P}) &= \alpha d S_0 + \beta(1 + r). && (7) \end{aligned}$$

Note that if we can find values of α and β such that the two payoffs 'match', that is, such that

$$
\begin{aligned}
C_{1,u} &= C_{1,u}(\mathcal{P}) && (8)\\
C_{1,d} &= C_{1,d}(\mathcal{P}), && (9)
\end{aligned}
$$

then since their payoffs are the same, their prices must be the same: *The price* C_0 *(price of the option) must be equal to* $C_0(\mathcal{P})$ *(price of the portfolio) when their payoffs are the same.*

In essence they become the same investment hence must have the same price.

But we know the price of the portfolio, $C_0(\mathcal{P}) = \alpha S_0 + \beta$, so by finding the appropriate matching values for α and β we obtain the price of the option! We thus must solve the following two linear equations with two unknowns (α, β)

$$
\begin{aligned}
C_{1,u} &= \alpha u S_0 + \beta(1+r) && (10)\\
C_{1,d} &= \alpha d S_0 + \beta(1+r). && (11)
\end{aligned}
$$

The solution which we denote by (α^*, β^*) is easily found by basic algebra:

$$
\begin{aligned}
\alpha^* &= \frac{C_{1,u} - C_{1,d}}{S_0(u-d)} && (12)\\
\beta^* &= \frac{uC_{1,d} - dC_{1,u}}{(1+r)(u-d)}. && (13)
\end{aligned}
$$

Plugging these values into the portfolio price then yields

$$
\begin{aligned}
C_0 &= C_0(\mathcal{P}) && (14)\\
&= \alpha^* S_0 + \beta^* && (15)\\
&= \frac{C_{1,u} - C_{1,d}}{(u-d)} + \frac{uC_{1,d} - dC_{1,u}}{(1+r)(u-d)}. && (16)
\end{aligned}
$$

But by some clever rearranging (algebraically, try to work it out!) of this price it can be re-written in a very elegant way:

$$
\begin{aligned}
C_0 &= \frac{1}{1+r}(p^* C_{1,u} + (1-p^*)C_{1,d}) && (17)\\
\text{where} & && (18)\\
p^* &\stackrel{\text{def}}{=} \frac{1+r-d}{u-d} && (19)\\
1-p^* &= \frac{u-(1+r)}{u-d}. && (20)
\end{aligned}
$$

Since $0 < d < 1+r < u$ (by assumption), we see that $0 < p^* < 1$ can be viewed as a probability.

Thus recalling (5), the expression $p^*C_{1,u} + (1-p^*)C_{1,d}$ in (17) is in fact the expected value $E(C_1)$ but with p replaced by p^*. We denote this by $E^*(C_1)$, the $*$ referring to the use of p^* in the stocks up/down probability (instead of the original p):

$$E^*(C_1) \stackrel{\text{def}}{=} p^*C_{1,u} + (1-p^*)C_{1,d}.$$

Thus we finally end up with the elegant formula for the option price:

$$C_0 = \frac{1}{1+r}E^*(C_1), \tag{21}$$

which says that the price is the present value of the expected payoff (but when using p^* instead of p). p^* is called the *risk-neutral* probability.

The point is that for the purpose of pricing options, we simply replace the original p by the risk-neutral probability p^* for the BLM $\{S_n\}$.

Note that something very mysterious seems to have taken place: Neither p^* nor C_1 nor r contain the original value p; hence the option price C_0 does not either; p appears to have vanished!

1.2.3 Examples

1. *European call option:*

$$C_0 = \frac{1}{1+r}E^*(C_1) = \frac{1}{1+r}(p^*(uS_0 - K)^+ + (1-p^*)(dS_0 - K)^+).$$

   Suppose that $r = 0.05$, $S_0 = 50$, $K = 51$, $u = 1.20$ and $d = 1.01$.
   Then $C_{1,u} = (uS_0 - K)^+ = ((1.20)50 - 51)^+ = 9$ and
   $C_{1,d} = (dS_0 - K)^+) = ((1.01)50 - 51)^+ = (-0.5)^+ = 0$. Meanwhile

$$p^* = \frac{1+r-d}{u-d} = 4/19.$$

   Thus

$$C_0 = \frac{1}{1+r}E^*(C_1) = \frac{1}{1.05}((4/19)(9) + (15/19)(0)) = \frac{1}{1.05}(36/19) = 1.801.$$

2. *The stock itself, $C_1 = S_1$:* We already know that if $C_1 = S_1$, then $C_0 = S_0$, but let's check to make sure our formula works. The first thing is to see that in fact $E^*(S_1) = p^*(uS_0) + (1-p^*)(dS_0) = S_0(p^*u + (1-p^*)d) = (1+r)S_0$. To see this, note that (compute, do the algebra algebra) $p^*u + (1-p^*)d = 1+r$, because

$$p^* = \frac{1+r-d}{u-d}.$$

   Thus indeed

$$S_0 = \frac{1}{1+r}E^*(S_1).$$

In other words $E^*(Y) = 1+r$, and so if we use $p = p^*$ for the BLM, then $E^*(S_n) = S_0 E^*(Y_1 \times \cdots \times Y_n) = S_0[E^*(Y)]^n = (1+r)^n S_0$. This says that *under p^*, the stock on average yields exactly the same payoff as if placing the initial amount S_0 in the bank at interest rate r.*

In fact this is if and only if: p^* is the unique probability (for the up/down probability of the BLM) such that $E(Y) = 1+r$. That is how to remember its value: Just set $pu + (1-p)d = 1+r$ and solve for p; you get $p = p^* = \frac{1+r-d}{u-d}$.

1.2.4 Pricing options when the expiration date is $T \geq 2$

If the expiration date of the option is $n = T$, then we denote the payoff at time T by the random variable C_T. For example, $C_T = (S_T - K)^+$ for the European call option. The various payoff values at time T depend on the outcomes over the T time units.

For example if $T = 2$, then there are the four values for the payoff C_2: $C_{2,uu}$, $C_{2,ud}$, $C_{2,du}$, $C_{2,dd}$ reflecting the up and down outcomes over the two time periods. The probabilities of these are $p^2,\ p(1-p), (1-p)p, (1-p)^2$ respectively, and so the expected payoff is given by

$$E(C_2) = p^2 C_{2,uu} + p(1-p)C_{2,ud} + (1-p)pC_{2,du} + (1-p)^2 C_{2,dd}.$$

For the European call option, order does not matter; $C_{2,ud} = C_{2,du} = (udS_0 - K)^+$, so the above reduces to just using the binomial $(2,p)$ probabilities:

$$E(S_2 - K)^+ = p^2(u^2 S_0 - K)^+ + 2p(1-p)(udS_0 - K)^+ + (1-p)^2(d^2 S_0 - K)^+.$$

In general, however, order will matter. More generally, for the European call option, the payoff at time T is always of the form $(u^i d^{T-i} S_0 - K)^+$ for some $0 \leq i \leq T$ and does not depend on the order in which the ups and downs occurred; for other options order may matter; they are called *path-dependent* options. Examples include an Asian call option with payoff $C_T = (\frac{1}{T}\sum_{n=1}^{T} S_n - K)^+$.

Because order does not matter for the payoff of the European call option, the expected value of the payoff is simply determined by the binomial (T,p) distribution:

$$E(S_T - K)^+ = \sum_{i=0}^{T} \binom{T}{i} p^i (1-p)^{T-i} (u^i d^{T-i} S_0 - K)^+.$$

For path dependent options, however, computing the expected value $E(C_T)$ can be very difficult, and instead one would use numerical estimations.

But in any case: The following is a generalization of (21):

Theorem 1.1 *Under the Binomial lattice model for stock pricing, the price of an option with expiration date $n = T$ and payoff C_T at time T is given by*

$$C_0 = \frac{1}{(1+r)^T} E^*(C_T). \tag{22}$$

E^* denotes expected value using the risk-neutral probability p^* for the up/down movement in the BLM (defined in (19)). In words: "the price of the option is equal to the present value of the expected payoff of the option under the risk-neutral probability".

Applying Theorem 1.1 to a European call option where the order of the ups and downs is irrelevant yields the discrete-time analog of the famous *Black-Scholes-Merton* pricing formula (for European call options):

Corollary 1.1 *[Black-Scholes-Merton] Under the Binomial lattice model for stock pricing, the price of a European call option with strike price K and expiration date $n = T$ is given by*

$$\begin{aligned} C_0 &= \frac{1}{(1+r)^T} E^*(C_T) && (23)\\ &= \frac{1}{(1+r)^T} E^*(S_T - K)^+ && (24)\\ &= \frac{1}{(1+r)^T} \sum_{i=0}^{T} \binom{T}{i} (p^*)^i (1-p^*)^{T-i} (u^i d^{T-i} S_0 - K)^+. && (25) \end{aligned}$$

Proof :[Theorem 1.1] We will prove Theorem 1.1 for $T = 2$, since the $T > 2$ case is analogous. To this end we must show that

$$\begin{aligned} C_0 &= \frac{1}{(1+r)^2} E^*(C_2) && (26)\\ &= \frac{1}{(1+r)^2} \Big[C_{2,uu}(p^{*2}) + C_{2,ud}(p^*(1-p^*)) + C_{2,du}(p^*(1-p^*)) + && (27)\\ &\quad C_{2,dd}(1-p^*)^2 \Big]. && (28) \end{aligned}$$

The key idea: Although we can't exercise the option at the earlier time $n = 1$, we can sell it, so it does have a "price" at that time which we can view as a potential "payoff". At time $n = 1$, we would know what the new price of the stock is, S_1, and we thus could sell the option which then would have an expiration date of $T = 1$ (the remaining time until $T = 2$).

For example, if $S_1 = uS_0$, then we use the $T = 1$ price formula in (21) with the two known payoff outcomes $C_{2,u,u}$ and $C_{2,ud}$ yielding the price (denoted by $C_{1,u}$, the price of the option at time $n = 1$ if the stock went up at time $n = 1$)

$$C_{1,u} = \frac{1}{1+r} [p^* C_{2,uu} + (1-p^*) C_{2,ud}].$$

Similarly, if $S_1 = dS_0$, then

$$C_{1,d} = \frac{1}{1+r} [p^* C_{2,du} + (1-p^*) C_{2,dd}].$$

But now we can go one more time step back to $n = 0$: We have these known "payoff" values at time $n = 1$ of $C_{1,u}$ and $C_{1,d}$, which we just computed, and thus we can now use them in the

$T = 1$ formula (21) again to obtain

$$C_0 = \frac{1}{1+r}[p^* C_{1,u} + (1-p^*)C_{1,d}] \tag{29}$$

$$= \frac{1}{(1+r)^2}\Big[C_{2,uu}(p^{*2}) + C_{2,ud}(p^*(1-p^*)) + C_{2,du}(p^*(1-p^*)) + \tag{30}$$

$$C_{2,dd}(1-p^*)^2\Big]. \tag{31}$$

In general, the proof proceeds by starting at time T and moving back in time step-by-step to each node on the lattice until finally reaching time $n = 0$. This procedure yields not only C_0 but all the intermediary prices as well. ■

The importance of the Black-Scholes-Merton formula is that it is explicit and thus one can just plug in the parameters to get the price. That is because computing $E^*(S_T - K)^+$ can be explicitly computed.

In general, for more complex options, computing $E^*(C_T)$ explicitly is not possible.

It is because of this explicit result and the general elegant result in Theorem 1.1 (but a much more complicated continuous-time version using Brownian motion) that the Nobel Prize in Economics was awarded to Robert C. Merton and Myron Scholes in 1997 (Fischer Black had died earlier, in 1995, hence his absence in the award).

Robert C. Merton was an undergraduate student at Columbia University in the 1960s, earning his BS degree in the School of Engineering and Applied Sciences in 1966 in what is now called the department of Industrial Engineering and Operations Research (IEOR).

1.3 Monte Carlo Simulation for estimating option prices

Recall that we can always estimate the expected value (mean) of a random variable, $E(X)$ by first taking a large number n of independent identically distributed copies, $X_1, X_2, \ldots, X_n$, and averaging the values:

$$E(X) \approx \frac{1}{n}\sum_{i=1}^{n} X_i.$$

(Here $\approx$ denotes "is approximately equal to".) The justification is from the Strong Law of Large Numbers (SLLN) which asserts that indeed with probability one,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = E(X).$$

In other words, as the size of samples n gets larger and larger, the exact answer is reached.

For example, when $X_i = B_i$ are iid Bernoulli (p) rvs, then $E(B) = p$ and indeed, with probability one,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} B_i = p;$$

the long-run proportion of trials that are a success is equal exactly to $p =$ the probability of success. For example, a "fair" coin ($p = 1/2$ case), in the long run, would land H half the time and T the other half.

In our option pricing formula in Theorem 1.1, we need to compute the expected value $E^*(C_T)$. If we can get our hands on n independent copies (n large) of C_T, say $C_{T,1}, C_{T,2}, \ldots, C_{T,n}$, (where they were constructed using p^* for the BLM instead of p) then we can obtain an estimate of $E^*(C_T)$ via the SLLN estimate:

$$E^*(C_T) \approx \frac{1}{n} \sum_{i=1}^{n} C_{T,i}.$$

Thus we get an estimate of the option price C_0 via

$$C_0 \approx \frac{1}{(1+r)^T} \frac{1}{n} \sum_{i=1}^{n} C_{T,i}.$$

This is the essence of *Monte Carlo Simulation*; estimating an expected value $E(X)$ by simulating iid copies of X and averaging them.

In general, simulating iid copies of C_T is easy. For example, suppose (just for illustration) we want to simulate iid copies of $C_2 = (S_2 - K)^+$. We would simulate a first copy of S_2 by simulating the BLM out to time 2 and thus obtain our first copy of C_2, denoted by $C_{2,1}$.

Then we would simulate yet another copy of S_2 and obtain our second copy of C_2, denoted by $C_{2,2}$.

Continuing onwards, we in the end obtain n iid such copies, and we can choose n as large as we want, such as $n = 1000$ or $n = 10,000$ or even $n = 1000,000$.

Since we in fact have an explicit Black-Scholes-Merton formula for $C_0 = \frac{1}{(1+r)^T} E^*(S_T - K)^+$ for the European call option, we of course would not need to use this Monte Carlo method for a European call option.

But as part of your project, you will compare the exact answer given by the Black-Scholes-Merton formula, with its estimate using Monte Carlo, just so that you can see for yourself how accurate the Monte Carlo method can be.

Then, you will use Monte Carlo to price more complicated options for which computing $E^*(C_T)$ explicitly is not possible.