Properties and Applications of the 2D Fourier Transform

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Objectives

01 Explore more properties and applications of Fourier Transform

02 Investigate and explain the rotation property of FT

03 Explore different applications of filtering in Fourier space

04 Revisit the concepts of Convolution Theorem

Activity 2.2.1. Rotation Property of the FT

Background

This activity investigates the rotation property of the 2D Fourier Transform. The theorem states that the FT of a rotated function is equal to a rotated version of the FT of that function [1]. We will not go into detail with the mathematical proof of this theorem. Instead, I will show the visual proof of this theorem in the following slides by using sinusoidal apertures.

Fourier Transform

I created a function FFT(aperture) to implement the following:

Apply fft2( ) on the image and compute the intensity values using abs( ).	Display the FFT magnitude as an intensity image.	Use fftshift( ) to make the output zero-centered. Display the magnitude of the shifted image.	Display the magnitude of the shifted image in logarithmic scale for better visualization.

2D Sinusoidal Aperture

I defined a function Sinusoid(f,d) to create the sinusoids.

frequency

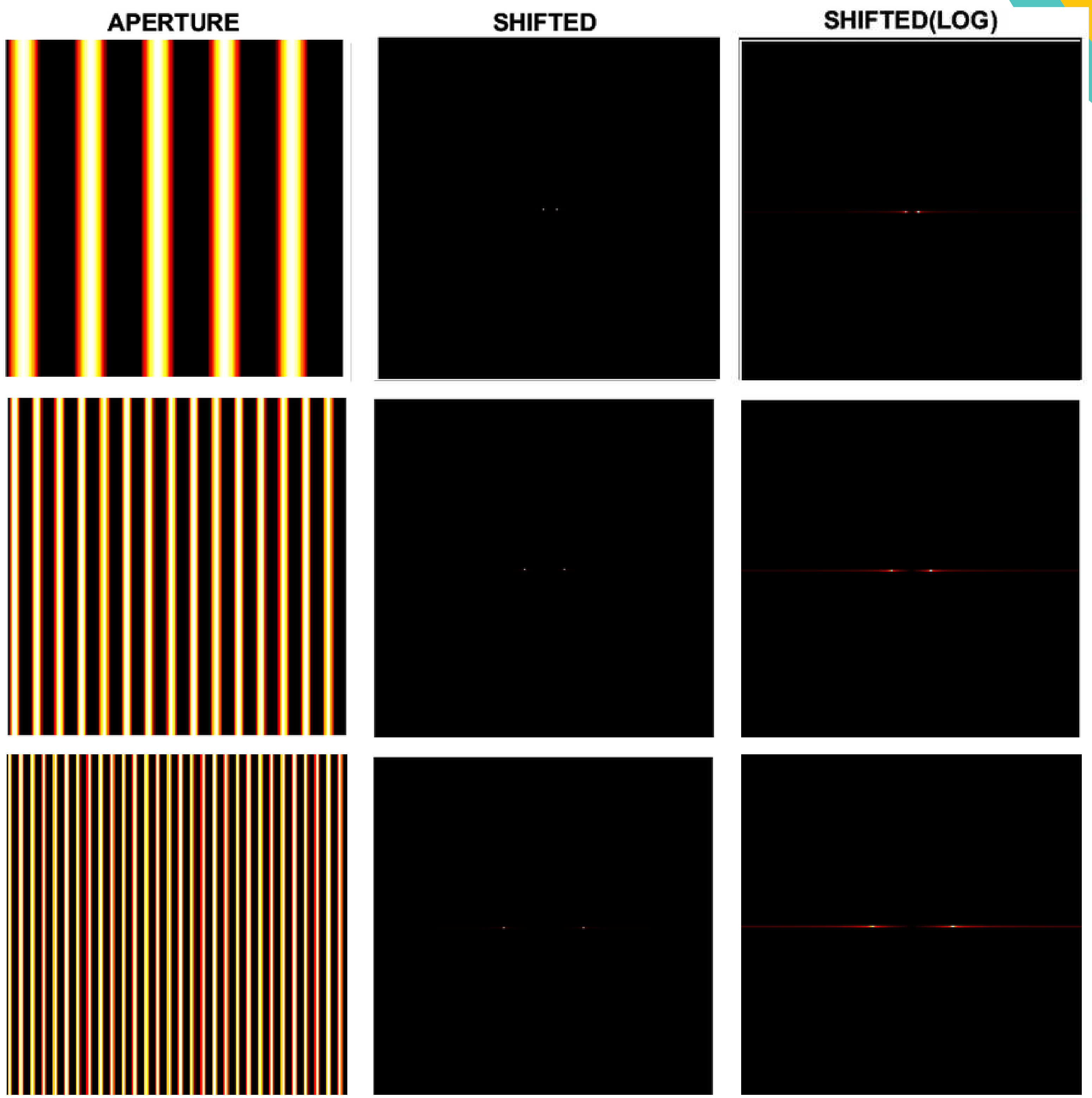
degree of rotation

```
function A = Sinusoid(f, d)
    N = 256;
    x = linspace(0,pi,N);
    y = x;
    [X,Y] = meshgrid(x,y);
    A = sin(2*f*(sin(d)*Y + cos(d)*X));
end
```

equation

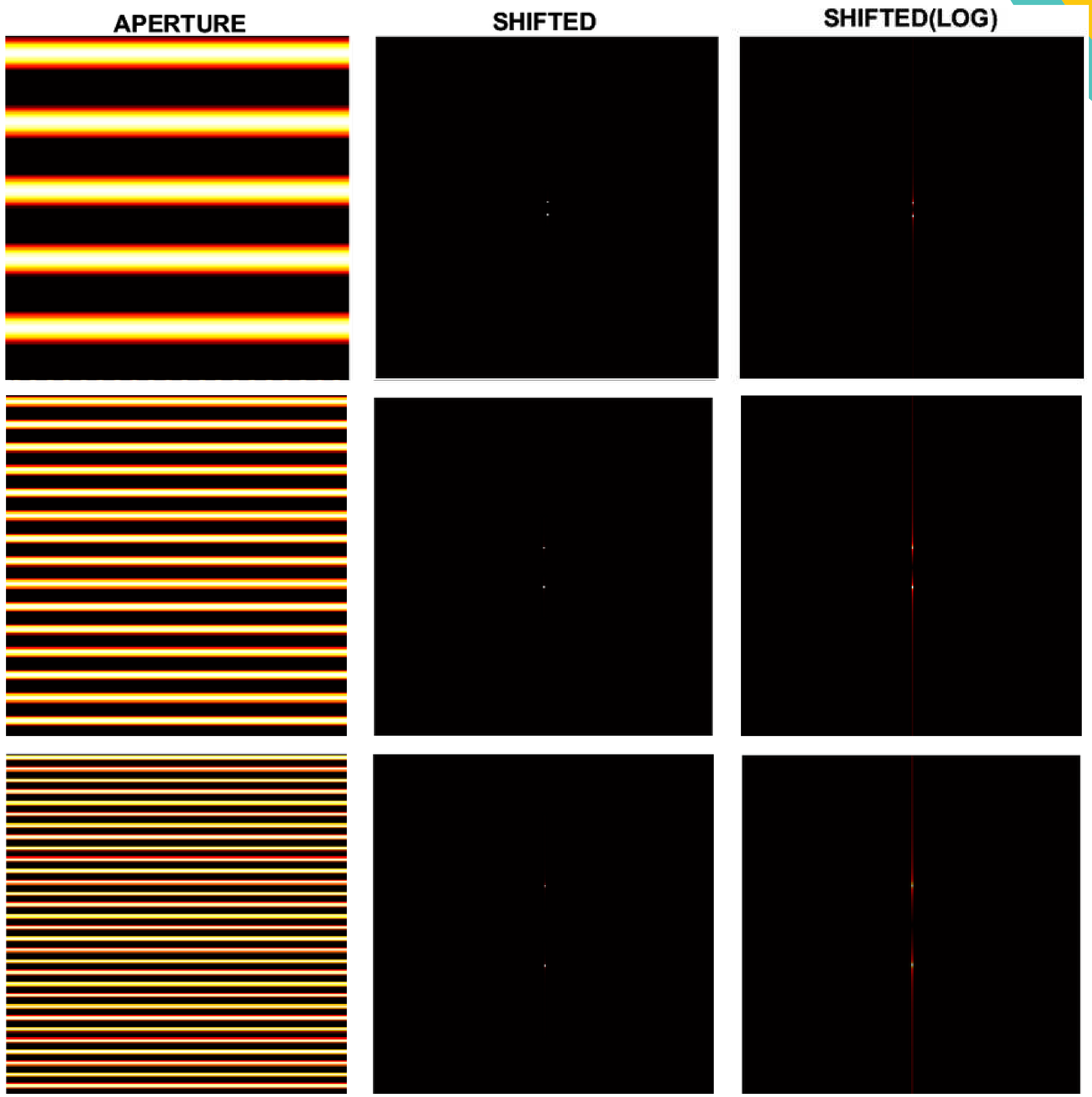
Increasing Frequency (along x)

In this part, I used f = 5, 15, 30 and d = 0. The shifted FT of the sinusoids appear to be two dots along the x-axis symmetric about the center. These dots represent the positive and the negative frequencies (dirac delta peaks) which is the FT of a sinusoid. Using the log scale of the shifted FT, the dots become lines of intensity equally spaced about the center position of the observing screen [2]. As the frequency is increased, the distance between the dots also increases.



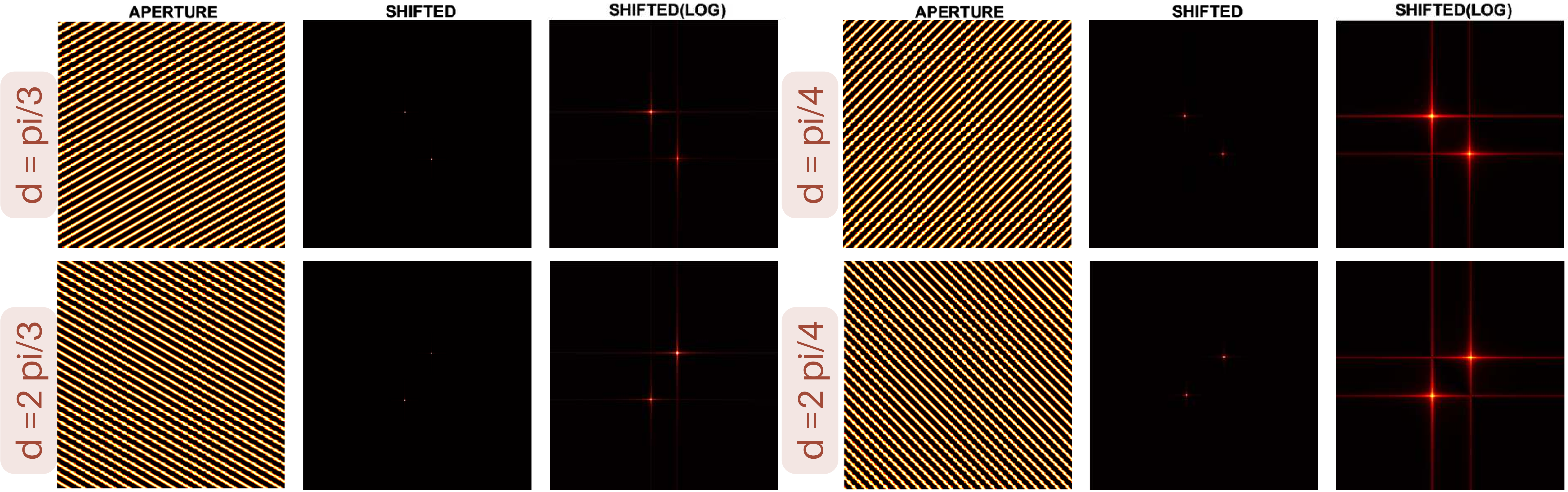
Increasing Frequency (along y)

Using the same frequencies, and setting d = pi/2, the resulting sinusoid is now along the y-direction. As the theorem of the rotation of FT states, as the sinusoid is rotated, the resulting FT also follows the direction of the rotation. Both the shifted FT and the log scale shows rotated FTs along the y-axis. Similar to the previous slide, as the frequency of the sinusoid is increased, the distance between the dots also increases. Hence, the theorem is proved.



Changing degree of rotation

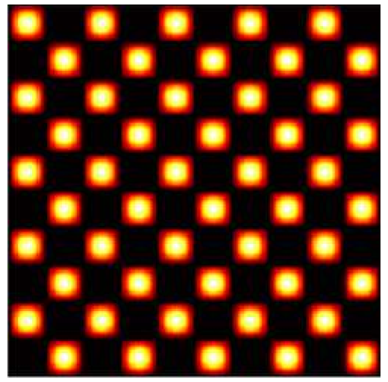
Keeping the frequency (f=30) constant and changing the degree of rotation d, the following will be observed. The shifted FT and its log scale are both rotated following the degree of rotation of the sinusoids. Moreover, the log scale now doesn't only show two separate lines with intensity only along one axis. Now it shows two intersecting lines centered on each of the dots. The intensity lines extend on both axis.



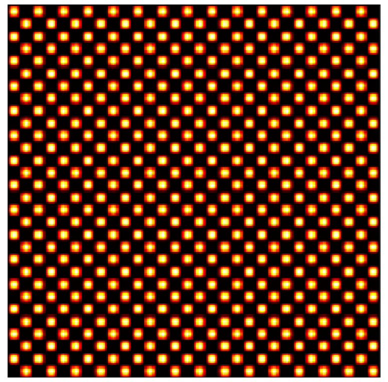
Combination of Sinusoids in x and y

We can also make a combination of the sinusoids. Here, 2 sinusoids are combined; one along the x-axis and the other along the y-axis. The frequency is also varied and I used f = 5, 15, and 30. Let's take a guess of what the FT of these sinusoids would look like.

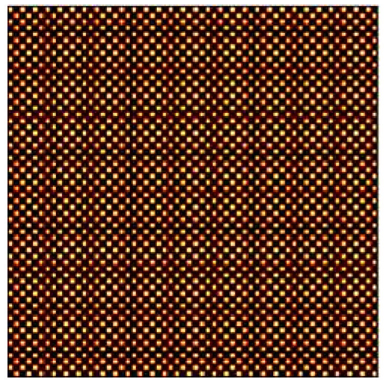
f = 5



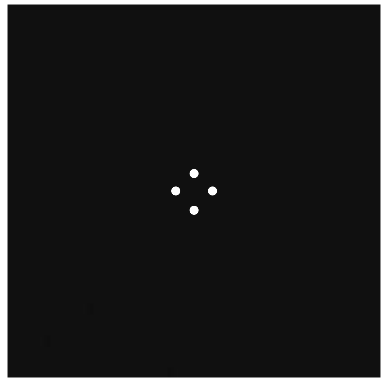
f = 15

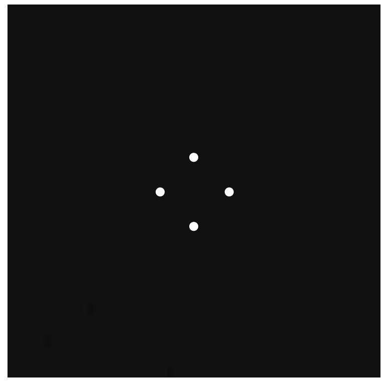


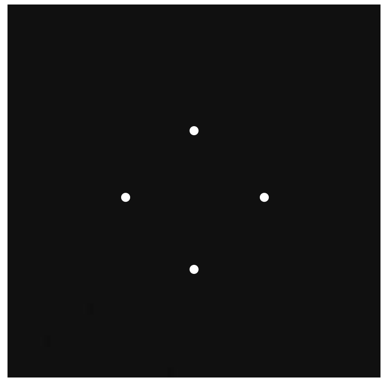
f = 30



The Fourier Transform follows the superposition principle. If it directly follows the superposition principle, then, we can use it to predict the pattern of the resulting FT of the sinusoids.

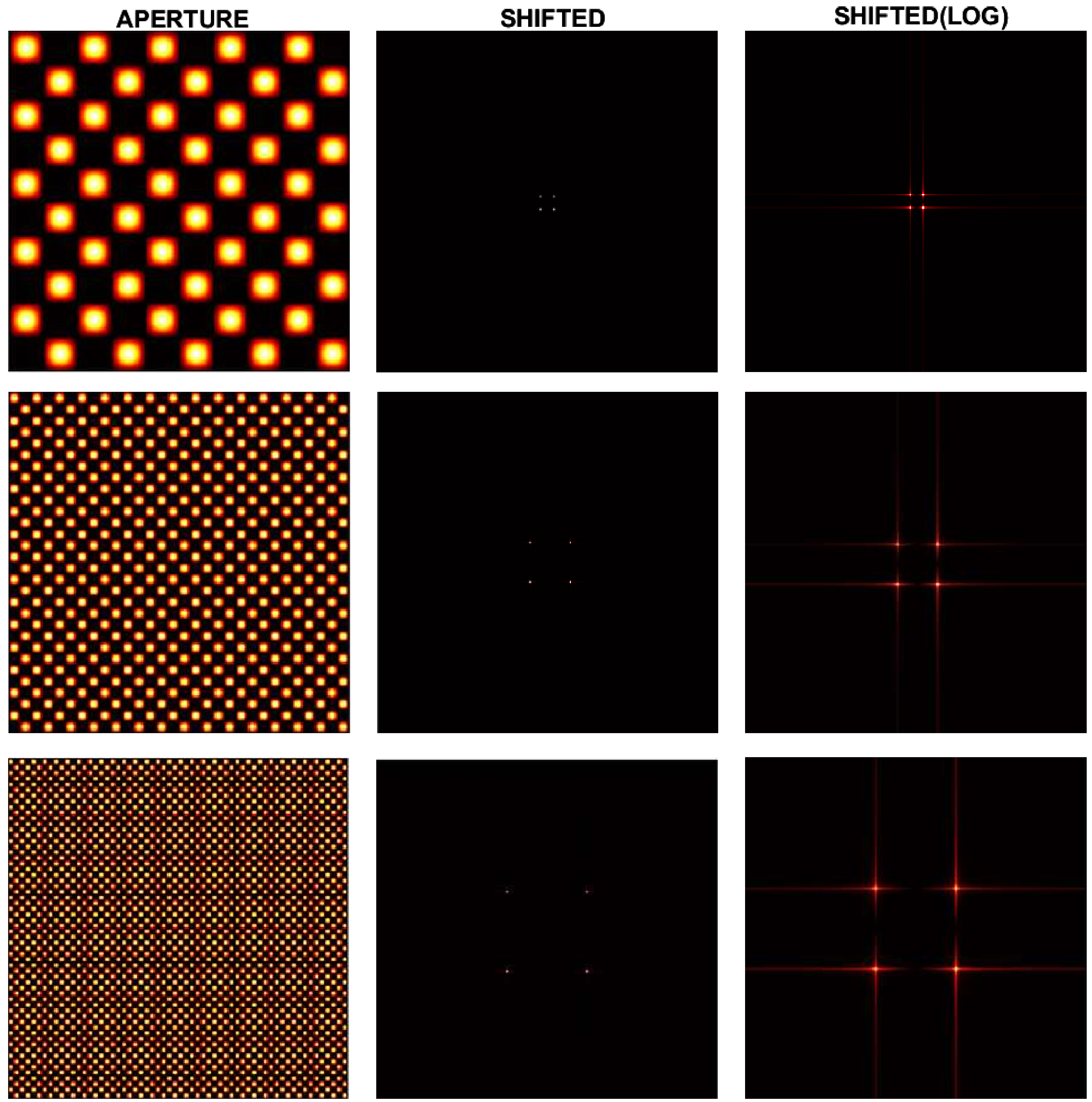






next slide will confirm whether these predictions are correct

Combination of Sinusoids in x and y

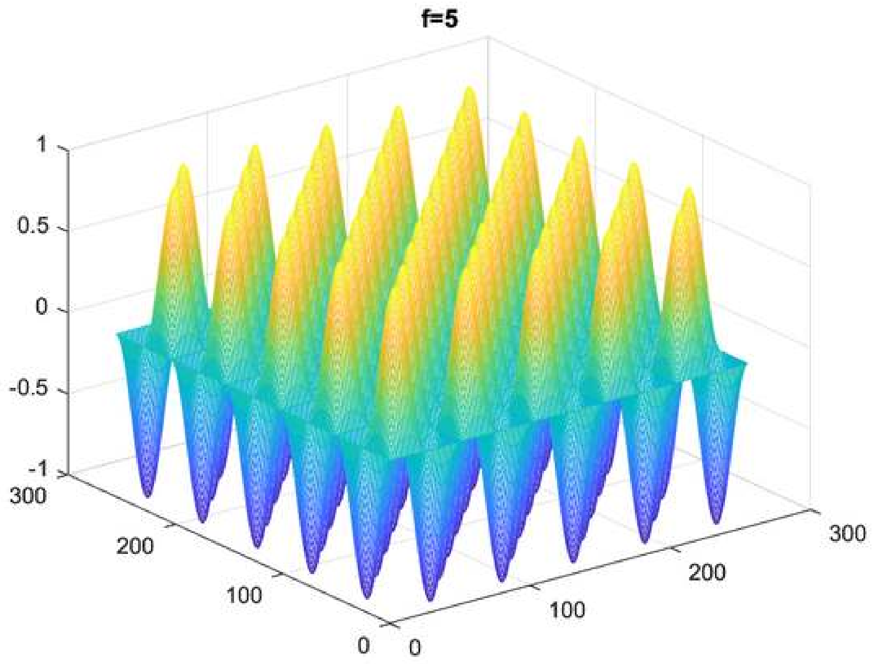


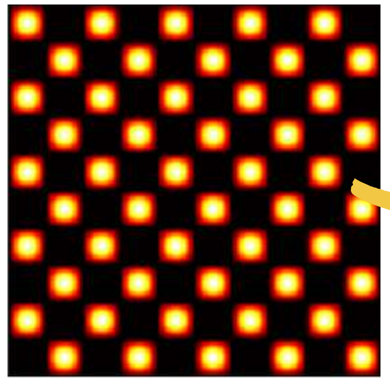
Here we show the resulting shifted FT of the combination of the two sinusoids. As we can see, we were right that it will be composed of four dots. However, the orientation of the dots differ to our predictions. It somehow forms the corner of a box. Increasing the frequency increases the "dimensions" of the box.

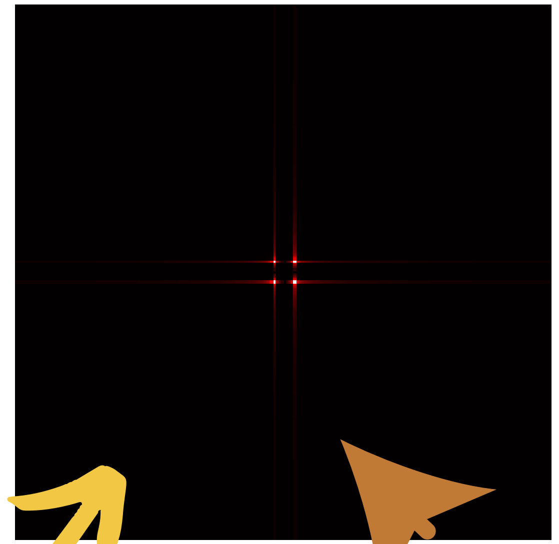
turns out our predictions were wrong. so what happened then?

Convolution in the Frequency Domain

Here is a surface plot of the combination of 2 sinusoids along x and y with the same frequency (f=5). It looks like an egg carton with positive and negative peaks. The bright parts in the 2D image corresponds to the positive peaks and the dark parts corresponds to the negative peaks.

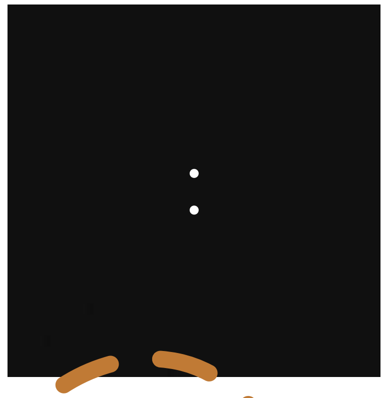




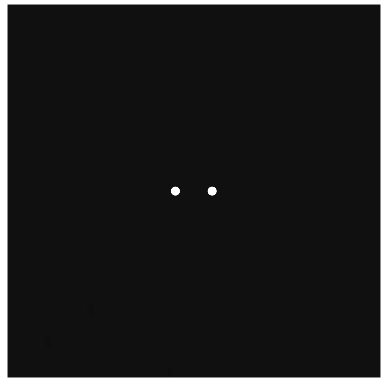


The FT of the product of two sinusoids is the same as convolving the FT of each of the sinusoids [3].

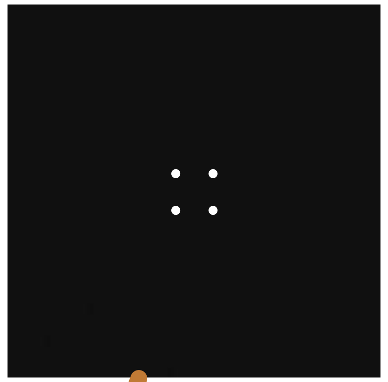
FT along y



FT along x



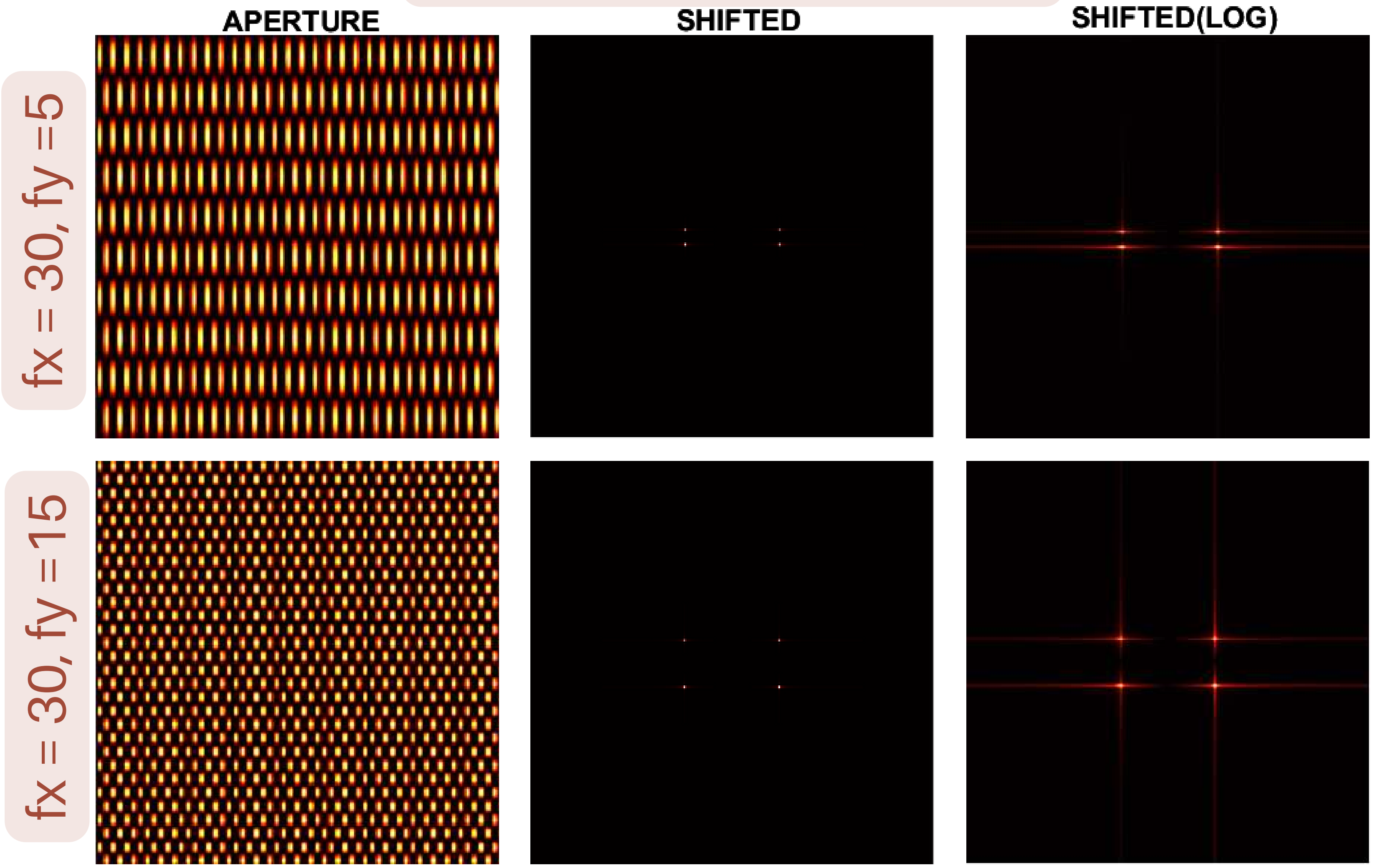
convolved



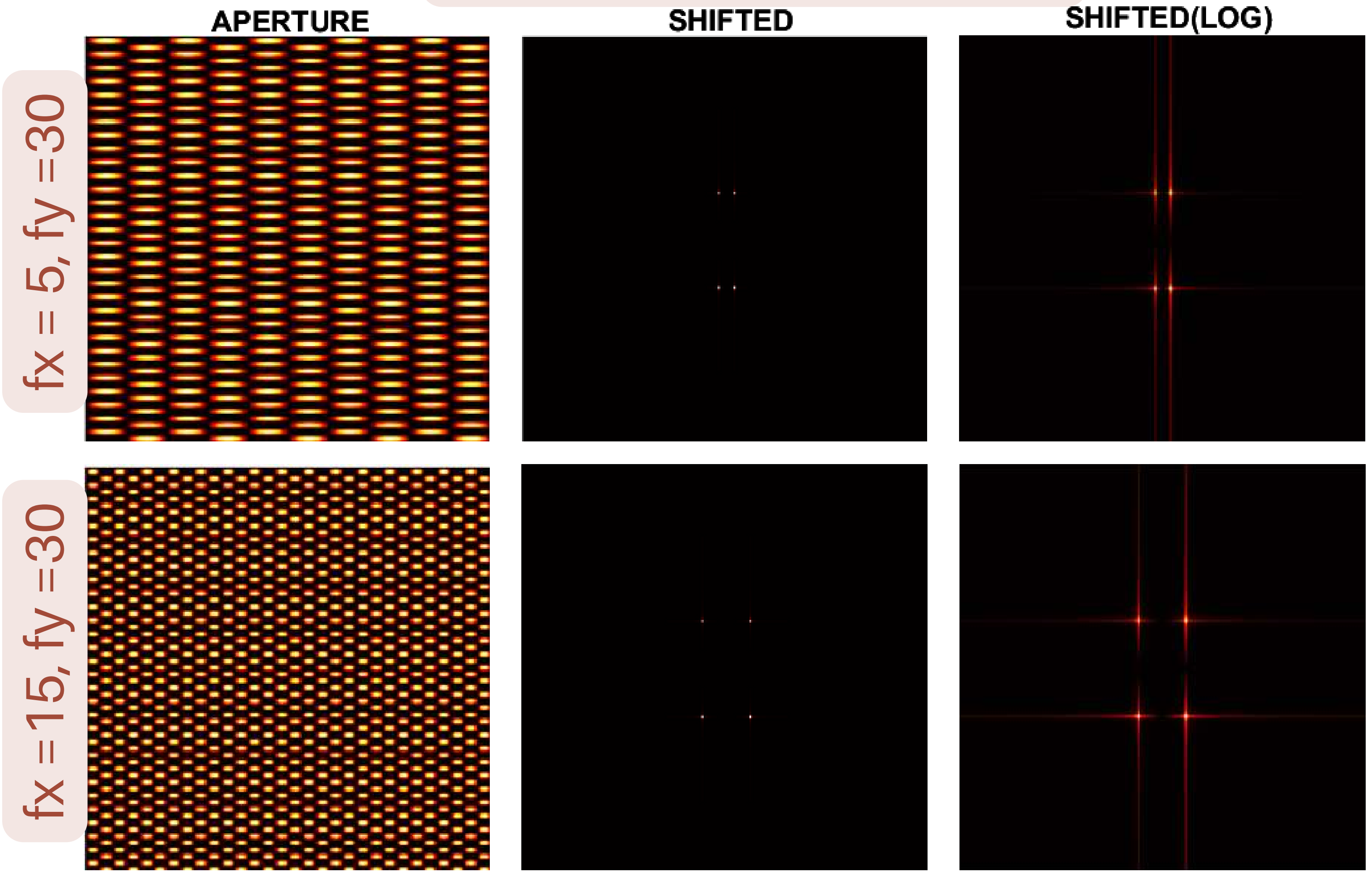
Varying the Frequencies

Here we vary the frequency on one axis only. When frequency is varied along y, the resulting FT is still four dots but now it doesn't make up the corners of a square; it now forms the corners of a horizontal rectangle. Increasing the frequency increases the width of the rectangle. The same can be observed when frequency is varied along x. Only this time, the FT forms the corners of a vertical rectangle.

varying frequency along y

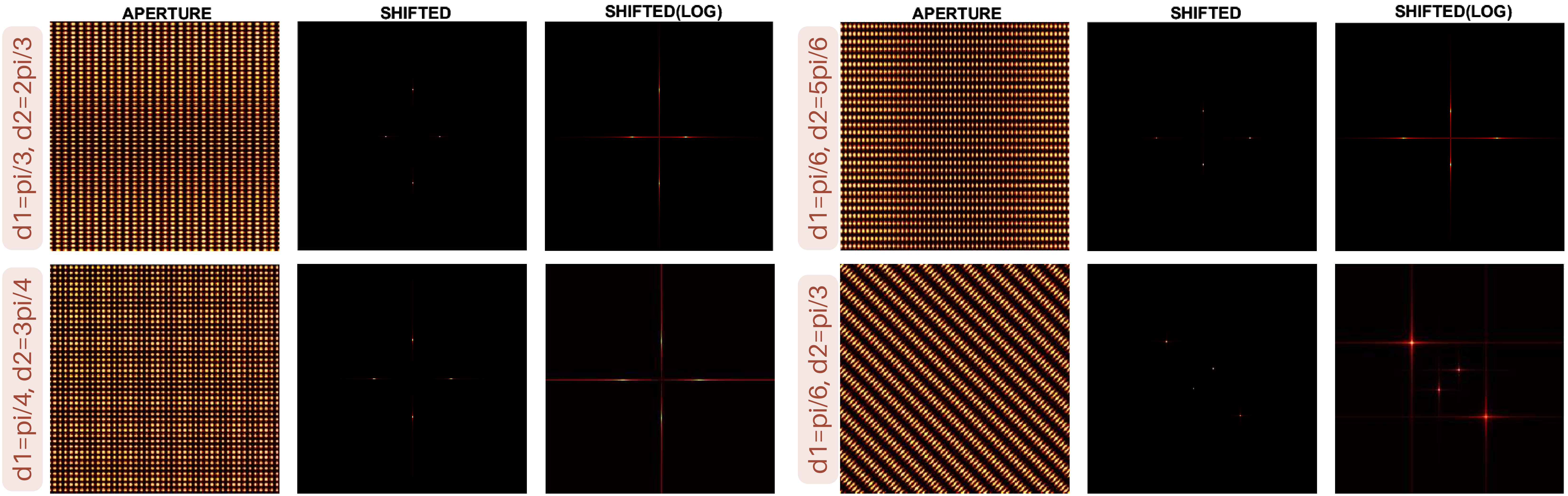


varying frequency along x

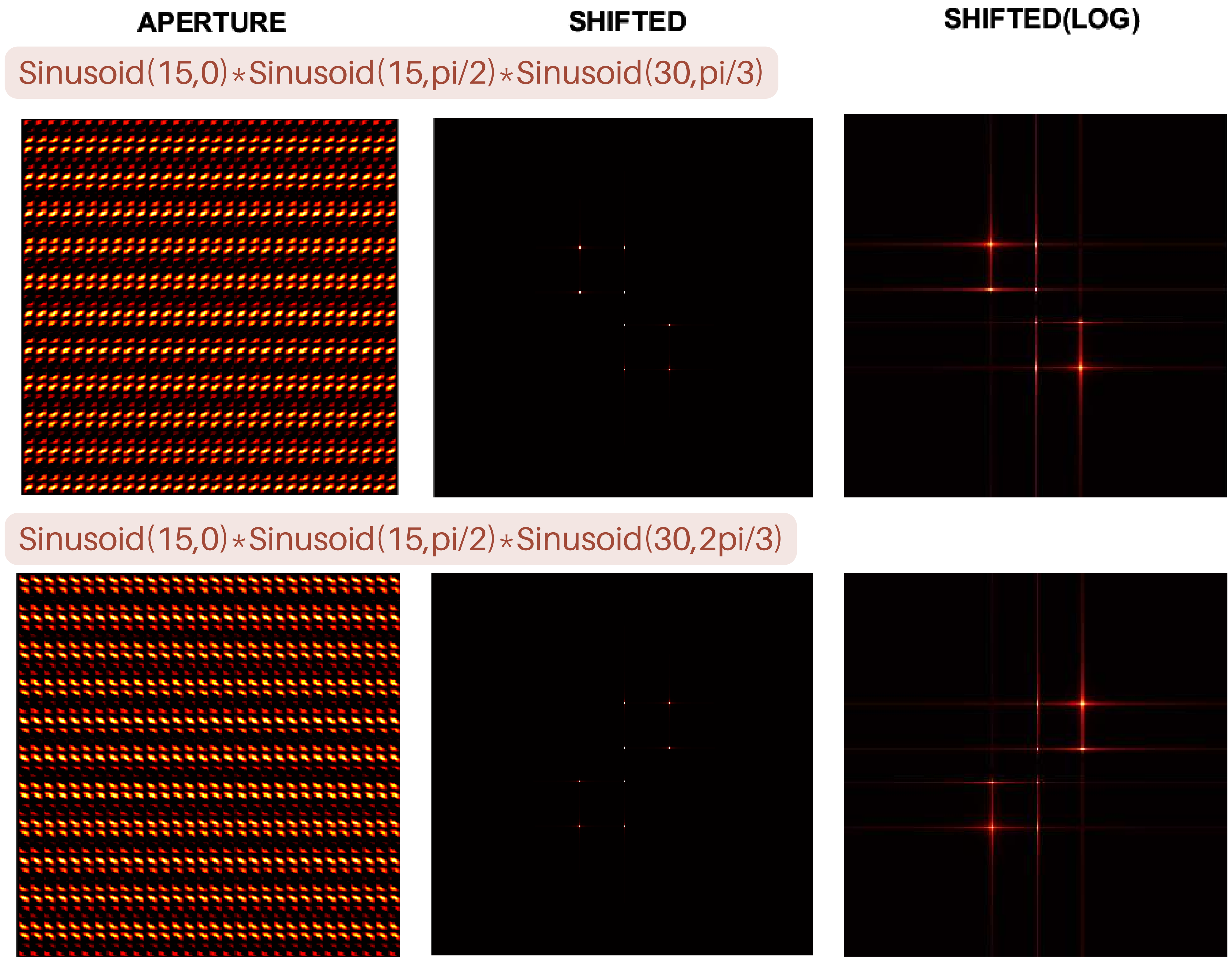


Varying the degree of rotation

Multiplying two sinusoids of the same frequency but different degrees of rotation yields an FT that looks like the corners of a diamond. Here, I used f = 30.



Combining 3 or more sinusoids



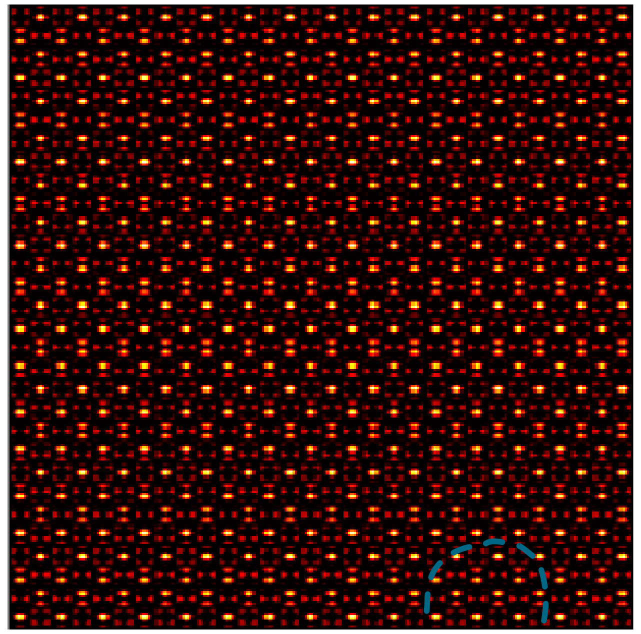
We already know that the FT of the product of two sinusoids is the same as convolving each of their FTs. So, if another sinusoid is multiplied, a similar process will be done. The FT of the 3rd sinusoid will be convolved with the FTs of the first two.

The FT of the product of the sinusoids of the same frequency along x and y is four dots forming the corners of a square. Convolving this to the FT of the third sinusoid is like copying the square of 4 dots in two different locations (since the FT of the third sinusoid is two dots). The location where the 4 dots will be placed is dependent on the frequency and degree of rotation of the third sinusoid.

Combining 3 or more sinusoids

APERTURE

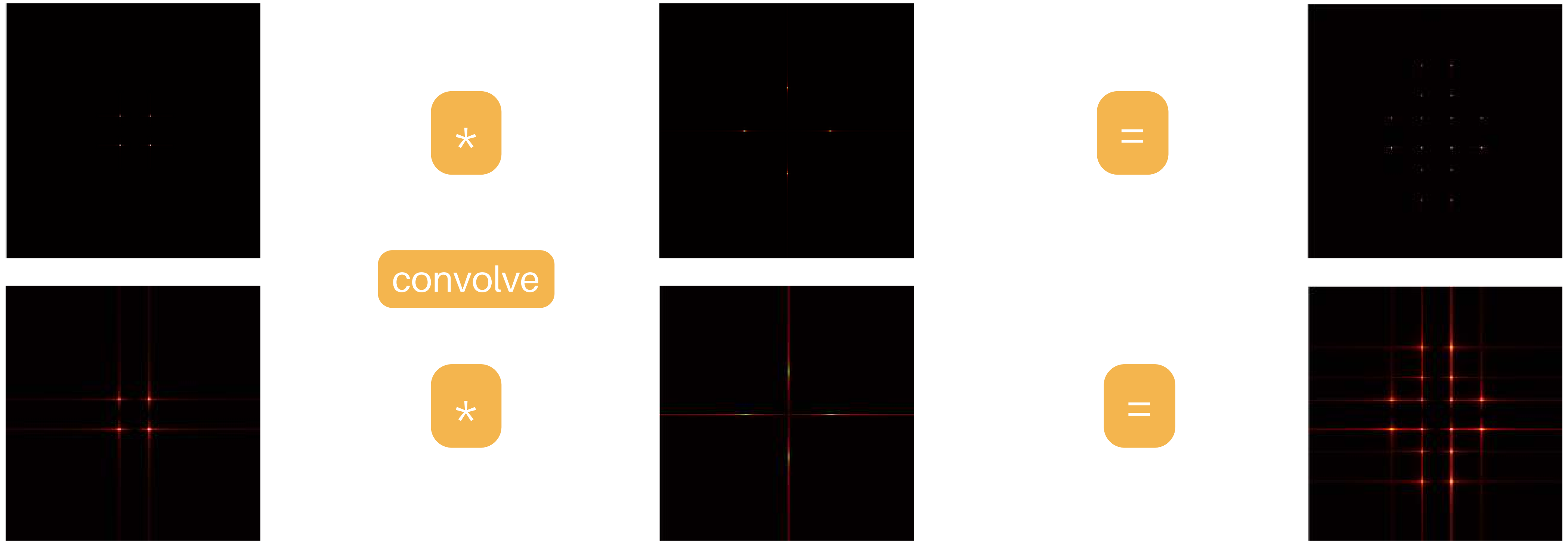
Sinusoid(15,0)*Sinusoid(15,pi/2)*Sinusoid(30,pi/3)*Sinusoid(30,2pi/3)



The same process is done when 4 sinusoids are multiplied together. Here, we already know the FTs of the two pairs of sinusoids. Convolving their FTs yields the FT of the product of the 4 sinusoids.

The process will be the same if more than 4 sinusoids are mutiplied together. The resulting pattern of the FT will depend on the frequency and degree of rotation of the sinusoids.

Sinusoid(15,0)*Sinusoid(15,pi/2) | Sinusoid(30,pi/3)*Sinusoid(30,2pi/3) | PRODUCT



The process should go like this. The resulting FT is like copying the square of 4 dots in 4 different locations.

Activity 2.2.2. Application: Canvas Weave Modeling and Removal

Background

Filtering images can be done using Fourier Transform. FT will decompose the image into its sine and cosine components. The resulting FT of an image is its equivalent in the frequency domain [4]. Masking these frequencies in the frequency domain can either remove or enhance the repetitive patterns present in the image [5]. Convolving the mask and the original image will yield the filtered image.

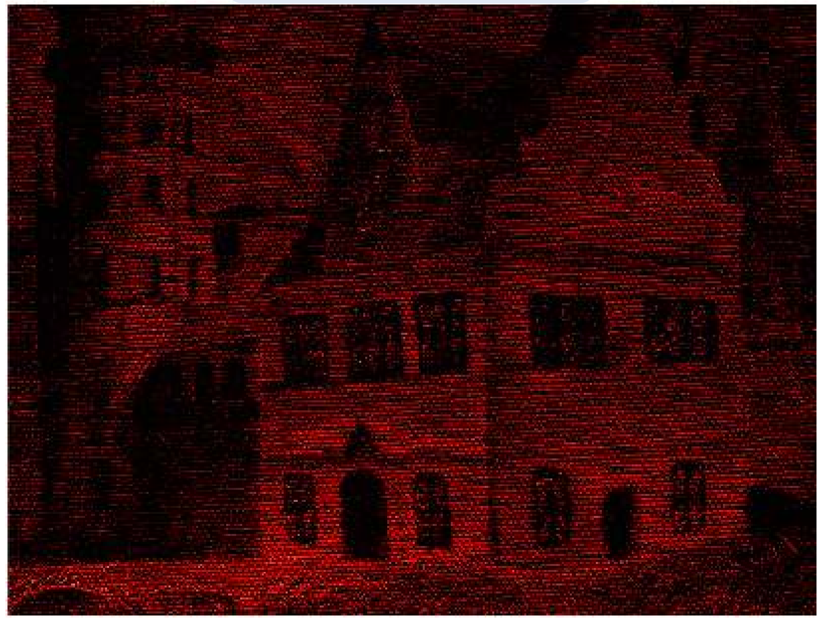
In this part of the activity, we'll try to use filter masks to enhance and eliminate patterns in an image.

Filtering: Painting

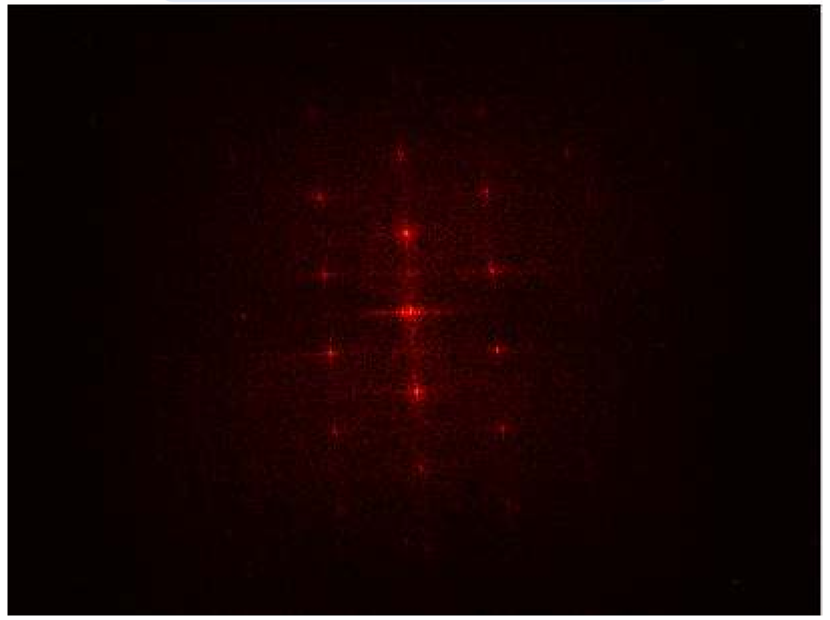
Here is the painting that we will filter by eliminating the texture of the canvas. In this way, we can investigate the brush strokes used by the painter. The first thing to do is to get the mean-center of the grayscale of the image by subtracting the mean grayscale to the original image. This eliminates the DC bias in the image [5]. Then, we get the FT of the mean-centered image. Here, I used the log scale to better see the frequency peaks. We will be using this to create the mask.

original

mean-centered

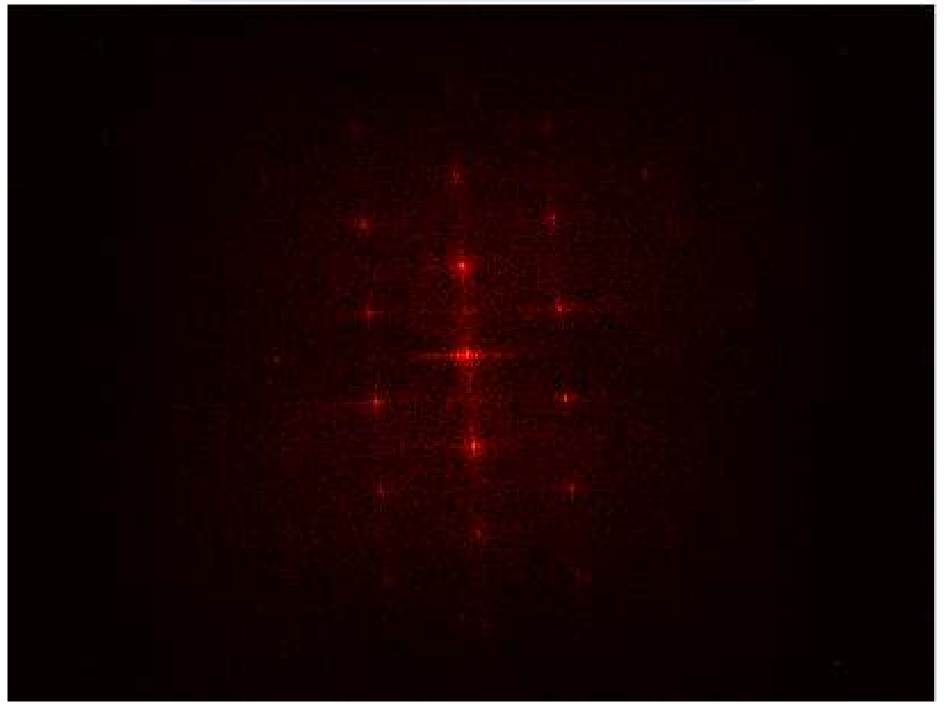


shifted FFT (log scale)

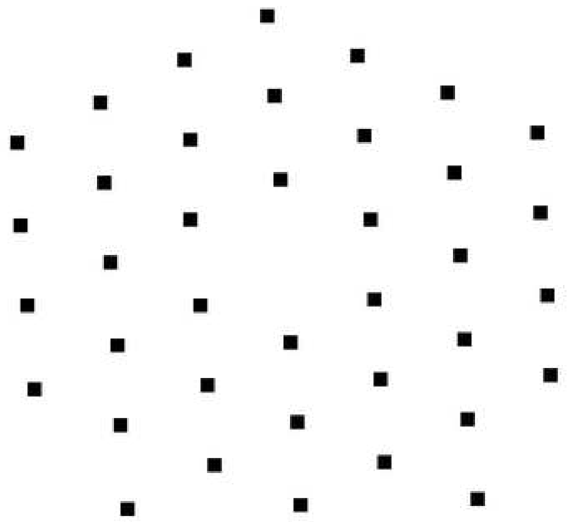


Filtering: Painting

shifted FFT (log scale)



mask



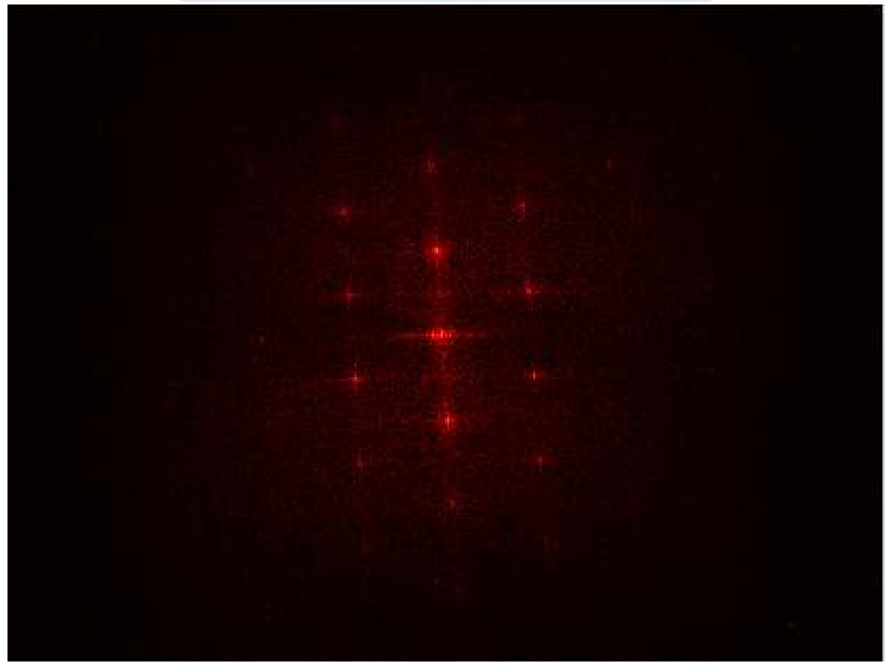
Notice that the filter is all 1's but are zero at the locations of the peaks. If we do the opposite, what would happen? We'll find out later on.

I manually created the mask by carefully plotting the location of the frequency peaks. The center frequency of the shifted FFT should not be masked because it contains the information of the image itself. Since we shifted the FT, the information of the image is centered.

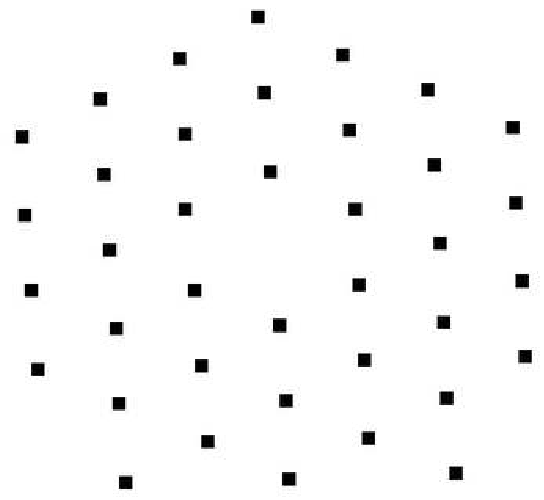
The mask will be convolved with the RGB channels of the original image. This will filter out or erase the peaks. Then, the filtered RBG channels will be overlayed to create the filtered image.

Filtering: Painting

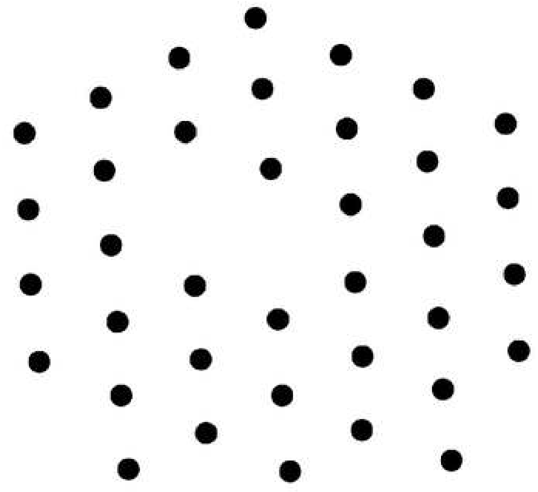
shifted FFT (log scale)



mask



mask



Let's examine the FT of the painting and the mask for a while here. From 2.2.1 we showed that the FT of a sinusoid is two dirac delta peaks (or a pair of dots). Now, the FT of the painting seems to be paired intensity peaks. This is because the repeating pattern of the canvas looks like combined sinusoids. We need to "mask" these peaks to filter the pattern of the canvas. Here, I tried two different masks with square and circle.

Filtering: Painting

Tada! We successfully filtered the image by eliminating the texture of the canvas. The filtered painting now shows more detailed brush strokes by the artist.

original

filtered (square)

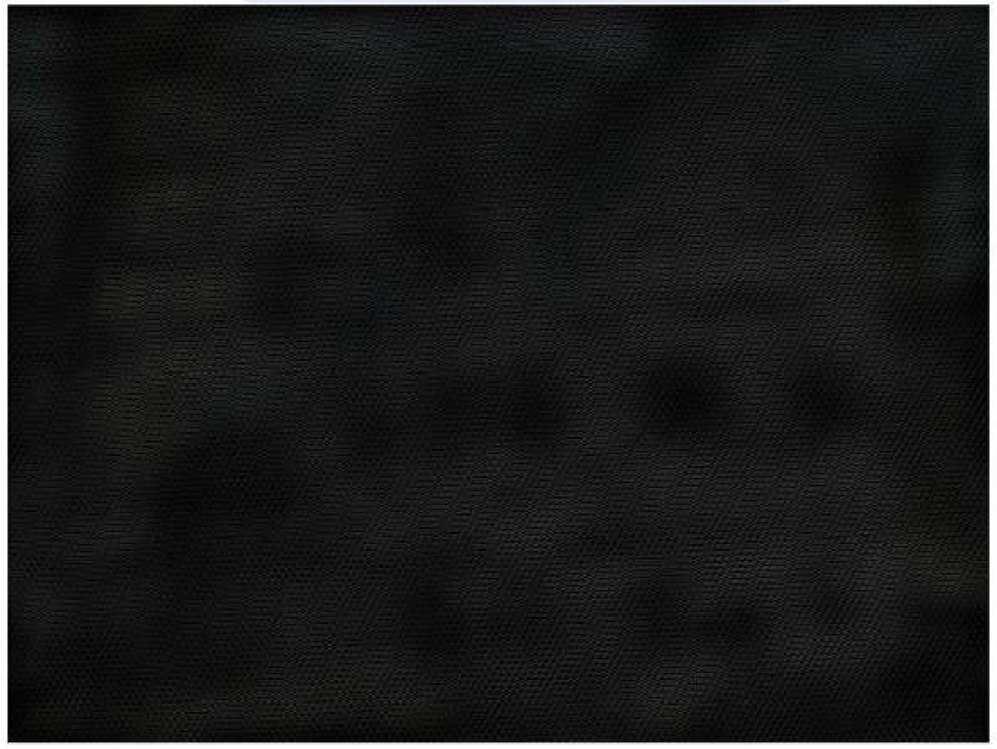
filtered (circle)

Filtering: Painting

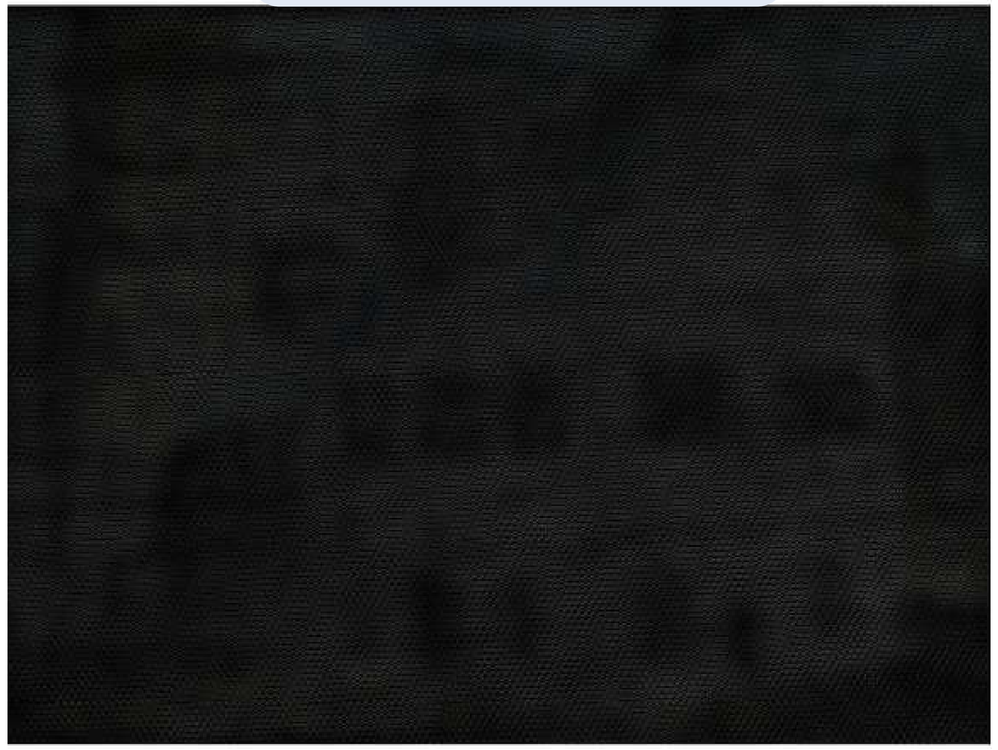
Using an inverted mask, the detailed pattern of the canvas weave will be the resulting image.

original

filtered (square)



filtered (circle)

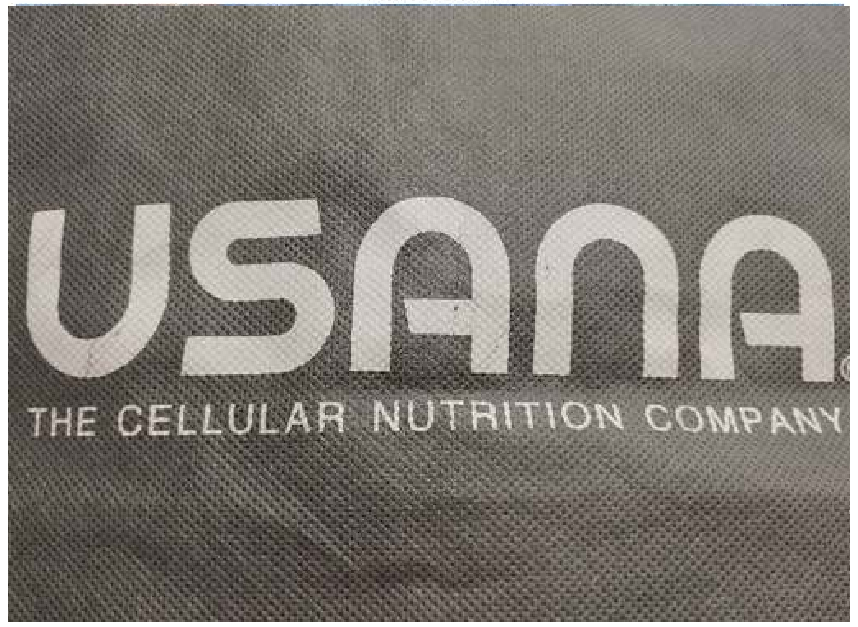


NOTE: Filtering in the frequency domain is faster than in the spatial domain. Frequency domain filters focused on the very high or very low frequencies of an image. These filters are used for smoothing and sharpening [6].

Extra Challenge 1

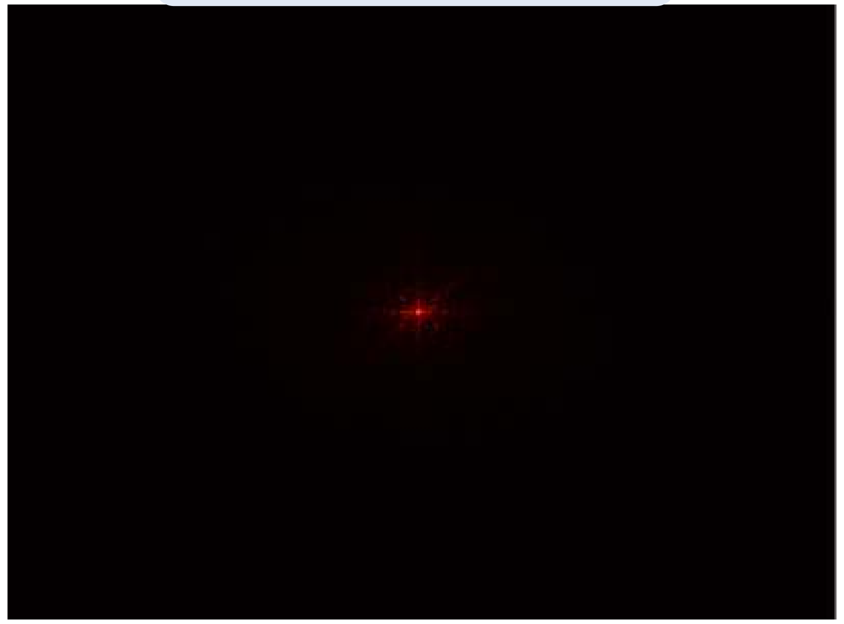
The challenge is to filter out the texture of a repeating material to yield an image of just the logo printed on it. Here, I used an ecobag with the logo of the USANA company. Similar to what I did with the painting, get the mean-centered grayscale of the image then display the shifted FT.

original



mean-centered

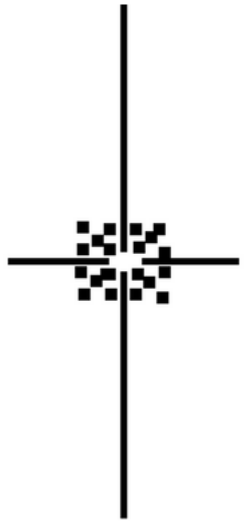
shifted FFT (log scale)



Extra Challenge 1

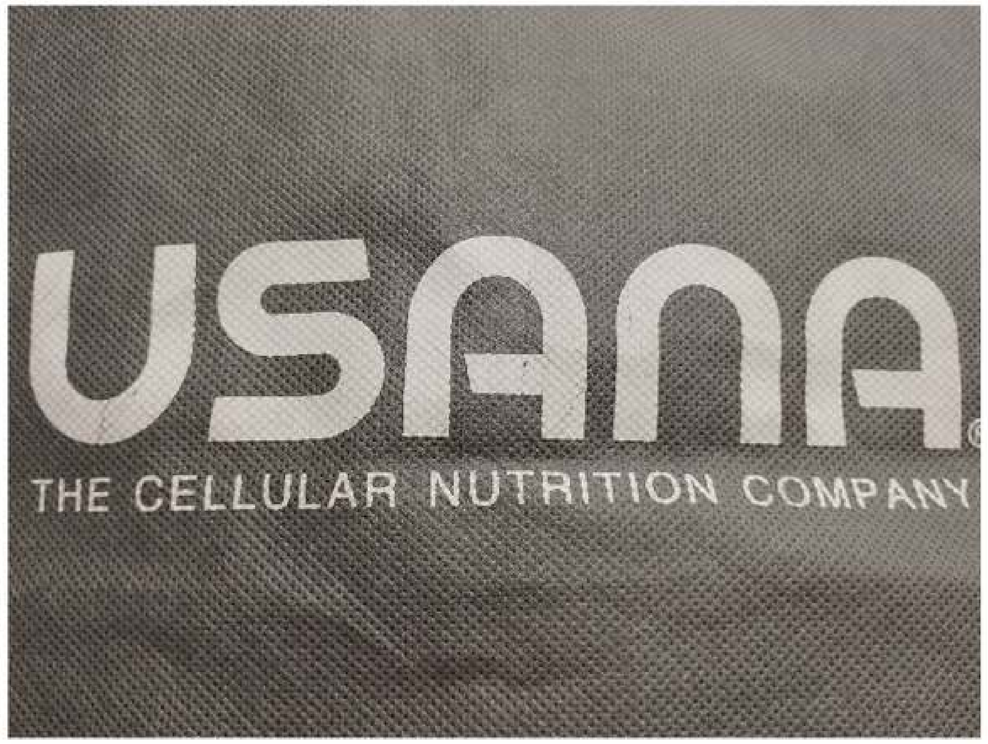
shifted FFT (log scale)

mask

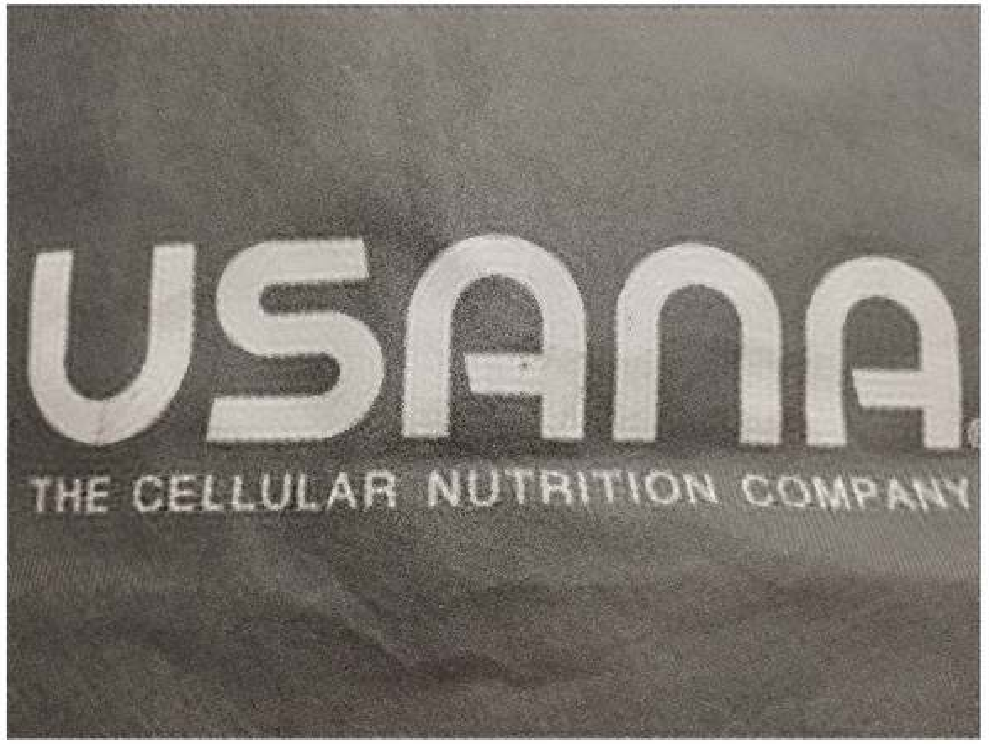


The filtered image contains only the logo. Therefore, the filtering process i s successful in eliminating the texture of the ecobag.

ORIGINAL



FILTERED



Extra Challenge 2

The challenge is to create an enhanced version of the weaving pattern of a piece of fabric, following the *Kaketsugi* tradition or invisible mending in Japan that studies the weaving pattern of fabrics to repair holes in garments [5]. Here, I used two types of fabrics, denim and cheesecloth, that I found at home.

denim

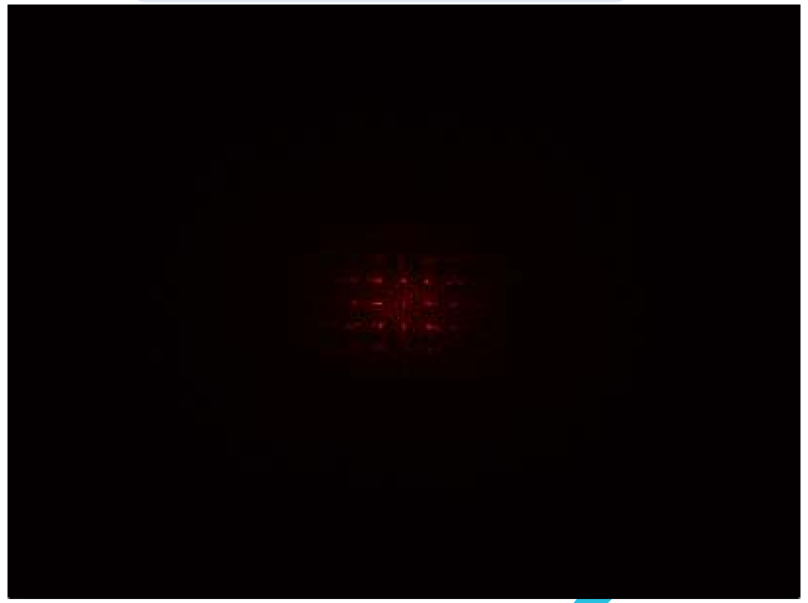
cheesecloth

Extra Challenge 2

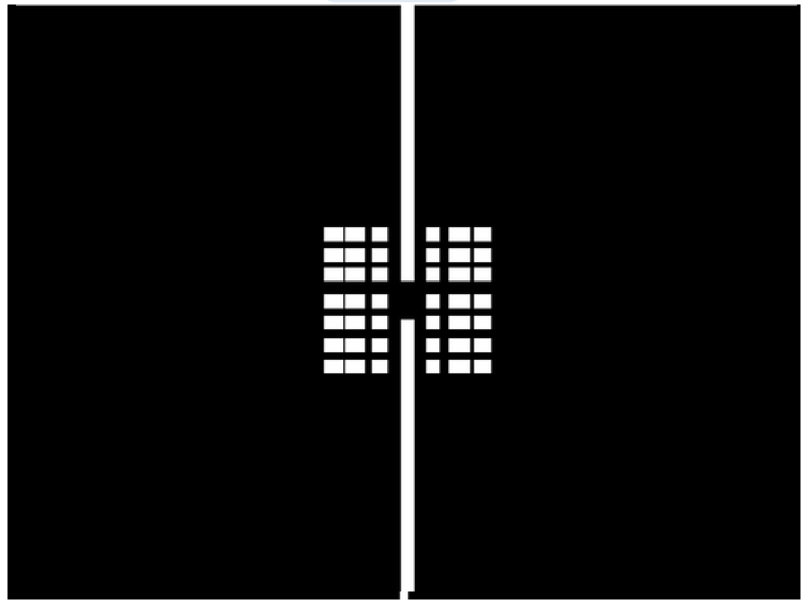
To enhance the weaving pattern of the denim cloth, an inverted mask was used. This will erase any other details and will only emphasize the pattern of the fabric.

Extra Challenge 2

shifted FFT (log scale)



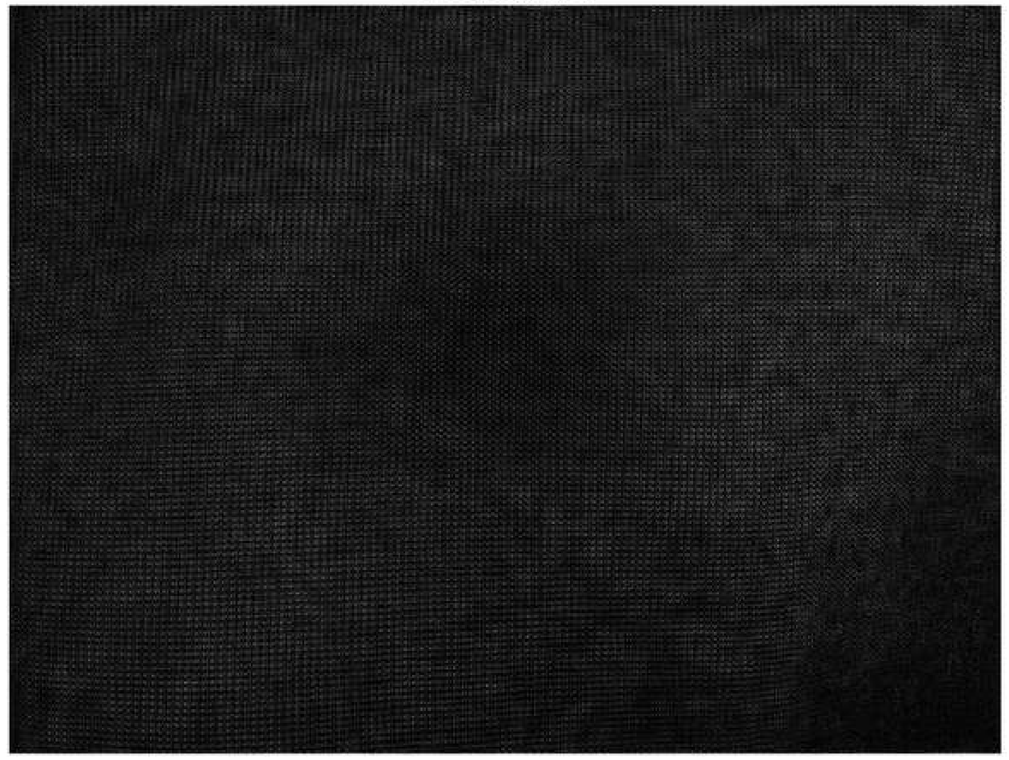
mask



An inverted mask was also used. The resulting image is the enhanced weaving pattern of the cheesecloth.

ORIGINAL

FILTERED



Activity 2.2.3. Convolution Theorem Redux

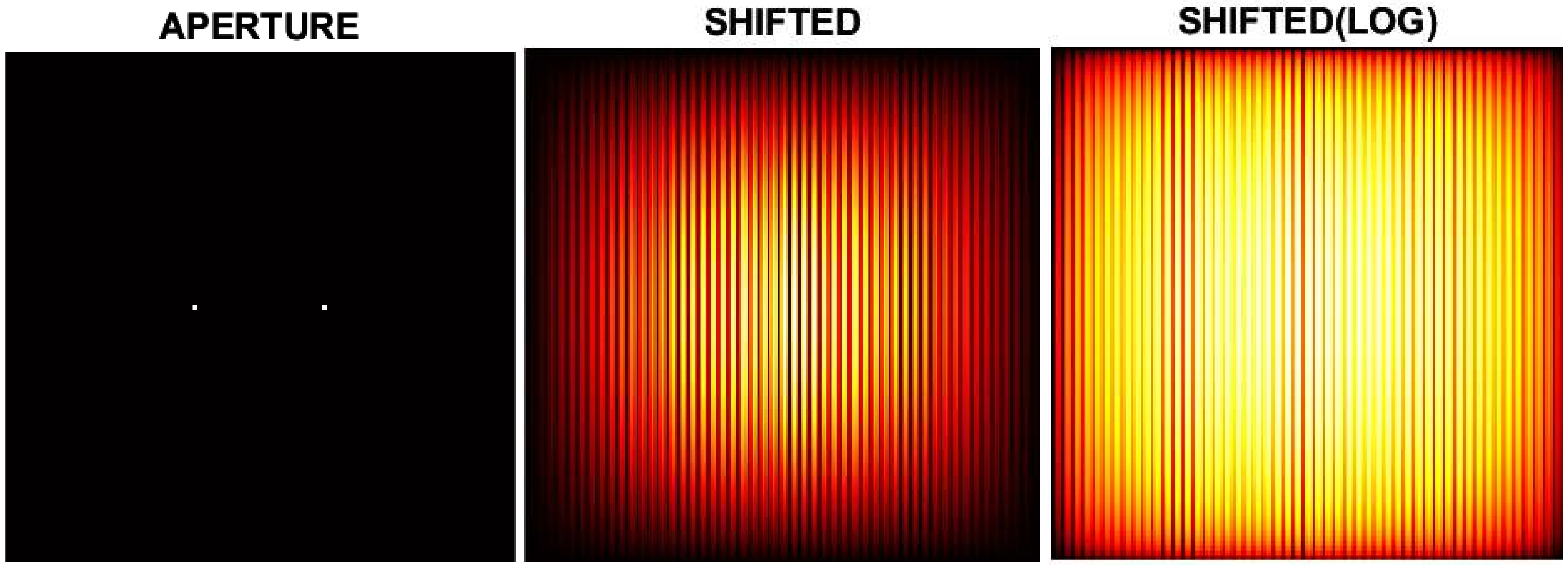
Background

The Convolution Theorem states that Fourier transform of the convolution of two functions in space is equal to the product of the Fourier transforms of the two functions in the Fourier domain [7]. Moreover, it is important to note that the convolution of a dirac delta function at position x0,y0 and a function f(x,y) yields a replication of f(x,y) in the location of the dirac delta [5].

In this part of the activity, we'll revisit the convolution theorem and how it is applied.

Two dots

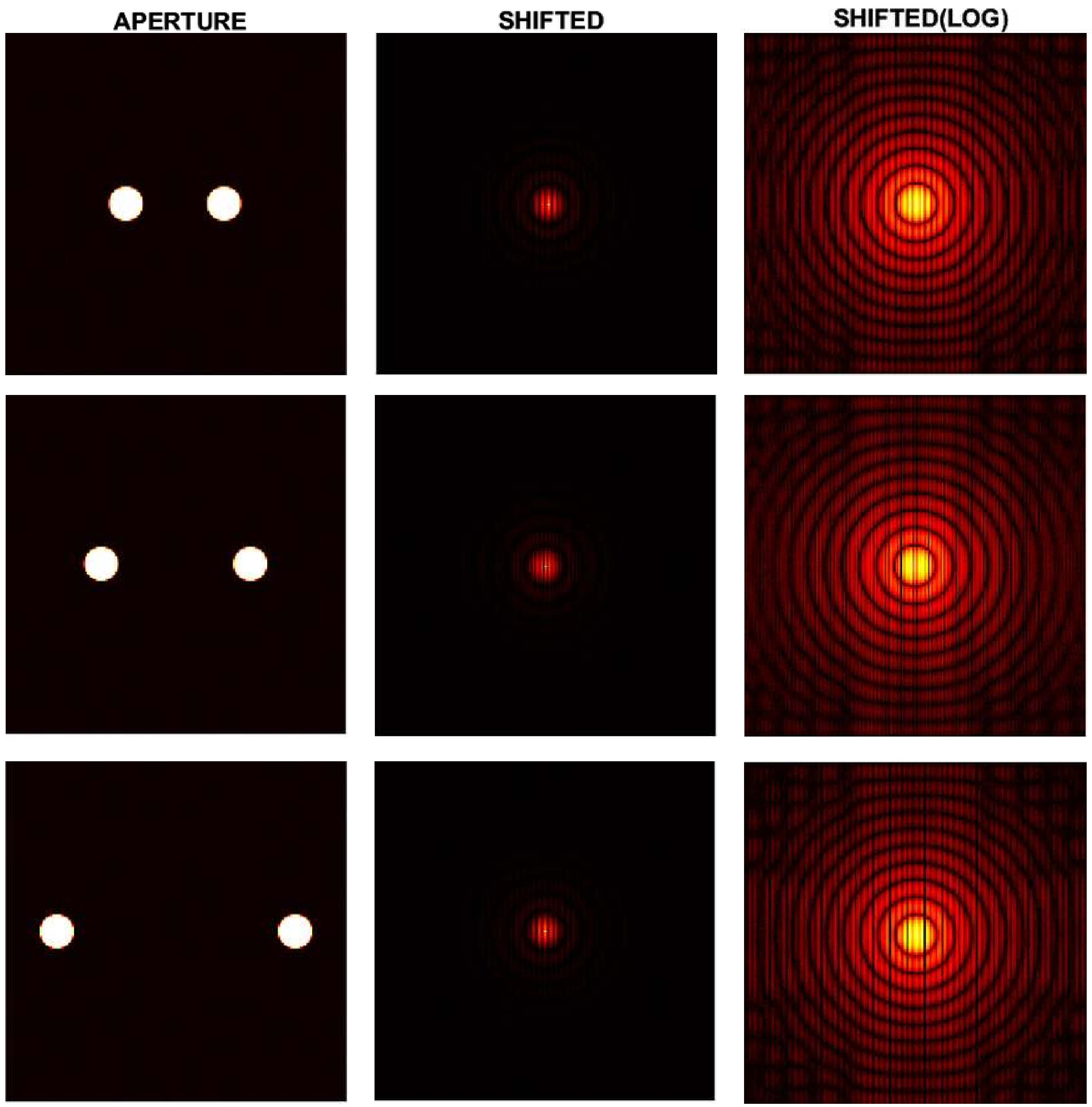
Here we take the FT of a binary image of two dots along the x-axis symmetric about the center. The shifted FT shows a sinusoid along the x-axis. The Fourier transform is a reversible process. From Activity 2.2.1, we know that the FT of a sinusoid is two diract delta peaks (or two dots). Taking the FT of the two dots, as we did here, yields a sinusoid. Therefore, what we did here is like taking the inverse FT of a sinusoid, which yields the original function [8].



Two Circles (Varying Distance)

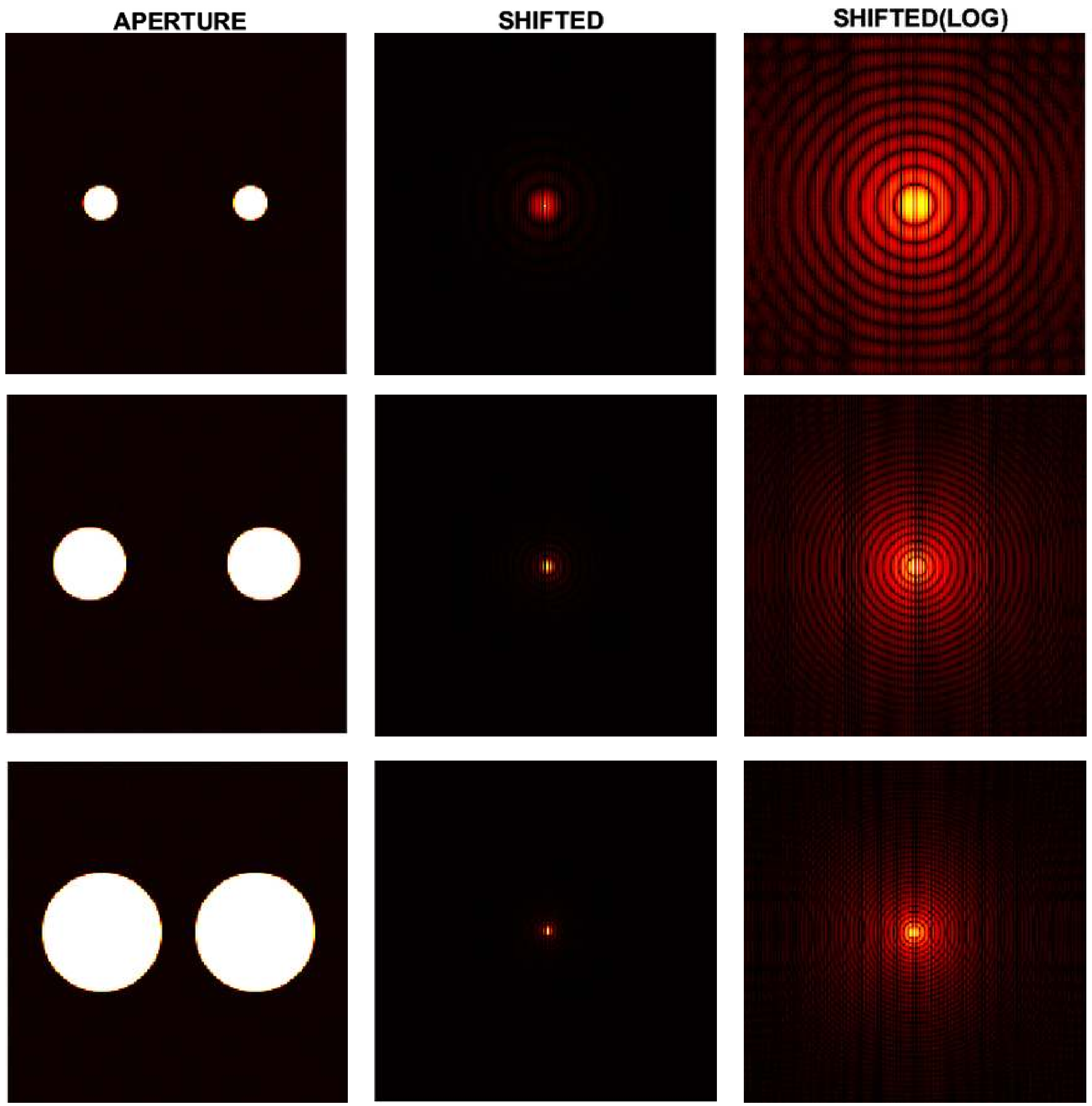
Increasing the radius of the two dots to two symmetric circles yields a different FT. The shifted FT looks like a combination of a Bessel function and a sinusoid along the x-axis. From Activity 2 Part 1, we discovered that the FT of a circle is a Bessel function. Moreover, we already know that the FT of two dots is a sinusoid. Hence, we can infer that the shifted FT of two symmetric circles is a convolution of the FTs of two dots and a circle.

Therefore, as the distance between the circles is increased, the frequency of the sinusoid in the shifted FT also increases, which stays true to what we have observed in Activity 2.2.1.



Two Circles (Varying Radius)

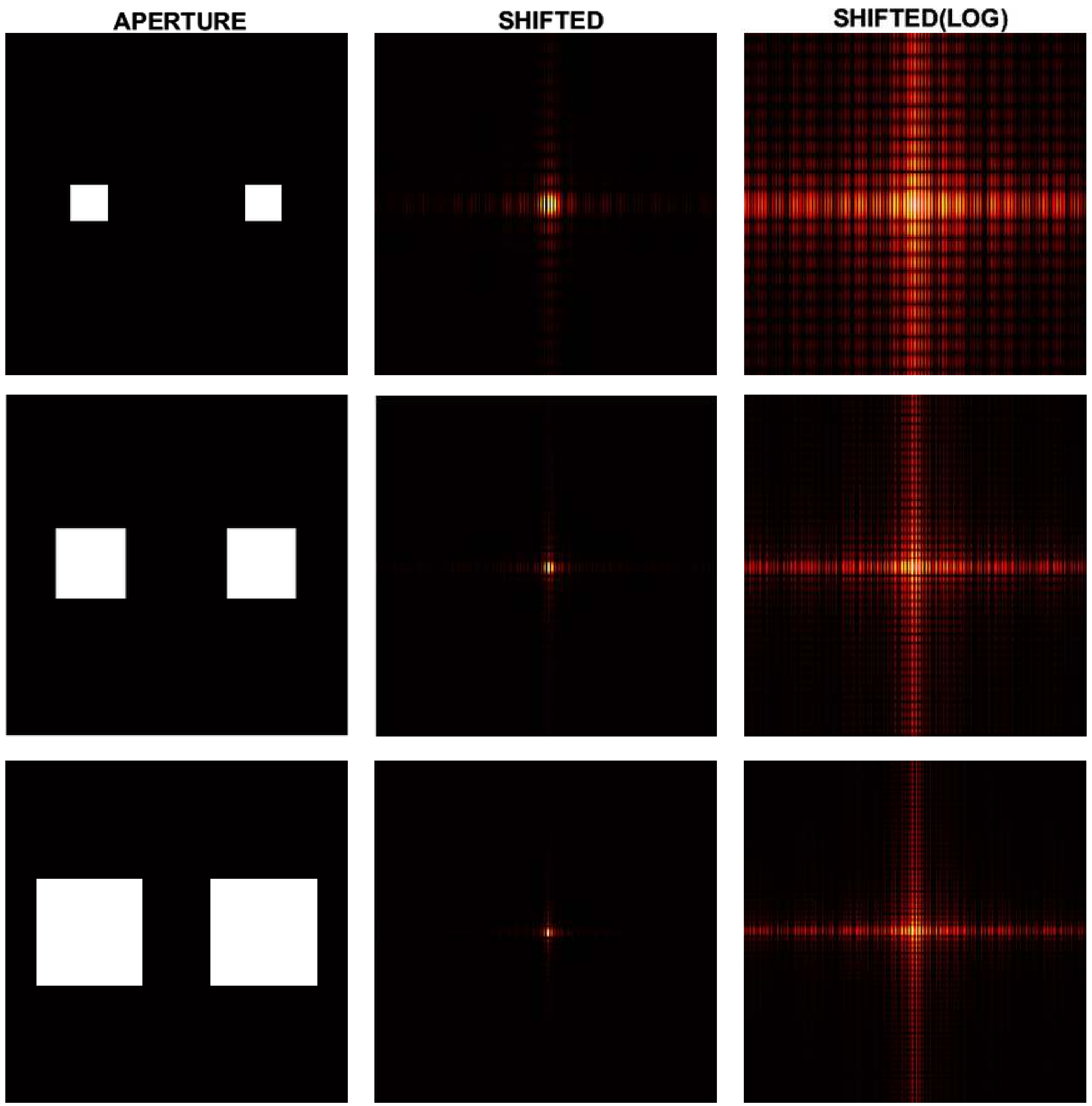
We vary the radius of the two symmetric circles. As we can see, as the radius of the circles are increased, the resulting FT becomes smaller. This shows the anamorphic property of the Fourier Transform. This means that a signal of X dimension in the spatial domain transforms to a signal of 1/X dimension in the frequency domain [9].



Two Squares (Varying Width)

We change the aperture from two circles to two squares of varying width. The shifted FT is a combination of a sinc function and a sinusoid. The FT of a box function is a sinc function while the FT of two dots is a sinusoid. Hence, the resulting FT is a convolution of the FTs of a box function and two dots.

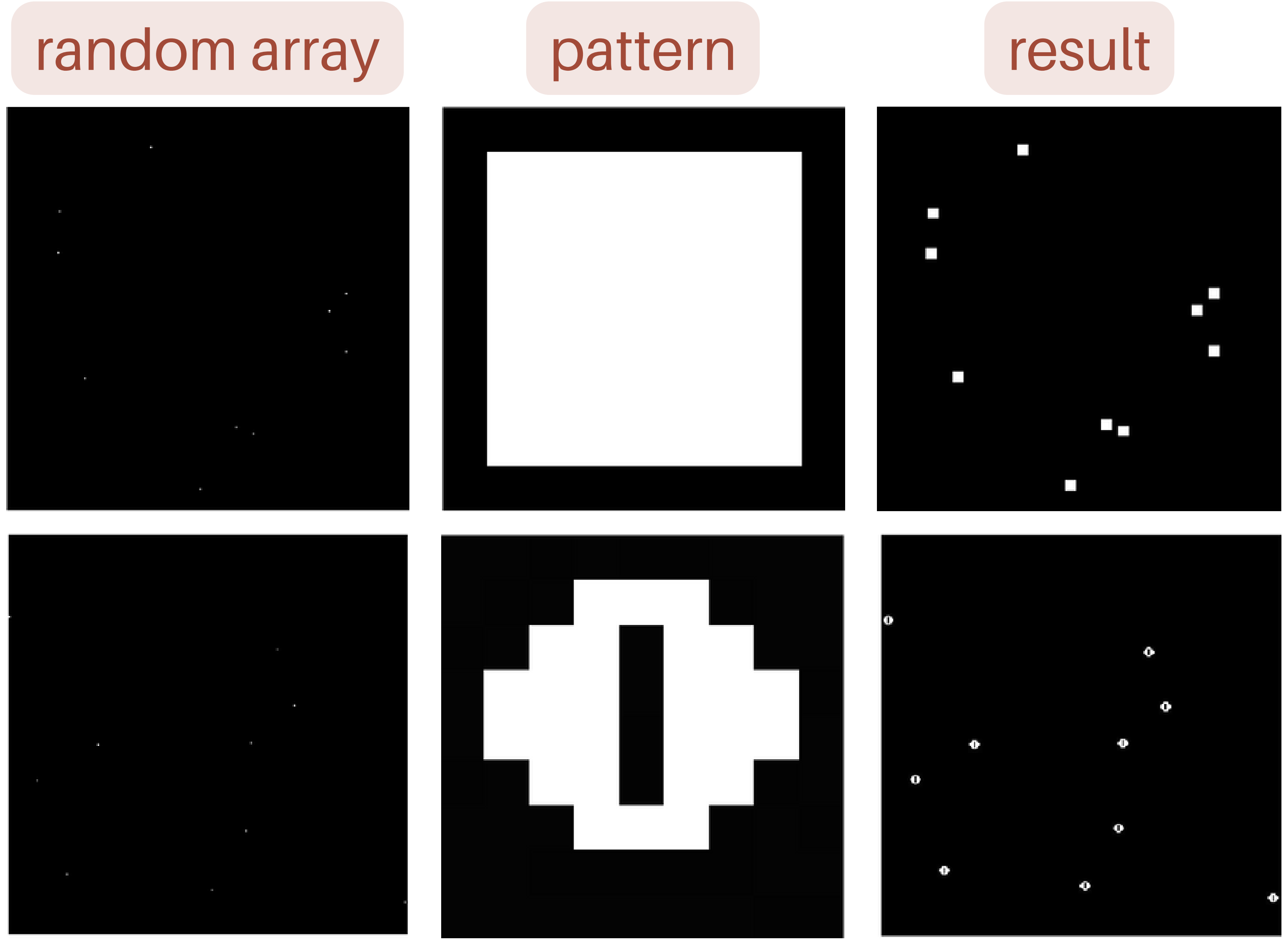
We can also observe anamorphism when the width of the boxes are varied. As the width increases, the dimensions of the resulting FT decreases.



Convolution

For this part, we are tasked to create a 200x200 array with 10 random 1's. These random 1's represent the dirac delta peaks. Then, we convolve the random array to a 9x9 pattern. I used the conv2() method in MATLAB for the convolution process.

The resulting image is the pattern in the same position as the random 1's. It's like copying the pattern to the position of the random points! Amazing :> .

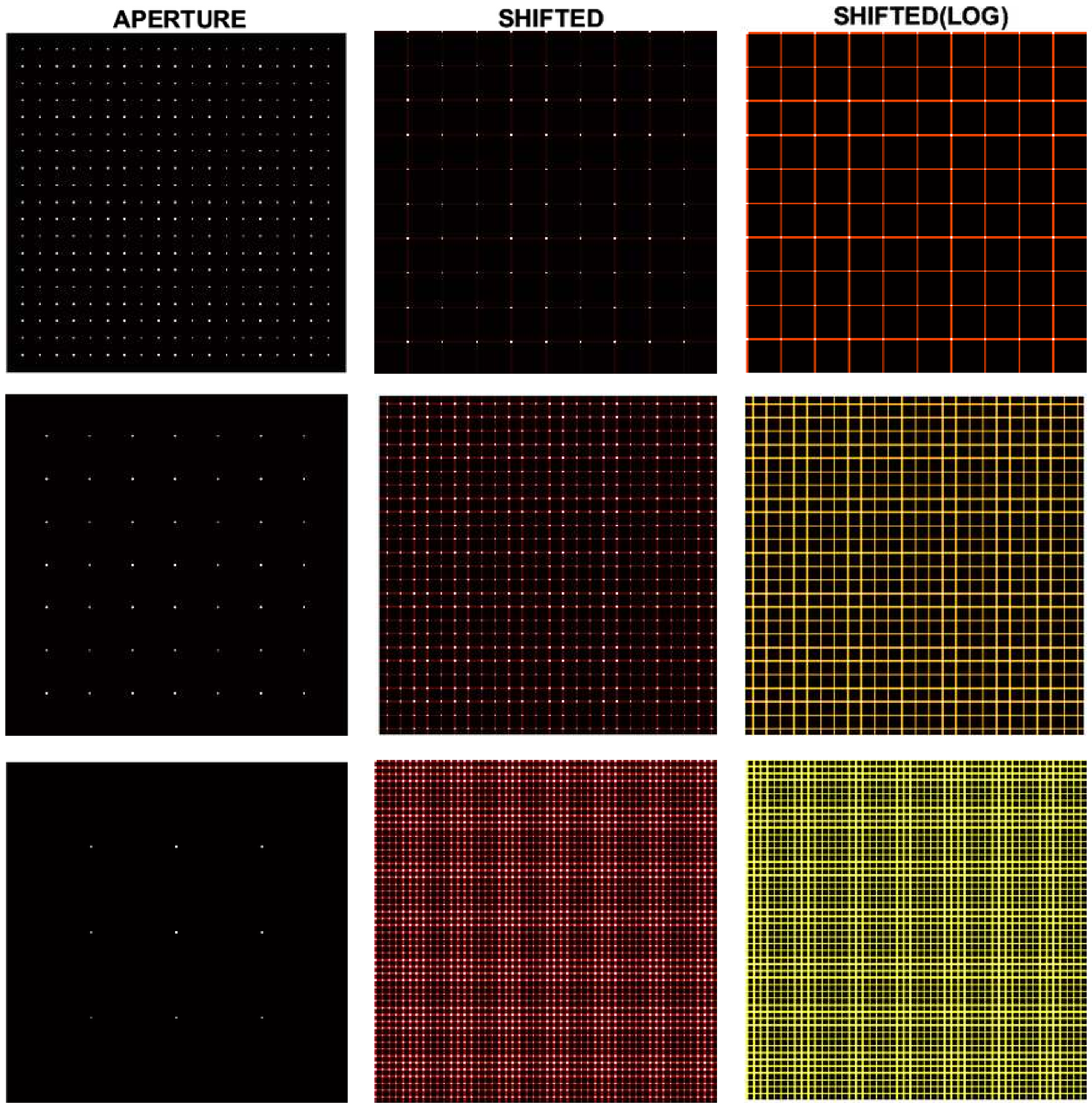


Recall that the convolution of a dirac delta and a function results in a replication of that function (in this case, the pattern) in the location of the dirac delta [5].

Equally Spaced 1's

For this part, we create a 200x200 array of zeros with equally spaced 1's along the x- and y-axis. Here, I used a 10, 25, and 50 spacing. The FT of the dots are also dots but with different spacing. If we look closely in the FT log scale, the dots appear to be gridlines with frequency depending on the spacing of the dots in the real space.

Another important thing to note here is that as the spacing of the dots in the real space increases, the spacing of the dots in the Fourier space decreases (more dots appear!). This also shows the anamorphism of the Fourier transform.



Activity 2.2.4. Fingerprints: Ridge Enhancement

Background

Another application of the filtering using Fourier transform is the ridge enhancement of fingerprints. The ridge arrangement on every finger of any person is unique. It does not change with age. Hence, it is one of the tools used for personal identification [11].

In this part of the activity, we'll try to determine the Fourier transform of fingerprints and enhance the ridges using filtering.

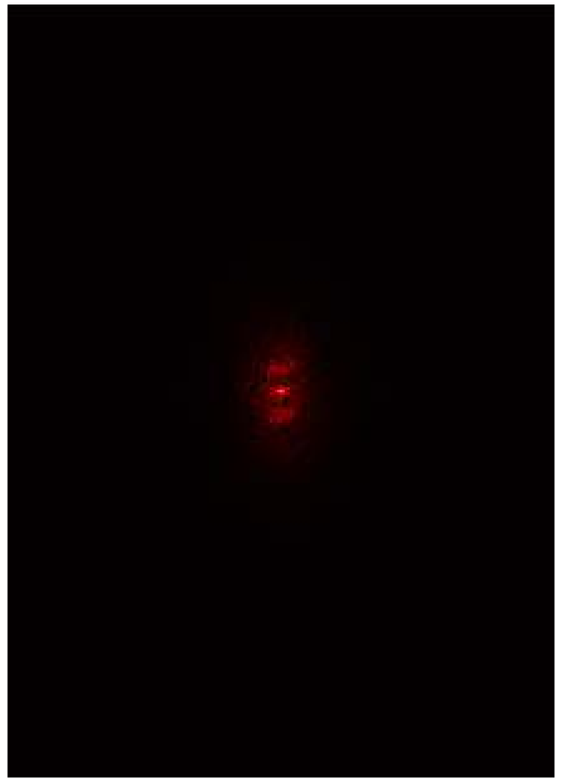
Preparing the Fingerprint

I've had several failed attempts in obtaining a good sample of my own fingerprints because I only used a permanent marker and a white paper. I also tried to get my parent's fingerprints so I will have enough samples to choose from. The fingerprint below is my mother's, and it's one of the best of the bunch. I get the mean-centered of the grayscale of the image and obtained its FT in log scale to show the frequency peaks.

fingerprint

mean-centered

shifted FFT (log scale)



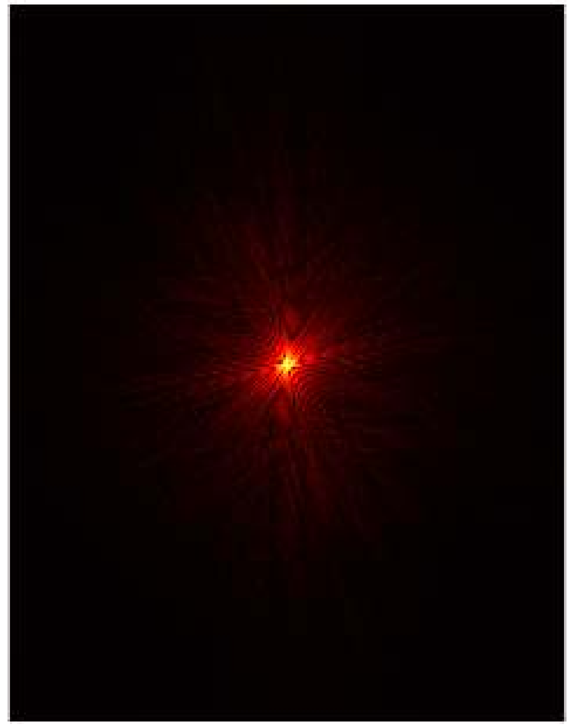
Preparing the Fingerprint

Below is my own fingerprint. I used a stamp pad on this one so it looks better than the previous fingerprint. I also got the mean-centered of the grayscale of the image and obtained its FT in the log scale.

fingerprint

mean-centered

shifted FFT (log scale)

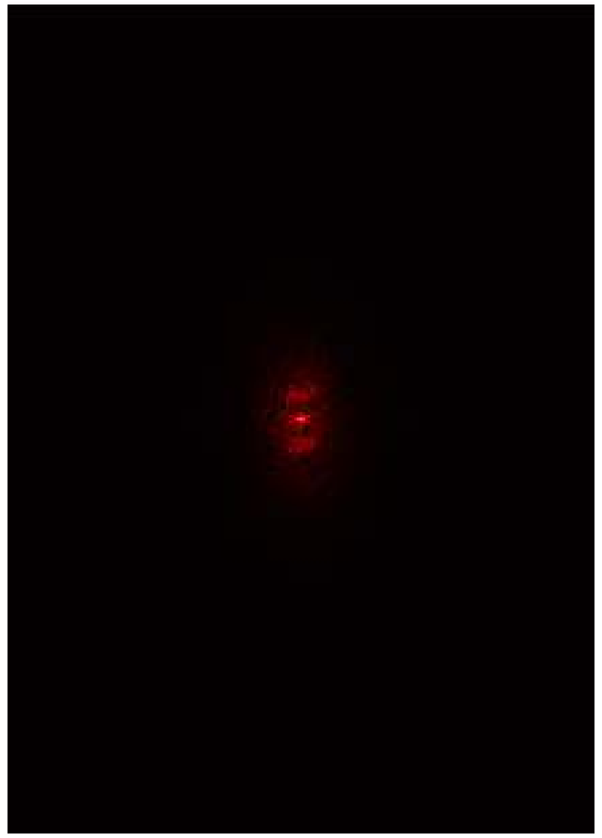


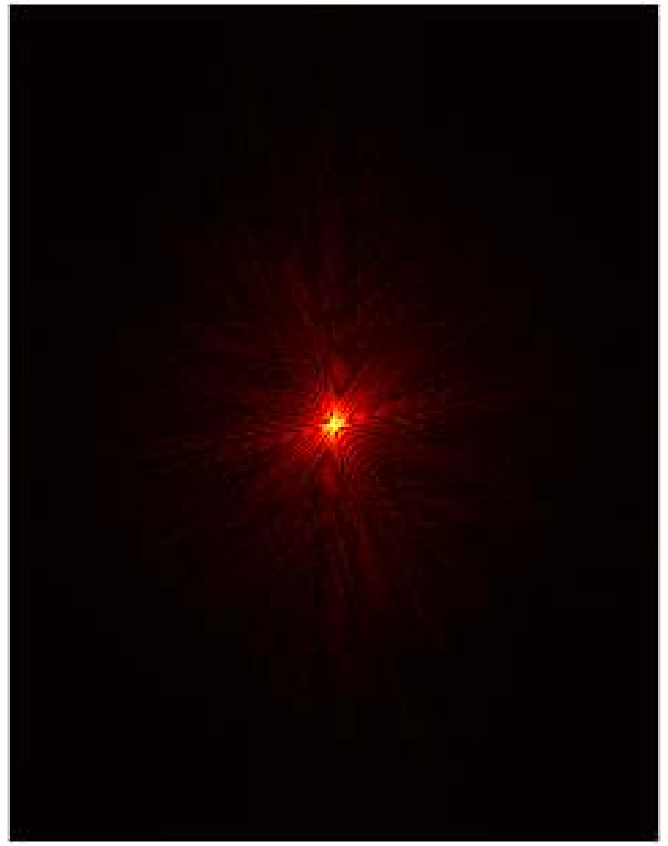
The FT of Fingerprints

Let's examine the FT of the fingerprints. Both the shifted FFT log scale shows a bright intensity at the center. This is the manifestation of the ridges which are the elevated strips of the skin. These appear to be black patterns in a fingerprint image [12].

If we compare the two fingerprint images, we can already see that the better image produces a better FT. It is brighter and more defined. Hence, we can say that the resulting FT also depends on the quality of the fingerprint image used.

shifted FFT (log scale)





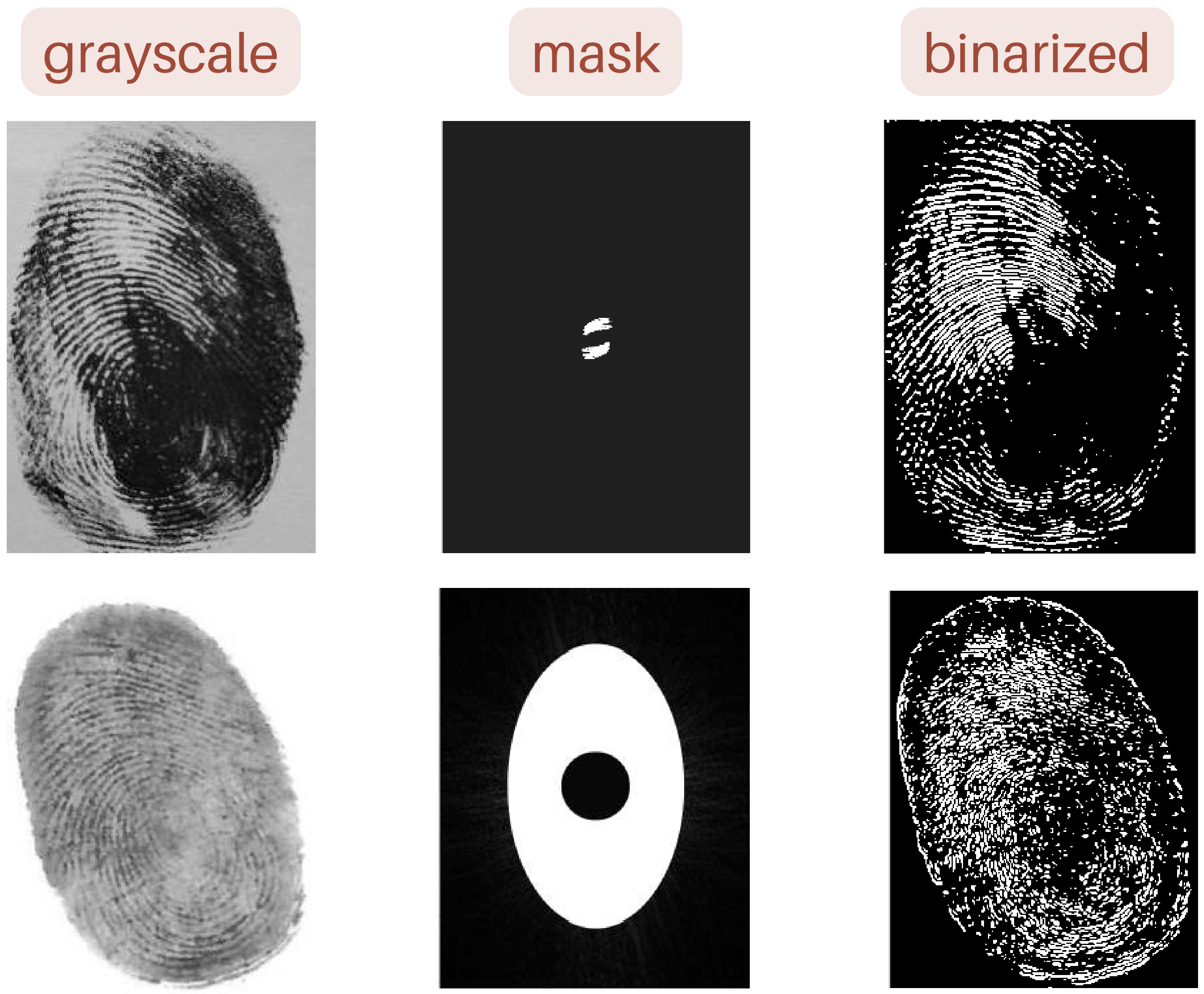
my mom's

mine

The FT of Fingerprints

I manually made the mask based on the information from the FT log scale. I convolved the mask with the grayscale of the fingerprint image and binarized the result.

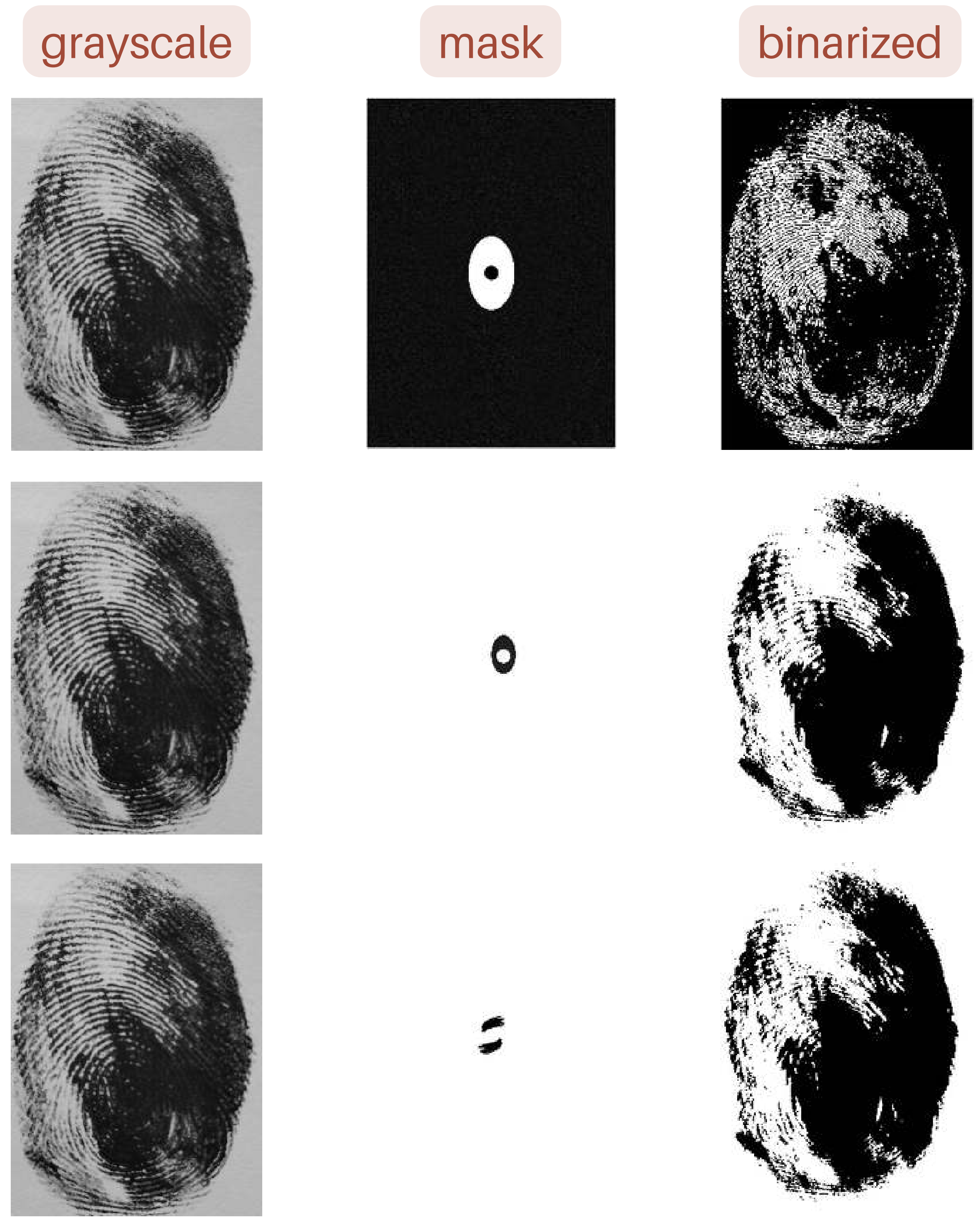
As shown here, the ridge of the fingerprints were enhanced after convolution. We can clearly see the pattern. Since fingerprints of all humans are unique, the patterns of the fingerprints used here are also unique.



The FT of Fingerprints

I also tried using different masks for my mom's fingerprint and investigated the resulting binarized image. The first mask was still able to enhance the ridges of the fingerprint.

For the second and last masks, I tried to invert the masks (white background this time). Since my mom's fingerprint was made by using ink only, the inverted mask was able to recover the smudge of inks that are included in the fingerprint image.



Activity 2.2.5. Lunar Landing Scanned Pictures: Line Removal

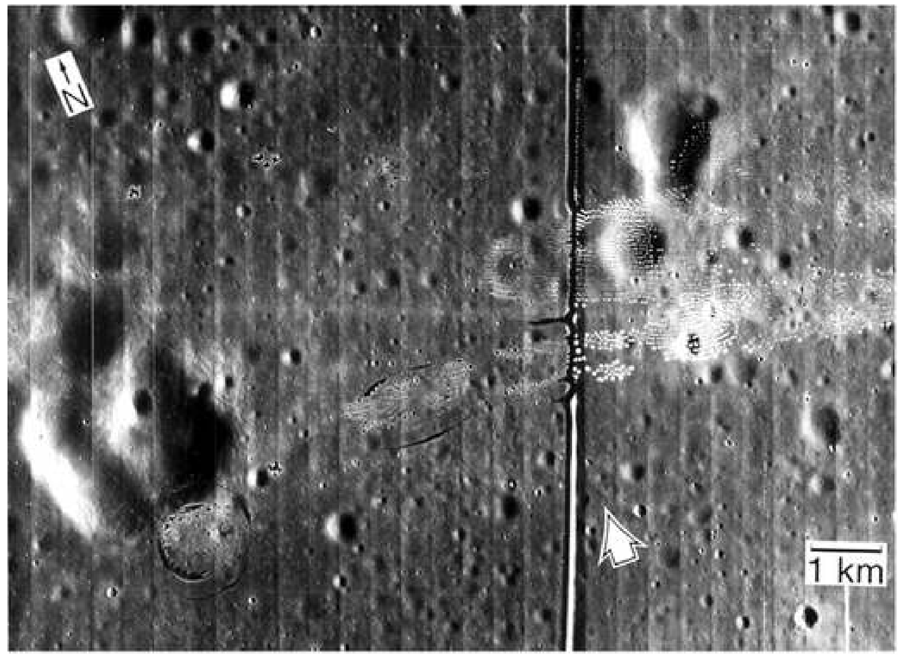
Background

With the advancement of science, we are able to capture images of different things in space. The right techniques in image processing compensate for the limitations of our cameras in space. For instance, the regularly spaced vertical lines of the images captured by the unmanned Lunar Orbiter V spacecraft caused by combining individually digitized framelets to make a composite photograph can be filtered out using filtering in Fourier Transform [5]. This is what we will do in this part of the activity.

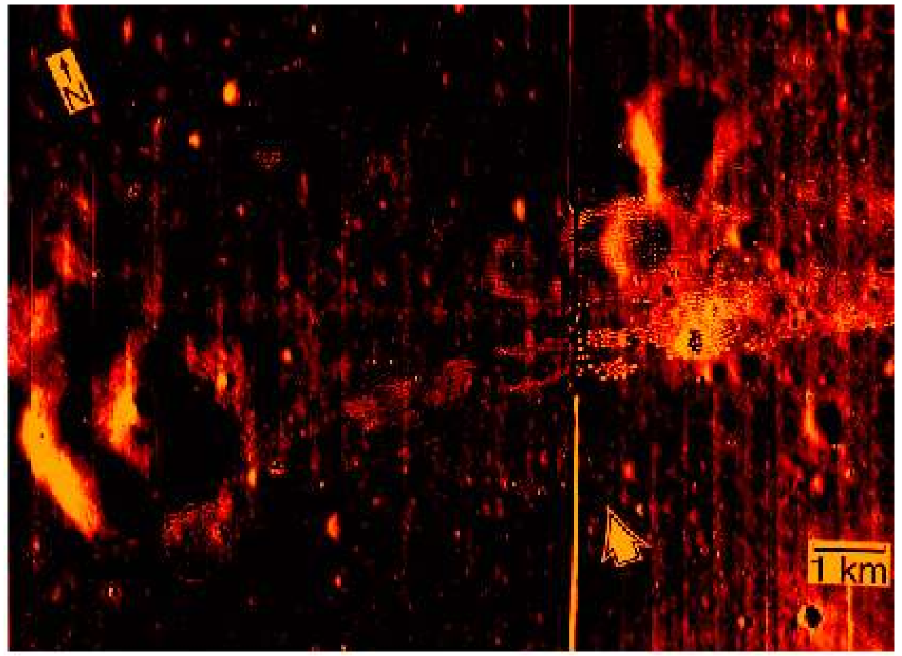
FT of the Lunar Landing Image

Shown below is a Lunar Landing Scanned image. Similar to what we did in the previous activity, we also get the mean-centered image and its FT. This will help us in the creation of the mask that we will use to filter the image later.

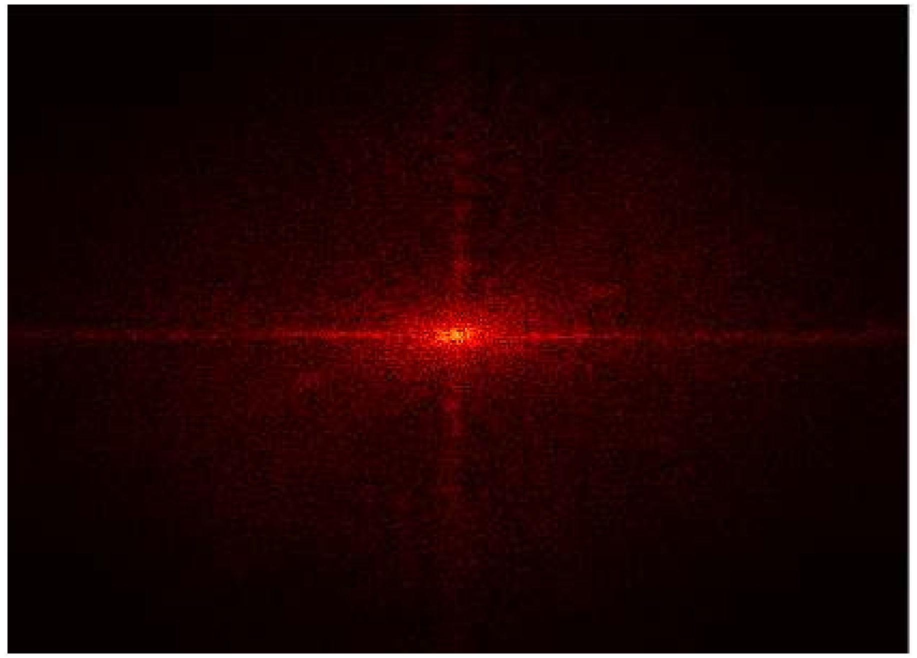
original



mean-centered

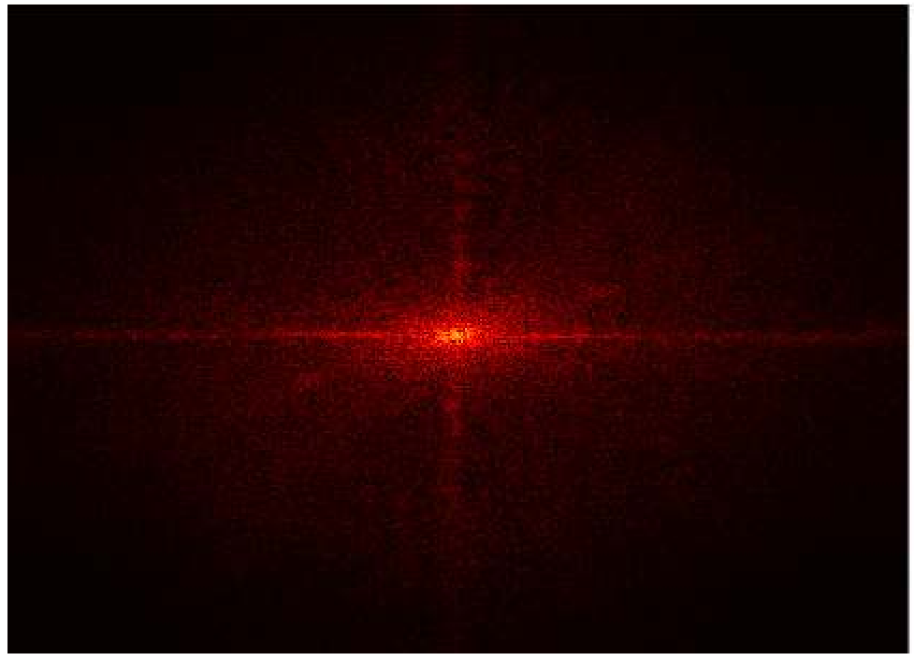


FFT(log scale)

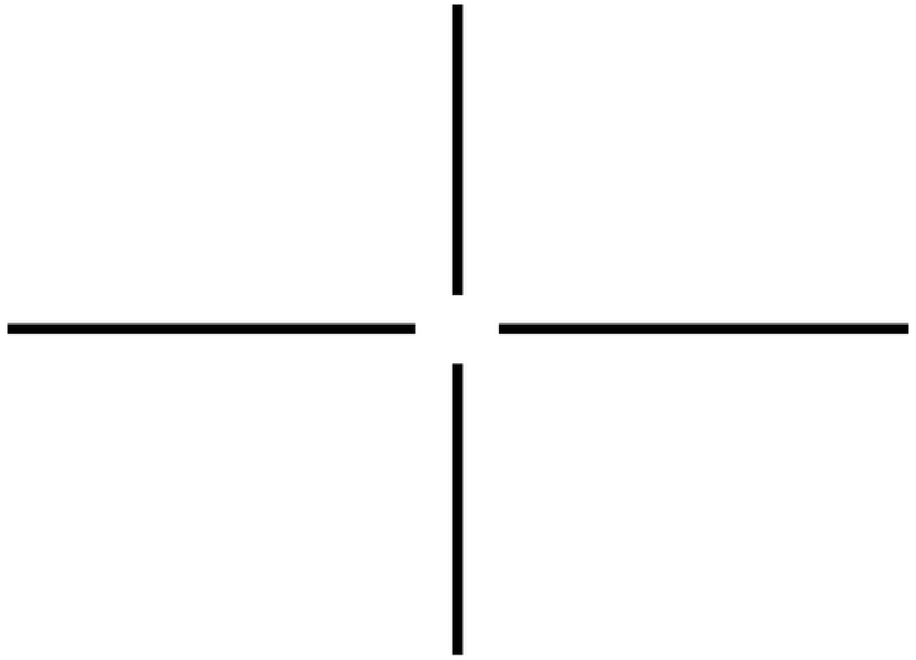


Mask

FFT(log scale)



mask

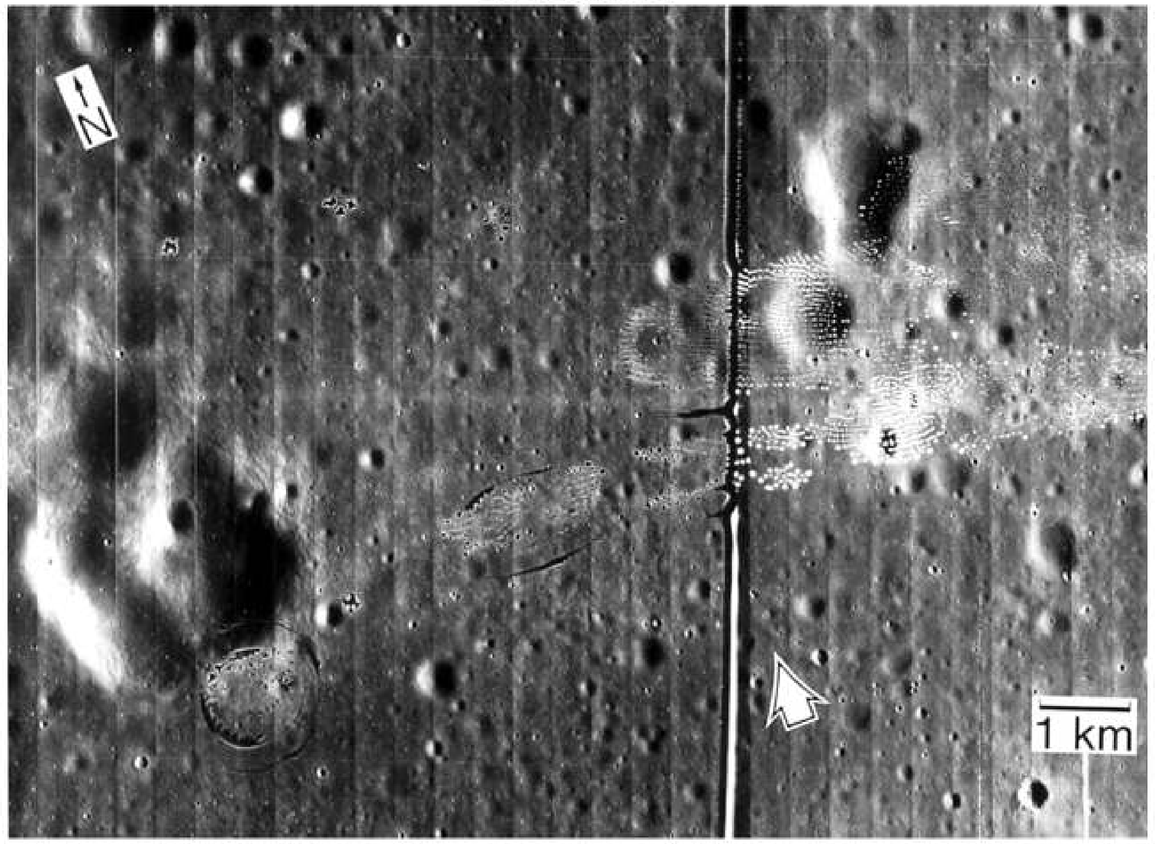


Shown above is the mask that I manually created. Since the bright center of the FT image is contains the details of the image, I left the middle part of the mask blank as to not distort the image itself. The horizontal and vertical lines correspond to the vertical lines (and some horizontal lines) in the image. These are what we want to filter out so we will mask these intensities.

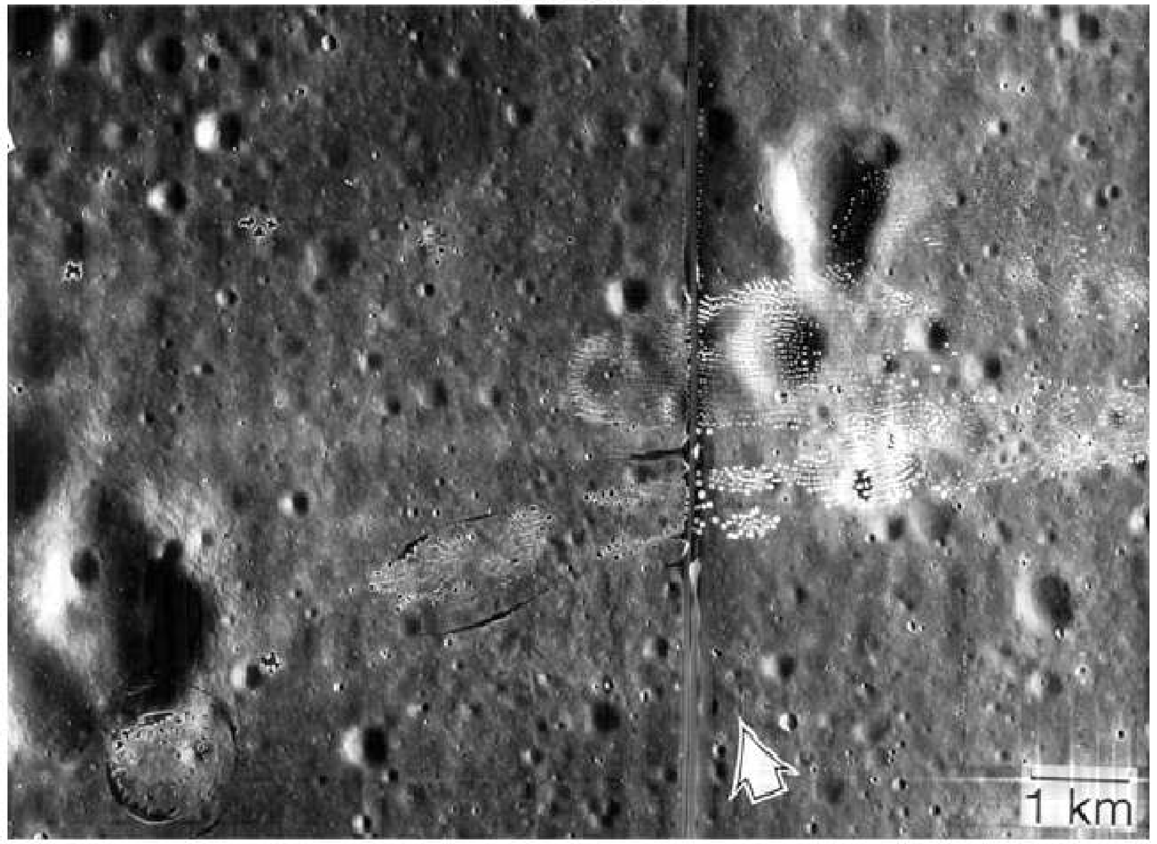
Filtered Image

Tada! We successfully filtered out the lines and we are left with a clearer image. It's like the image is not stitched together at all!

ORIGINAL

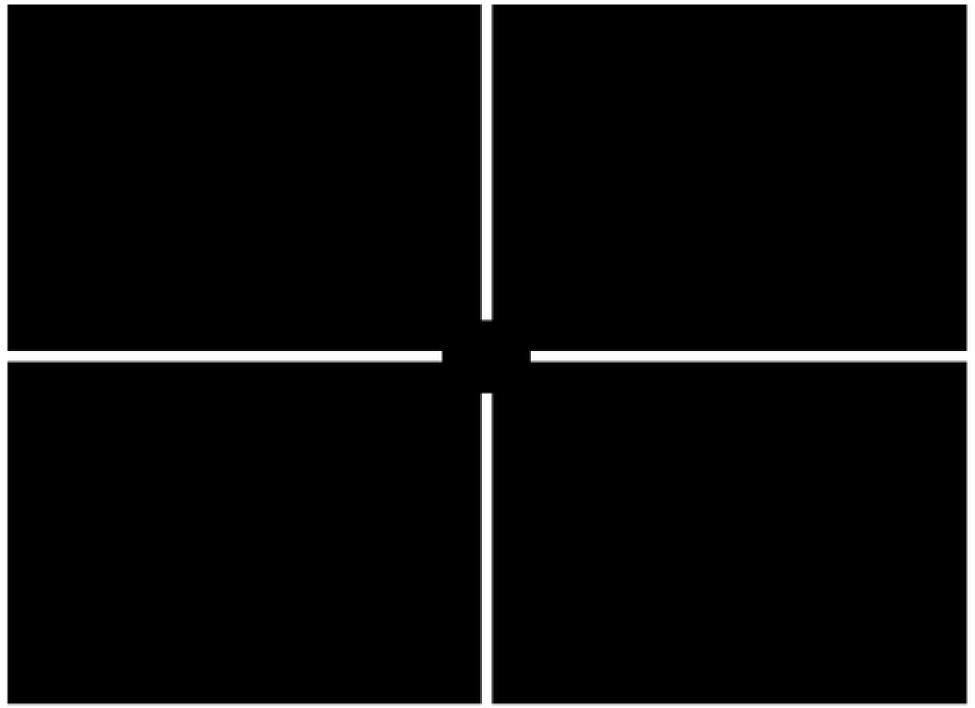


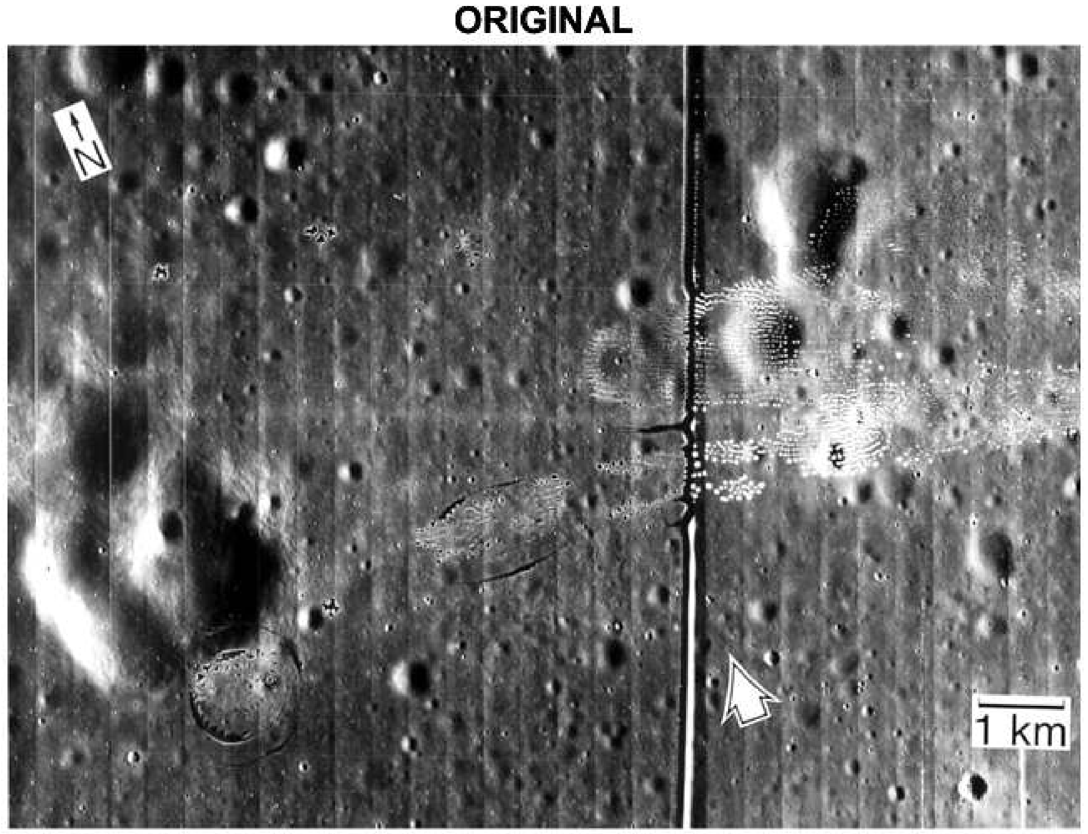
FILTERED

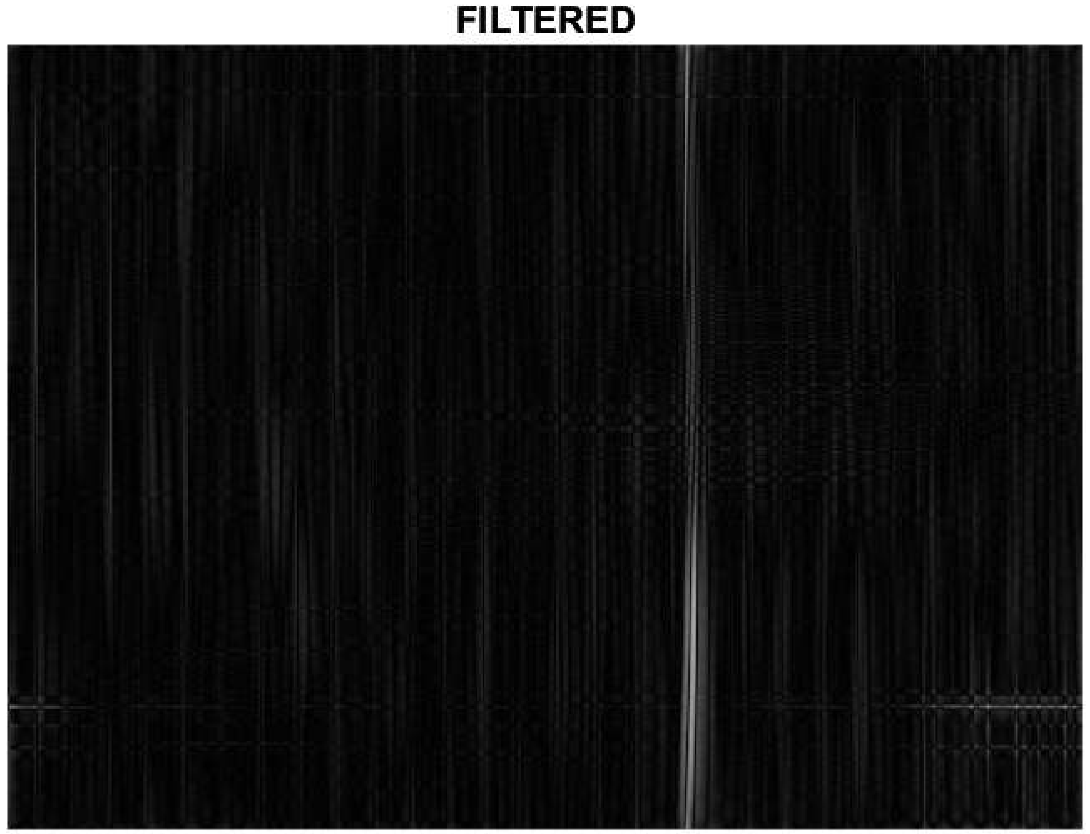


Inverted Mask

I tried to invert the mask and the resulting image contains the equally spaced vertical lines that we tried to filter out before.







Reflection

Similar with the Activity 1, this activity also involves working with images. It is a fun experience over all because I get to see how the images unfold. I already have a visualization of whether the code I made worked or did not work.

The coding part of this activity is not that cumbersome; as well as the report. However, the report part took much of my time because I wanted to find reasonable explanations from credible sources to confirm whether the resulting images that I got are theoretically correct. There is that "kilig" factor when I see that the results that I have matches the theory behind them.

Personally, the part that I particularly enjoyed is the filtering of the images. Getting the filtered image of the painting and seeing the brush strokes of the painter, and being able to see the pattern of the fabrics clearly is the best feeling in this activity.

Also, I wouldn't be able to finish this without the help of my VIP labmates who answered my questions when I don't know what to do next or when my code isn't working.

References

1. Jakowatz, C. V. J., Wahl, D. E., Eichel, P. H., Ghiglia, D. C., Thompson, P. A. (2012). Spotlight-Mode Synthetic Aperture Radar: A Signal Processing Approach. United States: Springer US.
2. Rogers, A. J. (2008). Polarization in Optical Fibers. United States: Artech House.
3. Thyagarajan, K. (2018). Introduction to Digital Signal Processing Using MATLAB with Application to Digital Communications. Germany: Springer International Publishing.
4. Mendel 2015: Recent Advances in Soft Computing. (2015). Germany: Springer International Publishing.
5. Module ni Ma;am Jing
6. https://www.geeksforgeeks.org/frequency-domain-filters-and-its-types/
7. Humar, J. (2002). Dynamics of structures. Netherlands: Taylor & Francis.
8. Pinkus, A., Zafrany, S. (1997). Fourier Series and Integral Transforms. United Kingdom: Cambridge University Press.
9. Linear Canonical Transforms: Theory and Applications. (n.d.). Germany: Springer New York.
10. https://www.thefouriertransform.com/pairs/box.php
11. https://www.britannica.com/topic/fingerprint
12. https://www.researchgate.net/publication/269588995_Biometric_quality_a_review_of_fingerprint_iris_and_face

REPORT GRADE

Criteria	Score
Technical Correctness	35
Quality of Presentation	35
Self Reflection	30
Initiative	10
TOTAL	100 (+10)

EVALUATION

Overall, I know that I have accomplished all the tasks included in this activity. Moreover, I did some extra experiments and made some comparisons in the resulting images. I also put a lot of effort in creating this presentation and cited a lot of sources for additional information in the report.

p.s. pahingi po plus points hehe