AMATH 481 / 581 Fall 2025
Homework 1 - Initial Value Problems

Submission open until 11:59:59pm Thursday October 16, 2025

1. Consider the ODE

$$\frac{dy(t)}{dt} = -3y(t)\sin t, \quad y(t=0) = \frac{\pi}{\sqrt{2}},$$

which has the exact solution $y(t) = \pi e^{3(\cos t - 1)}/\sqrt{2}$ (you can verify that). Implement the methods forward Euler and Heun's for this ODE to test the error as a function of Δt . In particular:

(a) Solve the ODE numerically using the forward Euler method:

$$y(t_{n+1}) = y(t_n) + \Delta t f(t_n, y(t_n))$$

with $t = \texttt{arange}(0, 5+\Delta t, \Delta t)$, where $\Delta t = 2^{-2},\ 2^{-3},\ 2^{-4}, \ldots, 2^{-8}$. For each of these Δt values calculate the error $E = \texttt{mean}(\texttt{abs}(y_{true} - y_{num}))$ of the numerical method. Plot $\log(\Delta t)$ on the x axis and $\log(E)$ on the y axis. Using `polyfit`, find the slope of the best fit line through this data. This is the order of the forward Euler method.

ANSWER: Save your last numerical solution ($\Delta t = 2^{-8}$) as a column vector in A1. Save the error values in a row vector with seven components in A2. Save the slope of the line in A3.

(b) Solve the ODE numerically using Heun's method:

$$y(t_{n+1}) = y(t_n) + \frac{\Delta t}{2}[f(t_n, y(t_n)) + f(t_n + \Delta t, y(t_n) + \Delta t f(t_n, y(t_n)))]$$

with $t = \texttt{arange}(0, 5+\Delta t, \Delta t)$, where $\Delta t = 2^{-2},\ 2^{-3},\ 2^{-4}, \ldots, 2^{-8}$. For each of these Δt values, calculate the error $E = \texttt{mean}(abs(y_{true} - y_{num}))$ of the numerical method. Plot $\log(\Delta t)$ on the x axis and $\log(E)$ on the y axis. Using `polyfit`, find the slope of the best fit line through this data. This is the order of the Heun's method.
ANSWER: Save your last numerical solution ($\Delta t = 2^{-8}$) as a column vector in A4. Save the error values in a row vector with seven components in A5. Save the slope of the line in A6.

2. Consider the van der Pol oscillator

$$\frac{d^2y(t)}{dt^2} + \epsilon[y^2(t) - 1]\frac{dy(t)}{dt} + y(t) = 0$$

with ϵ being a parameter.
(a) With $\epsilon = 0.1$, solve the equation for $t = \texttt{arange}(0, 32.5, 0.5)$ using `solve_ivp` from `scipy.integrate`. The initial conditions are $y(t=0) = \sqrt{3}$ and $dy(t=0)/dt = 1$. Repeat this for $\epsilon = 1$ and $\epsilon = 20$.
ANSWER: Save the solutions $y(t)$ for different ϵ as a matrix of 3 columns in A7.

(b) Using the time span $t = [0, 32]$ (the step size for displaying the result is not specified), solve the van der Pol's equation with `solve_ivp`. Use $\epsilon = 1$ and the initial conditions $y(t=0) = 2$ and

$dy(t = 0)/dt = \pi^2$.
Below is an example on how to control the error tolerance `TOL` in `solve_ivp`:

```
TOL = 1e-4
sol = solve_ivp(rhs, tspan, y0, atol=TOL, rtol=TOL)
T = sol.t
Y = sol.y
```

Using the `diff` and `mean` commands on the vector `T` shown above, calculate the average step-size t needed to solve the problem for each of the following tolerance values: $10^{-4}, 10^{-5}, \ldots, 10^{-10}$. Plot $\log(\Delta t)$ on the x axis and log(`TOL`) on the y axis. Using `polyfit`, find the slope of the best fit line through this data. This is the order of the local truncation error of `solve_ivp`. Repeat this with `solve_ivp(..., method="RK23")` and `solve_ivp(..., method="LSODA")`.
ANSWER: The slopes should be saved in variables A8 - A10 for `solve_ivp`, `solve_ivp(..., method="RK23")`, and `solve_ivp(..., method="LSODA")` respectively.

3. To explore interaction between neurons, implement two Fitzhugh neurons coupled via linear coupling:

$$\begin{aligned}
\frac{dv_1}{dt} &= -v_1^3 + (1 + a_1)v_1^2 - a_1 v_1 - w_1 + I + d_{12} v_2 \\
\frac{dw_1}{dt} &= b v_1 - c w_1 \\
\frac{dv_2}{dt} &= -v_2^3 + (1 + a_2)v_2^2 - a_2 v_2 - w_2 + I + d_{21} v_1 \\
\frac{dw_2}{dt} &= b v_2 - c w_2
\end{aligned}$$

with parameters $a_1 = 0.05, a_2 = 0.25$, $b = c = 0.01$ and, $I = 0.1$. Start the simulations with the initial condition of $(v_1(0), v_2(0)) = (0.1, 0.1)$ and $(w_1(0), w_2(0)) = (0, 0)$ and use the `solve_ivp(..., method="BDF")` solver. Set the interaction parameters such that d_{12} is negative and d_{21} is positive. What do you observe from the different graphical representations of the solutions?

ANSWERS: Set the interaction parameters to 5 different values
(d_{12}, d_{21}): (0,0), (0,0.2), (-0.1,0.2), (-0.3,0.2), (-0.5,0.2). For each interaction value solve the system for $t =$ `arange`$(0, 100.5, .5)$ and save the computed solution, (v_1, v_2, w_1, w_2), in a 201×4 matrix. Write out the 5 different solutions, each of which corresponds to an interaction parameter, in A11 - A15.