AMATH 482/582: HOMEWORK 1

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Abstract. Underwater movements of a submarine can be tracked if it emits a unique frequency that influences the submarine's surrounding acoustic pressure. Acoustic pressure data collected over 24 hours in the Puget Sound was analyzed using the Fast Fourier Transform to identify the most dominant frequency present. A Gaussian filter was applied to eliminate noise surrounding this center frequency and then the data was inverse transformed back into the spatial domain to plot the location of the center frequency's source. This process can be incredibly useful in identifying key features of a system and its interactions with its surroundings.

1. Introduction and Overview

Given 3-dimensional acoustic pressure data from a region of the Puget Sound over a 24-hour period, we intend to track the movements of an unknown submarine. We have no information about the submarine's component equipment and its cumulative acoustic signature. However, the submarine's signature frequency will influence the acoustic pressure underwater and will naturally propagate away from the submarine with decreasing amplitude. Therefore, identifying the location of the highest amplitude of the signature frequency will inherently locate the precise location of the submarine.

We intend to use the Fast Fourier Transform (FFT) to identify the submarine's signature frequency. The FFT relates frequencies to their prominence (amplitude) in a N-dimensional space at a point in time. We can apply a Gaussian filter to remove unwanted frequencies (i.e., noise) from the FFT to produce a clean signal which only communicates information from the signature frequency. This clean signal can be inverse transformed to reveal the location of the submarine in the spatial domain.

2. Theoretical Background

The Fourier Transform converts a continuous signal function of time into a function of frequency represented by a sum of sine and cosine basis functions[1]. The Fourier Transform can be extended as needed to evaluate a signal in N-dimensional space. The Fourier Transform can also be inverted to return the amplitude as a function of time rather than as a function of frequency using the Inverse Fourier Transform. The below equations are the 1-dimensional forward(1) and inverse(2) Fourier Transforms respectively.

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\mathrm{i}kt} \mathrm{d}t \tag{1}$$

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$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{\mathrm{i}kt} \mathrm{d}k \tag{2}$$

Similarly, the Discrete Fourier Transform (DFT) applies the Fourier Transform to discrete signal date. This is useful because unlike known signal functions that can be evaluated analytically, real-world data always includes a discrete sampling rate. The below equations are the 1-dimensional(3) and 3-dimensional(4) DFT functions respectively.

$$F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) e^{-\frac{\mathrm{i}2\pi kn}{N}} \tag{3}$$

$$F[K] = \frac{1}{N_x N_y N_z} \sum_{n_x=0}^{N_x-1} \sum_{n_y=0}^{N_y-1} \sum_{n_z=0}^{N_z-1} f(x_n, y_n, z_n) e^{-(\frac{\mathrm{i}2\pi kn_x}{N_x})-(\frac{\mathrm{i}2\pi kn_y}{N_y})} - (\frac{\mathrm{i}2\pi kn_z}{N_z}) \tag{4}$$

The FFT is the algorithm invented by John W. Tukey and James W. Cooley in the 1960s to quickly calculate the DFT. This is done by separating even signals from odd signals to reduce the total number of calculations. The FFT costs computations in the order of $Nlog(N)$, whereas the DFT costs significantly more computations in the order of N^2. Similarly to the Fourier Transform, the FFT can be inverted. After using the FFT, we reorder the even and odd signals by using the FFT shift function. FFT shift improves visualization by centering the FFT on the zero frequency.

The Gaussian filter is a simple function applied to data in the fourier domain to remove noise. It does this by applying a bell curve centered on the signal of interest k_0 and eliminating all frequencies outside of a standard deviation denoted by σ. To apply the Gaussian filter, the equation is solved for every value of k in the fourier domain mesh grid. Then the resulting array is multiplied element wise with the unfiltered FFT data. The filtered data is unshifted with the IFFT shift and then inverse transformed with IFFT. This process is conducted at every time point independently. The below equations are the 1-dimensional(5) and 3-dimensional(6) Gaussian filters respectively.

$$g(kx) = e^{\frac{(kx-kx_0)^2}{2\sigma^2}} \tag{5}$$

$$g(kx, ky, kz) = e^{\frac{(kx-kx_0)^2+(ky-ky_0)^2+(kz-kz_0)^2}{2\sigma^2}} \tag{6}$$

3. Algorithm Implementation and Development

The data was provided as a 262,144x49 array where each column was a flattened 64x64x64 vector representing 3-dimensional data at a single time point. The data was neither normalized nor did we receive information on the sensing equipments observable range. The code was written in Python using the pathlib, Numpy, Matplotlib, and Scipy libraries. The below pseudocode summarizes the general process of identifying the signature frequency, filtering the data using the Gaussian filter centered on the signature frequency, and locating the submarine by inverse transforming the filtered data at each time point.

Pseudocode

(1) Find the signature frequency:
  - Reshape the flattened data for each time point into a 64x64x64 array
  - Apply the N-dimensional FFT and FFT shift

- Sum the FFT arrays and find the time-averaged FFT
- Identify the most prominant frequencies in all 3 dimensions

(2) Create the Gaussian Filter using the signature frequencies
(3) Identify the location of the submarine:
- Reshape the flattened data for each time point into a 64x64x64 array
- Apply the N-dimensional FFT and FFT shift
- Multiply the resultant array by the Gaussian filter array
- Apply the IFFT shift and the N-dimensional IFFT
- Identify the spatial position of the most prominant data point

4. Computational Results

Using the unfiltered, time-averaged data from the FFT, the signature frequency was detected Table 1. The unfiltered data is noisy as shown below in the kz=0 cross sections in Figure 1. Figure 1 also includes the same cross section at kz=0 when the Gaussian filter is applied using $\sigma = 1$, 2, and 3. Because the higher values of sigma obscure the central frequency, we used sigma = 1 for the Gaussian Filter when plotting the submarine's path.

Coordinate plane	Signature frequency
kx	-5.341
ky	-2.199
kz	6.912

Table 1. Signature frequencies emitted by the submarine in the 3D frequency domain.

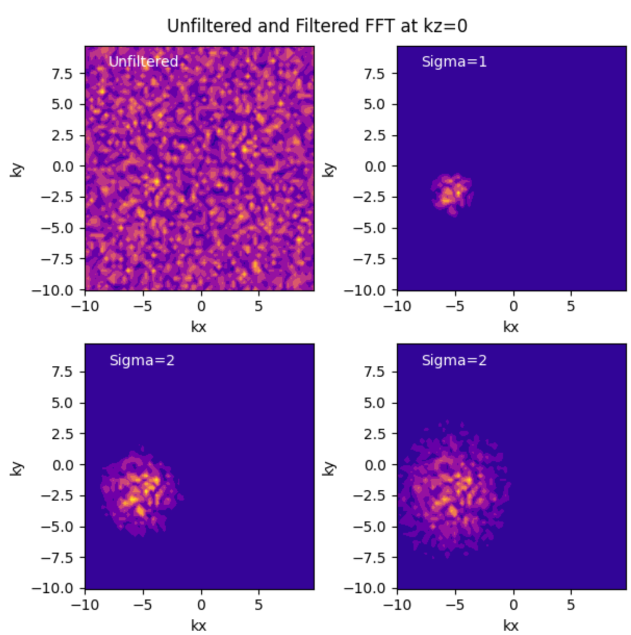

Figure 1. Graphs of the time-averaged amplitude of FFT at kz=0. The top left graph is unfiltered. The unfiltered graph has no clearly discernable peaks and clearly has noise. The remaining graphs apply Gaussian filters centered at the signature frequency using $\sigma = 1$, 2, and 3.

Using the Inverse FFT on the filtered data at the first and last time point, we returned the following coordinate tuples.

Coordinate plane	Start point	End point
x	2.92	-5.08
y	-7.85	6.00
z	-0.15	0.77

TABLE 2. Submarine start and end points

The 3-dimensional path of the submarine during the 24-hour period is shown below in Figure 2. The path using the noisy (unfiltered) data is red; and the filtered path is green. There are two clearly visible outliers effectively removed by the Gaussian filter, showing its effectiveness at removing unwanted noise from the data.The x,y coordinates of the submarine are shown below in Figure 3. This 2-dimensional path provides aircraft the ability to overfly and potentially observe the submarine.

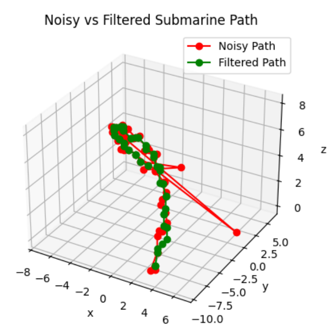

FIGURE 2. 3D submarine path using unfiltered (red) and filtered (green) data.

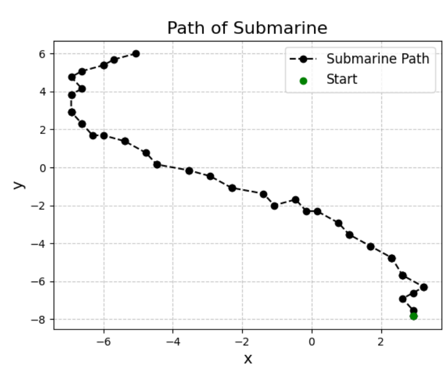

FIGURE 3. 2D submarine path in the xy-plane showing the overhead view.

5. Summary and Conclusions

Given noisy 3-dimensional acoustic pressure data, the path of the submarine and the submarine's signature frequencies were clearly determined using the Fast Fourier Transform and the Gaussian filter. The method eliminated clear outliers showing the effectiveness of the method.

Future improvements to the method should be focused on using knowledge about the sensing equipment to remove bad data that is outside the sensitivity of the equipment and normalizing the data using baseline acoustic pressure information. Additionally, the signature frequency should converted to Hertz adn be compared to known submarine signature frequencies to inform data filtering methods. Lastly, a more robust filter should be added to separate direction frequencies (e.g., tail fins) from omnidirectional frequencies (e.g., communication equipment). This could improve the predictive nature of submarine movement detection.

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References

[1] J. Kutz. *Data-Driven Modeling & Scientific Computation: Methods for Integrating Dynamics of Complex Systems and Big Data*. OUP Oxford, 2013.