Homework 3

AMATH 582/482, Winter 2026

Due on Feb 16, 2026 at midnight.

Directions, Reminders and Policies

Read these instructions carefully:

- **You are required to upload a PDF report to Gradescope along with your code in the form of a Jupyter notebook or a zip file of your main scripts and functions. The report and the code are to be submitted under separate assignments. DO NOT INCLUDE YOUR CODE IN THE REPORT.**
- The report should be a maximum of 10 pages in accordance with the template on Canvas including references. Minimum font size 11pts and margins of at least 1inch on A4 or standard letter size paper. The report should be formatted as follows:
  - Title/author/abstract: Title, author/address lines, and short (100 words or less) abstract. This is not meant to be a separate title page.
  - Sec. 1. Introduction and Overview
  - Sec. 2. Theoretical Background
  - Sec. 3. Algorithm Implementation and Development
  - Sec. 4. Computational Results
  - Sec. 5. Summary and Conclusions
  - Acknowledgments. Mention other students if you were part of a study group.
  - References
- **Recall our late policy: If you plan to use your tokens please send an email to Saba (`heravi@uw.edu`). If you do not use a token you will lose 1/3 of your grade for each day the submission is late. WE WILL NOT AUTOMATICALLY APPLY TOKENS. YOU MUST INFORM US IF YOU PLAN TO USE THEM.**

Problem Description: Video Background Subtraction using Randomized PCA

Your goal in this homework is to design an algorithm for video background subtraction using randomized PCA. Background subtraction is a common task in computer vision where the goal is to separate the static background from moving objects (foreground) in a video. Since video files are often large, using standard PCA is computationally expensive and so randomized PCA techniques can be used to reduce the computational burden. This homework has two parts: (1) Implementing and validating randomized PCA; and (2) Applying it to video background subtraction. For task (2) you will need to townload the video file `video.avi` from Canvas alongside `HW3_Helper.ipynb` which helps you load the video file and do some basic manipulations.

Tasks

Below is a list of tasks to complete in this homework and discuss in your report.

1. (Randomized PCA implementation and validation) Your goal in this task is to implment and validate the randomized PCA algorithm which replaces the full SVD used in PCA with a randomized SVD approximation based on the randomized range finding method outlined in (Martinsson and Tropp, "Randomized numerical linear algebra: Foundations and algorithms", Acta Numerica, 2020). The randomized range finder algorithm is summarized as follows:

Algorithm 1 Randomized Range Finder

Input: Data matrix $X \in \mathbb{R}^{m \times n}$, target rank k oversampling parameter p.
Output: Orthonormal matrix $Q \in \mathbb{R}^{m \times (k+p)}$

1: Draw a Gaussian random matrix $W \in \mathbb{R}^{n \times (k+p)}$.
2: Form the sample matrix $Y = XW \in \mathbb{R}^{m \times (k+p)}$.
3: Compute the QR factorization $Y = QR$.

The orthonormal matrix Q gives a randomized, low-rank approximation to the range of X. Note that this computation is cheap whenever $k + p \ll \min(m, n)$. Given the matrix Q we can now compute an approximate SVD of X as follows:

Algorithm 2 Randomized SVD

Input: Data matrix $X \in \mathbb{R}^{m \times n}$ and matrix $Q \in \mathbb{R}^{m \times (k+p)}$ from the randomized range finder.
Output: Approximate rank-k SVD of X: U_k, Σ_k, V_k.

1: Form the matrix $B = Q^T X \in \mathbb{R}^{(k+p) \times n}$.
2: Compute the SVD of the small matrix $B = \widehat{U}\widehat{\Sigma}\widehat{V}^T$.
3: Form the approximate left singular vectors of X as $U = Q\widehat{U}$.
4: Truncate $U, \widehat{\Sigma}, \widehat{V}$ to retain only the top k components.

Hint: the matrix U_k output by the above algorithm is an approximate basis for the the column space of X which you can use to perform PCA. Applying the algorithm to X^T will give you an approximate basis for the row space of X. In fact, if both m, n are large you can use the above algorithm twice: once on X and once on X^T to get approximate bases for both the column and row spaces of X.

(1.1) Implement the above randomized SVD algorithm in Python.

(1.2) Validate your implementation on the synthetic data set provided in `HW3_Helper.ipynb` by comparing the results of your randomized SVD with the full SVD with $k = 5, 10, 20$ and $p = 0, 5, 10$.

(1.3) Report the approximation errors $\|X - U_k \Sigma_k V_k^T\|_F$ for different values of k and p in a table or a figure. Since the algorithm is randomized, run it 10 times for each pair of (k, p) and report the mean and standard deviation of the errors. Discuss your results.

2. (Video background subtraction) Your goal in this task is to use PCA to remove the background in the video file `video.avi`. You will implement a method using both full PCA and randomized PCA and compare their performance empirically. In the report you may present a few sample frames of the original video alongside the background-subtracted frames obtained by both methods. Discuss your results and compare the computational cost of both methods.

   Hint: the video file can be large so first import a few frames and rescale them to low-resolution to benchmark your code. Once you have a working implementation you can try it on a larger protion of the video. You can also apply the method to a video of your own since your phone can produce a longer video at higher resolution to further test your implementation.